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A CHARACTERIZATION OF THE ε-NORMAL SET AND ITS APPLICATION IN ROBUST CONVEX OPTIMIZATION PROBLEMS

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Abstract. Let $C := \{x \in \mathbb{R}^n : g(x,v) \leqq 0, \forall v \in \mathscr{V}\}$, where $g: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ is a continuous function such that, for all $v \in \mathbb{R}^p$, $g(\cdot, v)$ is a convex function, and $\mathscr{V} \subset \mathbb{R}^p$ is some uncertain set. In this paper, under the satisfaction of the robust characteristic cone constraint qualification, we first propose a represented form of the ε-normal set to the convex set C at a considered point $\bar{x} \in C$. Then, the proposed result is applied to formulate a (necessary and sufficient) approximate optimality theorem for a quasi (α, ε)-solution to the robust counterpart of a convex optimization problem in the face of data uncertainty.

Keywords. Approximate optimality conditions; ε-normal set; Generalized approximate solutions; Robust characteristic cone constraint qualification; Robust optimization.

1. Introduction

Minimizing a numerical function over a given set is an interesting and important mathematical problem in the view of both mathematical theory and real-world applications. In particular, if the given numerical function is convex and the given set is also convex, it is then known as a *convex optimization problem*; see, for example, [1, 2, 3] for more details. Besides, the given convex set is usually described by an inequality system of convex functions. In other words, a convex optimization problem reads as follows:

$$\min\ f(x) \quad \text{subject to} \quad g_i(x) \leqq 0,\ i = 1, \ldots, m, \tag{CP}$$

where $f,\ g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$ are convex functions. The convex optimization problem (CP) in the face of data uncertainty in the constraints can be captured by the one:

$$\min\ f(x) \quad \text{subject to} \quad g_i(x, v_i) \leqq 0,\ i = 1, \ldots, m, \tag{UCP}$$

where $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ is continuous, $g_i(\cdot, v_i)$ is convex and $v_i \in \mathbb{R}^q$ is an uncertain parameter which is in the set $\mathscr{V}_i \subset \mathbb{R}^q$, $i = 1, \ldots, m$.

In this paper, we explore problem (UCP) by examining its robust (worst-case) counterpart [4, 5]:

$$\min\ f(x) \quad \text{subject to} \quad g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathscr{V}_i,\ i = 1, \ldots, m. \tag{RCP}$$

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Let F be its feasible set of problem (RCP), where

$$F := \{x \in \mathbb{R}^n : g_i(x, v_i) \leqq 0, \forall v_i \in \mathscr{V}_i, i = 1, \ldots, m\}. \quad (1.1)$$

It is well-known that an optimal solution to a mathematical optimization problem may not be exact but very near to it. This fact leads to the concept of approximate solutions, which play an important role in algorithmic study of optimization problems; see, for example, [6, 7, 8, 9, 10] and the references therein. Noting that the notion of the so-called *quasi* ε*-solution* introduced by Loridan [11] is motivated by the celebrated Ekeland Variational Principle [12]. Based on this motivation, numerous researchers studied the approximate solutions in convex or nonconvex optimization problems, and established approximate necessary conditions for them under various constraint qualifications; see, for example, [11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] and the references therein.

In 2008, Beldiman *et al.* [23] introduced a new class of approximate solutions (including the one that is studied in this paper in details) in scalar/vector optimization problems, and discussed the relationship among the introduced approximate solutions. However, they did not examine approximate optimality conditions for such a class of approximate solutions. In this paper, we aim to study approximate optimality conditions for a quasi (α, ε)-solution (one of the generalized approximate solutions introduced by Beldiman *et al.* [23]) to the robust convex optimization problem (RCP). In fact, many results on robust optimality conditions were investigated over the years; see, for example, [24, 25, 26, 27, 28] and the references therein.

It is worth mentioning that for certain approximate solutions in the robust convex optimization, Lee and Lee [27] and Lee and Jiao [26] did some works. To be more precise, they employed the robust version of Farkas's lemma to study some characterizations of ε-solutions and quasi ε-solutions, respectively; see also the works [18, 19] by Sun and his collaborators. In particular, Jiao and Lee [25] also proposed approximate optimality conditions for quasi ε-solutions in the robust convex semidefinite program due to its special structure. In addition, Strodiot *et al* [20] pointed out an effective method, that is, by analyzing the ε-normal set when the feasible set was explicitly expressed in terms of (convex) inequality systems under Slater's constraint qualification; then they formulated approximate optimality for a convex programming problem.

In the present paper, we focus on the study of the quasi (α, ε)-solution due to [23] for problem (RCP). We mainly make contributions to robust convex optimization as follows:

- We study the representation of the ε-normal set to set C at a considered point $\bar{x} \in C$, where C is explicitly expressed in terms of a robust (convex) inequality system, under a constraint qualification named *robust characteristic cone constraint qualification* [24], which is weaker than Slater's constraint qualification.
- By using the proposed result on representation of the ε-normal set, we examine (necessary and sufficient) approximate optimality theorems for the quasi (α, ε)-solution to problem (RCP).
- As a byproduct, we also have the following assertion: for any $\varepsilon \geqq 0$, if $\bar{x} = 0$, then $\partial_\varepsilon \|\bar{x}\| = \partial \|\bar{x}\| = \mathbb{B}$.

The organization of this paper is as follows. Section 2 states some notations and preliminaries. In Section 3, we introduce the approximate solution concepts. In Section 4, we examine the approximate optimality theorem for the generalized approximate solution to problem (RCP). Finally, concluding remarks are given in Section 5.

2. PRELIMINARIES

This section gives some notations and preliminary results that are used in the paper. We abbreviate $(x_1, x_2, \ldots, x_n)$ by x. The Euclidean space $\mathbb{R}^n$ is equipped with the usual Euclidean norm $\|\cdot\|$. The nonnegative orthant of $\mathbb{R}^n$ is defined by $\mathbb{R}^n_+ := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \geqq 0, i = 1, \ldots, n\}$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. We say that a set $A \subset \mathbb{R}^n$ is convex whenever $\mu a_1 + (1-\mu)a_2 \in A$ for all $\mu \in [0,1]$, $a_1, a_2 \in A$. For a given set $A \subset \mathbb{R}^n$, we denote the interior, closure, and convex hull generated by A, by $\operatorname{int} A$, $\operatorname{cl} A$, and $\operatorname{co} A$, respectively.

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [-\infty, +\infty]$. Here, f is said to be proper if, for all $x \in \mathbb{R}^n$, $f(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by $\operatorname{dom} f$, that is, $\operatorname{dom} f := \{x \in \mathbb{R}^n \colon f(x) < +\infty\}$. The epigraph f is defined by $\operatorname{epi} f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \colon f(x) \leqq r\}$. The function f is said to be convex if for all $\mu \in [0,1]$, $f((1-\mu)x + \mu y) \leqq (1-\mu)f(x) + \mu f(y)$ for all $x, y \in \mathbb{R}^n$; equivalently, $\operatorname{epi} f$ is convex. The function f is said to be concave whenever $-f$ is convex. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function, the (convex) subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in \mathbb{R}^n \colon \langle x^*, y - x\rangle \leqq f(y) - f(x),\ \forall y \in \mathbb{R}^n\}, \text{ if } x \in \operatorname{dom} f, \\ \emptyset, \text{ otherwise.} \end{cases}$$

More generally, for any $\varepsilon \geqq 0$, the ε-subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial_\varepsilon f(x) = \begin{cases} \{x^* \in \mathbb{R}^n \colon \langle x^*, y - x\rangle \leqq f(y) - f(x) + \varepsilon,\ \forall y \in \mathbb{R}^n\}, \text{ if } x \in \operatorname{dom} f, \\ \emptyset, \text{ otherwise.} \end{cases}$$

We say f is a lower semicontinuous function if $\liminf\limits_{y \to x} f(y) \geqq f(x)$ for all $x \in \mathbb{R}^n$. Let δ_C be the *indicator function* with respect to a convex and closed subset C of $\mathbb{R}^n$, that is, $\delta_C(x) = 0$ if $x \in C$, and $\delta_C(x) = +\infty$ if $x \notin C$.

Definition 2.1 (ε-normal set)**.** Let $C \subset \mathbb{R}^n$ be a convex and closed set and $x \in C$. Then $N_\varepsilon(x, C) := \{\xi \in \mathbb{R}^n \colon \langle \xi, y - x\rangle \leqq \varepsilon,\ \forall y \in C\}$ is called the ε-*normal set* to C at x.

The following two lemmas are the *sum rule* and the *scalar product rule* of the ε-subdifferential that are used in the sequel.

Lemma 2.1. [2, Theorem 2.115] *Consider two proper convex functions $f_1, f_2 : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that* $\operatorname{ri}\operatorname{dom} f_1 \cap \operatorname{ri}\operatorname{dom} f_2 \neq \emptyset$. *Then, for $\varepsilon > 0$,*

$$\partial_\varepsilon (f_1 + f_2)(\bar{x}) = \bigcup_{\substack{\varepsilon_1 \geqq 0, \varepsilon_2 \geqq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} (\partial_{\varepsilon_1} f_1(\bar{x}) + \partial_{\varepsilon_2} f_2(\bar{x})).$$

Lemma 2.2. [2, Theorem 2.117] *For a proper convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and any $\varepsilon \geqq 0$,*

$$\partial_\varepsilon (\lambda f)(\bar{x}) = \lambda \partial_{\varepsilon/\lambda} f(\bar{x}), \forall \lambda > 0.$$

The following remark is useful in the sequel, and its proof is easy by definition.

Remark 2.1. Let δ_C be the indicator function with respect to a convex and closed subset C of $\mathbb{R}^n$, and let $\varepsilon \geqq 0$ be given. If $\bar{x} \in C$, then $\partial_\varepsilon \delta_C(\bar{x}) = N_\varepsilon(\bar{x}, C)$.

3. Approximate Solution Concepts

Recall the robust counterpart of problem (UCP) introduced in Section 1,

$$\min\ f(x) \quad \text{subject to} \quad g_i(x,v_i) \leqq 0,\ \forall v_i \in \mathscr{V}_i,\ i=1,\dots,m, \tag{RCP}$$

with its feasible set F defined in (1.1).

Definition 3.1. Let $\alpha \geqq 0$ and $\varepsilon \geqq 0$ be given. Then $\bar{x}$ is said to be

(i) an *ε-solution* to problem (RCP) if $f(\bar{x}) \leqq f(x)+\varepsilon,\ \forall x \in F$;
(ii) a *quasi α-solution* to problem (RCP) if $f(\bar{x}) \leqq f(x)+\alpha\|x-\bar{x}\|,\ \forall x \in F$;
(iii) a *regular (α,ε)-solution* to problem (RCP) if, for any $x \in F$, $\bar{x}$ is an ε-solution to problem (RCP) as well as a quasi α-solution.

Now, we introduce a generalized approximate solution, i.e., the so-called *quasi (α,ε)-solution*, to problem (RCP).

Definition 3.2. [23] Let $\alpha \geqq 0$ and $\varepsilon \geqq 0$ be given. Then $\bar{x}$ is said to be a *quasi (α,ε)-solution* to problem (RCP) if

$$f(\bar{x}) \leqq f(x)+\alpha\|x-\bar{x}\|+\varepsilon,\ \forall x \in F.$$

Remark 3.1. (i) If $\alpha=0$, a quasi (α,ε)-solution to problem (RCP) coincides with an ε-solution, Lee and Lee [27] studied some characterizations of ε-solutions to problem (RCP).
(ii) If $\varepsilon=0$, a quasi (α,ε)-solution to problem (RCP) coincides with a quasi α-solution, Lee and Jiao [26] explored some characterizations of quasi α-solutions to problem (RCP). In addition, they analyzed the difference between ε-solutions and quasi α-solutions.
(iii) If both $\alpha=0$ and $\varepsilon=0$, then the quasi (α,ε)-solution $\bar{x}$ deduces to be an exact minimizer (if exists) to problem (RCP).
(iv) In order to make the quasi (α,ε)-solution to problem (RCP) meaningful, hereafter, we always assume $\alpha>0$ and $\varepsilon>0$.

Below, we analyze the geometric meanings of ε-solutions, quasi α-solutions, regular (α,ε)-solutions, and quasi (α,ε)-solutions to problem (RCP); and demonstrate the differences among them.

- *Geometric interpretation of an ε-solution.* By definition of an ε-solution to problem (RCP), $f(\bar{x}) \leqq \inf_{x\in F} f(x)+\varepsilon$, we can easily see that the ε-solution set of problem (RCP) coincides with the intersection of the $(\inf_{x\in F} f(x)+\varepsilon)$-level set and the feasible set, that is, $\{\bar{x} \in F:\ f(\bar{x}) \leqq \inf_{x\in F} f(x)+\varepsilon\}$.
- *Geometric interpretation of a quasi α-solution.* By definition, $\bar{x}$ is a quasi α-solution to problem (RCP) if
$$f(\bar{x}) \leqq f(x)+\alpha\|x-\bar{x}\|,\ \ \forall x \in F,$$
in other words, a quasi α-solution says $f(x) \geqq g(x)$ for all $x \in F$ in the view of their graphs, where $g(x)=f(\bar{x})-\alpha\|x-\bar{x}\|$; moreover, if f is differentiable over its domain, one has, $\|\nabla f(\bar{x})\| \leqq \alpha$, see [11]. It is also worth noting that if a function $f: F \to \mathbb{R}$ has a quasi α-solution at $\bar{x}$, then f is calm from below at $\bar{x}$ in the view of Rockafellar and Wets [29].

- *Geometric interpretation of a quasi (α,ε)-solution.* By definition, $\bar{x}$ is a quasi (α,ε)-solution to problem (RCP) if

$$f(\bar{x}) \leqq f(x) + \alpha\|x-\bar{x}\| + \varepsilon,\ \forall x \in F.$$

Setting $g(x) := f(\bar{x}) - \alpha\|x-\bar{x}\| - \varepsilon$, we have $g(\bar{x}) = f(\bar{x}) - \varepsilon$. A quasi (α,ε)-solution says $f(x) \geqq g(x)$ for all $x \in F$ in the view of their graphs.

Remark 3.2. (i) For a given (possibly nonconvex) function, it is crucial to use the (local) concept as the following one: a point $\bar{x}$ is a quasi ε-solution of f over F, then $\bar{x}$ is a (local) minimum of the function $x \mapsto f(x) + \sqrt{\varepsilon}\|x-\bar{x}\|$ over F.

(ii) Very similarly but essentially different to the quasi ε-solution, a point $\bar{x}$ is a quasi (α,ε)-solution of f over F, then $\bar{x}$ is an ε-solution of the function $x \mapsto f(x) + \sqrt{\varepsilon}\|x-\bar{x}\|$ over F. This plays a key role when we study the quasi (α,ε)-solution to problem (RCP).

Below, we give some simple examples, in which the differences among the mentioned approximate solutions can be verified well.

Example 3.1. Consider the following problem

$$\min\ f(x) = x^2 \quad \text{subject to} \quad x \in \mathbb{R}. \tag{P$_1$}$$

Let $\alpha > 0$ and $\varepsilon > 0$ be given.

(i) $\bar{x}$ is an ε-solution to problem (P$_1$) if $f(\bar{x}) \leqq f(x) + \varepsilon, \forall x \in \mathbb{R}$, which is equivalent to say $f(\bar{x}) \leqq \inf_{x\in\mathbb{R}} f(x) + \varepsilon$, and the ε-solution set is $[-\sqrt{\varepsilon}, \sqrt{\varepsilon}]$;

(ii) $\bar{x}$ is a quasi α-solution to problem (P$_1$) if $f(\bar{x}) \leqq f(x) + \alpha\|x-\bar{x}\|, \forall x \in \mathbb{R}$, by a geometrical calculation, we obtain that the quasi α-solution set is $[-\frac{\alpha}{2}, \frac{\alpha}{2}]$;

(iii) the regular (α,ε)-solution set is $[\max\{-\sqrt{\varepsilon}, -\frac{\alpha}{2}\}, \min\{\sqrt{\varepsilon}, \frac{\alpha}{2}\}]$;

(iv) the quasi (α,ε)-solution set is $[-\sqrt{\varepsilon} - \frac{\alpha}{2}, \sqrt{\varepsilon} + \frac{\alpha}{2}]$.

Example 3.2. Consider the following problem

$$\min\ f(x) = \max\{x^2 + x, x^2 - x\} \quad \text{subject to} \quad x \in \mathbb{R}. \tag{P$_2$}$$

Let $\alpha > 0$ and $\varepsilon > 0$ be given.

(i) $\bar{x}$ is an ε-solution to problem (P$_2$) if $f(\bar{x}) \leqq f(x) + \varepsilon, \forall x \in \mathbb{R}$, which is equivalent to say $f(\bar{x}) \leqq \inf_{x\in\mathbb{R}} f(x) + \varepsilon$, and the ε-solution set is $[\frac{1-\sqrt{1+4\varepsilon}}{2}, \frac{\sqrt{1+4\varepsilon}-1}{2}]$;

(ii) $\bar{x}$ is a quasi α-solution to problem (P$_2$) if $f(\bar{x}) \leqq f(x) + \alpha\|x-\bar{x}\|, \forall x \in \mathbb{R}$, since $f(x) = \max\{x^2 + x, x^2 - x\}$ is an even function over $\mathbb{R}$, by a geometrical calculation, we obtain that the quasi α-solution set is

$$\begin{cases} \{0\}, & 0 < \alpha \leqq 1, \\ \left[\frac{1-\alpha}{2}, \frac{\alpha-1}{2}\right], & \alpha > 1; \end{cases}$$

(iii) the regular (α,ε)-solution set is

$$\begin{cases} \{0\}, & 0 < \alpha \leqq 1, \\ \left[\max\{\frac{1-\sqrt{1+4\varepsilon}}{2}, \frac{1-\alpha}{2}\}, \min\{\frac{\alpha-1}{2}, \frac{\sqrt{1+4\varepsilon}-1}{2}\}\right], & \alpha > 1; \end{cases}$$

(iv) the quasi (α,ε)-solution set is

$$\begin{cases} \left[\frac{1-\sqrt{1+4\varepsilon}}{2}, \frac{\sqrt{1+4\varepsilon}-1}{2}\right], & 0<\alpha\leqq 1, \\ \left[\frac{1-\alpha-2\sqrt{\varepsilon}}{2}, -\sqrt{\varepsilon}\right] \cup \left[\frac{1-\sqrt{1+4\varepsilon}}{2}, \frac{\sqrt{1+4\varepsilon}-1}{2}\right] \cup \left[\sqrt{\varepsilon}, \frac{\alpha-1+2\sqrt{\varepsilon}}{2}\right], & \alpha>1. \end{cases}$$

4. MAIN RESULTS

In this section, we establish approximate optimality theorem for a quasi (α,ε)-solution to problem (RCP) under a robust characteristic cone constraint qualification [24], that is, the cone

$$\bigcup_{v_i\in\mathscr{V}_i,\lambda_i\geqq 0} \operatorname{epi}\big(\sum_{i=1}^{m}\lambda_i g_i(\cdot,v_i)\big)^*$$

is closed and convex.

Note that $D := \bigcup_{v_i\in\mathscr{V}_i,\lambda_i\geqq 0} \operatorname{epi}\big(\sum_{i=1}^{m}\lambda_i g_i(\cdot,v_i)\big)^*$ is a cone in $\mathbb{R}^{m+1}$, which is called the *robust characteristic cone* [24].

Definition 4.1. We say a *robust characteristic cone constraint qualification* (RCCCQ) holds for problem (RCP) if the robust characteristic cone D is closed and convex.

Remark 4.1. Indeed, Jeyakumar and Li [24] shown that the robust characteristic cone D is convex whenever $g_i(x,\cdot)$ is concave and $\mathscr{V}_i\subseteq\mathbb{R}^q$, $i=1,\ldots,m$, is convex (see the following Proposition 4.1). In addition, they also proved that the robust characteristic cone D is closed whenever the robust slater condition holds, that is, $\{x\in\mathbb{R}^m\colon g_i(x_0,v_i)<0,\ \forall v_i\in\mathscr{V}_i,\ i=1,\ldots,m\}\neq\emptyset$ and $\mathscr{V}_i\subseteq\mathbb{R}^q$, $i=1,\ldots,m$, is convex and compact (see the following Proposition 4.2).

Proposition 4.1. [24] *Let $g_i:\mathbb{R}^n\times\mathbb{R}^q\to\mathbb{R}$, $i=1,\ldots,m$ be continuous functions. Suppose that each $\mathscr{V}_i\subseteq\mathbb{R}^q$, $i=1,\ldots,m$, is convex, for all $v_i\in\mathbb{R}^q$, $g_i(\cdot,v_i)$ is a convex function, and for each $x\in\mathbb{R}^n$, $g_i(x,\cdot)$ is concave on $\mathscr{V}_i$. Then D is convex.*

Proposition 4.2. [24] *Let $g_i:\mathbb{R}^n\times\mathbb{R}^q\to\mathbb{R}$, $i=1,\ldots,m$, be continuous functions such that for all $v_i\in\mathbb{R}^q$, $g_i(\cdot,v_i)$ is a convex function. Suppose that each $\mathscr{V}_i$, $i=1,\ldots,m$, is compact and convex, and there exists $x_0\in\mathbb{R}^n$ such that $g_i(x_0,v_i)<0$, $\forall v_i\in\mathscr{V}_i$, $i=1,\ldots,m$. Then D is closed.*

4.1. **Approximate optimality conditions for unconstrained problems.**

Lemma 4.1. *Consider the unconstrained convex optimization problem:*

$$\min\ f(x)\quad \text{subject to}\quad x\in\mathbb{R}^n. \tag{CP$_u$}$$

Then $\bar{x}$ is an ε-solution to problem (CP$_u$) if and only if $0\in\partial_\varepsilon f(\bar{x})$.

Proof. The proof is trivial. Thus the proof is omit here. □

Theorem 4.1. *Let $\bar{x}$ be a quasi (α,ε)-solution to problem (CP$_u$). Then there exist $\bar{\varepsilon}_f\geqq 0$ and $\bar{\varepsilon}_b\geqq 0$ with $\bar{\varepsilon}_f+\bar{\varepsilon}_b=\varepsilon$, such that*

$$0\in\partial_{\bar{\varepsilon}_f} f(\bar{x})+\alpha\partial_{\bar{\varepsilon}_b/\alpha}\|\cdot-\bar{x}\|(\bar{x}).$$

Proof. Since $\bar{x}$ is a quasi (α,ε)-solution to problem (CP$_u$), then $f(\bar{x}) \leqq f(x)+\alpha\|x-\bar{x}\|+\varepsilon,\ \forall x\in\mathbb{R}^n$, which is equivalent to

$$f(\bar{x})+\alpha\|\bar{x}-\bar{x}\| \leqq f(x)+\alpha\|x-\bar{x}\|+\varepsilon,\ \forall x\in\mathbb{R}^n.$$

In other words, by Remark 3.2 (ii), $\bar{x}$ is an ε-solution to the problem:

$$\min\ f(x)+\alpha\|x-\bar{x}\| \ \text{ subject to } \ x\in\mathbb{R}^n.$$

Along with Lemma 4.1, we have

$$0\in\partial_\varepsilon(f+\alpha\|\cdot-\bar{x}\|)(\bar{x}).$$

As $ridomf = ridom\|\cdot-\bar{x}\| = \mathbb{R}^n$, it then follows from Lemmas 2.1 and 2.2 that there exist $\bar{\varepsilon}_f \geqq 0$ and $\bar{\varepsilon}_b \geqq 0$ with $\bar{\varepsilon}_f+\bar{\varepsilon}_b=\varepsilon$ such that

$$0\in\partial_{\bar{\varepsilon}_f} f(\bar{x})+\alpha\partial_{\bar{\varepsilon}_b/\alpha}\|\cdot-\bar{x}\|(\bar{x}).$$

Thus, the proof is completed. □

Lemma 4.2. *For any $\varepsilon\geqq 0$, $\partial_\varepsilon\|\cdot-\bar{x}\|(\bar{x})=\mathbb{B}$.*

Proof. By definition of ε-subdifferential, one has

$$\partial_\varepsilon\|\cdot-\bar{x}\|(\bar{x})=\{\xi\in\mathbb{R}^n\colon\ \|x-\bar{x}\|\geqq\langle\xi,x-\bar{x}\rangle-\varepsilon,\ \forall x\in\mathbb{R}^n\}. \tag{4.1}$$

Taking $x=\lambda z,\ \forall\lambda>0$, and $\forall z\in\mathbb{R}^n$, then $x\in\mathbb{R}^n$, along with (4.1) we have,

$$\begin{aligned}\partial_\varepsilon\|\cdot-\bar{x}\|(\bar{x}) &= \{\xi\in\mathbb{R}^n\colon\ \|\lambda z-\bar{x}\|\geqq\langle\xi,\lambda z-\bar{x}\rangle-\varepsilon,\ \forall z\in\mathbb{R}^n,\ \forall\lambda>0\}\\ &= \{\xi\in\mathbb{R}^n\colon\ \|z-\frac{\bar{x}}{\lambda}\|\geqq\langle\xi,z-\frac{\bar{x}}{\lambda}\rangle-\frac{\varepsilon}{\lambda},\ \forall z\in\mathbb{R}^n,\ \forall\lambda>0\}\\ &= \{\xi\in\mathbb{R}^n\colon\ \|z\|\geqq\langle\xi,z\rangle,\ \forall z\in\mathbb{R}^n\} \qquad (4.2)\\ &= \mathbb{B}. \qquad (4.3)\end{aligned}$$

By taking $\lambda\to\infty$, (4.2) is attained, and (4.3) is followed by the well-known Cauchy–Schwarz inequality. □

Thanks to Lemma 4.2, we have the following result, which is the restatement of Theorem 4.1. Since its proof is trivial, we omit it here.

Theorem 4.2. *Let $\bar{x}$ be a quasi (α,ε)-solution to problem (CP$_u$). Then there exist $\bar{\varepsilon}_f\geqq 0$ and $\bar{\varepsilon}_b\geqq 0$ with $\bar{\varepsilon}_f+\bar{\varepsilon}_b=\varepsilon$ such that*

$$0\in\partial_{\bar{\varepsilon}_f} f(\bar{x})+\alpha\mathbb{B}. \tag{4.4}$$

Remark 4.2. Note that, in Theorem 4.2, inclusion (4.4) is unrelated to $\bar{\varepsilon}_b$, and this is reasonable due to Lemma 4.2.

4.2. **Representation of the ε-normal set.** In order to obtain the approximate optimality condition in terms of the constraint functions $g_i(x,v_i)\leqq 0,\ \forall v_i\in\mathscr{V}_i,\ i=1,\ldots,m$, the *$\varepsilon$-normal set* (see Definition 2.1) must be explicitly expressed in their terms.

Below, we present such a result, which modifies the one studied by Strodiot *et al* [20], under the robust characteristic cone constraint qualification (see Definition 4.1) rather than the Slater's constraint qualification.

Proposition 4.3. *Let $\varepsilon \geqq 0$ be given. Let $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ be continuous function such that, for all $v \in \mathscr{V} \subset \mathbb{R}^p$, where $\mathscr{V}$ is some uncertain set, $g(\cdot, v)$ is a convex function. Suppose that* (RCCCQ) *holds. Let $\bar{x} \in C := \{x \in \mathbb{R}^n \colon g(x,v) \leqq 0, \ \forall v \in \mathscr{V}\}$. Then, $\xi \in N_\varepsilon(\bar{x}, C)$ if and only if there exist $\bar{\lambda} \geqq 0$, $\bar{v} \in \mathscr{V}$ and $\bar{\varepsilon} \geqq 0$ such that*

$$\bar{\varepsilon} \leqq \bar{\lambda} g(\bar{x}, \bar{v}) + \varepsilon \ \text{ and } \ \xi \in \partial_{\bar{\varepsilon}}(\bar{\lambda} g)(\bar{x}, \bar{v}).$$

Proof. By the definition of ε-normal set (see Definition 2.1), we have

$$\begin{aligned} N_\varepsilon(\bar{x}, C) &= \{\xi \in \mathbb{R}^n \colon \langle \xi, x - \bar{x} \rangle \leqq \varepsilon, \ \forall x \in C\} \\ &= \{\xi \in \mathbb{R}^n \colon \langle -\xi, \bar{x} \rangle \leqq \langle -\xi, x \rangle + \varepsilon, \ \forall x \in C\}. \end{aligned}$$

In other words, $\bar{x}$ is an ε-optimal solution to the following robust optimization problem with linear objective function:

$$\min \quad \langle -\xi, x \rangle \quad \text{subject to} \quad g(x,v) \leqq 0, \ \forall v \in \mathscr{V}. \tag{LP}$$

Since the condition (RCCCQ) holds, that is, the cone $\bigcup_{v \in \mathscr{V}, \lambda \geqq 0} \operatorname{epi}(\lambda g(\cdot, v))^*$ is closed and convex, it follows from [27, Theorem 2.2] that $\bar{x}$ is an ε-optimal solution to problem (LP) if and only if there exist there exist $\bar{\lambda} \geqq 0$, $\bar{v} \in \mathscr{V}$ and $\bar{\varepsilon} \geqq 0$ such that

$$\bar{\varepsilon} \leqq \bar{\lambda} g(\bar{x}, \bar{v}) + \varepsilon \ \text{ and } \ \xi \in \partial_{\bar{\varepsilon}}(\bar{\lambda} g)(\bar{x}, \bar{v}).$$

Thus the desired result follows. □

Remark 4.3. We mention here that if $\varepsilon = 0$, then the *ε-normal set* at $\bar{x}$ to C becomes the *normal cone* to C at $\bar{x} \in C$. With the same condition (RCCCQ), Jiao *et al.* [30] obtained a result, which was the representation of the normal cone to C at $\bar{x} \in C$.

4.3. **Approximate optimality condition.** Now, we are ready to give the main theorem in this subsection, which is the approximate optimality condition for a quasi (α, ε)-solution to problem (RCP) under the fulfilment of the robust characteristic cone constraint qualification.

Theorem 4.3 (approximate optimality theorem)**.** *Let $\alpha > 0$ and $\varepsilon > 0$ be given. Let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$ be continuous functions such that, for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is convex on $\mathbb{R}^n$, and $v_i \in \mathbb{R}^q$ is an uncertain parameter which is in the set $\mathscr{V}_i \subset \mathbb{R}^q$, $i = 1, \ldots, m$. Suppose that* (RCCCQ) *holds for problem* (RCP)*. Then the following statements are equivalent*:

(i) *$\bar{x}$ is a quasi (α, ε)-solution to problem* (RCP);

(ii) *there exist $\bar{\varepsilon}_f \geqq 0$, $\bar{\varepsilon}_b \geqq 0$, $\bar{\varepsilon}_i \geqq 0$, $\bar{v}_i \in \mathscr{V}_i$, and $\bar{\lambda}_i \geqq 0$, $i = 1, \ldots, m$ such that*

$$0 \in \partial_{\bar{\varepsilon}_f} f(\bar{x}) + \sum_{i=1}^m \partial_{\bar{\varepsilon}_i}(\bar{\lambda}_i g_i(\cdot, \bar{v}_i))(\bar{x}) + \alpha \mathbb{B}, \tag{4.5}$$

$$\bar{\varepsilon}_f + \bar{\varepsilon}_b + \sum_{i=1}^m \bar{\varepsilon}_i - \varepsilon \leqq \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i). \tag{4.6}$$

Proof. [(i) ⇒ (ii)] Let $\bar{x}$ be a quasi (α, ε)-solution to problem (RCP). then

$$f(\bar{x}) \leqq f(x) + \alpha \|x - \bar{x}\| + \varepsilon, \ \forall x \in F,$$

which is equivalent to say that $\bar{x}$ is an ε-solution to the following unconstrained problem:

$$\min \ (f + \alpha \| \cdot - \bar{x} \| + \sum_{i=1}^m \delta_{F_i})(x) \quad \text{s.t.} \ \ x \in \mathbb{R}^n,$$

where $F_i = \{x \in \mathbb{R}^n \colon g_i(x, v_i) \leqq 0 \text{ for all } v_i \in \mathscr{V}_i\}$, $i = 1, \ldots, m$, observe that $F = \cap_{i=1}^m F_i$. Thanks to Theorem 4.2, there exist $\varepsilon_f \geqq 0$, $\varepsilon_b \geqq 0$, $\varepsilon_i \geqq 0$, $i = 1, \ldots, m$ with

$$\varepsilon_f + \varepsilon_b + \sum_{i=1}^m \varepsilon_i = \varepsilon, \tag{4.7}$$

such that $0 \in \partial_{\varepsilon_f} f(\bar{x}) + \alpha\mathbb{B} + \sum_{i=1}^m \partial_{\varepsilon_i} \delta_{F_i}(\bar{x})$. This, along with Remark 2.1, yields

$$0 \in \partial_{\varepsilon_f} f(\bar{x}) + \alpha\mathbb{B} + \sum_{i=1}^m N_{\varepsilon_i}(\bar{x}, F_i).$$

Now, applying Proposition 4.3 to F_i, $i = 1, \ldots, m$, one sees that there exist $\bar{\lambda}_i \geqq 0$ and $\bar{\varepsilon}_i \geqq 0$, $i = 1, \ldots, m$ such that

$$0 \in \partial_{\bar{\varepsilon}_f} f(\bar{x}) + \sum_{i=1}^m \partial_{\bar{\varepsilon}_i} (\bar{\lambda}_i g(\cdot, \bar{v}_i))(\bar{x}) + \alpha\mathbb{B}, \tag{4.8}$$

$$\bar{\varepsilon}_i - \varepsilon_i \leqq \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \leqq 0, \; i = 1, \ldots, m, \tag{4.9}$$

where $\bar{\varepsilon}_f = \varepsilon_f$ and $\bar{\varepsilon}_b = \varepsilon_b$. [Note that ε_b is unrelated to (4.8), but related to (4.7), where ε_b is a member for controlling ε]. Now summing (4.9) over $i = 1, \ldots, m$ and using the condition $\varepsilon_f + \varepsilon_b + \sum_{i=1}^m \varepsilon_i \leqq \varepsilon$ [in fact (4.7)] with $\bar{\varepsilon}_f = \varepsilon_f$ and $\bar{\varepsilon}_b = \varepsilon_b$ leads to (4.6) as desired.

[(ii) $\Rightarrow$ (i)] Since (4.5) and (4.6) holds for some $\bar{\varepsilon}_f \geqq 0$, $\bar{\varepsilon}_b \geqq 0$, $\bar{\varepsilon}_i \geqq 0$, $\bar{v}_i \in \mathscr{V}_i$ and $\bar{\lambda}_i \geqq 0$, $i = 1, \ldots, m$, then, there exist $\xi_f \in \partial_{\bar{\varepsilon}_f} f(\bar{x})$, $\xi_i \in \partial_{\bar{\varepsilon}_i} (\bar{\lambda}_i g(\cdot, \bar{v}_i))(\bar{x})$, and $b \in \mathbb{B}$ such that

$$0 = \xi_f + \sum_{i=1}^m \xi_i + \alpha b. \tag{4.10}$$

Moreover, we have

$$f(x) - f(\bar{x}) \geqq \langle \xi_f, x - \bar{x} \rangle - \bar{\varepsilon}_f, \; \forall x \in \mathbb{R}^n, \tag{4.11}$$

$$\bar{\lambda}_i g_i(x, \bar{v}_i) - \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \geqq \langle \xi_i, x - \bar{x} \rangle - \bar{\varepsilon}_i, \; \forall x \in \mathbb{R}^n. \tag{4.12}$$

Summing (4.12) over $i = 1, \ldots, m$ and adding it to (4.11) yields that, for all $x \in \mathbb{R}^n$,

$$[f(x) - f(\bar{x})] + [\sum_{i=1}^m \bar{\lambda}_i g_i(x, \bar{v}_i) - \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i)] \geqq \langle \xi_f + \sum_{i=1}^m \xi_i, x - \bar{x} \rangle - \bar{\varepsilon}_f - \sum_{i=1}^m \bar{\varepsilon}_i.$$

This, together with (4.6) and (4.10), arrives at

$$f(x) - f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i g_i(x, \bar{v}_i) \geqq -|\alpha b| \|x - \bar{x}\| - \varepsilon, \; \forall x \in \mathbb{R}^n.$$

Particularly, if $x \in F$, we have $f(x) - f(\bar{x}) \geqq -\alpha \|x - \bar{x}\| - \varepsilon$ for all $x \in F$. Hence $\bar{x}$ is a quasi (α, ε)-solution to problem (RCP). Thus, the proof is completed. □

5. Conclusions

In this paper, we mainly studied the approximate optimality theorem for a quasi (α, ε)-solution to the robust convex optimization problem (RCP) under the fulfilment of the (RCCCQ) condition. We do this by exploring the representation of the ε-normal set to a convex set (see Proposition 4.3). Examples are given to illustrate the differences among approximate solutions.

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