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OPTIMALITY CONDITIONS IN OPTIMIZATION UNDER UNCERTAINTY

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Abstract. Most optimization problems involve uncertain data due to measurement errors, unknown future developments, and modeling approximations. In this paper, we consider scalar optimization problems under uncertainty with infinite scenario sets. We apply methods from vector optimization in general spaces, set-valued optimization, and scalarization techniques to derive necessary optimality conditions for solutions of robust optimization problems.
Keywords. Nonlinear scalarization; Necessary optimality conditions; Optimization under uncertainty; Robustness; Set optimization; Vector optimization.

1. Introduction and Previous Work

Most optimization problems involve uncertain data due to measurement errors, unknown future developments and modeling approximations. For companies, these uncertainties could be future demands that have to be predicted in order to adapt the production process. In risk theory, assets are naturally affected by uncertainty due to market changes, changing preferences of customers and unforeseeable events. Consequently, it is highly important to introduce uncertain parameters in optimization problems.

In the literature, one can find two main approaches regarding optimization problems under uncertainty:

- Stochastic Optimization: This idea goes back to Dantzig (1955). Stochastic optimization assumes that the uncertain parameter is probabilistic. Usually, one optimizes some cost function using the expected value of the uncertain parameter (cf. Birge and Louveaux [1]).
- Robust Optimization: Robustness, pursues a distinctively different approach to optimization problems with uncertainties not relying on a probability distribution but only using the uncertainty set. Typically, one wishes to optimize the worst-case scenario (strict robustness: Ben-Tal, Ghaoui and Nemirovski [2]).

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Soyster introduced robust optimization problems in [3]. This approach to optimization under uncertainty is intensively studied in the literature, see Ben-Tal, Ghaoui and Nemirovski [2], Goerigk and Schöbel [4], Kouvelis and Yu [5] and the references therein.

In stochastic optimization, the expected performance of a solution is usually involved in the objective functions or the preferences are described by stochastic dominance. For a general introduction to stochastic programming, see Birge and Louveaux [1].

In Klamroth, Köbis, Schöbel and Tammer [6, 7], it is shown that different approaches to optimization under uncertainty (for both stochastic and robust optimization) can be put in a unifying context using vector optimization, set optimization and nonlinear scalarizing functionals, assuming that the uncertainty set consists of finitely / infinitely many elements.

Our goal is to employ these unifying concepts for deriving necessary optimality conditions for solutions of robust counterpart problems. For doing this, we express robust and stochastic optimization problems by using vector optimization problems in general spaces, set-valued optimization and nonlinear scalarizing functionals. These results will be used to derive necessary optimality conditions for different kind of robust solutions.

There exist several papers investigating optimality conditions for robust solutions of uncertain optimization problems (see [8], and the recent publications [9, 10]). For example, in [11], the authors explore a robust nonsmooth multiobjective optimization problem involving data uncertainty, presenting two types of generalized convex functions and establishing robust optimality conditions for weakly and properly robust efficient solutions, along with formulating dual problems and deriving robust duality results. However, to the best of our knowledge, none uses the expression of robust problems using the nonlinear scarlarizing functional while deploying a generic approach to subdifferentials.

Our paper is organized as follows: In Section 2, we introduce the scalar optimization problems under uncertainty that we consider in our paper. Three unifying approaches to optimization under uncertainty (vector optimization, set optimization and a nonlinear scalarizing functional) are recalled in Section 3. Section 4 is devoted to a unified characterization of the concepts of strict robustness, regret robustness, ε-constraint robustness and proper robustness through the unifying approaches (vector optimization, set optimization and nonlinear scalarization) including a discussion of the corresponding assumptions. In Section 5, we introduce a generic approach to subdifferentials in order to derive necessary optimality condition for solutions of robust counterpart problems using the unifying approaches studied before. Finally, we give some conclusions for further research in Section 6.

2. Scalar Optimization under Uncertainty

We consider a scalar optimization problem $(Q(\xi))$ which depends on uncertain parameters ξ that belong to a given uncertainty set $\mathscr{U}$:

$$\begin{aligned} & f(x,\xi) \to \inf \\ & \text{s.t. } h_i(x,\xi) \leq 0,\ i=1,\ldots,m, \\ & x \in \mathbb{R}^n, \end{aligned} \qquad (Q(\xi))$$

where $f : \mathbb{R}^n \times \mathscr{U} \to \mathbb{R}$, $h_i : \mathbb{R}^n \times \mathscr{U} \to \mathbb{R}$, $i = 1,\ldots,m$.

The set of feasible solutions of $Q(\xi)$ is given by

$$\mathscr{X}(\xi) = \{x \in \mathbb{R}^n \mid h_i(x,\xi) \leq 0,\ i = 1,\ldots,m\}.$$

We assume that $\mathscr{U} \neq \emptyset$ is a not necessarily finite set, $\mathscr{X}(\xi) \neq \emptyset$ for all $\xi \in \mathscr{U}$.

An uncertain optimization problem $P(\mathscr{U})$ is defined as a family of parametrized optimization problems

$$(Q(\xi), \xi \in \mathscr{U}). \tag{2.1}$$

There are a lot of papers and books dealing with optimization under uncertainty; see, for example, Birge and Louveaux [1], Ben-Tal and Nemirovski [12, 13], Ben-Tal, Ghaoui and Nemirovski [2], Rockafellar and Royset [14, 15, 16, 17], Klamroth, Köbis, Schöbel and Tammer [6, 7] and the references therein.

3. Three Unifying Approaches to Optimization under Uncertainty

The unifying approaches to optimization under uncertainty presented in this section are derived by Klamroth, Köbis, Schöbel and Tammer in [6, 7]. For a discussion of the assumptions concerning the involved spaces and sets; see [18, Chapter 6].

3.1. **Vector optimization as unifying approach.**

A unified approach for a finite uncertainty set is derived in [6], where the set of uncertain parameters is given by $\mathscr{U} = \{\xi_1, \ldots, \xi_N\}$. Then, each scenario can be interpreted as an objective function. For some point x, we then obtain a vector $F_x \in \mathbb{R}^{|\mathscr{U}|}$ which contains $f(x, \xi_i)$ in its ith coordinate. In [6], it is shown that robust solutions can be characterized in terms of multiobjective optimization for many robustness concepts.

Klamroth, Köbis, Schöbel and Tammer [7] considered the case where $\mathscr{U}$ is a not necessarily finite set. Then, one obtains not vectors F_x but functions, i.e., $F_x : \mathscr{U} \to \mathbb{R}$ where $F_x(\xi) := f(x, \xi)$ contains the objective value of x in scenario ξ.

In order to compare two points x and y, we are looking for certain order relations in the real linear functional space $Y := \mathbb{R}^{\mathscr{U}}$ of all functions $F : \mathscr{U} \to \mathbb{R}$.

Let $(Q(\xi), \xi \in \mathscr{U})$ be the given optimization problem under uncertainty. For some fixed $x \in \mathbb{R}^n$, we define outcome functions by

$$F_x \in Y : \; F_x(\xi) := f(x, \xi).$$

To compare elements of Y, we consider different orderings on the space Y denoted by α. Let C be a proper pointed closed convex cone. Such a cone C induces the partial ordering $\alpha := \leq_C$

$$y_1 \in y_2 - C \iff y_1 \leq_C y_2.$$

Example 3.1. The natural order relation α_N on Y is induced by the cone

$$C_Y := \{F \in Y \mid \forall \xi \in \mathscr{U} : \; F(\xi) \geq 0\} :$$

$$\forall \, F, G \in Y : \; F \; \alpha_N \; G \; :\iff \; G \in F + C_Y \; \iff F(\xi) \leq_{C_Y} G(\xi) \text{ for all } \xi \in \mathscr{U}.$$

We use the following notion of minimality in Y well known in vector optimization.

Definition 3.1. Let $\mathscr{F}$ be a nonempty subset of Y. An element $F \in \mathscr{F}$ is a minimal element of $\mathscr{F}$ in Y w.r.t. α if

$$\text{for } G \in \mathscr{F} : \; G \; \alpha \; F \; \Longrightarrow F \; \alpha \; G.$$

If α is induced by a proper cone C in Y with $\operatorname{int} C \neq \emptyset$, an element $F \in \mathscr{F}$ is a weakly minimal element of $\mathscr{F}$ in Y w.r.t. α if

$$(F - \operatorname{int} C) \cap \mathscr{F} = \emptyset.$$

If α is induced by a cone, then an element $F \in \mathscr{F}$ is a minimal element of $\mathscr{F}$ in Y w.r.t. α if and only if $(F - C) \cap \mathscr{F} \subseteq F + C$.

Remark 3.1. Rockafellar and Royset [14, 15, 16] introduced an unifying framework for handling uncertainty in a decision making process. They considered $f(x,\cdot)$ as random variable and by means of risk measures, different models would be possible that address the issue how to treat that random variable. Since random variables are also functions, the connection to the vector approach is evident.

3.2. **Set-based optimization as unifying approach.** We are interested in all possible objective values which can appear if a feasible element $x \in \mathbb{R}^n$ is chosen. These outcome sets are given by

$$B_x := f(x,\mathscr{U}) := \{f(x,\xi) \mid \xi \in \mathscr{U}\}.$$

In order to compare two feasible elements x and y in this setting we have to define order relations between their corresponding outcome sets B_x and B_y.

Consider $Z := Pot(\mathbb{R})$. For a given $x \in \mathbb{R}^n$ we have

$$B_x \in Z: \; B_x = \operatorname{img}(F_x)$$

(the image of the mapping F_x under $\mathscr{U}$). $B_x \subseteq \mathbb{R}$ is an interval in case that $f(x,\cdot)$ is a continuous function.

In order to compare elements of Z we consider certain set relations denoted by β.

Example 3.2. (Lower-type set-relation (Kuroiwa [19, 20]) Let $A, B \in Z$ be arbitrarily chosen nonempty closed sets. Then the l-type set-relation $\beta := \preceq$ is defined by

$$A \;\beta\; B :\Longleftrightarrow B \subseteq A + \mathbb{R}_+ \Longleftrightarrow \forall b \in B\; \exists a \in A : a \leq b$$

which is equivalent to $\inf A \leq \inf B$. Note that β is induced by the cone $\mathbb{R}_+$ in Z. In order to show $B \subseteq A + \mathbb{R}_+ \Longleftrightarrow \forall b \in B\; \exists a \in A : a \leq b \Longleftrightarrow \inf A \leq \inf B$, we need the closedness of the involved sets A and B (see [18, Chapter 6]).

Let $\mathscr{B}$ be a nonempty subset of Z. Now, we are using the minimality notion well known in set optimization; see Jahn [21] and Khan, Tammer and Zălinescu [22] and the references therein.

Definition 3.2. $A \in \mathscr{B}$ is a minimal element in $\mathscr{B}$ w.r.t. β if

$$\text{for } B \in \mathscr{B}: \; B \;\beta\; A \implies A \;\beta\; B.$$

Remark 3.2. Given an ordering β and a set $\mathscr{B}$, set-valued optimization asks for minimal elements of $\mathscr{B}$ in Z w.r.t. β. Some concepts for uncertain optimization can be interpreted as solving such a set-valued problem. Furthermore, every set order relation β induces a concept for handeling uncertainty.

Remark 3.3. For the special case that the uncertainty set is given by $\mathscr{U} = \{\xi_1,\ldots,\xi_q\}$ ($q \in \mathbb{N}$), we obtain the outcome sets

$$B_x = \{f(x,\xi) \mid \xi \in \mathscr{U}\} = \{f(x,\xi_1),\ldots,f(x,\xi_q)\} \subset \mathbb{R}.$$

3.3. **A nonlinear scalarizing functional as unifying approach.** Let Y be a linear topological space, $k \in Y \setminus \{0\}$ and let $\mathscr{F}, B$ be proper subsets of Y, B closed. We assume that

$$B + [0, +\infty) \cdot k \subseteq B. \tag{3.1}$$

We consider the functional $z^{B,k} : Y \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\} =: \bar{\mathbb{R}}$ defined by

$$z^{B,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - B\}. \tag{3.2}$$

This functional was introduced as nonlinear separating functional with the corresponding properties and applications in vector optimization by Gerstewitz [23], see Göpfert, Riahi, Tammer, Zălinescu [24, Theorem 2.3.1], Khan, Tammer, Zălinescu [22, Chapter 5] for an overview on important properties of the functional (3.2). Pascoletti and Serafini [25] introduced an optimization problem, where the functional (3.2) is involved. Krasnosel'skiĭ [26], Rubinov [27] studied a functional of type (3.2) in the context of operator theory. In economics, the *Shortage Function* associated to the production possibility set $\mathscr{Y} \subset \mathbb{R}^m$ and $g \in \mathbb{R}^m_+ \setminus \{0\}$:

$$\sigma(g; y) := \inf\{\xi \in \mathbb{R} \mid y - \xi g \in \mathscr{Y}\},$$

and the *Benefit Function* are discussed by Luenberger in [28]. Clearly, these functions are of type (3.2). Furthermore, in Mathematical Finance, *Coherent Risk Measures* associated to the set of random variables corresponding to acceptable investments are functions of type (3.2), see Artzner et al. [29]. *Topical Functionals* studied by Rubinov and Singer in [30] are related to the function (3.2).

In the approach based on nonlinear scalarization, we will use the following notion of minimality.

Definition 3.3. An element $F \in \mathscr{F}$ is a minimal element of $\mathscr{F}$ in Y w.r.t. (3.2) if F solves the problem

$$z^{B,k}(y) \to \inf_{y \in \mathscr{F}}. \tag{$P_{k,B,\mathscr{F}}$}$$

4. Unified Characterization of Uncertain Optimization Concepts

The unified characterizations of uncertain optimization concepts discussed in this section were developed by Klamroth, Köbis, Schöbel and Tammer in [6, 7]; compare also [18, Chapter 6] for a discussion of the assumptions concerning the involved spaces and sets.

4.1. **Strict robustness.** The concept of strict robustness is introduced and studied by Soyster [3], Ben-Tal, Nemirovski [13], Ben-Tal, El Ghaoui, Nemirovski [2].

The idea of the concept of strict robustness is that the worst possible objective function value is minimized in order to get a solution that is "good enough" even in the worst case scenario. The strictly robust counterpart of $(Q(\xi), \xi \in \mathscr{U})$ is given by

$$\begin{aligned} &\rho_{RC}(x) = \sup_{\xi \in \mathscr{U}} f(x, \xi) \to \inf \\ &\text{s.t.} \forall \xi \in \mathscr{U} : \; h_i(x, \xi) \le 0, \; i = 1, \dots, m, \\ &x \in \mathbb{R}^n. \end{aligned} \tag{RC}$$

We call a feasible solution of (RC) strictly robust. The set of strictly robust solutions is

$$\mathfrak{A}_1 := \{x \in \mathbb{R}^n \mid \forall \xi \in \mathscr{U} : \; h_i(x, \xi) \le 0, \; i = 1, \dots, m\}. \tag{4.1}$$

Vector optimization

The strictly robust counterpart problem (*RC*) can be formulated as a vector optimization problem in the infinite dimensional space $Y = \mathbb{R}^{\mathscr{U}}$ (see [7, Section 3.1] and [18, Section 6.3.1]). The set of strictly robust outcome functions in Y is given by

$$\mathscr{F}_1 := \{F_x \in Y \mid x \in \mathfrak{A}_1\}. \tag{4.2}$$

Let two functions F_x, $F_y \in Y$ be given. Consider the following sup-order order relation on Y:

$$F_x \; \alpha_1 \; F_y :\Longleftrightarrow \sup_{\xi \in \mathscr{U}} F_x(\xi) \leq \sup_{\xi \in \mathscr{U}} F_y(\xi).$$

The following lemma (see [7, Lemma 2]) gives a relationship between the solution concept based on the sup-order relation α_1 and the concept based on the natural ordering α_N (see Example 3.1).

Lemma 4.1. *Consider $Y = C(\mathscr{U}, \mathbb{R})$. Suppose that every $F \in \mathscr{F}_1$ attains its supremum on $\mathscr{U}$. If $F \in \mathscr{F}_1$ is a minimal element of $\mathscr{F}_1$ w.r.t. α_1, then F is a weakly minimal element of $\mathscr{F}_1$ w.r.t. the natural order relation α_N of Y induced by the ordering cone C_Y.*

The assertion in the following theorem was shown in [7, Theorem 1].

Theorem 4.1. *A strictly robust solution $x \in \mathfrak{A}_1$ is an optimal solution of (RC) if and only if the corresponding strictly robust outcome function $F_x \in \mathscr{F}_1$ is a minimal element of $\mathscr{F}_1$ w.r.t. the sup-order relation α_1.*

Remark. In the light of Lemma 4.1, for each optimal solution x of the strictly robust counterpart problem (*RC*), F_x is a weakly minimal element of $\mathscr{F}_1$ w.r.t. the natural ordering α_N in Y.

Employing Lemma 4.1 together with Theorem 4.1, we get that F_x (with x is an optimal solution to the strictly robust counterpart (*RC*)) is a weakly minimal element of $\mathscr{F}_1$ w.r.t. α_N as shown in ([7, Corollary 1]).

Corollary 4.1. *Consider $Y = C(\mathscr{U}, \mathbb{R})$ and suppose that every $F \in \mathscr{F}_1$ attains its supremum on $\mathscr{U}$ for every solution $x \in \mathfrak{A}_1$. Then, for every optimal solution x to the strictly robust counterpart (RC), F_x is a weakly minimal element of $\mathscr{F}_1$ w.r.t. the natural order relation α_N in Y.*

Set-valued Optimization

Analogously, we interpret the strictly robust counterpart problem (*RC*) as a set-valued optimization problem. The set of strictly robust outcome sets in the power set Z is

$$\mathscr{B}_1 := \{B_x \in Z \mid x \in \mathfrak{A}_1\}.$$

For outcome sets B_x, $B_y \in Z$, let the upper-type set-relation β_1 be defined as

$$B_x \; \beta_1 \; B_y :\Longleftrightarrow B_x \subseteq B_y - \mathbb{R}_+.$$

We have

$$B_x \; \beta_1 \; B_y \Longleftrightarrow \sup B_x \leq \sup B_y$$

for *closed* outcome sets B_x, $B_y \in Z$. This equivalence is mentioned in [7, (3.13)], however, one needs the closedness of the outcome sets B_x, $B_y \in Z$ in order to show $B_x \subseteq B_y - \mathbb{R}_+ \iff \sup B_x \leq \sup B_y$ (see [18, Chapter 6]).

For the proof of the characterization of optimal solutions to the strictly robust counterpart problem (*RC*) in the next theorem (see [7, Theorem 2]), the closedness assumption concerning the outcome sets B_x for all $x \in \mathfrak{A}_1$ is again essential (see [18, Chapter 6]).

Theorem 4.2. *Suppose that the outcome sets B_x are closed for all $x \in \mathfrak{A}_1$. A strictly robust solution $x \in \mathfrak{A}_1$ is an optimal solution of (RC) if and only if the corresponding strictly robust outcome set $B_x \in \mathscr{B}_1$ is a minimal element of $\mathscr{B}_1$ with respect to the order relation β_1.*

Nonlinear Scalarization

Now, we give an interpretation of the strictly robust counterpart problem (*RC*) using the nonlinear scalarizing functional (3.2)

$$z^{B,k}(y) = \inf\{t \in \mathbb{R} \mid y \in tk - B\},$$

where $k \in Y \setminus \{0\}$, $B \subset Y$ proper and closed, $B + [0, +\infty) \cdot k \subseteq B$.

The relationship between optimal solutions of the strictly robust counterpart problem (RC) and solutions of a scalar optimization problem involving the functional (3.2) in the next theorem is shown in [7, Theorem 2], however, one needs the assumptions that Y is a linear topological space and C_Y is closed (see [18, Chapter 6]).

Theorem 4.3. *Consider a linear topological space Y of functions $F : \mathscr{U} \to \mathbb{R}$, $B_1 := C_Y$, $k_1 :\equiv 1 \in Y$ and $\mathscr{F}_1 := \{F_x \in Y \mid x \in \mathfrak{A}_1\}$. Suppose that C_Y is closed. Then,*

$$x \in \mathbb{R}^n \text{ solves } (RC) \iff F_x \text{ is a solution of } (P_{k_1,B_1,\mathscr{F}_1}).$$

4.2. **Regret robustness.** The regret robust counterpart of the optimization problem under uncertainty $(Q(\xi), \xi \in \mathscr{U})$ is given by

$$\begin{aligned} &\rho_{\text{rRC}}(x) = \sup_{\xi \in \mathscr{U}} \left(f(x,\xi) - f^*(\xi)\right) \to \inf \\ &\text{s.t. } \forall \xi \in \mathscr{U} : \; h_i(x,\xi) \leq 0, \; i = 1, \ldots, m, \\ &x \in \mathbb{R}^n, \end{aligned} \tag{rRC}$$

where $f^*(\xi)$ is defined below. It is important to mention that we require $x \in \mathfrak{A}_1$, i.e., we only permit strictly robust solutions as admissible solutions for the regret robust counterpart. So, $\mathscr{F}_2 := \{F_x \in Y \mid x \in \mathfrak{A}_1\}$.

Now, we consider a function $f^* \in Y$, $f^* : \mathscr{U} \to \mathbb{R}$ defined by

$$f^*(\xi) := \inf\{F_x(\xi) \mid x \in \mathfrak{A}_1\}. \tag{4.3}$$

We supposed that for every fixed scenario ξ an optimal solution in (4.3) exists. Then, the inf in (4.3) can be replaced by min.

The dominating set $B \subset Y$ is now shifted by the ideal solution $f^* \in Y$ of problem ($Q(\xi)$). Let $B_2 := C_Y - f^*$, and $k_2 :\equiv 1 \in Y$.

Now, we consider the scalar optimization problem

$$z^{B_2,k_2}(y) \to \inf_{y \in \mathscr{F}_2}. \tag{$P_{k_2,B_2,\mathscr{F}_2}$}$$

In the next theorem, we show that $x \in \mathbb{R}^n$ is an optimal solution to (rRC) if and only if F_x solves problem $(P_{k_2,B_2,\mathscr{F}_2})$ (see [7, Theorem 10]). In order to prove this result, we have to suppose that C_Y is closed, see [18, Chapter 6].

Theorem 4.4. *Consider a linear topological space Y of functions $F: \mathscr{U} \to \mathbb{R}$, $B_2 = C_Y - f^*$ and $k_2 :\equiv 1 \in Y$ and $\mathscr{F}_2 = \{F_x \in Y \mid x \in \mathfrak{A}_1\}$. Suppose that C_Y is closed.*

Then,

$$x \in \mathbb{R}^n \text{ solves (rRC)} \iff F_x \text{ is a solution of } (P_{k_2,B_2,\mathscr{F}_2}).$$

4.3. **ε-constraint robustness.** The concept of ε-constraint robustness can be described based on the nonlinear scalarization approach and it is introduced in [7, Section 4.3].

Let Y be a linear topological space of functions $F: \mathscr{U} \to \mathbb{R}$. Consider $\varepsilon: \mathscr{U} \to \mathbb{R}$.

We fix $\overline{\xi} \in \mathscr{U}$. The ε-constraint robust counterpart problem of the optimization problem under uncertainty $(Q(\xi), \xi \in \mathscr{U})$ is given by

$$\begin{aligned} &\inf f(x,\overline{\xi}) - \varepsilon(\overline{\xi}) \\ &\text{s.t. } \forall\, \xi \in \mathscr{U}:\ h_i(x,\xi) \le 0,\ i = 1,\dots,m, \\ &x \in \mathbb{R}^n, \\ &\forall\, \xi \in \mathscr{U} \setminus \{\overline{\xi}\}:\ f(x,\xi) \le \varepsilon(\xi). \end{aligned} \tag{εRC}$$

Let

$$\mathscr{F}_3 := \{F_x \in Y \mid x \in \mathfrak{A}_1\},$$

with $\mathfrak{A}_1 \subseteq \mathbb{R}^n$, $F_x(\xi) = f(x,\xi)$. Furthermore, let $k_3: \mathscr{U} \to \mathbb{R}$,

$$k_3 := \begin{cases} 1 & \text{for } \xi = \overline{\xi}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Again, we use

$$C_Y = \{F \in Y \mid \forall \xi \in \mathscr{U}:\ F(\xi) \ge 0\}.$$

Moreover, let

$$B_3 := \{y \in Y \mid y \in C_Y - \varepsilon\}. \tag{4.5}$$

Now, we consider the scalar optimization problem

$$z^{B_3,k_3}(y) \to \inf_{y \in \mathscr{F}_3}. \tag{$P_{k_3,B_3,\mathscr{F}_3}$}$$

The next Theorem is shown in [7, Theorem 23]. In order to prove this result, we have to suppose the closedness of C_Y (see [18, Chapter 6]).

Theorem 4.5. *Consider a linear topological space Y of functions $F: \mathscr{U} \to \mathbb{R}$. Let $\varepsilon: \mathscr{U} \to \mathbb{R}$, B_3 given by (4.5), k_3 given by (4.4) and $\mathscr{F}_3 = \{F_x \in Y \mid x \in \mathfrak{A}_1\}$. Suppose that C_Y is closed. Then,*

$$x \in \mathbb{R}^n \text{ solves } (\varepsilon RC) \iff F_x \text{ is a solution of } (P_{k_3,B_3,\mathscr{F}_3}).$$

4.4. **Proper robustness.** The properly robust counterpart of the optimization problem under uncertainty $(Q(\xi), \xi \in \mathscr{U})$ is defined by the problem of finding properly minimal elements of $\mathscr{F}_1$ (given by (4.2)) w.r.t. a nontrivial ($C \neq \{0\}$ and $C \neq Y$) convex cone $C \subset Y$:

$$\text{PMin}(\mathscr{F}_1, C) := \{F_x \in \mathscr{F}_1 \mid \exists \text{ dilating cone } D \text{ such that } (F_x - \text{intD}) \cap \mathscr{F}_1 = \emptyset\}.$$

A convex cone $D \subset Y$ ($D \neq Y$) is called dilating cone of C if $C \setminus \{0\} \subset \text{int} D$.

The concept of properly robust counterparts is introduced in [7, Section 4.1] and related to the vector approach, see Section 3.1. The advantage of this concept is that one obtains in general a smaller set of robust solutions in comparison with the concept based on the natural order relation α_N (see Example 3.1).

We consider two given functions $F, G \in Y$ and the following order relation α_2, where a dilating cone D is employed:

$$F \; \alpha_2 \; G \quad :\Longleftrightarrow \quad \exists \text{ a dilating cone } D \subset Y \text{and } G \in F + D. \tag{4.6}$$

Using the order relation α_2 in (4.6), a properly robust solution x is defined as follows:

$$x \in \mathfrak{A}_1 \text{ is } \textit{properly robust} :\Longleftrightarrow F_x \text{ is a minimal element of } \mathscr{F}_1 \text{ w.r.t } \alpha_2.$$

The *properly robust counterpart* of the optimization problem under uncertainty $(Q(\xi), \xi \in \mathscr{U})$ is given by the problem of searching properly minimal elements of $\mathscr{F}_1$ w.r.t. C:

$$\text{Compute} \quad \text{PMin}(\mathscr{F}_1, C). \tag{pRC}$$

An element $F_x \in \text{PMin}(\mathscr{F}_1, C)$ is called properly minimal element of $\mathscr{F}_1$ w.r.t. C.

5. Optimality Conditions for Solutions of Robust Counterpart Problems

The aim of this section is to derive necessary optimality condition for optimal solutions of robust counterpart problems using the unifying approaches to optimization problems under uncertainty described in Sections 3 and 4. Especially, by employing the approach via nonlinear scalarization (see Section 3.3), it is possible to show useful necessary optimality conditions. The main tool for deriving the necessary optimality conditions is a generic approach to subdifferentials such that our results hold for many classes of generalized subdifferentials.

5.1. **Generic approach to subdifferentials.** In this section, we derive necessary optimality conditions for solutions of robust counterpart problems employing a generic approach to subdifferentials (see e.g. Dolecki and Malivert [31], Durea and Tammer [32], Durea, Strugariu and Tammer [33]).

Throughout this section, let Y be a Banach space.

We now introduce an abstract subdifferential ∂: A map which associates to every lower semicontinuous (lsc) function $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$ and to every $y \in Y$ a (possible empty) subset $\partial\varphi(y) \subset Y^*$. We use the notation $\text{Dom}\,\varphi := \{y \in Y \mid \varphi(y) \neq +\infty\}$.
(H1) If φ is convex and lower semicontinuous, then $\partial\varphi(y)$ coincides with the Fenchel subdifferential.
(H2) If $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz, $\Omega \subset Y$ is a nonempty and closed set and $\bar{y}$ is a minimum point for φ over Ω, then

$$0 \in \partial\varphi(\bar{y}) + \partial I_\Omega(\bar{y}).$$

For a closed set $\Omega \subset Y$, the set $\partial I_\Omega(y)$ is denoted by $N_\partial(y;\Omega)$ and is called the set of normal directions to Ω at $y \in \Omega$ with respect to ∂.

Under suitable assumptions concerning the involved spaces and functions (see, e.g., [32, 33] and the references therein), these axioms are fulfilled for well-known subdifferentials by Clarke, Ioffe, Kruger, Mordukhovich and several other important subdifferentials.

In some applications, for deriving necessary optimality conditions for solutions of the ε-constraint robust counterpart problem in Section 5.4, we replace axiom (H2) in the axiomatic approach to a subdifferential by the following axiom:
(H3) If $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous, $\Omega \subset Y$ is a closed set and $\bar{y}$ is a minimum point for φ over Ω, then

$$0 \in \partial\varphi(\bar{y}) + \partial I_\Omega(\bar{y}).$$

We formulate the necessary optimality conditions for solutions of robust counterpart problems in terms of subgradients of the nonlinear scalarizing functional $z^{B,k}$ given by (3.2). For these assertions, the structure of the subdifferential $\partial z^{B,k}$ in the following theorem and corollary (derived by Durea and Tammer in [32]) are very helpful.

Theorem 5.1. *Let $B \subset Y$ be a closed convex proper set and $k \in Y \setminus \{0\}$ s.t. (3.1) holds. Consider the functional $z^{B,k}$ in (3.2) and let $\bar{y} \in \operatorname{Dom}\varphi$. Then*

$$\partial z^{B,k}(\bar{y}) = \{y^* \in Y^* \mid y^*(k) = 1, \forall b \in B : y^*(b) + y^*(\bar{y}) - z^{B,k}(\bar{y}) \geq 0\}.$$

We denote the dual cone (to a cone C in the Banach space space Y) by

$$C^+ := \{y^* \in Y^* \mid \forall y \in C : \quad y^*(y) \geq 0\}.$$

Corollary 5.1. *Let $C \subset Y$ be a closed convex cone with nonempty interior. Then, for every $k \in \operatorname{int} C$ the functional $z^{B,k}$ in (3.2) is continuous, sublinear, strictly-$\operatorname{int} C$-monotone and for every $\bar{y} \in Y$, $\partial z^{B,k}(\bar{y})$ is nonempty and*

$$\partial z^{B,k}(\bar{y}) = \{y^* \in C^+ \mid y^*(k) = 1,\ y^*(\bar{y}) = z^{B,k}(\bar{y})\}.$$

In particular,

$$\partial z^{B,k}(0) = \{y^* \in C^+ \mid y^*(k) = 1\}. \tag{5.1}$$

5.2. **Necessary conditions for solutions of the strictly robust counterpart problem.** In this section, we derive necessary conditions for solutions of the strictly robust counterpart problem (RC) using the unifying approach based on nonlinear scalarization by functionals (3.2) (see Theorem 4.3) as well as the approach based on vector optimization (see Theorem 4.1). We consider an abstract subdifferential ∂ such that (H1) and (H2) (see Section 5.1) are fulfilled.

Theorem 5.2. *Let Y be a Banach space of functions $F : \mathscr{U} \to \mathbb{R}$, $B_1 = C_Y$, $k_1 \equiv 1 \in \operatorname{int} C_Y$ and $\mathscr{F}_1 = \{F_x \in Y \mid x \in \mathfrak{A}_1\}$. Suppose that $\mathscr{F}_1$ and C_Y are closed sets. Consider an optimal solution x to the strictly robust counterpart problem (RC) and the abstract subdifferential ∂ such that (H1) and (H2) are fulfilled. Then, there exists an element $y^* \in C_Y^+$ with $y^*(k_1) = 1$ and $y^*(F_x) = z^{B_1,k_1}(F_x)$ such that*

$$-y^* \in N_\partial(F_x;\mathscr{F}_1).$$

Proof. Let x be an optimal solution to the strictly robust counterpart problem (RC). Taking into account Theorem 4.3, the corresponding strictly robust outcome function F_x solves problem $(P_{k_1,B_1,\mathscr{F}_1})$ with $B_1 = C_Y$, $k_1 \equiv 1$ and $\mathscr{F}_1 = \{F_x \in Y \mid x \in \mathfrak{A}_1\}$:

$$z^{B_1,k_1}(y) \to \inf_{y\in\mathscr{F}_1}, \qquad (P_{k_1,B_1,\mathscr{F}_1})$$

where the functional $z^{B_1,k_1} : Y \to \bar{\mathbb{R}}$ with

$$z^{B_1,k_1}(y) := \inf\{t \in \mathbb{R} \mid y \in tk_1 - B_1\}$$

(compare that (3.2)) is involved. In this setting, condition (3.1) is satisfied. From [24, Corollary 2.3.5], we obtain that $z^{B_1,k_1}(\cdot)$ is finite-valued, sublinear and continuous, and so Lipschitz continuous. Taking into account (H1) and (H2), we obtain

$$0 \in \partial z^{B_1,k_1}(\cdot)(F_x) + N_\partial(F_x;\mathscr{F}_1) \tag{5.2}$$

since the constraint set $\mathscr{F}_1$ is assumed to be closed. Because of the structure of the subdifferential of z^{B_1,k_1} given by Corollary 5.1, we obtain

$$\partial z^{B_1,k_1}(F_x) \quad = \quad \{y^* \in C_Y^+ \mid y^*(k_1) = 1,\ y^*(F_x) = z^{B_1,k_1}(F_x)\}.$$

Together with the inclusion (5.2), we find a subgradient $y^* \in \partial z^{B_1,k_1}(F_x)$ with $-y^* \in N_\partial(F_x;\mathscr{F}_1)$, which justifies the necessary conditions. The proof is complete. □

Remark 5.1. In order to get a simpler structure of the subdifferential, we suppose that $F_x = 0$ for a minimal solution of problem $(P_{k_1,B_1,\mathscr{F}_1})$. Using (5.1) in Corollary 5.1, the necessary condition for an optimal solution x to the strictly robust counterpart problem (RC) in Theorem 5.2 reads in this case: There exists $y^* \in C_Y^+$ with $y^*(k_1) = 1$ and

$$-y^* \in N_\partial(F_x;\mathscr{F}_1).$$

Remark In the case of a finite number of scenarios (q scenarios), we have $y^* \in \mathbb{R}^q_+$ in the conditions of Theorem 5.2.

Furthermore, we can use the characterization of solutions of the strictly robust counterpart problem (RC) by vector optimization (see Theorem 4.1) for deriving necessary conditions for solutions of the strictly robust counterpart problem.

Theorem 5.3. *Consider $Y = C(\mathscr{U},\mathbb{R})$ and suppose that every $F_x \in \mathscr{F}_1$ attains its supremum on $\mathscr{U}$ for every solution $x \in \mathfrak{A}_1$. Let $B_1 = C_Y$, $k_1 \equiv 1$ and $\mathscr{F}_1 = \{F_x \in Y \mid x \in \mathfrak{A}_1\}$. Suppose that $\mathscr{F}_1$ is closed. Let x be an optimal solution to the strictly robust counterpart problem (RC) and the abstract subdifferential ∂ be such that (H1) and (H2) are fulfilled. Then, there exists an element $y^* \in C_Y^+$ with $y^*(k_1) = 1$ such that*

$$-y^* \in N_\partial(F_x;\mathscr{F}_1).$$

Proof. Let x be an optimal solution of the strictly robust counterpart problem (RC). By Theorem 4.1, $F_x \in \mathscr{F}_1$ is a minimal element of $\mathscr{F}_1$ w.r.t. α_1. From Corollary 4.1, we obtain that F_x is a weakly minimal element of $\mathscr{F}_1$ in Y w.r.t. the natural ordering α_N of Y induced by the ordering cone C_Y. Then, we consider the functional $z^{B_1,k_1} : Y \to \bar{\mathbb{R}}$ given by (3.2) with

$$z^{B_1,k_1}(y - F_x) := \inf\{t \in \mathbb{R} \mid y - F_x \in t_1k - B_1\}.$$

In this setting, condition (3.1) is satisfied and the functional $z^{B_1,k_1}(\cdot - F_x)$ is finite-valued, continuous and convex (see [24, Proposition 2.3.4]) and so Lipschitz continuous. From [34, Lemma 5.2], we know that F_x solves

$$z^{B_1,k_1}(y - F_x) \to \inf_{y \in \mathscr{F}_1} . \qquad (P_{k_1,B_1,\mathscr{F}_1})$$

Now, we can follow the line of Theorem 5.2 taking into account (5.1) in Corollary 5.1. □

5.3. **Necessary conditions for solutions of the regret robust counterpart problem.** In this section, we derive necessary conditions for solutions of the regret robust counterpart problem (rRC) using the characterization of solutions to (rRC) by the unifying approach based on nonlinear scalarization in Theorem 4.4. We consider an abstract subdifferential ∂ such that (H1) and (H2) (see Section 5.1) are fulfilled.

Theorem 5.4. *Let Y be a Banach space of functions $F : \mathscr{U} \to \mathbb{R}$, $\mathscr{F}_2 = \{F_x \in Y \mid x \in \mathfrak{A}_1\}$, with $\mathfrak{A}_1 \subseteq \mathbb{R}^n$. Furthermore, let $C_Y = \{F \in Y \mid \forall \xi \in \mathscr{U} : F(\xi) \geq 0\}$, $k_2 \equiv 1 \in \operatorname{int} C_Y$, and $B_2 = \{y \in Y \mid y \in C_Y - f^*\}$. Suppose that $\mathscr{F}_2 = \{F_x \in Y \mid x \in \mathfrak{A}_1\}$ and C_Y are closed. Consider an optimal solution x to the regret robust counterpart problem (rRC) and the abstract subdifferential ∂ such that (H1) and (H2) are fulfilled. Then, there exists an element $y^* \in Y^*$ with $y^*(k_2) = 1$ and $y^*(F_x) + y^*(b) \geq z^{B_2,k_2}(F_x)$ for every $b \in B_2$ such that*

$$-y^* \in N(F_x; \mathscr{F}_2).$$

Proof. Consider an optimal solution x to the regret robust counterpart problem (rRC). From Theorem 4.4, we obtain that the corresponding regret robust outcome function F_x solves problem

$$z^{B_2,k_2}(y) \to \inf_{y \in \mathscr{F}_2} , \qquad (P_{k_2,B_2,\mathscr{F}_2})$$

where the functional $z^{B_2,k_2} : Y \to \bar{\mathbb{R}}$

$$z^{B_2,k_2}(y) = \inf\{t \in \mathbb{R} \mid y \in tk_2 - B_2\}$$

(with $B_2 = \{y \in Y \mid y \in C_Y - f^*\}$, $k_2 \equiv 1 \in \operatorname{int} C_Y$ and $\mathscr{F}_2 = \{F_x \in Y \mid x \in \mathfrak{A}_1\}$) is involved, i.e., $F \in \mathscr{F}_2$ is a minimal element over $\mathscr{F}_2$ in Y w.r.t. the functional z^{B_2,k_2} (compare (3.2)). Under these specifications, condition (3.1) is satisfied and the functional z^{B_2,k_2} is finite-valued, continuous and convex (see [24, Proposition 2.3.4]) and so Lipschitz continuous. From (H1) and (H2), we obtain

$$0 \in \partial z^{B_2,k_2}(\cdot)(F_x) + N_\partial(F_x; \mathscr{F}_2) \qquad (5.3)$$

since the constraint set $\mathscr{F}_2$ is assumed to be closed.

In compliance with the structure of the subdifferential of z^{B_2,k_2} (see Theorem 5.1), we obtain

$$\partial z^{B_2,k_2}(F_x) = \{y^* \in Y^* \mid y^*(k_2) = 1, \forall b \in B : y^*(b) + y^*(F_x) - z^{B_2,k_2}(F_x) \geq 0\}.$$

This yields together with inclusion (5.3) that there is a subgradient $y^* \in \partial z^{B_2,k_2}(F_x)$ with $-y^* \in N_\partial(F_x; \mathscr{F}_2)$ such that the necessary conditions are fulfilled. □

5.4. **Necessary conditions for solutions of the ε-constraint robust counterpart problem.** For deriving a necessary optimality condition for optimal solutions to the ε-constraint robust counterpart problem (εRC), we consider an abstract subdifferential such that (H1) and (H3) (see Section 5.1) are fulfilled. Furthermore, we are using the notations introduced in Section 4.3.

Theorem 5.5. *Let Y be a Banach space of functions $F:\mathscr{U}\to\mathbb{R}$. Consider $\varepsilon:\mathscr{U}\to\mathbb{R}$, $\mathscr{F}_3=\{F_x\in Y\mid x\in\mathfrak{A}_1\}$. Furthermore, let $k_3:\mathscr{U}\to\mathbb{R}$,*

$$k_3=\begin{cases}1 & \text{for } \xi=\overline{\xi},\\ 0 & \text{otherwise,}\end{cases} \tag{5.4}$$

$C_Y=\{F\in Y\mid\forall\xi\in\mathscr{U}:\ F(\xi)\geq 0\}$ and $B_3=\{y\in Y\mid y\in C_Y-\varepsilon\}$. Suppose that $\mathscr{F}_3$ and C_Y are closed. Consider an optimal solution x to the ε-constraint robust counterpart problem (εRC) and the abstract subdifferential ∂ such that (H1) and (H3) are fulfilled. Then, there exists an element $y^*\in Y^*$ with $y^*(k_3)=1$ and $y^*(F_x)+y^*(b)\geq z^{B_3,k_3}(F_x)$ for every $b\in B_3$ such that

$$-y^*\in N(F_x;\mathscr{F}_3).$$

Proof. Let x be an optimal solution to the ε-constraint robust counterpart problem (εRC). Taking into account Theorem 4.5, the corresponding ε-constraint robust outcome function F_x solves problem $(P_{k_3,B_3,\mathscr{F}_3})$ with $B_3=\{y\in Y\mid\ y\in C_Y-\varepsilon\}$, $k_3:\mathscr{U}\to\mathbb{R}$,

$$k_3=\begin{cases}1 & \text{for } \xi=\overline{\xi},\\ 0 & \text{otherwise}\end{cases} \tag{5.5}$$

and $\mathscr{F}_3=\{F_x\in Y\mid x\in\mathfrak{A}_1\}$, i.e., $F_x\in\mathscr{F}_3$ is a minimal element on $\mathscr{F}_3$ w.r.t. the functional z^{B_3,k_3} (compare (3.2)):

$$z^{B_3,k_3}(y)\to\inf_{y\in\mathscr{F}_3}, \tag{$P_{k_3,B_3,\mathscr{F}_3}$}$$

where the functional $z^{B_3,k_3}:Y\to\bar{\mathbb{R}}$

$$z^{B_3,k_3}(y)=\inf\{t\in\mathbb{R}\mid y\in tk_3-B_3\}.$$

In this setting, condition (3.1) is satisfied. Since z^{B_3,k_3} is a lower semicontinuous convex functional (see [24, Theorem 2.3.1]), we get from (H1) and (H3)

$$0\in\partial z^{B_3,k_3}(\cdot)(F_x)+N_\partial(F_x;\mathscr{F}_3) \tag{5.6}$$

since the constraint set $\mathscr{F}_3$ is assumed to be closed.

Taking into account the structure of the subdifferential of z^{B_3,k_3} given by Theorem 5.1, we obtain

$$\partial z^{B_3,k_3}(F_x)=\{y^*\in Y^*\mid y^*(k_3)=1,\forall b\in B:y^*(b)+y^*(F_x)-z^{B_3,k_3}(F_x)\geq 0\}.$$

Together with the inclusion (5.6), there is a subgradient $y^*\in\partial z^{B_3,k_3}(F_x)$ with $-y^*\in N_\partial(F_x;\mathscr{F}_3)$ such that the necessary conditions are satisfied. ☐

5.5. **Necessary conditions for solutions of the properly robust counterpart.** The aim of this section is to derive necessary optimality conditions for properly robust solutions in the sense of Section 4.4. We are using the Henig dilating cone in the concept of a properly robust counterpart problem for deriving necessary optimality conditions because this cone has a very useful structure. Suppose that Y is a Banach space and $C \subset Y$ a nontrivial convex cone. We suppose that C has a closed convex base $B_C \subset Y$ such that

$$C = \bigcup_{\lambda \geq 0} \{\lambda y \mid y \in B_C\} \text{ and } 0 \notin B_C.$$

Furthermore, we assume w.l.o.g. that the base B_C is given by a functional $y^* \in C^{\#} := \{y^* \in Y^* \mid \forall y \in C \setminus \{0\} : \ y^*(y) > 0\}$, this is $B_C = \{y \in C \mid y^*(y) = 1\}$. By normalization, we get $\| y^* \|_{Y^*} = 1$ (see Jadamba, Khan, Lopez and Sama [35]).

For $\varepsilon \in (0,1)$, the Henig dilating cone is defined by

$$D(\varepsilon) := \text{cl}[\text{cone}(B_C + \varepsilon B_Y^o)], \tag{5.7}$$

where B_C is the base of C and $B_Y^o := \{y \in Y \mid \| y \|_Y \leq 1\}$ is the closed unit ball in Y. It is well known that $D(\varepsilon)$ in (5.7) is a pointed, closed, convex cone with nonempty interior, $C = \bigcap_{0<\varepsilon<1} D(\varepsilon)$ (see [36, Theorem 1.1] and [35]) and $C \setminus \{0\} \subset \text{int} D(\varepsilon)$ for every $\varepsilon \in (0,1)$. Interesting applications of dilating cones for regularization methods are derived by Khan and Sama in [37].

The dual cone of the Henig dilating cone is given by (see Durea and Dutta [38, Lemma 3.7] and Jadamba, Khan, Lopez and Sama [35])

$$D(\varepsilon)^+ = \{0\} \cup \{y^* \in C^+ \setminus \{0\} \mid y^*(y) \geq \varepsilon \| y^* \|_{Y^*} \text{ for all } y \in B_C\}.$$

In the next theorem, we give a necessary condition for a properly robust solution w.r.t. the order relation α_2 introduced in (4.6) with the dilating cone (5.7).

Theorem 5.6. *Let Y be a Banach space of functions $F : \mathscr{U} \to \mathbb{R}$, $C \subset Y$ a nontrivial convex cone with $k_1 \equiv 1 \in C$ and $\mathscr{F}_1 = \{F_x \in Y \mid x \in \mathfrak{A}_1\}$ given by (4.2). Suppose that $\mathscr{F}_1$ is closed. Consider properly robust solution x, i.e., F_x is a properly minimal element of $\mathscr{F}_1$ w.r.t. α_2 with the dilating cone (5.7) of the properly robust counterpart problem (pRC), and the abstract subdifferential ∂ such that (H1) and (H2) are fulfilled.*
Then, there exists an element

$$\bar{y}^* \in D(\varepsilon)^+ = \{0\} \cup \{y^* \in C^+ \setminus \{0\} \mid y^*(y) \geq \varepsilon \| y^* \|_{Y^*} \textit{ for all } y \in B_C\} \textit{ with } \bar{y}^*(k_1) = 1$$

such that

$$-\bar{y}^* \in N(F_x; \mathscr{F}_1).$$

Proof. The proof is analogously to the proof of Theorem 5.3 using the functional (3.2) with $B = D(\varepsilon)$, $k = k_1 \equiv 1 \in C \setminus \{0\}$ (such that $k \in \text{int} D(\varepsilon)$) and taking into account the structure of the dual cone $D(\varepsilon)^+$ to the dilating cone $D(\varepsilon)$. □

6. Conclusions and Further Research

We derived necessary optimality conditions for solutions of strictly/regret/ε-constraint/properly robust counterpart problems using the unifying approaches to robustness and stochastic programming shown in [7] as well as a generic approach to the subdifferential. Corresponding results can also be shown for other types of robustness and stochastic programming (for instance reliability, adjustable robustness, minimizing the expectation, stochastic dominance) discussed in the unifying framework in [7]. For deriving the necessary optimality conditions, the unifying approach based on nonlinear scalarization is very unseful. However, it would be of interest to derive necessary optimality conditions using the approach based on vector optimization. Especially, employing Lemma 4.1 and Theorem 4.1, necessary conditions for optimal solutions $x \in \mathfrak{A}_1$ of (RC) in terms of vector variational inequality could be derived where a mapping $W : X \to L(X,Y)$, $W(x) \in L(X,Y)$ is involved:

$$\text{Find } x \in \mathfrak{A}_1 \text{ such that } (W(x))(u-x) \notin \operatorname{int} C_Y \text{ for every } u \in \mathfrak{A}_1.$$

For an overview of the theory on vector variational equalities, which may be useful here, we refer to Ansari, Köbis and Yao [39], and for a detailed analysis on existence results, one can consult [40]. Another approach for deriving necessary optimality conditions for solutions of the robust counterpart problems (RC) is based on set optimization, especially, a characterization of solutions with respect to the upper set relation using the results by Bao and Tammer in [41, Theorem 4.1].

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