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EXISTENCE RESULTS AND OPTIMIZATION OVER THE SET OF EFFICIENT SOLUTIONS IN VECTOR-VALUED APPROXIMATION THEORY

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Abstract. Continuity of the objective functions and compactness of their domain are classical assumptions widely used to obtain existence results for solutions of optimization problems. Due to the lack of compactness in general spaces, under some moderate assumptions concerning the objective function and the feasible set, we derive existence results for vector-valued optimization problems and corresponding results for associated scalarized problems in this paper. Furthermore, we apply our results to special vector-valued approximation problems, especially to multi-objective location problems where the whole set of efficient solutions can be generated by a geometric primal-dual algorithm. Moreover, by using the nonlinear scalarizing functional introduced by Gerstewitz, we perform an optimization according to the preferences of a decision maker on the generated set of efficient solutions from which we derive a single solution of this set that corresponds to the preferences of the decision maker.
Keywords. Decision process; Multi-objective optimization; Optimization over the set of efficient solutions; Regularization problems; Vector-valued approximation problems.

1. Introduction and Motivation

In industry and finance, a variety of applications occur, where several objective functions are desired to be optimized concurrently. For instance, for an exquisite economical production plan, the ultimate common goal would be to maximize quality while minimizing the cost of the product at the same time. This example already demonstrates that the several objectives typically contradict each other, and therefore do not have identical optima (since a high-quality product does not have the lowest price). Thus, a set of optimal compromises between the objectives has to be determined. This set is called the Pareto optimal set or simply the Pareto set, named after the Italian engineer, sociologist, and economist Vilfredo Pareto (1848 - 1923).

Basically, multi–objective optimization can be understood as the process of determining the set of optimal compromises between two or more conflicting objectives. From the preceding scenario, it is clear that the solution to the multi–objective optimization problem is not given by a single point but a whole set of optimal compromises (efficient solutions) which may be infinite, unbounded, and can in general be a nonconvex even in the linear case. When this set is

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generated, a decision problem arises for the decision maker to select a single preferred solution from it. An important and efficient approach to address this is to perform optimization over the set of efficient solutions. This particular problem has been given attention in [1, 2, 3, 4, 5, 6] and the references therein.

It is important to point out that in the theory of multi–objective optimization problems, assumptions and conditions guaranteeing the existence of efficient solutions are of paramount importance because they are not always assured in applications. The study of the existence of solutions for multi–objective optimization problems has been attracting lots of attention in the literature. However, a bulk of available literature has only been devoted to deriving results under some restricted assumptions concerning the feasible set of solutions, objective functions, and ordering cones, so also the spaces under consideration; see [4, 6, 7, 8, 9, 10, 11, 12].

The main contributions of this paper are as follows: Under some suitable moderate assumptions, we derive the existence result for vector–optimization problems in a case where the preimage space E is a reflexive Banach space and the image space is q–dimensional Euclidean space $\mathbb{R}^q$. Moreover, we also obtain the existence result for the associated scalarized problem under additional mild assumptions. Additionally, we apply our new existence results in vector–valued approximation problems where as a special case, the whole set of efficient solutions of the multi–objective location problem is derived. Furthermore, for application of the location problem in town planning, by using the so-called nonlinear scalarization functional introduced by Gerstewitz [13], a single preferred solution from the set of efficient solutions which corresponds to the preferences of the decision maker is also generated.

We organize the subsequent presentation of this paper as follows: Some important concepts to be used throughout the forthcoming sections are recalled in Section 2. In Section 3, we present formulation and solution concepts for the vector–valued optimization problem. The existence results for the vector–valued optimization problems and associated scalarized problems are presented in Section 4. In Section 5, we perform optimization over the set of efficient elements of a multi–objective location problem. By applying the characterization of the nonlinear scalarizing functional introduced by Gerstewitz [13] in [14], a single solution from the set of efficient elements which corresponds to the preferences of the decision maker is generated. Finally, the paper concludes in Section 6 with some remarks and future research goals.

2. Notations and Preliminaries

In this section, we recall some notations, definitions, and useful results that are relevant throughout the subsequent sections.

For any non–negative integer q, let the q–dimensional Euclidean space $\mathbb{R}^q$ and a subset D of $\mathbb{R}^q$ be given. We denote the interior, closure, and complement of D by $\operatorname{int} D$, $\operatorname{cl} D$, and D^c, respectively. Also, for a given set of real numbers $\mathbb{R}$, we denote by $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ the extended set of real numbers. Let a function $F : D \to \overline{\mathbb{R}}$, where a set $D \subset \mathbb{R}^q$ is nonempty, be given. The function F is said to be proper if the domain of F, denoted by $\operatorname{dom} F = F^{-1}(\mathbb{R}) := \{x \in D \mid F(x) \in \mathbb{R}\}$, is not empty. We begin to recall some notations concerning cones in Definition 2.1 whose general form for arbitrary topological spaces can be seen in [15].

Definition 2.1. (See, e.g., [15]) Let $\mathbb{R}^q$ be a q–dimensional Euclidean space. A nonempty subset K of $\mathbb{R}^q$ is said to be

(i) *A cone* if for all $y \in K$ and $\lambda \geq 0$, $\lambda y \in K$.

(ii) *A convex cone* if $K+K \subseteq K$.
(iii) *Pointed* if $K \cap (-K) = \{0_{\mathbb{R}^q}\}$.
(iv) *Proper* if $K \neq \mathbb{R}^q$,
(v) *A nontrivial cone* if it is proper and $K \neq \{0\}$.

The next example illustrates a cone with some of the properties in Definition 2.1.

Example 2.1. Consider the Euclidean space $\mathbb{R}^q$, the non–negative orthant

$$\mathbb{R}^q_+ := \{(y_1, \dots, y_q) \in \mathbb{R}^q \mid \forall i \in \{1, \dots, q\} : y_i \geq 0\}$$

consisting of all vectors of $\mathbb{R}^q$ with non–negative coordinates is closed, convex and pointed.

Throughout this paper, unless otherwise specified, the space E is supposed to be a reflexive Banach space and the cone $K \subset \mathbb{R}^q$ is supposed to be nontrivial, closed, and convex.

Given a convex cone $K \subset \mathbb{R}^q$, for $y_1, y_2 \in \mathbb{R}^q$, we define the order "$\geq_K$" by $y_1 \geq_K y_2$ if and only if $y_1 - y_2 \in K$, and "$>_K$" by $y_1 >_K y_2$ whenever $y_1 \geq_K y_2$ and $y_1 \neq y_2$. The relation "$\geq_K$" is reflexive, and since K is convex is also transitive, but not necessarily antisymmetric. However, if K is pointed then "$\geq_K$" is antisymmetric.

The following concepts of convexity of functions, whose details can be found in [16], will be used in the sequel.

Definition 2.2. Let D be a convex subset of $\mathbb{R}^q$. A function $F : D \to \overline{\mathbb{R}}$ is said to be convex if for all $x, y \in D$

$$F(\alpha x + (1-\alpha)y) \leq \alpha F(x) + (1-\alpha)F(y) \text{ for all } \alpha \in [0,1].$$

Definition 2.3. Let D be a nonempty subset of $\mathbb{R}^q$ and a given function $F : D \to \overline{\mathbb{R}}$. The set denoted by

$$\operatorname{epi} F := \left\{(x,t) \in D \times \mathbb{R} \mid F(x) \leq t\right\},$$

is called epigraph of F.

A convex function can be characterized by its epigraph. Indeed, the following lemma, whose general details can be found in [17], is evident.

Lemma 2.1. *([17, Theorem 2.6]) Suppose that D is a nonempty subset of $\mathbb{R}^q$. The function $F : D \to \overline{\mathbb{R}}$ is convex if and only if* $\operatorname{epi} F$ *is a convex set.*

One very important example of an extended real–valued function is the indicator function defined in the next definition. The indicator function is often used as a tool (penalization function) for transforming constrained optimization problems into unconstrained problems.

Definition 2.4. Let D be a nonempty set and $\Omega \subseteq D$. *The indicator function* δ_Ω of Ω is a mapping $\delta_\Omega : D \to \overline{\mathbb{R}}$ defined by

$$\delta_\Omega(x) = \begin{cases} 0 & \text{for } x \in \Omega, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

Remark 2.1. It is clear that the indicator function is proper and convex if the set Ω is nonempty and convex.

The notion of sublevel sets of extended real–valued functions given in Definition 2.5 will play an important role throughout the subsequent sections.

Definition 2.5. Let $F : D \to \overline{\mathbb{R}}$ be a function defined over a set D subset of $\mathbb{R}^q$. The lower sublevel set of F with level $t \in \mathbb{R}$ is given by

$$L(F;t) := \{y \in D \mid F(y) \leq t\}. \tag{2.2}$$

In the next definition, whose general form can be seen in [18, 19, 20], the lower semicontinuity of extended real–valued functions is characterized in terms of lower sublevel sets.

Definition 2.6. ([18, 19], and [20, Definition 38.5]) Let D be a closed subset of $\mathbb{R}^q$. The function $F : D \subset \mathbb{R}^q \to \overline{\mathbb{R}}$ is *lower semicontinuous* if for any $t \in \mathbb{R}$ the lower sublevel set in (2.2) is closed.

Remark 2.2. The indicator function in (2.1) is lower semicontinuous if the set Ω is closed.

In Definition 2.7 we recall the notion of K–semicontinuous vector–valued functions which shall be used in the sequel.

Definition 2.7. (See, e.g., [18, 19]) Let E be a reflexive Banach space and $\mathbb{R}^q$ ordered by a nontrivial pointed, closed, and convex cone K. The function $G : E \to \mathbb{R}^q$ is said to be K–*semicontinuous* if for every $y \in \mathbb{R}^q$ the set

$$G^{-1}(\operatorname{cl} K + y) := \{x \in E \mid G(x) - y \in \operatorname{cl} K\},$$

is closed in E.

Lemma 2.2, whose more details can be seen in [21, Lemma 2.17]), establishes K–semicontinuity as a proper generalization of lower semicontinuity of real–valued functions.

Lemma 2.2. *(*[21, Lemma 2.17]*) Suppose that E is a reflexive Banach space. A vector–valued function $G : E \to \mathbb{R}^q$ is K–semicontinuous if and only if for all $i = 1, 2, \ldots, q$, the component functions $G_i : E \to \mathbb{R}$ are lower semicontinuous.*

Remark 2.3. (i) If in Definition 2.7 the space $\mathbb{R}^q$ is considered for $q = 1$, and E the set of real numbers $\mathbb{R}$, then K–semicontinuity of G reduces to classical lower and upper semicontinuity for $K = (-\infty, 0]$ and $K = [0, +\infty)$, respectively. Indeed, the function G is lower semicontinuous if for every $y \in \mathbb{R}$ the set

$$\begin{aligned} G^{-1}\big([-\infty, 0] + y\big) &= G^{-1}\big([-\infty, y]\big) \\ &= \{x \in \mathbb{R} \mid G(x) \in [-\infty, y]\} \\ &= \{x \in \mathbb{R} \mid G(x) \leq y\} \end{aligned}$$

is closed. Similarly, G is upper semicontinuous if for every $y \in \mathbb{R}$ the set

$$G^{-1}\big([0, +\infty] + y\big) = \{x \in \mathbb{R} \mid y \leq G(x)\}$$

is closed.

(ii) Evidently, continuous functions are K–semicontinuous, but the converse does not hold in general, since the function $G : \mathbb{R} \to \mathbb{R}$ defined by $G(x) = [x]$, where $[\cdot]$ denotes the greatest integer function, is $[0, +\infty)$–semicontinuous but not continuous.

In the next proposition which is a special case of [18, Theorem 4.1], [21, Proposition 2.18], we present a characterization of K–semicontinuous vector–valued functions in terms of K–semicompactness of subsets of $\mathbb{R}^q$. The concept of K–semicompactness which is considered a weaker form of compactness is given Definition 2.12.

Proposition 2.1. *(*[18, Theorem 4.1]*,* [21, Proposition 2.18]*) Suppose that E is a reflexive Banach space and S is a compact subset of E. If a vector–valued function $G : S \to \mathbb{R}^q$ is K–semicontinuous then $G(S)$ is K–semicompact.*

Remark 2.4. We are going to apply the hypothesis of Proposition 2.1 in a case where the set S is endowed with a weak topology.

The following concept of *lower semicompactness* of extended real–valued functions will play an important role in the subsequent sections.

Definition 2.8. (See, e.g., [20, Definition 38.5]) Let D be a nonempty subset of $\mathbb{R}^q$. The function $F : D \subset \mathbb{R}^q \to \overline{\mathbb{R}}$ is said to be *lower semicompact (sometimes called inf–compact)* if it has compact lower sublevel sets.

In the following definitions, we state relationships between lower semicompactness and coerciveness of a function.

Definition 2.9. A function $F : \mathbb{R}^q \to \overline{\mathbb{R}}$ is called *coercive* if

$$\lim_{\|x\| \to +\infty} F(x) = +\infty.$$

It can be seen in Proposition 2.2 that real–valued lower semicontinuous and coercive functions are lower semicompact.

Proposition 2.2. *(See, e.g.,* [22, Theorem 1.2]*) Let $F : \mathbb{R}^q \to \overline{\mathbb{R}}$ be lower semicontinuous. The function F is coercive if and only if it has compact lower sublevel sets.*

Suppose that E is a reflexive Banach space. The following proposition, whose details can be found in [23, Corollary 3.22], characterizes the notion of weakly compact subsets of E.

Proposition 2.3. *(*[23, Corollary 3.22]*) Let E be a reflexive Banach space, $S \subset E$ nonempty, closed, convex, and bounded set. Then S is weakly compact.*

Proposition 2.4 below popularly known as Heine-Borel Theorem will be used in the sequel.

Proposition 2.4. *(See, e.g.,* [18]*) A set D subset of $\mathbb{R}^q$ is compact if and only if it is bounded and closed.*

Now, we recall the notions of monotonicity concerning sequences in $\mathbb{R}^q$ and some cone properties of subsets of $\mathbb{R}^q$.

Definition 2.10. (See, e.g., [24]) Suppose that D is a nonempty subset of $\mathbb{R}^q$ and K is a nonempty, convex cone in $\mathbb{R}^q$. Consider $L_K := K \cap -K$, the set of nonnegative integers $\mathbb{N}$, and $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^q$ the sequence in $\mathbb{R}^q$. Then

- (i) $\{y_n\}_{n\in\mathbb{N}}$ is said to be K–increasing if for all $n,m \in \mathbb{N}$, $n \le m$, $y_m - y_n \in K$.
- (ii) $\{y_n\}_{n\in\mathbb{N}}$ is said to be strictly K–increasing if for all $n,m \in \mathbb{N}$, $n \le m$, $n \ne m$, $y_m - y_n \in (K \setminus L_K)$.
- (iii) $\{y_n\}_{n\in\mathbb{N}}$ is $\big($strictly$\big)$ K–decreasing if $\{-y_n\}_{n\in\mathbb{N}}$ is $\big($strictly$\big)$ K–increasing.
- (iv) The set $D \subset \mathbb{R}^q$ is $\big(K$–lower$\big)$ upper bounded if $\big(D \subset y_0 + K\big)$ $D \subset y_0 - K$ for some $y_0 \in \mathbb{R}^q$.

Definition 2.11 below, whose general form is stated in [24], will be used in Proposition 2.13.

Definition 2.11. (See, e.g., [24]) A nonempty convex cone K in $\mathbb{R}^q$ is said to be *Daniell* if any K–decreasing sequence in $\mathbb{R}^q$ having a lower bound converges to its infimum.

In Remark 2.5 below which is motivated from [25, Example 3.3 (c)] we state some conditions that ensure Daniell property.

Remark 2.5. Every closed and pointed ordering cone K in the finite–dimensional space $\mathbb{R}^q$ is Daniell.

Example 2.2. For $q = 2$, the ordering cone $K \subset \mathbb{R}^2$ defined by

$$K := \{(y_1, y_2) \in \mathbb{R}^2 \mid \; y_2 \geq 0, \; y_1 \geq 0\},$$

is closed and pointed, therefore, Daniell by Remark 2.5.

In Definition 2.12, the notion of K–semicompactness which is a weaker form of compactness is defined for general topological spaces in [18, 19, 21] using similar finite subcover property of compact sets. K–semicompactness condition, which considers open covers with special sets will be used as a sufficient condition for K–completeness in Proposition 2.5. K–completeness of subsets of $\mathbb{R}^q$ which turns out to be a weaker form of K–semicompactness will be presented in Definition 2.13.

Definition 2.12. (See, e.g., [18, 19, 21]) Let K be a given cone in $\mathbb{R}^q$. A subset D of $\mathbb{R}^q$ is said to be *K–semicompact* if every open cover of D of the form

$$\big\{(y_i - \operatorname{cl} K)^c \mid \; y_i \in D, \; i \in \mathscr{I}, \text{ where } \mathscr{I} \text{ is the index set}\big\},$$

has a finite subcover. This means that whenever $D \subset \cup_{i \in \mathscr{I}} (y_i - \operatorname{cl} K)^c$ there exist $m \in \mathbb{N}$ and $\{i_1, \ldots, i_m\} \subset \mathscr{I}$ such that $D \subset \cup_{k=1}^{m} (y_{i_k} - \operatorname{cl} K)^c$.

Remark 2.6. Compact sets are K–semicompact for any ordering cone K. However, the converse does not hold in general, since for $q = 2$, the set

$$D := \big\{\, (x, y) \in \mathbb{R}^2 \mid \; 0 \leq x < 1, \; 0 \leq y < 1\big\},$$

with respect to $K = \mathbb{R}^2_+$ is K–semicompact but not compact.

The next definition, whose general form can be seen in [11], will be used in Proposition 2.6 to guarantee the nonemptiness of a set D subset of $\mathbb{R}^q$ with respect to a closed and convex cone K.

Definition 2.13. (See, e.g., [11]) Let K be a given cone in $\mathbb{R}^q$. A subset D of $\mathbb{R}^q$ is said to be *K–complete* if every Cauchy K–increasing sequence is convergent to an element of K.

Remark 2.7. In the sense of [26, Lemma 2.2], K–semicompact sets are K–complete. However, the converse fails in general, since for $q = 2$, the set D defined by

$$D := \left\{(x, y) \in \mathbb{R}^2 \mid \; x = \frac{1}{y}, \; 0 > y\right\} \cup \left\{(x, y) \in \mathbb{R}^2 \mid \; x = -y, \; 0 \geq y\right\},$$

where cone $K = \mathbb{R}^2_+$, is K–complete but not K–semicompact.

Sufficient conditions for K–completeness of a subset D of $\mathbb{R}^q$ are stated in Proposition 2.5 whose general form for arbitrary topological spaces is to find in [26].

Proposition 2.5. ([26, Lemma 2.2]) *Suppose that K is a closed and convex cone in $\mathbb{R}^q$. A set D subset of $\mathbb{R}^q$ is K–complete if one of the following assertions holds:*

(i) *D is K–semicompact.*
(ii) *D is compact.*
(iii) *D is closed and bounded, K is Daniell, and $\mathbb{R}^q$ has a monotone sequence property i.e., every bounded decreasing sequence in $\mathbb{R}^q$ has an infimum.*
(iv) *D is K–lower bounded and K is Daniell.*

Proposition 2.6, whose general form is in [26], generalizes many existence results for vector optimization problems, especially results where norm compactness and continuity assumptions are used, see, e.g., [19, 27, 28]. Proposition 2.6 which is a general form of existence results in [4, 6, 8] will be used in Section 4 to derive the existence result for multi–objective optimization problem (MOP).

Proposition 2.6. *(*[26, Theorem 2.6]*) Suppose that $\mathbb{R}^q$ is ordered by a closed and convex cone K. For a given nonempty subset D of $\mathbb{R}^q$, the set of efficient elements of D with respect to K is nonempty if D is K–complete.*

3. Vector-Valued Optimization Problems and Solution Concepts

Most of the optimization problems concerned with a real–life require the need to optimize several conflicting objective functions simultaneously under constraints, which are popularly known as *vector optimization problems*. In vector optimization, one considers optimization problems with a vector–valued objective map and thus, one has to compare elements in a linear space. If the linear space is finite–dimensional space $\mathbb{R}^q$, the comparison can be done component–wise. In case the linear space is arbitrary infinite–dimensional, a partial ordering which defines how elements are compared is introduced.

In this section, we study vector optimization problem (MOP) in a case where the preimage space E is a reflexive Banach space and the image space is $\mathbb{R}^q$.

Suppose that $G : E \to \mathbb{R}^q$ is a vector–valued function and K be a pointed, closed, convex cone in $\mathbb{R}^q$. Multi–objective optimization problem which is a problem of optimizing a vector–valued function G over the set of feasible solutions $S \subseteq E$ is formulated as follows:

$$\mathrm{Opt}_{x\in S}\, G(x), \tag{MOP}$$

where $G := \left(G_1,\ldots,G_q\right)^T$ is the vector of the objectives and the component functions $G_i : E \to \mathbb{R}$, $i = 1,2,\ldots,q$ are the conflicting objective functions to be optimized simultaneously.

It is important to explain that the notation 'Opt' appearing in the formulation of (MOP) signifies that, the vector–valued function G is to be minimized or maximized in the sense of Definition 3.1, see Remark 3.1. Of course, a minimization problem can be transformed into maximization problem when G and the ordering cone K are considered as $-G$ and $-K$, respectively.

Due to the loss of a total order of $\mathbb{R}^q$ for $q \geq 2$, the solution to the (MOP) is not a specific point, but the *set of non–dominated points*, also referred to as the *Pareto set*, or *set of optimal compromises*. Consider Problem (MOP), a point $\hat{x} \in S \subseteq E$ is said to be *non–dominated if there exists no $x \in S$ with $G_i(x) \leq G_i(\hat{x})$ for all $i = 1,2,\ldots,q$ and $G_j(x) < G_j(\hat{x})$ for at least one $j \in \{1,2,\ldots,q\}$.*

If $\hat{x}$ satisfies this property only in an open neighborhood $U(\hat{x}) \subset S \subseteq E$, then $\hat{x}$ is called locally non–dominated point (or a local Pareto optimal solution).

Definition 3.1 and Remark 3.1 formalize these solution concepts.

Definition 3.1. Let D be a nonempty subset of $\mathbb{R}^q$ and K a nontrivial pointed, convex cone in $\mathbb{R}^q$. An element of the set

$$\begin{aligned}\mathrm{Eff}(D;K) &:= \big\{y\in D \mid (\{y\}-K)\cap D=\{y\}\big\}\\ &= \big\{y\in D \mid (\{y\}-K)\cap(D\setminus\{y\})=\emptyset\big\},\end{aligned}$$

is called *efficient element of D with respect to K*. Moreover, suppose that $\operatorname{int}K\neq\emptyset$. An element $y\in D$ is called a *weakly efficient element* of D, if y belongs to the set

$$\mathrm{Eff_w}(D;K) := \big\{y\in D \mid (\{y\}-\operatorname{int}K)\cap D=\emptyset\big\}.$$

Remark 3.1. For a given nonempty subset S of a linear space E, we apply the solution concepts in Definition 3.1 to Problem (MOP) for $D := G(S)$, where $G(S) := \{y\in Y \mid G(x)=y,\ \text{for some } x\in S\}$ is called the image of S under G. In this paper, we call elements $x\in S$ with $G(x)\in \mathrm{Eff}(G(S);K)$ efficient solutions.

For properly efficient elements and more details about the solution concepts for vector optimization problems, see, [15, 17, 24, 29] and references therein.

Remark 3.2. By Definition 3.1, it is straightforward to see that it holds that

$$\mathrm{Eff}(D;K)\subseteq \mathrm{Eff_w}(D;K).$$

The condition for a set to have a domination property is recalled in Definition 3.2.

Definition 3.2. (See, e.g., [15, 24]) Let $D\subset\mathbb{R}^q$ be a given set. The *domination property* is said to be fulfilled for D, if for every point $y\in D$ there is some $y^*\in \mathrm{Eff}(D;K)$ such that $y^*\leq_K y$ (i.e., every element of D is dominated by a minimal element of D).

4. Existence Results for (MOP) in Reflexive Banach Preimage Spaces and Associated Scalarized Problem

Existence results for (MOP) have been studied in the literature, see, [4, 6, 8, 29] under the assumptions that the preimage space E is $\mathbb{R}^n$ with a special structure of the set of feasible elements S $:= \big\{x\in\mathbb{R}^q \;\big|\; g_i(x)\leq 0,\ i=1,2,\dots,d\big\}$, where for each $i=1,2,\dots,d$, $g_i:\mathbb{R}^q\to\mathbb{R}$ are convex.

Supposing compactness of the set of feasible elements, finite–valued, positive, and convex objective functions, the authors in [4] considered the existence of solutions of a special case namely, bicriteria problem of (MOP) i.e., for $q=2$. Moreover, authors in [8] also considered the existence result for the bicriteria problem using continuity assumption of the objective functions, convexity of the constraint functions, and compactness of the feasible set. Furthermore, in [30] the existence result for unconstrained (MOP) is considered for strictly convex and coercive objective functions.

Several versions of existence results for vector optimization problems with respect to more general feasible sets, special types of convex cones, and compactness notions with respect to cones have been considered in detail in Chapter 9 of [24].

Under relaxed continuity assumption concerning the objective functions, our aim is to derive existence result for (MOP) using a more general feasible set that does not involve constraint functions and without compactness assumption. Precisely, considering the more general image space $\mathbb{R}^q$ for the objective functions, assuming K–semicontinuity of the objective functions, we obtain the existence result in high dimensional preimage space. More specifically, the following assumptions concerning the preimage space E, set of feasible elements S, and vector–valued objective function G will be used to derive the existence of efficient elements of (MOP).

Assumption 4.1. (C1) E is real reflexive Banach space.
(C2) S is nonempty, closed, convex and bounded subset of E.
(C3) K is a nontrivial pointed, and closed, convex cone in $\mathbb{R}^q$.
(C4) $\forall\, i = 1,2,\dots,q$: the component function G_i is lower semicontinuous.

Theorem 4.1 ensures the nonemptiness of the set of efficient elements of (MOP).

Theorem 4.1. *Suppose that all the conditions of Assumptions 4.1 are satisfied, then the set of efficient elements of* (MOP) *is nonempty.*

Proof. Taking into account conditions $(C1)$, $(C2)$, and Proposition 2.3, the set S is weakly compact in E. Moreover, from Lemma 2.2, condition $(C4)$ implies that G is K–semicontinuous. Therefore, in light of Proposition 2.1 and Remark 2.4 with the set S endowed with weak topology, we have that $G(S)$ is K–semicompact in $\mathbb{R}^q$. Finally, by Proposition 2.5(i) and Proposition 2.6 we have that the set of efficient elements $\mathrm{Eff}(G(S);K)$ of (MOP) is nonempty. □

Remark 4.1. The existence result in Theorem 4.1 is shown under the Assumptions 4.1 where the compactness of the feasible set is not supposed. So, the existence result in Theorem 4.1 is shown under weaker assumptions as in corresponding existence results in the literature (see, e.g., [4, 6, 8, 12, 29]) and the references therein.

For generating an approximation of the set of efficient elements to the vector optimization problem (MOP), a large class of methods is based on scalarizations which generally, means the replacement of a vector optimization problem by a suitable scalar optimization problem involving possibly some parameters or additional constraints. Some examples of such scalarizations are the weighted sum [31] and the ε-constraint problem [32]. It is important to explain that these two examples of scalarization approach are special cases of the nonlinear scalarizing functional presented in (NLSF); see, e.g., [14, 29].

In the following and subsequent sections, we consider the following scalarized formulation for (MOP).

A scalarized problem associated with the (MOP) takes the form:

$$\mathrm{Opt}_{x\in \mathrm{S}}\, F\big((G(x)\big), \tag{MOP$_\mathrm{S}$}$$

where the composition of a scalarization function $F:\mathbb{R}^q\to\overline{\mathbb{R}}$ and the vector–valued objective function $G:E\to\mathbb{R}^q$ is a mapping $F\circ G:E\to\overline{\mathbb{R}}$.

Remark 4.2. A general nonlinear scalarization functional is presented in Example 4.3 while a particular example of scalarization functional is also given in Example 4.1.

In general, it is practically challenging to optimize over the whole feasible set of (MOP), this is due to the fact that it may be nonconvex and in general, an infinite set. Therefore, by nonemptiness of the set $\mathrm{Eff}(G(S);K)$ of efficient elements of (MOP), we consider the following corresponding scalar formulation over the set of efficient elements of (MOP):

$$\mathrm{Opt}\,\big(F(y^0)\mid\ y^0\in \mathrm{Eff}(G(S);K)\big), \tag{MOP$_\mathrm{SI}$}$$

where $F:\mathbb{R}^q\to\overline{\mathbb{R}}$ is the scalarization functional which is nonlinear in general.

Remark 4.3. (i) The explicit nonlinear scalarization function that befits our formulation is given in Example 4.2.

(ii) The optimal set of (MOP_{SI}) is defined by

$$\operatorname{argmin} F := \left\{ y^0 \mid F(y^0) \leq F(y) \; \textit{for all } y \in \text{Eff}(G(S);K) \right\}. \tag{OS}$$

(iii) For maximization problems the *argmin* notation in (OS) is replaced by *argmax*.

It is very important to point out that in the formulation of (MOP_{SI}), the decision maker already knows the set of efficient elements of (MOP) and he is interested in one element of this set that corresponds to his preferences. In general, nonlinear scalarization functionals usually appearing in applications involve additional constraints and parameters that are to be assorted in the solution strategy to suit the preferences of the decision maker.

Problem (MOP_{SI}) has been considered by many authors when the feasible set has some special structures and preimage and image spaces are finite–dimensional. For instance, authors in [8] studied (MOP_{SI}) for $q = 2$ over a compact set of efficient elements of a bi–criteria problem. For details about optimization over the set of efficient elements, see, e.g., [1, 2, 3, 4, 6, 8, 9, 10, 33] and references therein.

We will apply the following result popularly known as the generalized Weierstrass theorem, whose general form for arbitrary topological linear spaces can be seen in [20], to derive the existence result for (MOP_{SI}).

Theorem 4.2. *([20, Theorem 38B]) Suppose that D is a nonempty subset of $\mathbb{R}^q$. A minimal element of a functional $h : \mathbb{R}^q \to \overline{\mathbb{R}}$ over D exists if either of the following conditions is fulfilled:*

(i) *h is lower semicompact.*

(ii) *h is lower semicontinuous on the compact set D.*

For $q = 2$ in Example 4.1, we formulate an objective function and the feasible set that satisfy the hypothesis of Theorem 4.2.

Example 4.1. Suppose D is a rectangle in $\mathbb{R}^2$ defined by the points $(0,0)$ and $(1,1)$ as the set

$$D = \left\{ (x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 \right\}.$$

A function $h : D \to \mathbb{R}$ defined by

$$h(x,y) = \begin{cases} 1 & \text{for } y \neq 0, \\ \frac{1}{2} & \text{for } y = 0, \end{cases}$$

is lower semicontinuous on the compact set D. Therefore, condition (ii) of Theorem 4.2 is satisfied. Moreover, lower semicontinuity of h implies that h has closed lower sublevel sets that are compacts being subsets of the compact set D, hence, by Definition 2.8, condition (i) is also fulfilled.

4.1. **Existence results for solutions of (MOP_{SI})**

In this section, we establish the existence result for (MOP_{SI}) by applying the generalized Weierstrass result (Theorem 4.2) and the Assumptions 4.2 concerning the extended real–valued scalarizing functional F.

In Example 4.2, the nonlinear Gerstewitz scalarization functional [13] is considered as scalarizing functional that fulfills condition $(D1)$ of Assumption 4.2 under some additional assumptions concerning the ordering cone K and the scalarization direction $r \in K \setminus \{0\}$.

Example 4.2. Let D be a nonempty closed subset of $\mathbb{R}^q$, $r \in \mathbb{R}^q \setminus \{0\}$. Suppose that

$$D - [0,+\infty)r \subseteq D. \tag{4.1}$$

Consider the function $F : \mathbb{R}^q \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ defined by

$$F(y) := \inf\{t \in \mathbb{R} \mid \ y \in tr + D\} \ \text{ for all } y \in \mathbb{R}^q. \tag{NLSF}$$

We are using the convention $\inf \emptyset := +\infty$.

Remark 4.4. (i) In situations where $\inf \emptyset = +\infty$ in (NLSF), this either serves as a symbol for a function value at points that are not feasible for the problem or the related argument of the function is considered not belong to the domain of interest.

(ii) It is important to mention that the class of functions defined in (NLSF) coincides with the class of translation invariant functions.

Extended real–valued functions of type (NLSF) have been applied by many researchers of different fields of mathematics and economics theory, for instance, in optimization theory, these functions have been used for the representation of orders, preference relations, and other binary relations, especially of partial orders. In economics theory, similar functions of type (NLSF) have been used as the so-called shortage functions and in mathematical finance as coherent risk measure functions. Functions of the form (NLSF) are also appropriate for separating sets that are not necessarily convex. This has a deep impact on functional analysis and operator theory, where many proofs require separation theorems. For detailed bibliographical notes related to the historical facts about functions of type (NLSF); see [11, 24, 34] and the references contained therein.

Some essential properties for the usage of function in (NLSF) as scalarizing functional for vector optimization problems are outlined in the following result whose general form can be found in [11, 24, 34].

Proposition 4.1. *(See, e.g.,* [11, Theorem 2.3.1]*) Suppose that D is a proper and closed subset of $\mathbb{R}^q$ and $r \in \mathbb{R}^q \setminus \{0\}$ such that condition in* (4.1) *holds. Then,*

$$\text{the function } F \text{ defined in (NLSF) is lower semicontinuous,} \tag{4.2}$$

$\operatorname{dom}(F) = \mathbb{R}r + D$,

$$\forall\, \lambda \in \mathbb{R}, \quad L(F;\lambda) = \{y \in \mathbb{R}^q \mid F(y) \leq \lambda\} = \lambda r + D \tag{4.3}$$

and

$$\forall\, y \in \mathbb{R}^q,\ \forall\, \lambda \in \mathbb{R}, \quad F(y + \lambda r) = F(y) + \lambda.$$

Moreover,

(i) *F is convex if and only if D is convex.*

(ii) *F is positively homogeneous, i.e., $F(\lambda y) = \lambda F(y)$ for all $\lambda > 0$, $y \in \mathbb{R}^q$ if and only if D is a cone.*

(iii) *F is proper if and only if D does not contain lines parallel to r, i.e.,*

$$\forall\, y \in \mathbb{R}^q,\ \exists\, t \in \mathbb{R} : y + tr \notin D.$$

(iv) *F is finite-valued if and only if D does not contain lines parallel to r and*

$$\mathbb{R}r + D = \mathbb{R}^q.$$

(v) *Let* $B \subset \mathbb{R}^q$*;* F *is* B*–monotone if and only if* $D - B \subset D$*, where* F *is* B*–monotone if* y_1*,* $y_2 \in \operatorname{dom} F$*,* $y_1 - y_2 \in B$ *implies* $F(y_2) \leq F(y_1)$*.*

(vi) F *is subadditive if and only if* $D + D \subseteq D$*.*

As recalled in the foregoing, since binary relations such as partial orders can be represented by scalarizing functionals of type F in (NLSF), the monotonicity property of such functionals turns out to be of great importance due to the fact that, if D is taken as closed ordering cone K in $\mathbb{R}^q$, the corresponding order $\leq_K$ can be represented by F with an arbitrary $r \in K \setminus \{0\}$ since from (4.3) of Proposition 4.1, for all $y_1, y_2 \in \mathbb{R}^q$,

$$y_1 \leq_K y_2 \iff F(y_2 - y_1) \leq \{0\}.$$

In this case, it was shown in [11, Theorem 2.3.1] that the functional F is K–monotone i.e., for all $y_1, y_2 \in \operatorname{dom} F$

$$y_1 \leq_K y_2 \implies F(y_1) \leq F(y_2).$$

In what follows, we apply the conditions of Assumptions 4.2 in proving the existence result in Theorem 4.3.

Assumption 4.2. Suppose that conditions $(C2)$, $(C3)$, and $(C4)$ of Assumptions 4.1 are satisfied. Further, let

(D1) $F : \mathbb{R}^q \to \overline{\mathbb{R}}$ is lower semicontinuous.

(D2) $G(S) \subset \mathbb{R}^q$ is bounded.

Remark 4.5. (1) If, for instance, all lower sublevel sets of F are closed and the set $G(S)$ satisfies Definition 2.10(iv), then conditions $(D1)$ and $(D2)$ are satisfied, respectively.

(2) Condition $(D2)$ implies that lower sublevel sets of F on $\operatorname{Eff}(G(S); K)$ are bounded.

In the next example, a particular scalarization function is defined so that the conditions in Assumptions 4.2 are fulfilled.

Example 4.3. If $q = 2$, a closed unit disk is a set given by

$$D := \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\right\},$$

then the function $\hat{F} : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\hat{F}(x,y) = \begin{cases} 0 & \text{for } x, y \in D, \\ x^2 + y^2 - 1 & \text{for } x, y \in \mathbb{R}^2 \setminus D, \end{cases}$$

is lower semicontinuous with a bounded image. Therefore, conditions $(D1)$ and $(D2)$ satisfied since $\hat{F}$ is a restriction of F.

In the last result of this section, we provide a generalization of the existence results in [8, 4] with weaker compactness and continuity assumptions.

Theorem 4.3. *Suppose that the conditions in Assumptions 4.2 are satisfied. Then, Problem* (MOP_{SI}) *admits at least one solution.*

Proof. By setting the set $D := \operatorname{Eff}(G(S); K)$ in Definition 2.6, condition $(D1)$ of Assumptions 4.2 guarantees that the function F has closed lower sublevel sets on $\operatorname{Eff}(G(S); K) \subseteq \mathbb{R}^q$. Therefore, since the lower sublevel sets of F are subsets of finite–dimensional image space $\mathbb{R}^q$, their closedness together with condition $(D2)$ and Remark 4.5(2) imply that F has compact lower sublevel sets on $\mathrm{Eff}(G(S);K)$ by Proposition 2.4. Hence, by Definition 2.8 we obtain that F is lower semicompact. Thus, applying Theorem 4.2(i) by considering $D = \mathrm{Eff}(G(S);K)$, $h := F$, and $E := \mathbb{R}^q$, a minimal element of F on $\mathrm{Eff}(G(S);K)$ exists. □

Remark 4.6 characterizes the optimal set of the scalarized problem ($\mathrm{MOP_{SI}}$).

Remark 4.6. We deduce from (OS) that the set of optimal solutions to ($\mathrm{MOP_{SI}}$) is the set $\operatorname{argmin} F := \{y^0 \mid F(y^0) \leq \alpha\}$, where $\alpha := F(y)$ for all $y \in \mathrm{Eff}(G(S);K)$. Condition $(D1)$ of Assumptions 4.2 and Remark 4.5(2) imply that the scalarizing functional F in (OS) has compact lower sublevel sets. Therefore, ensuring the compactness of the optimal set $\operatorname{argmin} F := \{y^0 \mid F(y^0) \leq \alpha\}$.

In the preceding sections, under some suitable moderate assumptions, results guaranteeing the existence of solutions of vector optimization problem (MOP) and the corresponding scalarized problem ($\mathrm{MOP_{SI}}$) are derived. The difficulty of handling problem (MOP) arises from the presence of several conflicting objectives. Therefore, a unique optimal solution can not be obtained without incorporating preference information from the decision maker (DM). Consequently, in order to address the fundamental question concerning how to choose one element from the solution set of (MOP) that corresponds to the preferences of the (DM), we studied the associated scalarized problem ($\mathrm{MOP_{SI}}$) over the set of efficient elements of (MOP). Real-life scenario to demonstrate the applicability for the existence results of (MOP) and ($\mathrm{MOP_{SI}}$) will be presented in the next section.

5. Applications

If the scalar objective function of ($\mathrm{MOP_{SI}}$) involves some complex features such as non–convexity and discontinuity, even a small change of the involved parameters could lead to a significant change to the objective function value. The regularization framework contributes to the stabilization of such a function. In the Tikhonov regularization method [35], which is arguably the most popular regularization technique, a scalar objective function h is formulated as

$$h := f + \alpha g, \tag{TR}$$

where $f : \mathbb{R}^q \to \mathbb{R}$ is the objective function which is mostly assumed to be the square of a norm, $g : \mathbb{R}^q \to \mathbb{R}$ is the regularization function, and $\alpha > 0$ is the corresponding regularization parameter.

The main difficulty associated with optimization of the function in (TR) is that the solution depends upon the choice of the regularization parameter α, and most of the research reports only demonstrate results based on a limited number of *a priori* selected values, see, for instance, [36, 37] and references therein.

In order to find solutions independent of the regularization parameter α when dealing with the objective function of the form (TR) and to generate the whole set of solutions, we consider the associated formulation as a multi–objective optimization problem i.e., (MOP) for $q = 2$. More precisely, the general constrained bi–objective optimization problem associated with optimization of the function in (TR) takes the form:

$$\mathrm{Opt}_{x \in S}\, G(x), \tag{BOP}$$

where $G := (f,\ g)^T$ and f, g are real–valued conflicting objective functions to be optimized simultaneously over the feasible set S subset of some topological linear spaces and the notation 'Opt' is to understand in the sense of Definition 3.1 with respect to $K = \mathbb{R}^2_+$.

Motivated by [38, Example 3.6], the following example illustrates how conflicting objective functions are optimized.

Example 5.1. Consider the following simple constrained bi–objective optimization problem, which for a given vector–valued function $G : \mathbb{R} \to \mathbb{R}^2$ takes the form:

$$\mathrm{Opt}_{x\in S}\, G(x), \tag{$\mathrm{MOP_{q=2}}$}$$

where the feasible set $S := \left[-\frac{8}{3},\ 2\right]$ and $G(x) := \big(f(x),\ g(x)\big)^T = \big((x+2)^2,\ (x-2)^2\big)^T$.
It is straightforward to see that the efficient solutions of Problem ($\mathrm{MOP_{q=2}}$) with respect to $K = \mathbb{R}^2_+$ in the sense of Definition 3.1 are given by all points $\hat{x} \in [-2,\ 2]$.

To apply the existence results from Section 4, we consider a corresponding scalarized formulation of Problem ($\mathrm{MOP_{q=2}}$) which takes the form:

$$\mathrm{Opt}\,\big(F(\hat{y}) \mid\ \hat{y} \in \mathrm{Eff}(G(S);K)\big), \tag{$\mathrm{SMOP_{q=2}}$}$$

where $F : \mathbb{R}^2 \to \overline{\mathbb{R}}$ is the scalarizing functional of the form (NLSF) which is to be optimized over the set of efficient elements of Problem ($\mathrm{MOP_{q=2}}$).

Remark 5.1. Problem ($\mathrm{SMOP_{q=2}}$) is well–defined since the vector–valued function G is continuous and the feasible set S is a compact subset of $\mathbb{R}$.

Taking into account (4.2) of Proposition 4.1, the scalarizing functional in ($\mathrm{SMOP_{q=2}}$) is assumed to be lower semicontinuous. Moreover, the set of efficient solutions of ($\mathrm{MOP_{q=2}}$) is a closed and bounded subset of $\mathbb{R}$, hence, its image under G is a compact subset of $\mathbb{R}^2$ since G is continuous. Therefore, the set $\mathrm{Eff}(G(S);K)\big)$ which is the set of efficient elements in the image space is compact (due to the fact that the continuous image of a compact set is compact). Consequently, by lower semicontinuity of the scalarizing functional in ($\mathrm{SMOP_{q=2}}$) and the compactness of the set $\mathrm{Eff}(G(S);K)\big)$, Theorem 4.2(ii) guarantees the nonemptiness of the optimal set of Problem ($\mathrm{SMOP_{q=2}}$).

5.1. **Regularization problems.** In the study of inverse problems of identifying variable parameters in variational equations, the famous optimization framework is through the output least-squares objective. These types of problems have been attracting lots of attention in the literature, especially, in the recent works [39, 40, 41, 42, 43, 44, 45, 46] and references therein.

Considering (TR) with $f := \|Ax-b\|_2^2$ and $g := \|Lx\|_2^2$, where $\|\cdot\|_2^2$ denotes the square of the Euclidean norm, the regularized least square problem

$$\min_{x\in\mathbb{R}^n} J_\alpha(x) = \|Ax-b\|_2^2 + \alpha\|Lx\|_2^2, \tag{5.1}$$

for an arbitrary regularization parameter $\alpha > 0$, $A \in \mathbb{R}^{q\times n}$, $n \le q$, $b \in \mathbb{R}^q$, and matrix $L \in \mathbb{R}^{k\times n}$, $k \le n$, was considered in [47] as the unconstrained multi–objective optimization problem with two objectives

$$\min_{x\in\mathbb{R}^n} \left(\|Ax-b\|_2^2, \|Lx\|_2^2\right)^T. \tag{5.2}$$

Remark 5.2. For arbitrary regularization parameter $\alpha > 0$, a solution x_α of Problem (5.1) is efficient with respect to $K = \mathbb{R}^2_+$ (in the sense of Definition 3.1) of Problem (5.2) as shown in [47, Theorem 1].

The elastic net regularized problem of the form

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\left\|Ax-b\right\|_2^2 + \frac{\alpha}{2}\|x\|_2^2 + \beta\|x\|_1, \tag{5.3}$$

where $A \in \mathbb{R}^{q\times n}$, $b \in \mathbb{R}^q$, $\|\cdot\|_2^2$ denotes the square of the Euclidean norm, $\|\cdot\|_1$ denotes the l_1-(or Manhattan-)norm in $\mathbb{R}^n$ and the regularization parameters are α, $\beta \in \mathbb{R}_+ := [0,+\infty)$, was also studied in [30, 48, 49] as the unconstrained three–objective optimization problem

$$\min_{x\in\mathbb{R}^n} \left(\left\|Ax-b\right\|_2^2,\ \|x\|_2^2,\ \|x\|_1\right)^T. \tag{5.4}$$

Remark 5.3. Observe that the objective function of the elastic net regularized Problem (5.3) is strictly convex, continuous, and coercive (see [30] for details). Hence, by Proposition 2.2 and Definition 2.8, the objective function is lower semicompact. Therefore, by Theorem 4.2(i) there exists a minimal solution to the Problem (5.3) over $\mathbb{R}^n$. Thus, by [30, Proposition 2.1], the corresponding three–objective optimization Problem (5.4) has an efficient element with respect to $K = \mathbb{R}^3_+$ (in the sense of Definition 3.1).

5.2. **Vector-valued approximation problems** The problems studied by the authors in [30, 47, 49] and those presented in (5.1) and (5.3) lead us to the formulation of following vector–valued approximation problem as a special case of (MOP):

$$\mathrm{Opt}_{x\in S}\left(G_1(x), G_2(x), \dots, G_q(x)\right)^T, \tag{VAP}$$

where the set of feasible elements S is a nonempty, closed, convex, and bounded subset of $\mathbb{R}^n$,

$$\left(G_1(x),\ G_2(x),\ \dots,\ G_q(x)\right)^T = \left(\|A_1x-a^1\|_1^{\beta_1},\ \|A_2x-a^2\|_2^{\beta_2},\ \dots\ ,\ \|A_qx-a^q\|_q^{\beta_q}\right)^T,$$

and for each $i \in \{1,2,\dots,q\}$: $A_i \in \mathbb{R}^{k\times n}$, $a^i \in \mathbb{R}^k$, $\beta_i \geq 1$ and $\|\cdot\|_i$ denotes a norm in $\mathbb{R}^k$.

In (VAP), the optimization is to understand in the sense of Definition 3.1 with respect to $K = \mathbb{R}^q_+$.

Remark 5.4. (i) Clearly, Problem (5.3) is a special case of Problem (VAP) for $a^1 := b$, $a^2 = a^3 := 0$, $\beta_1 = \beta_2 := 2$, $\beta_3 := 1$, $A_1 := A$, A_2 and A_3 identity matrices.

(ii) Observe that for Problem (VAP) Assumptions 4.1 are fulfilled such that we can apply Theorem 4.1. So, we get from Theorem 4.1 that there exist efficient elements of (VAP).

5.3. **Application to multi-objective location problems.** In what follows, we study a special case of a vector-valued approximation problem (VAP), namely a multi–objective location problem.

5.3.1. *An algorithm for solving multi-objective location problems.* In order to apply the existence results from Section 4 to a special case of the approximation problem (VAP), we study the following multi–objective location problem.

Consider a finite family $P := \{a^1,\dots,a^q\}$ of q existing facilities in the plane $\mathbb{R}^2$, given by

$$a^1 := (a^1_1, a^1_2)^T, \dots, a^q := (a^q_1, a^q_2)^T \in \mathbb{R}^2.$$

In many problems of location analysis, the decision maker is looking for new facilities such that distances between the new facilities and existing facilities are minimal in the sense of multi-objective optimization.

In view of the formulation studied by the authors in [50], the distances will be induced by the l_1-norm (also known as Manhattan-norm) in our illustrating example. The Manhattan-norm is defined for any $(x_1,x_2) \in \mathbb{R}^2$ by

$$\|(x_1,x_2)\|_1 := |x_1| + |x_2|.$$

For a given vector–valued function $G : \mathbb{R}^2 \to \mathbb{R}^q$, the multi–objective location problem associated to the points $P \subset \mathbb{R}^2$ takes the form

$$\min_{x\in\mathbb{R}^2} G(x) = \left(\|x-a^1\|_1,\ \ldots\ ,\ \|x-a^q\|_1\right)^T. \tag{MOLP}$$

Clearly, Problem (MOLP) is a special case of Problem (VAP) with $S := \mathbb{R}^2$ and for all $i \in \{1,2,\ldots,q\}$: $\|\cdot\|_i := \|\cdot\|_1$, $\beta_i := 1$, $A_i := I$, where I is the identity matrix.

Remark 5.5. Consider the weighted-sum scalarization

$$\min_{x\in\mathbb{R}^2} \sum_{i=1}^{q} \alpha_i \|x-a^i\|_1, \tag{5.5}$$

of Problem (MOLP). For similar reasons to that in Remark 5.3, continuity and coercivity of the objective functions in (5.5) guarantee the existence of solution of the weighted-sum problem in (5.5). Therefore, for arbitrary $\alpha_i > 0$, $i \in \{1,2,\ldots,q\}$, the corresponding multi–objective location problem (MOLP) has efficient solutions with respect to $\mathbb{R}^q_+$ (in the sense of Definition 3.1); see [17, 51].

For applications in town planning, it is important that we can choose different norms in the formulation of (MOLP). The decision on the type of the norm that is used depends on the course of the roads in the city or in the district.

The following preparation allows us to characterize the set $\chi_{\mathrm{Eff}}(G(\mathbb{R}^2);\mathbb{R}^q_+)$ of efficient solutions of the multi–objective location problem (MOLP). In the literature, several characterizations of the set efficient solutions of location problems have been obtained; see, e.g., [50, 52, 53]. We will adapt here the approach proposed by the authors in [53] which is based on the dual norm to the Manhattan-norm, namely the maximum-norm defined for any $(x_1,x_2) \in \mathbb{R}^2$ by

$$\|(x_1,x_2)\|_\infty := \max\{|x_1|,|x_2|\}.$$

The authors in [53] characterized the set of efficient solutions of multi–objective location problems, where the distances are given by the Manhattan-norm, using the smallest sublevel set of the dual norm to the Manhattan-norm (i.e., the maximum-norm) containing the points $a^i \in \mathbb{R}^2$, $i = 1,2,\ldots,q$, denoted by $\mathscr{N}$.

In order to introduce a geometric primal-dual algorithm for solving Problem (MOLP), we consider the following sets with respect to the existing facilities $a^i \in \mathbb{R}^2$ $(i = 1,2,\ldots,q)$, which are related to the structure of the subdifferential of the Manhattan-norm:

$$S_1(a^i) := \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 < a_1^i,\ x_2 < a_2^i\},$$
$$S_2(a^i) := \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 > a_1^i,\ x_2 > a_2^i\},$$
$$S_3(a^i) := \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 > a_1^i,\ x_2 < a_2^i\},$$
$$S_4(a^i) := \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 < a_1^i,\ x_2 > a_2^i\}.$$

Moreover, for $p \in \{1,2,3,4\}$, we consider the set

$$S_p \quad := \quad \{x \in \mathscr{N} \mid \exists\, i \in \{1,2,\dots,q\}\ :\ x \in S_p(a^i)\} = \mathscr{N} \cap \big(\cup_{i=1}^{q} S_p(a^i)\big).$$

The following result from [50] is a counterpart of the characterization of solutions $x \in \mathbb{R}^2$ with $G(x) \in \mathrm{Eff}(G(\mathbb{R}^2);\mathbb{R}^q_+)$ (see Remark 3.1) proposed by the authors in [53], where multi–objective location problems defined by the maximum-norm are considered.

Proposition 5.1. *(See, e.g., [50, 53]) The set of solutions $x \in \mathbb{R}^2$ with $G(x) \in \mathrm{Eff}(G(\mathbb{R}^2);\mathbb{R}^q_+)$ of the multi–objective location Problem (MOLP) admits the following representation:*

$$\chi_{\mathrm{Eff}}(G(\mathbb{R}^2);\mathbb{R}^q_+) := \big[\big(\mathrm{cl}\, S_1 \cap \mathrm{cl}\, S_2\big) \cup \big((\mathscr{N} \setminus S_1) \cap (\mathscr{N} \setminus S_2)\big)\big]$$
$$\cap \big[\big(\mathrm{cl}\, S_3 \cap \mathrm{cl}\, S_4\big) \cup \big((\mathscr{N} \setminus S_3) \cap (\mathscr{N} \setminus S_4)\big)\big]. \tag{5.6}$$

Remark 5.6. It is important to mention that the set $\chi_{\mathrm{Eff}}(G(\mathbb{R}^2);\mathbb{R}^q_+)$ in (5.6) is in general not convex and always contained in the smallest sublevel set of the dual norm to the Manhattan-norm containing the existing facilities, i.e., $\chi_{\mathrm{Eff}}(G(\mathbb{R}^2);\mathbb{R}^q_+) \subset \mathscr{N}$.

The representation in (5.6) allows us to generate the whole set of efficient solutions of the multi–objective location Problem (MOLP).

5.3.2. *Optimization over the set of efficient solutions.* Given the set of solutions to the multi–objective location problem (MOLP) in (5.6), in order to find a solution of Problem (MOLP) that corresponds to the preferences of the decision maker, we consider a scalar surrogate problem of the form

$$\min_{x \in \cup_{j=1}^{n} \mathscr{R}_j} F(G(x)), \tag{SSP}$$

where $G(x) := \Big(\|x-a^1\|_1,\dots,\|x-a^q\|_1\Big)^T$, $F : \mathbb{R}^q \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ is the nonlinear scalarization functional of the form (NLSF), $\cup_{j=1}^{n} \mathscr{R}_j = \chi_{\mathrm{Eff}}(G(\mathbb{R}^2);\mathbb{R}^q_+)$, and for each $j = 1,2,\dots,n$, $\mathbb{R}_j$ is the convex subset of the solution set $\chi_{\mathrm{Eff}}(G(\mathbb{R}^2);\mathbb{R}^q_+)$.

Under certain assumptions concerning the parameters in the scalarizing functional (NLSF) that are to be chosen by the decision maker corresponding to his preferences, we adapt here the preamble presented in [14] as follows:

In order to get a simple and useful description of the functional (NLSF), we consider a set B given by a system of linear inequalities.

$$B := \big\{y \in \mathbb{R}^q \mid \langle b^i,\, y\rangle \le \alpha_i,\ b^i \in \mathbb{R}^q,\ \alpha_i \in \mathbb{R},\ i \in \{1,\dots,m\}\big\}. \tag{5.7}$$

With the vectors b^i in the definition of the set B in (5.7), a set A is defined by

$$A := \big\{y \in \mathbb{R}^q \mid \langle b^i, y\rangle \le \alpha_i,\ i \in I\big\}, \tag{5.8}$$

where I is the index set:

$$I := \{i \in \{1,\dots,m\} \mid \{y \in \mathbb{R}^q \mid \langle b^i, y\rangle = \alpha_i\} \cap B \cap \operatorname{int}\mathbb{R}^q_+ \neq \emptyset\}.$$

This means that the set I is exactly the set of indices $i \in \{1,\dots,m\}$ for that the hyperplanes $\langle b^i, y\rangle = \alpha_i$ are active in the non–negative orthant.

Consider B and the corresponding set A given in (5.8), vectors $w \in \mathbb{R}^q$ and $r \in -0^+A \setminus \{0\}$, where 0^+A denotes the recession cone of A defined by:

$$0^+A := \{r \in \mathbb{R}^q \mid \forall y \in A,\ \lambda > 0 : \ y + \lambda r \in A\}.$$

In the following, we study a special case of the nonlinear scalarization functional F (see (NLSF)) with $D := w + A$, i.e., we consider the functional $F_{r,w}$ of type (NLSF) given by:

$$F_{r,w}(y) := \inf\{t \in \mathbb{R} \mid \ y \in tr + w + A\}, \qquad y \in \mathbb{R}^q. \tag{5.9}$$

Clearly, the functional in (5.9) depends on the set A and the parameters r and w.

In what follows, we make use of the following proposition, whose general form can be seen in [14], to solve Problem (SSP) with $q = 8$.

Proposition 5.2. *Let the set B be given by (5.7) where $\{b^1,\dots,b^8\}$ is the canonical base in $\mathbb{R}^8$ (the set of vectors whose components are all zero, except the i-th component that equals 1) and $w \in \mathbb{R}^8$ arbitrarily chosen. Consider the function $F_{r,w}$ in (5.9) with the set A in (5.8) taken as $A := \mathbb{R}^8_- = \{y \in \mathbb{R}^8 \mid \ \langle b^i, y\rangle \leq 0,\ i \in \{1,\dots,8\}\}$, $r \in \mathbb{R}^8_+ \setminus \{0\}$. Assume that $\langle b^i, r\rangle \neq 0$ for all $i \in \{1,\dots,8\}$. Then, the nonlinear functional in (5.9) is convex and $\mathbb{R}^8_+$–monotone. Furthermore, $F_{r,w}$ has the structure:*

$$F_{r,w}(G(x)) = \max_{i \in \{1,\dots,8\}} \frac{\langle b^i, G(x)\rangle - \langle b^i, w\rangle}{\langle b^i, r\rangle}. \tag{5.10}$$

By considering Problem (SSP) for $q = 8$, we generate a solution in $\chi_{\text{Eff}}(G(\mathbb{R}^2); \mathbb{R}^8_+)$ that corresponds to the preferences of the decision maker by choosing suitable parameters r and w in (5.10) in the next subsection.

5.3.3. *Application to a multi–objective location problem.* In the following example, we compute the whole set of solutions of Problem (MOLP) for $q = 8$ and generate a candidate solution that corresponds to the preferences of the decision maker.

Example 5.2. Consider a vector–valued function $G : \mathbb{R}^2 \to \mathbb{R}^8$ defined by

$$G(x) = \left(\|x - a^1\|_1,\ \dots\ ,\ \|x - a^8\|_1\right)^T,$$

where the existing facilities are located at the points
$a^1 = (0.5, 4.5)$; $a^2 = (4, 3.5)$; $a^3 = (3.5, 1.5)$; $a^4 = (1.5, 3)$; $a^5 = (2.5, 2)$; $a^6 = (6, 1.5)$; $a^7 = (3.5, 5)$; and $a^8 = (5.5, 2.5)$. We want to find new facilities (efficient solutions) $x \in \mathbb{R}^2$ with $G(x) \in \text{Eff}(G(\mathbb{R}^2); \mathbb{R}^8_+)$ (see Remark 3.1) such that the distances between the new facilities and the existing facilities are minimal in the sense of the multi–objective location problem (MOLP):

$$\min_{x \in \mathbb{R}^2} \left(\|x - a^1\|_1, \dots, \|x - a^8\|_1\right)^T. \tag{MOLP$_{q=8}$}$$

We obtain the set of solutions $x \in \mathbb{R}^2$ with $G(x) \in \mathrm{Eff}(G(\mathbb{R}^2);\mathbb{R}^8_+)$ of the location Problem ($\mathrm{MOLP}_{q=8}$) via the characterization in (5.6) using the MATLAB–based software Facility location optimizer (FLO) [54] as follows:

$$\begin{aligned}\chi_{\mathrm{Eff}}(G(\mathbb{R}^2);\mathbb{R}^8_+) \;=\;& \mathrm{conv}\{(0.5,4.5),\ (1.5,4.5)\}\\ &\cup\mathrm{conv}\{(1.5,3),\ (1.5,4.5),\ (2.5,3),\ (2.5,4.5)\}\\ &\cup\mathrm{conv}\{(2.5,2),\ (2.5,4.5),\ (3.5,2),\ (3.5,4.5)\}\\ &\cup\mathrm{conv}\{(3.5,1.5),\ (3.5,3.5),\ (4,1.5),\ (4,3.5)\}\\ &\cup\mathrm{conv}\{(3.5,4.5),\ (3.5,5)\}\\ &\cup\mathrm{conv}\{(4,1.5),\ (4,2.5),\ (5.5,1.5),\ (5.5,2.5)\}\\ &\cup\mathrm{conv}\{(5.5,1.5),\ (6,1.5)\}.\end{aligned}$$

Figure 1 displays the graphical representation of the solution set $\chi_{\mathrm{Eff}}(G(\mathbb{R}^2);\mathbb{R}^8_+)$. Due to the

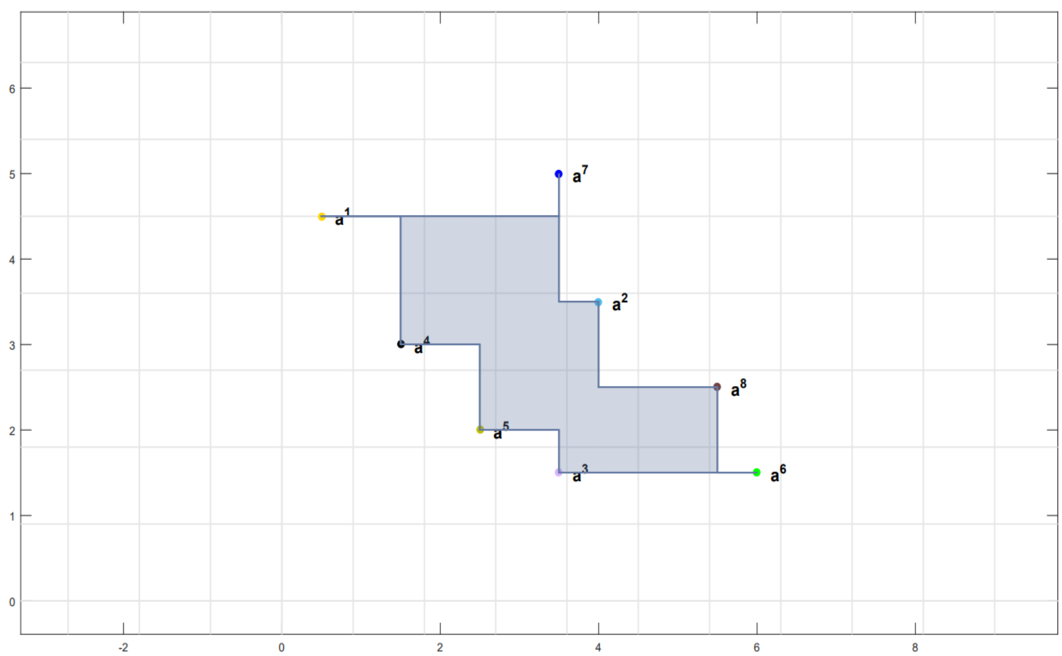

FIGURE 1. The graphical representation of the solution set $\chi_{\mathrm{Eff}}(G(\mathbb{R}^2);\mathbb{R}^8_+)$ of Problem ($\mathrm{MOLP}_{q=8}$) using the software Facility Location Optimizer [54].

large size and nonconvexity of the set of efficient solutions in Example 5.2 (see Fig. 1) and the fact that its elements cannot be easily compared, we apply our results concerning the existence of solutions and optimization over the set of efficient solutions using Proposition 5.2 to generate a single solution from $\chi_{\mathrm{Eff}}(G(\mathbb{R}^2);\mathbb{R}^8_+)$ corresponding to the preferences of the decision maker. In order to easily analyze the set $\chi_{\mathrm{Eff}}(G(\mathbb{R}^2);\mathbb{R}^8_+)$, we employ an approach of segmenting it into convex regions. The method of segmentation allows us to use the scalarization functional in (NLSF) with a special parameter constellation to generate an element of $\chi_{\mathrm{Eff}}(G(\mathbb{R}^2);\mathbb{R}^8_+)$ that closely corresponds to the preferences of the decision maker.

The following Figures 2, 3, and 4 display the segmentation of the solution set $\chi_{\mathrm{Eff}}(G(\mathbb{R}^2);\mathbb{R}^8_+)$ as convex regions.

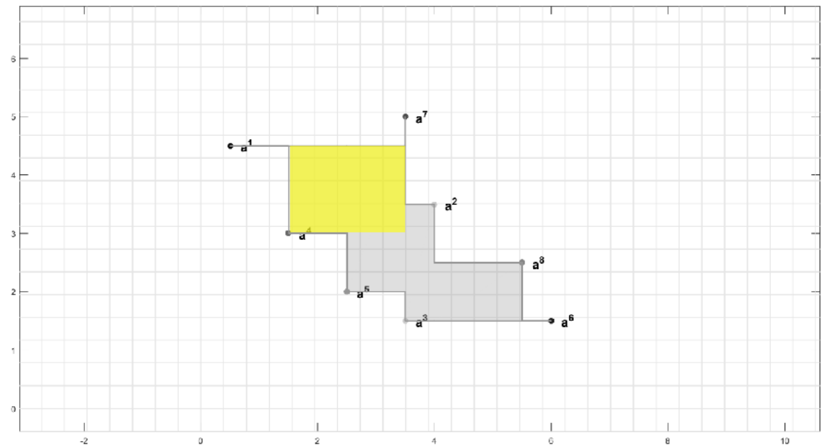

FIGURE 2. Yellow marked region $\mathscr{R}_1 := [1.5, 3] \times [3.5, 4.5]$ generated by the software image segmenter in MATLAB.

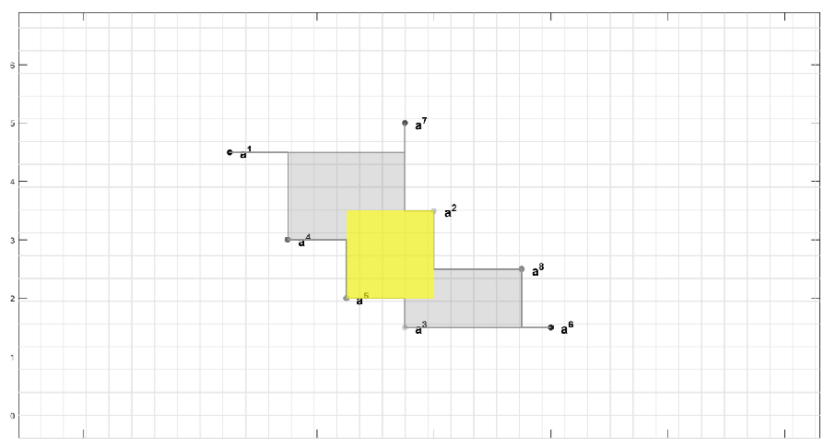

FIGURE 3. Yellow marked region $\mathscr{R}_2 := [2.5, 2] \times [4, 3.5]$ generated by the software image segmenter in MATLAB.

Now, by solving Problem (SSP) for $q = 8$ and $n = 3$ with the existing facilities considered in Example 5.2, we generate a solution in $\chi_{\text{Eff}}(G(\mathbb{R}^2); \mathbb{R}^q_+)$ that corresponds to the preferences of the decision maker. We suppose that the decision maker is not interested in solutions belonging to line segments of $\chi_{\text{Eff}}(G(\mathbb{R}^2); \mathbb{R}^8_+)$ such that we do not consider these elements in the further procedure.

We consider the scalarizing functional in (5.10), Problem (SSP) for $q = 8$ and $n = 3$, the regions $\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3$ (see Fig. 2, 3, and 4), the canonical base $\{b^1, \ldots, b^8\}$ in $\mathbb{R}^8$ (the set of vectors whose components are all zero, except the i-th component that equals 1), $r = (1, 1, 1, 1, 1, 1, 1, 1)^T$

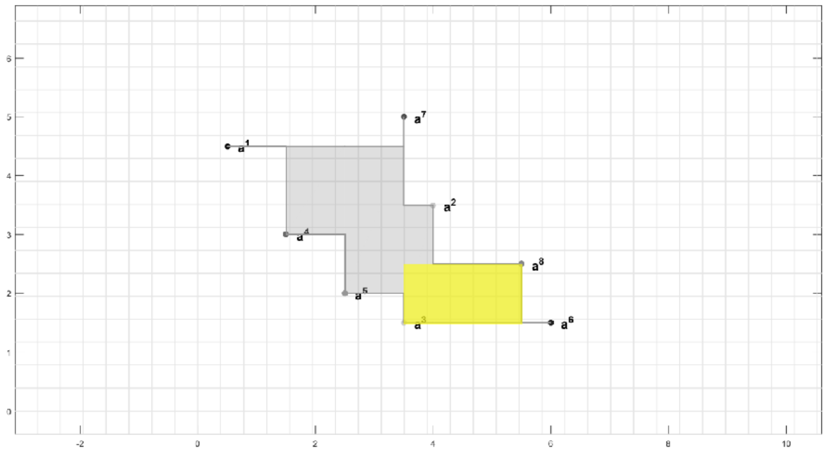

FIGURE 4. Yellow marked region $\mathscr{R}_3 := [3.5, 1.5] \times [5.5, 2.5]$ generated by the software image segmenter in MATLAB.

$\in \mathbb{R}^8_+$, and $w = 0_{\mathbb{R}^8}$. For these parameters, we minimize the functional $F_{r,w}$ in (5.10) over $\mathscr{R}_1$:

$$\begin{aligned}\min_{x\in\mathscr{R}_1} F_{r,w}(G(x)) &= \min_{x\in[1.5,3]\times[3.5,4.5]} \left\{ \max_{i\in\{1,\ldots,8\}} \langle b^i, G(x)\rangle \right\} \\ &= F_{r,w}(G(1.5967, 4.5)) \\ &= 1.0967.\end{aligned}$$

Similarly, suppose that the vectors b^i, r, and w are taken as in the case above, we minimize the functional $F_{r,w}$ in (5.10) over $\mathscr{R}_2$:

$$\begin{aligned}\min_{x\in\mathscr{R}_2} F_{r,w}(G(x)) &= \min_{x\in[2.5,2]\times[4,3.5]} \left\{ \max_{i\in\{1,\ldots,8\}} \langle b^i, G(x)\rangle \right\} \\ &= F_{r,w}(G(2.0950, 4)) \\ &= 1.5950.\end{aligned}$$

Finally, using the vectors b^i, r, and w as above, we minimize the functional $F_{r,w}$ in (5.10) over $\mathscr{R}_3$:

$$\begin{aligned}\min_{x\in\mathscr{R}_3} F_{r,w}(G(x)) &= \min_{x\in[3.5,1.5]\times[5.5,2.5]} \left\{ \max_{i\in\{1,\ldots,8\}} \langle b^i, G(x)\rangle \right\} \\ &= F_{r,w}(G(1.5975, 4.5)) \\ &= 1.0975.\end{aligned}$$

For b^i, r, and w as above, comparing these objective function values, we obtain that $(x_1^1, x_2^1) = (1.5967, 4.5)$ is a minimal solution of

$$\min_{x\in\cup_{j=1}^3 \mathscr{R}_j} F_{r,w}(G(x)), \tag{SSP$_{\text{n=3}}$}$$

with the objective function value

$$\begin{aligned}\min_{x\in\cup_{j=1}^3 \mathscr{R}_j} F_{r,w}(G(x)) &= F_{r,w}(G(1.5967, 4.5)) \\ &= 1.0967.\end{aligned}$$

As the result, the decision maker is choosing the location $(x_1, x_2) = (1.5967, 4.5)$ in the set $\chi_{\text{Eff}}(G(\mathbb{R}^2); \mathbb{R}^8_+)$ of solutions to ($\text{MOLP}_{q=8}$) because it corresponds to his preferences. More precisely, in Figure 5 the red marked location $(x_1^0, x_2^0) = (1.5967, 4.5) \in \chi_{\text{Eff}}(G(\mathbb{R}^2; \mathbb{R}^8_+))$ corresponds to the preferences of the decision maker.

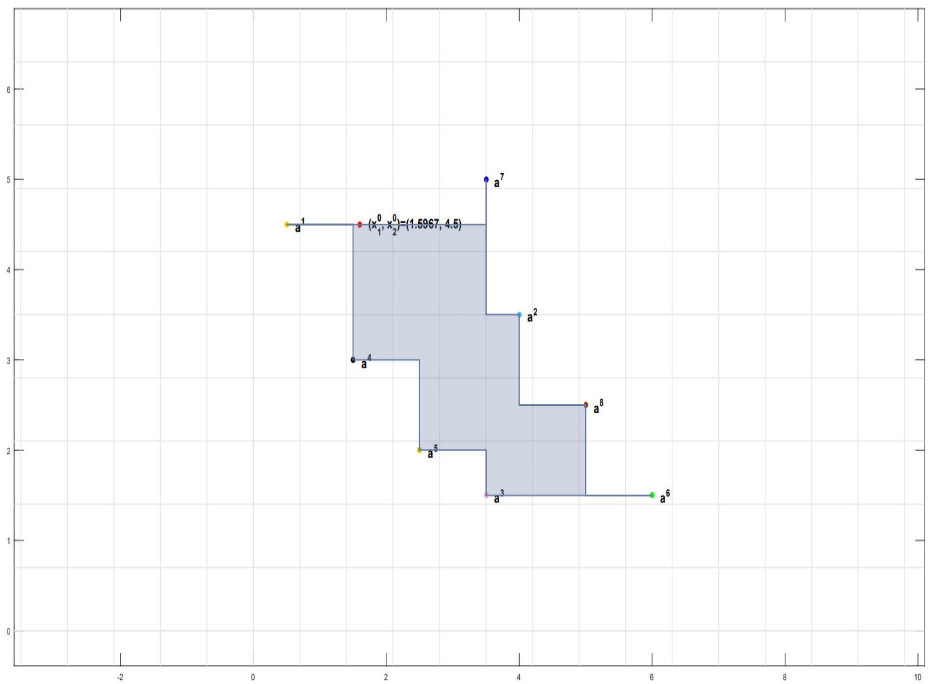

FIGURE 5. The red marked location $(x_1^0, x_2^0) = (1.5967, 4.5) \in \chi_{\text{Eff}}(G(\mathbb{R}^2; \mathbb{R}^8_+))$ corresponds to the preferences of the decision maker described by the choice of the parameters r, b, and w in the definition of the scalarizing functional (5.10). The whole set $\chi_{\text{Eff}}(G(\mathbb{R}^2; \mathbb{R}^8_+))$ is generated using the software Facility Location Optimizer (FLO) [54].

6. CONCLUSIONS

It is evident that in the theory of vector optimization problem certain assumptions, such as continuity of the objective functions and compactness of the feasible set easily guarantee the existence of solutions for such problems. However, due to the fact that compact sets are very limited in general, in this regard, under some moderate weaker assumptions, we presented existence results for vector optimization problems in reflexive preimage spaces as well as for the associated scalarized problems in this paper. Furthermore, we applied our results to vector–valued approximation problems. Especially, we employed our method to multi–objective location problems. For these problems, we generated the whole set of efficient elements and computed a single solution belonging to the set of efficient elements corresponding to the preferences of the decision maker using the nonlinear Gerstewitz functional (see Fig. 5).

It is our goal in further research to derive optimality conditions for the vector–valued approximation problems and apply our existence results and these optimality conditions in regularization techniques for deterministic and stochastic inverse problems.

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