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WEAK HENIG PROPER SOLUTION SETS FOR SET OPTIMIZATION PROBLEMS

YU HAN

School of Statistics and Data Science, Jiangxi University of Finance and Economics, Nanchang 330013, China

Abstract. Huerga, Jiménez, and Novo introduced the notion of weak Henig proper solution sets for set optimization problems (J. Optim. Theory Appl. 195 (2022), 878-902). This paper aims to establish some characterizations of weak Henig proper solution sets for set optimization problems. We first obtain some properties of the Henig dilating cone and the continuity of nonlinear scalarizing functions with respect to the Henig dilating cone. Then, we derive density and connectedness of weak Henig proper solution sets for set optimization problems under some suitable conditions.

Keywords. Connectedness; Density theorem; Henig dilating cone; Henig proper solution; Set optimization problem.

1. Introduction

It is well known that the concept of efficient solutions plays an important role in solving vector optimization problems. However, sometimes the efficient solution set includes some points of a certain anomalous type and has some undesirable properties. For example, the efficient solution set can not be characterized by linear scalarization. Thus various concepts of classic proper efficiency were introduced in the literature, such as Geoffrion proper efficiency [1], Benson proper efficiency [2], Borwein proper efficiency [3], Henig proper efficiency [4], super efficiency [5], and strict efficiency [6].

In recent years, set optimization problems have been intensively studied due to their wide applications in many fields, such as optimal control problems, vector variational inequalities, vector optimization problems, fuzzy optimization problems, viability theory, image processing problems, mathematical economics, and differential inclusions. However, the studies on set optimization problems are far from enough. As pointed out by Khan et al. [7], since set-valued mappings appear naturally in many practical problems, set optimization problems will remain an important and active research topic in both the near and foreseeable future. To the best of our knowledge, there are only three papers [8–10] considering proper efficiency for set optimization problems. Huerga et al. [9] introduced two notions of proper efficiency in the sense of Henig (named Henig proper solution and weak Henig proper solution) for set optimization problems. The authors investigated some properties and compared these concepts with the analogous notions by using the vector criterion. Moreover, they derived a Lagrange multiplier rule

E-mail address: hanyumath@163.com
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for Henig proper solutions of set optimization problems. Therefore, it is necessary to explore further Henig proper solutions (or weak Henig proper solutions) for set optimization problems. In this paper, we establish some characterizations of weak Henig proper solution sets for set optimization problems.

Henig [4] introduced a concept of proper efficiency for vector optimization based on the idea of replacing the ordering cone by Henig dilating cone, which has nonempty interior and is bigger then the ordering cone. Henig dilating cone plays a key role in the study of Henig proper efficiency. For some properties of the Henig dilating cone, we refer to [5, 11, 12]. However, in order to study weak Henig proper solution sets for set optimization problems, there is a need to further investigate the properties of the Henig dilating cone. The first aim of this paper is to derive some properties of the Henig dilating cone and establish the continuity of nonlinear scalarizing functions with respect to the Henig dilating cone. In the past decades, density theorems for various proper efficiency, especially the generalizations of the density theorem of Arrow, Barankin and Blackwell have been extensively studied in the literature (see, e.g., [13–25] and the references therein). Up to our knowledge, the set of Henig proper efficient points is dense in the set of efficient points under some suitable conditions (see, e.g., [4, 17, 18, 21, 23, 25]). Thus, it is natural to understand whether we can obtain that weak Henig proper solution set for set optimization problems is dense in minimal solution set for set optimization problems. The second aim of this paper is to make an attempt in this direction. On the other hand, among many desirable properties of the solution sets, the connectedness is of considerable interest, since it provides the possibility of continuously moving (transformations) from one solution to another. It is worth noting that the studies on connectedness of solution sets for set optimization problems are still in the initial stage. Until now, to the best of our knowledge, it seems that there are only five papers [26–30] considering connectedness of solution sets for set optimization problems. Naturally, there is a need to explore further the connectedness of solution sets for set optimization problems. The third aim of this paper is to establish connectedness of weak Henig proper solution sets for set optimization problems.

The rest of the paper is organized as follows. In Section 2, we present some necessary notations and lemmas. In Section 3, we derive some properties of the Henig dilating cone and establish the continuity of nonlinear scalarizing functions with respect to the Henig dilating cone. In Section 4, we establish a density result for weak Henig proper solution sets of set optimization problems by using some properties of the Henig dilating cone. In Section 5, we investigate connectedness and arcwise connectedness of weak Henig proper solution sets for set optimization problems by employing the continuity of nonlinear scalarizing functions with respect to the Henig dilating cone.

2. Preliminaries

From now on, unless otherwise specified, let X and Y be two normed vector spaces. Assume that $C \subseteq Y$ is a nonempty, convex, closed, and pointed cone with $\text{int}C \neq \emptyset$. We denote by $\text{int}A$ and $\text{cl}A$ the topological interior and the topological closure, respectively. The family of the neighborhoods of $0 \in Y$ is denoted by $N(0)$. We denote by B_Y the closed unit ball in Y. Let A and B be two nonempty subsets of Y. The lower relation "$\leq_C^l$" and the weak lower relation "$\ll_C^l$" are defined, respectively, by

$$A \leq_C^l B \Leftrightarrow B \subseteq A + C$$

and

$$A \ll^l_C B \Leftrightarrow B \subseteq A + \mathrm{int} C.$$

A nonempty and convex subset $\bar{B}$ of the cone C is called a base of C if $0 \notin \mathrm{cl}(\bar{B})$ and $C = \mathrm{cone}(\bar{B}) := \{tb : t \geq 0, b \in \bar{B}\}$. Let $\bar{B}$ be a base of C. Due to $0 \notin \mathrm{cl}\bar{B}$, we have

$$\delta := \inf\{\|b\| : b \in \bar{B}\} > 0.$$

For $\varepsilon \in [0, \delta)$, let $C_\varepsilon := \mathrm{cl}(\mathrm{cone}(\bar{B} + \varepsilon B_Y))$. C_ε is known as the Henig dilating cone whenever $\varepsilon \in (0, \delta)$ (see [4]). It is clear that $C_0 = C$ and $C_{\varepsilon_1} \subseteq C_{\varepsilon_2}$ for $0 \leq \varepsilon_1 \leq \varepsilon_2 < \delta$. By [31, Remarks 2.1 and 2.2], we obtain that C_ε is a convex, closed, and pointed cone for any $\varepsilon \in (0, \delta)$. In the sequel, we always assume that C has a base $\bar{B}$.

Let $\wp_0(Y)$ be the family of all nonempty subsets of Y. It is said that a set $A \in \wp_0(Y)$ is C-proper if $A + C \neq Y$; C-convex if $A + C$ is a convex set; C-closed if $A + C$ is a closed set; C-bounded if, for any neighborhood U of $0 \in Y$, there exists $t > 0$ such that $A \subseteq tU + C$, and C-compact if any cover of A of the form $\{U_\alpha + C\}_{\alpha \in I}$, where U_α is open for any $\alpha \in I$, admits a finite subcover.

Remark 2.1. If $A \subseteq Y$ is C-compact, then A is C-bounded and C-closed (see [32]).

Remark 2.2. It follows from [33, Lemma 2.3] that if A is C-bounded, then A is C-proper.

Remark 2.3. For any $\varepsilon \in [0, \delta)$, it is clear that if A is C-bounded, then A is C_ε-bounded. By Remark 2.2, we obtain that if A is C-bounded, then A is C_ε-proper for any $\varepsilon \in [0, \delta)$.

Remark 2.4. Let $\varepsilon \in (0, \delta)$. It is easy to see that if A is C_ε-proper, then A is C-proper.

We give the following counterexample to illustrate that the converse does not hold.

Example 2.1. Let $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$. Let

$$\bar{B} = \{(x, y) \in \mathbb{R}^2 : x + y = 2, 0 \leq x \leq 2\}.$$

It is clear that $\bar{B}$ is a base of C and $\delta = \inf\{\|b\| : b \in \bar{B}\} = \sqrt{2}$. Let $\varepsilon = 1$. Then we obtain

$$C_\varepsilon = \mathrm{cl}(\mathrm{cone}(\bar{B} + \varepsilon B_Y)) = \left\{(x, y) \in \mathbb{R}^2 : y \geq -\sqrt{3}x, y \geq -\frac{1}{\sqrt{3}}x\right\}.$$

Let $A = \{(x, y) \in \mathbb{R}^2 : x \leq 0, y = 0\}$. Thus, it is easy to see that A is C-proper. However, A is not C_ε-proper.

Lemma 2.1. *[9] Let C and P be two convex cones with $C \subseteq P$. If A is C-compact, then A is P-compact.*

Let $F : X \to 2^Y$ be a set-valued mapping and $K \subseteq X$ with $K \neq \emptyset$. We consider the following set optimization problem:

$$\text{(SOP)} \quad \min F(x) \quad \text{subject to} \quad x \in K.$$

Definition 2.1. An element $x_0 \in K$ is said to be

(i) l-minimal solution of (SOP) if, for $x \in K$, $F(x) \preceq^l_C F(x_0)$ implies $F(x_0) \preceq^l_C F(x)$.

(ii) weak l-minimal solution of (SOP) if, for $x \in K$, $F(x) \ll^l_C F(x_0)$ implies $F(x_0) \ll^l_C F(x)$.

(iii) [9] weak l-Henig proper solution of (SOP) if there exists $\varepsilon \in (0, \delta)$ such that for $x \in K$, $F(x) \ll^l_{C_\varepsilon} F(x_0)$ implies $F(x_0) \ll^l_{C_\varepsilon} F(x)$.

Let $E_l(F,K,C)$, $W_l(F,K,C)$, and $H_l(F,K,C)$ denote the l-minimal solution set of (SOP), the weak l-minimal solution set of (SOP), and the weak l-Henig proper solution set of (SOP), respectively.

Remark 2.5. It follows from [34, Proposition 2.7] that $E_l(F,K,C) \subseteq W_l(F,K,C)$.

Remark 2.6. It is easy to see that $H_l(F,K,C) = \bigcup_{\varepsilon\in(0,\delta)} W_l(F,K,C_\varepsilon)$.

Lemma 2.2. *Let $x_0 \in K$ and $0 \le \varepsilon < \delta$. If $F(x_0)$ is C-compact, then $x_0 \in W_l(F,K,C_\varepsilon)$ if and only if there is no $y \in K$ satisfying $F(y) \ll^l_{C_\varepsilon} F(x_0)$.*

Proof. It is clear that $C \subseteq C_\varepsilon$. Since $F(x_0)$ is C-compact, it follows from Lemma 2.1 that $F(x_0)$ is C_ε-compact. By [33, Lemma 4.1], we can see that $x_0 \in W_l(F,K,C_\varepsilon)$ if and only if there is no $y \in K$ satisfying $F(y) \ll^l_{C_\varepsilon} F(x_0)$. This completes the proof. □

Lemma 2.3. *Let $x_0 \in K$ and $0 \le \varepsilon_1 \le \varepsilon_2 < \delta$. If $F(x_0)$ is C-compact and $x_0 \in W_l(F,K,C_{\varepsilon_2})$, then $x_0 \in W_l(F,K,C_{\varepsilon_1})$.*

Proof. Suppose that $x_0 \notin W_l(F,K,C_{\varepsilon_1})$. It follows from Lemma 2.2 that there exists $y_0 \in K$ such that $F(y_0) \ll^l_{C_{\varepsilon_1}} F(x_0)$, so $F(x_0) \subseteq F(y_0) + \text{int}C_{\varepsilon_1}$. Combining this with $C_{\varepsilon_1} \subseteq C_{\varepsilon_2}$, we obtain $F(x_0) \subseteq F(y_0) + \text{int}C_{\varepsilon_2}$, which implies $F(y_0) \ll^l_{C_{\varepsilon_2}} F(x_0)$. We conclude from Lemma 2.2 that $x_0 \notin W_l(F,K,C_{\varepsilon_2})$, which contradicts $x_0 \in W_l(F,K,C_{\varepsilon_2})$. Therefore, we obtain $x_0 \in W_l(F,K,C_{\varepsilon_1})$. This completes the proof. □

From Lemma 2.3 , it is easy to obtain the following corollary.

Corollary 2.1. *Let $0 \le \varepsilon_1 \le \varepsilon_2 < \delta$. Assume that $F(x)$ is C-compact for any $x \in K$. Then $W_l(F,K,C_{\varepsilon_2}) \subseteq W_l(F,K,C_{\varepsilon_1})$. In particular, $W_l(F,K,C_\varepsilon) \subseteq W_l(F,K,C)$ for any $\varepsilon \in [0,\delta)$.*

By Remark 2.6 and Corollary 2.1, we can obtain the following lemma.

Lemma 2.4. *Assume that $F(x)$ is C-compact for any $x \in K$. Then $H_l(F,K,C) \subseteq W_l(F,K,C)$.*

Definition 2.2. *[12]* Let T and T_1 be two topological vector spaces, and let C be a cone of T_1. A set-valued mapping $\Phi : T \to 2^{T_1}$ is said to be

(i) upper semicontinuous (u.s.c.) at $u_0 \in T$ if, for any neighborhood V of $\Phi(u_0)$, there exists a neighborhood U of u_0 such that for every $u \in U$, $\Phi(u) \subseteq V$.
(ii) C-upper semicontinuous (C-u.s.c.) at $u_0 \in T$ if, for any neighborhood V of $\Phi(u_0)$, there exists a neighborhood U of u_0 such that for every $u \in U$, $\Phi(u) \subseteq V + C$.
(iii) lower semicontinuous (l.s.c.) at $u_0 \in T$ if, for any $x \in \Phi(u_0)$ and any neighborhood V of x, there exists a neighborhood U of u_0 such that for every $u \in U$, $\Phi(u) \cap V \neq \emptyset$.
(iv) C-lower semicontinuous (C-l.s.c.) at $u_0 \in T$ if, for any $x \in \Phi(u_0)$ and any neighborhood V of x, there exists a neighborhood U of u_0 such that for every $u \in U$, $\Phi(u) \cap (V - C) \neq \emptyset$.

We say that Φ is u.s.c., C-u.s.c., l.s.c. and C-l.s.c. on T if it is u.s.c., C-u.s.c., l.s.c. and C-l.s.c. at each point $u \in T$, respectively. We say that Φ is continuous on T if it is both u.s.c. and l.s.c. on T. We also say that Φ is C-continuous on T if it is both C-u.s.c. and C-l.s.c. on T.

Definition 2.3. Let D be a nonempty convex subset of X. A set-valued mapping $\Phi : X \to 2^Y$ is said to be

(i) [35] C-convex on D if, for any $x_1, x_2 \in D$ and for any $t \in [0,1]$,

$$t\Phi(x_1) + (1-t)\Phi(x_2) \subseteq \Phi(tx_1 + (1-t)x_2) + C.$$

(ii) [36] naturally quasi C-convex on D if, for any $x_1, x_2 \in D$ and for any $t \in [0,1]$, there exists $\lambda \in [0,1]$ such that

$$\lambda\Phi(x_1) + (1-\lambda)\Phi(x_2) \subseteq \Phi(tx_1 + (1-t)x_2) + C.$$

(iii) [37] strictly naturally quasi C-convex on D if, for any $x_1, x_2 \in D$ with $x_1 \neq x_2$ and for any $t \in (0,1)$, there exists $\lambda \in [0,1]$ such that

$$\lambda\Phi(x_1) + (1-\lambda)\Phi(x_2) \subseteq \Phi(tx_1 + (1-t)x_2) + \text{int}C.$$

Remark 2.7. It is clear that if Φ is C-convex on D, then Φ is naturally quasi C-convex on D. However, the converse does not hold.

Definition 2.4. [38] Let K be a nonempty subset of X. A set-valued mapping $\Phi : X \to 2^Y$ is said to be strictly quasi l-C-convexlike on K if, for any $x_1, x_2 \in K$ with $x_1 \neq x_2$, there exist $x_3 \in K$ and $t_0 \in [0,1]$ such that

$$\Phi(x_3) \ll_C^l t_0\Phi(x_1) + (1-t_0)\Phi(x_2).$$

Definition 2.5. *[39]* Let $F : X \to 2^Y$ be a set-valued mapping, and let K be a nonempty subset of X. The sublevel set of F at x in K is the set

$$Q_l(x,C) := \left\{u \in K : F(u) \leq_C^l F(x)\right\}.$$

Remark 2.8. It is easy to see that $W_l(F, Q_l(x,C), C) \subseteq W_l(F,K,C)$ for any $x \in K$.

3. Continuity of Nonlinear Scalarizing Functions with respect to the Henig Dilating Cone

In this section, we establish the continuity of nonlinear scalarizing functions with respect to the Henig dilating cone.

Definition 3.1. Let $e \in -\text{int}C$. With respect to C_ε, we define the Gerstewitz's function $\phi_e : Y \times Y \times [0,\delta) \to \mathbb{R}$ by

$$\phi_e(a,y,\varepsilon) = \min\{t \in \mathbb{R} : y \in te + a + C_\varepsilon\}, \ \forall (a,y,\varepsilon) \in Y \times Y \times [0,\delta).$$

Replacing a by a set $A \in \wp_0(Y)$, we obtain the function $\psi_e : \wp_0(Y) \times Y \times [0,\delta) \to \mathbb{R} \cup \{-\infty\}$ as follows

$$\psi_e(A,y,\varepsilon) = \min\{t \in \mathbb{R} : y \in te + A + C_\varepsilon\}, \ \forall (A,y,\varepsilon) \in \wp_0(Y) \times Y \times [0,\delta).$$

Remark 3.1. Let $(A,y,\varepsilon) \in \wp_0(Y) \times Y \times [0,\delta)$. It is easy to see that $\psi_e(A,y,\varepsilon) = \inf_{a \in A}\{\phi_e(a,y,\varepsilon)\}$ (see [34]).

Inspired by [34, Definition 3.1], we give the following definition.

Definition 3.2. Define the function $G_e : \wp_0(Y) \times \wp_0(Y) \times [0,\delta) \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$G_e(A,B,\varepsilon) = \sup_{b \in B}\{\psi_e(A,b,\varepsilon)\}, \ \ \forall (A,B,\varepsilon) \in \wp_0(Y) \times \wp_0(Y) \times [0,\delta).$$

Remark 3.2. If $\varepsilon = 0$, it is clear that $G_e(A,B,\varepsilon)$ coincides with $G_e(A,B)$ in the Definition 3.1 of [34].

Remark 3.3. Let $(A,B,\varepsilon) \in \wp_0(Y) \times \wp_0(Y) \times [0,\delta)$. It follows from Remark 3.1 that

$$G_e(A,B,\varepsilon) = \sup_{b\in B} \inf_{a\in A} \{\phi_e(a,b,\varepsilon)\}.$$

Remark 3.4. Let $(A,B,\varepsilon) \in \wp_0(Y) \times \wp_0(Y) \times [0,\delta)$. It follows from [34, Lemma 2.16] that if A is C_ε-proper, then $G_e(A,B,\varepsilon) > -\infty$.

Remark 3.5. Let $(A,B,\varepsilon) \in \wp_0(Y) \times \wp_0(Y) \times [0,\delta)$. By [34, Theorem 3.6] and Remarks 2.3 and 3.4, we obtain that if A and B are C-bounded, then $-\infty < G_e(A,B,\varepsilon) < +\infty$.

Next, we give an example to illustrate the scalarizing function G_e.

Example 3.1. Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+ = \left\{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\right\}$ and $e = (-1,-1) \in -\mathrm{int}C$. Let

$$\bar{B} = \left\{(x,y) \in \mathbb{R}^2 : x+y = 2, 0 \leq x \leq 2\right\}.$$

We can see that $\bar{B}$ is a base of C and $\delta = \inf\{\|b\| : b \in \bar{B}\} = \sqrt{2}$. For any $\varepsilon \in \left(0,\sqrt{2}\right)$, let $\beta = \frac{\sqrt{4-\varepsilon^2}}{\varepsilon}$. Then we get $\beta > 1$ and

$$C_\varepsilon = \mathrm{cl}\left(\mathrm{cone}\left(\bar{B} + \varepsilon B_Y\right)\right) = \left\{(x,y) \in \mathbb{R}^2 : y \geq -\beta x, y \geq -\frac{1}{\beta}x\right\}.$$

For any $a = (a_1,a_2) \in Y$ and $y = (y_1,y_2) \in Y$, we have

$$\begin{aligned}
y \in te + a + C_\varepsilon \quad &\Leftrightarrow \quad (y_1,y_2) \in (-t,-t) + (a_1,a_2) + C_\varepsilon \\
&\Leftrightarrow \quad (t+y_1-a_1, t+y_2-a_2) \in C_\varepsilon \\
&\Leftrightarrow \quad t+y_2-a_2 \geq -\beta(t+y_1-a_1) \text{ and } t+y_2-a_2 \geq -\frac{1}{\beta}(t+y_1-a_1) \\
&\Leftrightarrow \quad t \geq \frac{1}{1+\beta}(a_2-y_2) + \frac{\beta}{1+\beta}(a_1-y_1) \text{ and} \\
&\qquad\;\; t \geq \frac{\beta}{1+\beta}(a_2-y_2) + \frac{1}{1+\beta}(a_1-y_1).
\end{aligned}$$

Thus, it is easy to obtain that

$$\begin{aligned}
\phi_e(a,y,\varepsilon) &= \min\{t \in \mathbb{R} : y \in te + a + C_\varepsilon\} \\
&= \max\left\{\frac{1}{1+\beta}(a_2-y_2) + \frac{\beta}{1+\beta}(a_1-y_1), \frac{\beta}{1+\beta}(a_2-y_2) + \frac{1}{1+\beta}(a_1-y_1)\right\}.
\end{aligned}$$

Let $(A,B,\varepsilon) \in \wp_0(Y) \times \wp_0(Y) \times \left(0,\sqrt{2}\right)$. We conclude from Remark 3.3 that

$$\begin{aligned}
&G_e(A,B,\varepsilon) = \sup_{y\in B} \inf_{a\in A} \{\phi_e(a,y,\varepsilon)\} \\
&= \sup_{y\in B} \inf_{a\in A} \max\left\{\frac{1}{1+\beta}(a_2-y_2) + \frac{\beta}{1+\beta}(a_1-y_1), \frac{\beta}{1+\beta}(a_2-y_2) + \frac{1}{1+\beta}(a_1-y_1)\right\},
\end{aligned}$$

where $\beta = \frac{\sqrt{4-\varepsilon^2}}{\varepsilon}$.

Let Λ_1 and Λ_2 be two normed vector spaces, and let $A:\Lambda_1\to 2^Y$ and $B:\Lambda_2\to 2^Y$ be two set-valued mappings. We define $\varsigma:\Lambda_1\times\Lambda_2\times[0,\delta)\to\mathbb{R}\cup\{-\infty,+\infty\}$ by

$$\varsigma(\lambda,u,\varepsilon)=G_e(A(\lambda),B(u),\varepsilon)=\sup_{b\in B(u)}\{\psi_e(A(\lambda),b,\varepsilon)\},\quad \forall(\lambda,u,\varepsilon)\in\Lambda_1\times\Lambda_2\times[0,\delta).$$

Remark 3.6. Let $(\lambda,u,\varepsilon)\in\Lambda_1\times\Lambda_2\times[0,\delta)$. If $A(\lambda)$ and $B(u)$ are C-bounded, then it follows from Remark 3.5 that $\varsigma(\lambda,u,\varepsilon)\in\mathbb{R}$.

Next, we give four lemmas concerned with the Henig dilating cone, which will be used in the sequel.

Lemma 3.1. *Let $\{\varepsilon_n\}\subseteq[0,\delta)$ with $\varepsilon_n\to\varepsilon_0\in[0,\delta)$. Assume that $z_n\in C_{\varepsilon_n}$ with $z_n\to z_0$. Then $z_0\in C_{\varepsilon_0}$.*

Proof. Since $z_n\in C_{\varepsilon_n}=\mathrm{cl}(\mathrm{cone}(\bar{B}+\varepsilon_nB_Y))$, there exists

$$y_n\in\left(z_n+\frac{1}{n}B_Y\right)\cap\mathrm{cone}(\bar{B}+\varepsilon_nB_Y).\tag{3.1}$$

It follows from $z_n\to z_0$ and (3.1) that $y_n\to z_0$. If $z_0=0$, it is clear that $z_0\in C_{\varepsilon_0}$. If $z_0\neq 0$, we conclude from (3.1) that there exist $\lambda_n>0$ (because of $z_0\neq0$), $a_n\in\bar{B}$ and $b_n\in B_Y$ such that

$$y_n=\lambda_n(a_n+\varepsilon_nb_n)=\lambda_n(a_n+\varepsilon_0b_n)+\lambda_n(\varepsilon_n-\varepsilon_0)b_n.\tag{3.2}$$

Let $\eta=\frac{1}{2}(\delta-\varepsilon_0)$. It follows from $\varepsilon_0\in[0,\delta)$ that $\eta>0$. Noting that

$$\|a_n\|=\|a_n+\varepsilon_nb_n-\varepsilon_nb_n\|\leq\|a_n+\varepsilon_nb_n\|+\|\varepsilon_nb_n\|,$$

we have

$$\|a_n+\varepsilon_nb_n\|\geq\|a_n\|-\|\varepsilon_nb_n\|.\tag{3.3}$$

Due to $\varepsilon_n\to\varepsilon_0$ and $b_n\in B_Y$, we have

$$\|\varepsilon_nb_n\|\leq\varepsilon_n\leq\varepsilon_0+\eta\tag{3.4}$$

for n large enough. We conclude from $a_n\in\bar{B}$ that $\|a_n\|\geq\inf\{\|b\|:b\in\bar{B}\}=\delta$. Combining this with (3.3) and (3.4), we have

$$\|a_n+\varepsilon_nb_n\|\geq\|a_n\|-\|\varepsilon_nb_n\|\geq\delta-(\varepsilon_0+\eta)=\eta$$

for n large enough. Suppose that $\{\lambda_n\}$ is unbounded. Without loss of generality, we assume that $\lambda_n\to+\infty$. Then

$$\|y_n\|=\lambda_n\|a_n+\varepsilon_nb_n\|\geq\lambda_n\eta\to+\infty.\tag{3.5}$$

It follows from $y_n\to z_0$ that $\{y_n\}$ is bounded, which contradicts (3.5). Thus, we obtain that $\{\lambda_n\}$ is bounded. Let $x_n=\lambda_n(a_n+\varepsilon_0b_n)$. It is clear that $x_n\in\mathrm{cone}(\bar{B}+\varepsilon_0B_Y)$. Noting that $\{\lambda_n\}$ and $\{b_n\}$ are bounded, and $\varepsilon_n\to\varepsilon_0$, one has $\lambda_n(\varepsilon_n-\varepsilon_0)b_n\to0$. This together with (3.2) and $y_n\to z_0$ implies that $x_n\to z_0$, so $z_0\in\mathrm{cl}(\mathrm{cone}(\bar{B}+\varepsilon_0B_Y))=C_{\varepsilon_0}$. This completes the proof. □

Lemma 3.2. *Assume that $\{\varepsilon_n\}\subseteq[0,\delta)$ with $\varepsilon_n\to\varepsilon_0\in[0,\delta)$. Then, for any $\beta>0$, there exists $\xi>0$ and $n_0\in\mathbb{N}$ such that*

$$(\beta B_Y)\cap C_{\varepsilon_n}\subseteq\mathrm{cl}\left(\bigcup_{\lambda\in[0,\xi]}\lambda(\bar{B}+\varepsilon_nB_Y)\right),\quad\forall n\geq n_0.$$

Proof. Suppose on the contrary that there exists $\beta_0 > 0$ such that, for any $\xi > 0$ and for any $n \in \mathbb{N}$, there is $n' \geq n$ satisfying $(\beta_0 B_Y) \cap C_{\varepsilon_{n'}} \not\subset \mathrm{cl}\left(\bigcup_{\lambda\in[0,\xi]} \lambda\left(\bar{B}+\varepsilon_{n'} B_Y\right)\right)$. In particular, for any $n \in \mathbb{N}$, choosing $\xi = n$, one sees that there exists $n' \geq n$ such that

$$(\beta_0 B_Y) \cap C_{\varepsilon_{n'}} \not\subset \mathrm{cl}\left(\bigcup_{\lambda\in[0,n]} \lambda\left(\bar{B}+\varepsilon_{n'} B_Y\right)\right).$$

Without loss of generality, we assume that

$$(\beta_0 B_Y) \cap C_{\varepsilon_n} \not\subset \mathrm{cl}\left(\bigcup_{\lambda\in[0,n]} \lambda\left(\bar{B}+\varepsilon_n B_Y\right)\right), \quad \forall n \in \mathbb{N}.$$

Then there exists $y_n \in (\beta_0 B_Y) \cap C_{\varepsilon_n}$ such that

$$y_n \notin \mathrm{cl}\left(\bigcup_{\lambda\in[0,n]} \lambda\left(\bar{B}+\varepsilon_n B_Y\right)\right). \tag{3.6}$$

It follows from (3.6) that there exists $W_n \in N(0)$ such that

$$(y_n + W_n) \cap \left(\bigcup_{\lambda\in[0,n]} \lambda\left(\bar{B}+\varepsilon_n B_Y\right)\right) = \emptyset. \tag{3.7}$$

Let $U_n = W_n \cap B_Y$. Since $y_n \in C_{\varepsilon_n} = \mathrm{cl}\left(\mathrm{cone}\left(\bar{B}+\varepsilon_n B_Y\right)\right)$. Then there exists

$$z_n \in (y_n + U_n) \cap \mathrm{cone}\left(\bar{B}+\varepsilon_n B_Y\right).$$

Due to (3.7), we have

$$z_n \notin \bigcup_{\lambda\in[0,n]} \lambda\left(\bar{B}+\varepsilon_n B_Y\right).$$

Then there exist $\lambda_n > n$ and $b_n \in \bar{B}+\varepsilon_n B_Y$ such that $z_n = \lambda_n b_n$. By $\varepsilon_n \to \varepsilon_0 \in [0,\delta)$, it is easy to see that there exists $\eta > 0$ and $\bar{n} \in \mathbb{N}$ such that, for any $n \geq \bar{n}$ and for any $b \in \bar{B}+\varepsilon_n B_Y$, $\|b\| \geq \eta$, so $\|b_n\| \geq \eta$ for any $n \geq \bar{n}$. Then $\|z_n\| = \lambda_n \|b_n\| \geq n\eta$ for all $n \geq \bar{n}$, which yields $\|z_n\| \to +\infty$.

On the other hand, it follows from $y_n \in \beta_0 B_Y$ that $\|y_n\| \leq \beta_0$. This together with $z_n \in y_n + U_n$ implies that $\|z_n\| \leq \beta_0 + 1$, which contradicts $\|z_n\| \to +\infty$. This completes the proof. □

Corollary 3.1. *Let $\varepsilon_0 \in [0,\delta)$. Then, for any $\beta > 0$, there exists $\xi > 0$ such that*

$$(\beta B_Y) \cap C_{\varepsilon_0} \subseteq \mathrm{cl}\left(\bigcup_{\lambda\in[0,\xi]} \lambda\left(\bar{B}+\varepsilon_0 B_Y\right)\right).$$

Lemma 3.3. *Assume that $\{\varepsilon_n\} \subseteq [0,\delta)$ with $\varepsilon_n \to \varepsilon_0 \in [0,\delta)$. Then, for any $\alpha > 0$ and for any $\beta > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$(\beta B_Y) \cap C_{\varepsilon_0} \subseteq \alpha B_Y + C_{\varepsilon_n}, \quad \forall n \geq n_0.$$

Proof. It follows from Corollary 3.1 that, for any $\beta > 0$, there exists $\xi > 0$ such that

$$(\beta B_Y) \cap C_{\varepsilon_0} \subseteq \mathrm{cl}\left(\bigcup_{\lambda \in [0,\xi]} \lambda\,(\bar{B} + \varepsilon_0 B_Y) \right). \tag{3.8}$$

For any $\alpha > 0$, it is clear that

$$\mathrm{cl}\left(\bigcup_{\lambda \in [0,\xi]} \lambda\,(\bar{B} + \varepsilon_0 B_Y) \right) \subseteq \frac{\alpha}{2} B_Y + \bigcup_{\lambda \in [0,\xi]} \lambda\,(\bar{B} + \varepsilon_0 B_Y). \tag{3.9}$$

We claim that there exists $n_0 \in \mathbb{N}$ such that

$$\bigcup_{\lambda \in [0,\xi]} \lambda\,(\bar{B} + \varepsilon_0 B_Y) \subseteq \frac{\alpha}{2} B_Y + \bigcup_{\lambda \in [0,\xi]} \lambda\,(\bar{B} + \varepsilon_n B_Y), \quad \forall n \geq n_0. \tag{3.10}$$

In fact, for any $y \in \bigcup_{\lambda \in [0,\xi]} \lambda\,(\bar{B} + \varepsilon_0 B_Y)$, there exist $\lambda_y \in [0,\xi]$, $a_y \in \bar{B}$ and $b_y \in B_Y$ such that $y = \lambda_y\,(a_y + \varepsilon_0 b_y)$. Due to $\varepsilon_n \to \varepsilon_0$, it is clear that there exists $n_0 \in \mathbb{N}$ such that

$$|\varepsilon_0 - \varepsilon_n| < \frac{\alpha}{2} \cdot \frac{1}{\xi}, \quad \forall n \geq n_0.$$

Then we have

$$\begin{aligned} y = \lambda_y\,(a_y + \varepsilon_0 b_y) &= \lambda_y\,(a_y + \varepsilon_n b_y) + \lambda_y\,(\varepsilon_0 - \varepsilon_n)\,b_y \\ &\in \bigcup_{\lambda \in [0,\xi]} \lambda\,(\bar{B} + \varepsilon_n B_Y) + \frac{\lambda_y}{\xi} \cdot \frac{\alpha}{2} B_Y \\ &\subseteq \bigcup_{\lambda \in [0,\xi]} \lambda\,(\bar{B} + \varepsilon_n B_Y) + \frac{\alpha}{2} B_Y. \end{aligned}$$

This means that (3.10) holds. We conclude from (3.8), (3.9), and (3.10) that, for any $n \geq n_0$,

$$\begin{aligned} \beta B_Y \cap C_{\varepsilon_0} &\subseteq \frac{\alpha}{2} B_Y + \bigcup_{\lambda \in [0,\xi]} \lambda\,(\bar{B} + \varepsilon_0 B_Y) \\ &\subseteq \frac{\alpha}{2} B_Y + \frac{\alpha}{2} B_Y + \bigcup_{\lambda \in [0,\xi]} \lambda\,(\bar{B} + \varepsilon_n B_Y) \\ &\subseteq \alpha B_Y + \mathrm{cone}\,(\bar{B} + \varepsilon_n B_Y) \\ &\subseteq \alpha B_Y + \mathrm{cl}\,(\mathrm{cone}\,(\bar{B} + \varepsilon_n B_Y)) = \alpha B_Y + C_{\varepsilon_n}. \end{aligned}$$

This completes the proof. □

Lemma 3.4. *Assume that $\{\varepsilon_n\} \subseteq [0,\delta)$ with $\varepsilon_n \to \varepsilon_0 \in [0,\delta)$. Then, for any $\alpha > 0$ and for any $\beta > 0$, there exists $n_0 \in \mathbb{N}$ such that $(\beta B_Y) \cap C_{\varepsilon_n} \subseteq \alpha B_Y + C_{\varepsilon_0}$ for all $n \geq n_0$.*

Proof. By Lemma 3.2, we obtain that, for any $\beta > 0$, there exists $\xi > 0$ and $\bar{n} \in \mathbb{N}$ such that

$$(\beta B_Y) \cap C_{\varepsilon_n} \subseteq \mathrm{cl}\left(\bigcup_{\lambda \in [0,\xi]} \lambda\,(\bar{B} + \varepsilon_n B_Y) \right), \quad \forall n \geq \bar{n}. \tag{3.11}$$

For any $\alpha > 0$, it is easy to see that

$$\mathrm{cl}\left(\bigcup_{\lambda\in[0,\xi]}\lambda\left(\bar{B}+\varepsilon_n B_Y\right)\right)\subseteq\frac{\alpha}{2}B_Y+\bigcup_{\lambda\in[0,\xi]}\lambda\left(\bar{B}+\varepsilon_n B_Y\right),\quad\forall n\in\mathbb{N}.\tag{3.12}$$

It follows from $\varepsilon_n\to\varepsilon_0$ that there exists $n_0\in\mathbb{N}$ such that $|\varepsilon_n-\varepsilon_0|<\frac{\alpha}{2}\cdot\frac{1}{\xi},\quad\forall n\geq n_0$. For any $n\geq n_0$ and for any $y\in\bigcup_{\lambda\in[0,\xi]}\lambda\left(\bar{B}+\varepsilon_n B_Y\right)$, there are $\lambda_y\in[0,\xi]$, $a_y\in\bar{B}$, and $b_y\in B_Y$ such that $y=\lambda_y\left(a_y+\varepsilon_n b_y\right)$. Thus,

$$\begin{aligned} y &= \lambda_y\left(a_y+\varepsilon_n b_y\right)=\lambda_y\left(a_y+\varepsilon_0 b_y\right)+\lambda_y\left(\varepsilon_n-\varepsilon_0\right)b_y\\ &\in \bigcup_{\lambda\in[0,\xi]}\lambda\left(\bar{B}+\varepsilon_0 B_Y\right)+\frac{\lambda_y}{\xi}\cdot\frac{\alpha}{2}B_Y\\ &\subseteq \bigcup_{\lambda\in[0,\xi]}\lambda\left(\bar{B}+\varepsilon_0 B_Y\right)+\frac{\alpha}{2}B_Y. \end{aligned}$$

Hence, we have

$$\bigcup_{\lambda\in[0,\xi]}\lambda\left(\bar{B}+\varepsilon_n B_Y\right)\subseteq\frac{\alpha}{2}B_Y+\bigcup_{\lambda\in[0,\xi]}\lambda\left(\bar{B}+\varepsilon_0 B_Y\right),\quad\forall n\geq n_0.\tag{3.13}$$

Thanks to (3.11), (3.12), and (3.13), for any $n\geq\max\{\bar{n},n_0\}$, one has

$$\begin{aligned} \left(\beta B_Y\right)\cap C_{\varepsilon_n} &\subseteq \frac{\alpha}{2}B_Y+\bigcup_{\lambda\in[0,\xi]}\lambda\left(\bar{B}+\varepsilon_n B_Y\right)\\ &\subseteq \frac{\alpha}{2}B_Y+\frac{\alpha}{2}B_Y+\bigcup_{\lambda\in[0,\xi]}\lambda\left(\bar{B}+\varepsilon_0 B_Y\right)\\ &\subseteq \alpha B_Y+\mathrm{cone}\left(\bar{B}+\varepsilon_0 B_Y\right)\subseteq\alpha B_Y+C_{\varepsilon_0}. \end{aligned}$$

This completes the proof. □

Theorem 3.1. *ϕ_e is continuous on $Y\times Y\times[0,\delta)$.*

Proof. Let $(a_0,y_0,\varepsilon_0)\in Y\times Y\times[0,\delta)$. We prove that ϕ_e is l.s.c. at (a_0,y_0,ε_0). In fact, if not, then there exists $\beta_0>0$ such that, for any neighborhood $U(a_0)\times U(y_0)\times U(\varepsilon_0)$ of (a_0,y_0,ε_0), there exists $(a',y',\varepsilon')\in U(a_0)\times U(y_0)\times U(\varepsilon_0)$ satisfying $\phi_e(a',y',\varepsilon')\leq\phi_e(a_0,y_0,\varepsilon_0)-\beta_0$. This means that there exists a sequence $\{(a_n,y_n,\varepsilon_n)\}\subseteq Y\times Y\times[0,\delta)$ with $(a_n,y_n,\varepsilon_n)\to(a_0,y_0,\varepsilon_0)$ such that $\phi_e(a_n,y_n,\varepsilon_n)\leq\phi_e(a_0,y_0,\varepsilon_0)-\beta_0$ for all $n\in\mathbb{N}$. Let $\gamma=\phi_e(a_0,y_0,\varepsilon_0)-\beta_0$. It is clear that $\phi_e(a_n,y_n,\varepsilon_n)\leq\gamma<\gamma+\frac{1}{n},\quad\forall n\in\mathbb{N}$. By the definition of $\phi_e(a_n,y_n,\varepsilon_n)$, there exists $t_n\in\left[\phi_e(a_n,y_n,\varepsilon_n),\gamma+\frac{1}{n}\right)$ such that $y_n\in t_n e+a_n+C_{\varepsilon_n}$. Thus

$$\begin{aligned} y_n &\in \left(\gamma+\frac{1}{n}\right)e+\left(t_n-\left(\gamma+\frac{1}{n}\right)\right)e+a_n+C_{\varepsilon_n}\\ &\subseteq \left(\gamma+\frac{1}{n}\right)e+\mathrm{int}C+a_n+C_{\varepsilon_n}\\ &\subseteq \left(\gamma+\frac{1}{n}\right)e+a_n+C_{\varepsilon_n}. \end{aligned}\tag{3.14}$$

Let $z_n = y_n - \left(\gamma + \frac{1}{n}\right) e - a_n$. We conclude from (3.14) that $z_n \in C_{\varepsilon_n}$. Noting that $z_n \to y_0 - \gamma e - a_0$ and $\varepsilon_n \to \varepsilon_0$, it follows from Lemma 3.1 that $y_0 - \gamma e - a_0 \in C_{\varepsilon_0}$, so $y_0 \in \gamma e + a_0 + C_{\varepsilon_0}$. Then

$$\phi_e(a_0, y_0, \varepsilon_0) \leq \gamma = \phi_e(a_0, y_0, \varepsilon_0) - \beta_0,$$

which is a contradiction.

Next, we show that ϕ_e is u.s.c. at $(a_0, y_0, \varepsilon_0)$. In fact, if not, then there exist $\rho_0 > 0$ and a sequence $\{(a_n, y_n, \varepsilon_n)\} \subseteq Y \times Y \times [0, \delta)$ with $(a_n, y_n, \varepsilon_n) \to (a_0, y_0, \varepsilon_0)$ such that

$$\phi_e(a_n, y_n, \varepsilon_n) \geq \phi_e(a_0, y_0, \varepsilon_0) + \rho_0, \quad \forall n \in \mathbb{N}.$$

Let $\eta = \phi_e(a_0, y_0, \varepsilon_0) + \rho_0$. Then

$$\phi_e(a_n, y_n, \varepsilon_n) \geq \eta, \quad \forall n \in \mathbb{N}. \tag{3.15}$$

It is clear that there exists $\varphi \in \mathbb{R}$ such that $\phi_e(a_0, y_0, \varepsilon_0) < \varphi < \eta$. Then there exists $t_0 \in [\phi_e(a_0, y_0, \varepsilon_0), \varphi)$ such that

$$y_0 \in t_0 e + a_0 + C_{\varepsilon_0}. \tag{3.16}$$

Due to $0 \in (\varphi - t_0) e + \mathrm{int} C$, there exists a bounded neighborhood W_0 of $0 \in Y$ such that $W_0 \subseteq (\varphi - t_0) e + C$. This together with (3.16) implies that

$$\begin{aligned} y_0 - a_0 + W_0 &\subseteq t_0 e + C_{\varepsilon_0} + (\varphi - t_0) e + C \\ &\subseteq \varphi e + C_{\varepsilon_0} + C_{\varepsilon_0} \subseteq \varphi e + C_{\varepsilon_0}. \end{aligned} \tag{3.17}$$

Since $y_n \to y_0$ and $a_n \to a_0$, there exists $n_1 \in \mathbb{N}$ such that

$$y_n - a_n \in y_0 - a_0 + W_0, \quad \forall n \geq n_1. \tag{3.18}$$

It is clear that there exists $\beta_0 > 0$ such that

$$y_0 - a_0 + W_0 - \varphi e \subseteq \beta_0 B_Y. \tag{3.19}$$

Let $r_0 = \frac{\eta - \varphi}{2}$. Due to $e \in -\mathrm{int} C$, there exists $\theta_0 > 0$ such that $\theta_0 B_Y \subseteq e + C$, so

$$r_0 \theta_0 B_Y \subseteq r_0 e + C. \tag{3.20}$$

For $\beta_0 > 0$ and for $r_0 \theta_0 > 0$, it follows from Lemma 3.3 that there exists $n_2 \in \mathbb{N}$ such that

$$(\beta_0 B_Y) \cap C_{\varepsilon_0} \subseteq r_0 \theta_0 B_Y + C_{\varepsilon_n}, \quad \forall n \geq n_2. \tag{3.21}$$

By (3.20) and (3.21), one has $(\beta_0 B_Y) \cap C_{\varepsilon_0} \subseteq r_0 e + C + C_{\varepsilon_n} \subseteq r_0 e + C_{\varepsilon_n}, \quad \forall n \geq n_2$. This together with (3.17) and (3.19) implies that

$$y_0 - a_0 + W_0 - \varphi e \subseteq (\beta_0 B_Y) \cap C_{\varepsilon_0} \subseteq r_0 e + C_{\varepsilon_n},$$

so $y_0 - a_0 + W_0 \subseteq (r_0 + \varphi) e + C_{\varepsilon_n}$ for all $n \geq n_2$. Combining this with (3.18), for any $n \geq \max\{n_1, n_2\}$, we obtain $y_n \in (r_0 + \varphi) e + a_n + C_{\varepsilon_n}$. This means that $\phi_e(a_n, y_n, \varepsilon_n) \leq r_0 + \varphi < 2r_0 + \varphi = \eta$, which contradicts (3.15). This completes the proof. □

Similar to the proof of [40, Theorem 5.1], we can obtain from Theorem 3.1 the following theorem.

Theorem 3.2. *If A and B are C-continuous with nonempty and C-compact values, then $\varsigma(\cdot,\cdot,\cdot)$ is continuous on $\Lambda_1 \times \Lambda_2 \times [0, \delta)$.*

4. Density of the Weak Henig Proper Solution Set

In this section, we obtain a density result for weak Henig proper solution sets of set optimization problems.

Lemma 4.1. *[41] Assume that K is nonempty and closed, $x \in K$ and F is C-u.s.c. on K with nonempty and C-closed values. Then $Q_l(x,C)$ is closed.*

Lemma 4.2. *[42] Let K be a nonempty subset of X. If F is strictly quasi l-C-convexlike on K with nonempty C-compact values, then $W_l(F,K,C) = E_l(F,K,C)$.*

Lemma 4.3. *Assume that K is a nonempty subset of X and F is strictly quasi l-C-convexlike on K with nonempty C-compact values. If $x_0 \in W_l(F,K,C)$, then $Q_l(x_0,C) = \{x_0\}$.*

Proof. It is clear that $x_0 \in Q_l(x_0,C)$. Suppose that there exists $\bar{x} \in Q_l(x_0,C)$ such that $\bar{x} \neq x_0$. Since F is strictly quasi l-C-convexlike on K, there exist $y_0 \in K$ and $t_0 \in [0,1]$ such that

$$F(y_0) \ll_C^l t_0 F(\bar{x}) + (1-t_0) F(x_0). \tag{4.1}$$

It follows from $\bar{x} \in Q_l(x_0,C)$ that $F(\bar{x}) \leq_C^l F(x_0)$, so $F(x_0) \subseteq F(\bar{x}) + C$. This together with (4.1) implies that

$$\begin{aligned} F(x_0) &\subseteq t_0 F(x_0) + (1-t_0) F(x_0) \subseteq t_0 F(\bar{x}) + t_0 C + (1-t_0) F(x_0) \\ &\subseteq F(y_0) + \operatorname{int} C + t_0 C \subseteq F(y_0) + \operatorname{int} C, \end{aligned}$$

which yields $F(y_0) \ll_C^l F(x_0)$. This together with Lemma 2.2 implies that $x_0 \notin W_l(F,K,C)$, which contradicts $x_0 \in W_l(F,K,C)$. This completes the proof. □

From [43, Propositions 29 and 30] and $E_l(F,K,C) \subseteq W_l(F,K,C)$, we can obtain the following lemma.

Lemma 4.4. *If K is nonempty compact and F is C-u.s.c. on K, then $W_l(F,K,C) \neq \emptyset$.*

It is easy to see the following lemma.

Lemma 4.5. *If K is closed and F is C-u.s.c. on K, then $W_l(F,K,C)$ is closed.*

Lemma 4.6. *Let $\bar{B}$ be a bounded base of C and $\{\varepsilon_n\} \subseteq [0,\delta)$ with $\varepsilon_n \to \varepsilon_0 \in [0,\delta)$. Then, for any $\omega > 0$, there exists $\beta > 0$ and $n_0 \in \mathbb{N}$ such that*

$$(\omega B_Y - C_{\varepsilon_n}) \cap C_{\varepsilon_n} \subseteq \beta B_Y, \quad \forall n \geq n_0.$$

Proof. Suppose on the contrary that there exists $\omega_0 > 0$ such that for any $\beta > 0$ and for any $n \in \mathbb{N}$, there is $n' \geq n$ satisfying $\left(\omega_0 B_Y - C_{\varepsilon_{n'}}\right) \cap C_{\varepsilon_{n'}} \not\subseteq \beta B_Y$. In particular, for any $n \in \mathbb{N}$, choosing $\beta = n$, one sees that there exists $n' \geq n$ such that $\left(\omega_0 B_Y - C_{\varepsilon_{n'}}\right) \cap C_{\varepsilon_{n'}} \not\subseteq n B_Y$. Without loss of generality, we assume that

$$(\omega_0 B_Y - C_{\varepsilon_n}) \cap C_{\varepsilon_n} \not\subseteq n B_Y, \quad \forall n \in \mathbb{N}.$$

Then there exist $z_n \in \omega_0 B_Y$, $c_n \in C_{\varepsilon_n}$ and $u_n \in C_{\varepsilon_n}$ such that $z_n - c_n = u_n$ and $\|u_n\| > n$. It follows from [5, Theorem 1.1] that

$$C_{\varepsilon_n} = \operatorname{cone}\left(\operatorname{cl}\left(\bar{B} + \varepsilon_n B_Y\right)\right), \quad \forall n \in \mathbb{N}. \tag{4.2}$$

By (4.2), $c_n \in C_{\varepsilon_n}$, and $u_n \in C_{\varepsilon_n}$, there exist $t_n \geq 0$, $\lambda_n \geq 0$, $b_n \in \mathrm{cl}(\bar{B}+\varepsilon_n B_Y)$ and $s_n \in \mathrm{cl}(\bar{B}+\varepsilon_n B_Y)$ such that $c_n = t_n b_n$ and $u_n = \lambda_n s_n$. Due to the boundedness of $\bar{B}$ and $\varepsilon_n \to \varepsilon_0$, it is clear that there exists $\varphi > 0$ such that

$$\|b\| \leq \varphi, \quad \forall b \in \mathrm{cl}(\bar{B}+\varepsilon_n B_Y), \forall n \in \mathbb{N},$$

which implies that $\|s_n\| \leq \varphi$ for any $n \in \mathbb{N}$. Then $n < \|u_n\| = \|\lambda_n s_n\| = \lambda_n \|s_n\| \leq \lambda_n \varphi$, which yields $\lambda_n \to +\infty$. Noting that $\{\varepsilon_n\} \subseteq [0,\delta)$ with $\varepsilon_n \to \varepsilon_0 \in [0,\delta)$, it is easy to see that there exists $\phi > 0$ such that

$$\|b\| \geq \phi, \quad \forall b \in \mathrm{cl}(\bar{B}+\varepsilon_n B_Y), \forall n \in \mathbb{N}. \tag{4.3}$$

In view of $z_n = c_n + u_n = t_n b_n + \lambda_n s_n$, it follows from the convexity of $\mathrm{cl}(\bar{B}+\varepsilon_n B_Y)$ that

$$\frac{z_n}{t_n+\lambda_n} = \frac{t_n}{t_n+\lambda_n} b_n + \frac{\lambda_n}{t_n+\lambda_n} s_n \in \mathrm{cl}(\bar{B}+\varepsilon_n B_Y).$$

This together with (4.3) implies that $\left\|\frac{z_n}{t_n+\lambda_n}\right\| \geq \phi > 0$, so $\|z_n\| \geq \phi(t_n+\lambda_n) \geq \phi\lambda_n$. Combining this with $\lambda_n \to +\infty$, we obtain $\|z_n\| \to +\infty$, which contradicts $z_n \in \omega_0 B_Y$. This completes the proof. □

Corollary 4.1. *Let $\bar{B}$ be a bounded base of C and $\varepsilon \in [0,\delta)$. Then, for any $\omega > 0$, there exists $\beta > 0$ such that $(\omega B_Y - C_\varepsilon) \cap C_\varepsilon \subseteq \beta B_Y$.*

Lemma 4.7. *Let $\bar{B}$ be a bounded base of C and $\{\varepsilon_n\} \subseteq [0,\delta)$ with $\varepsilon_n \to \varepsilon_0 \in [0,\delta)$. Then, for any $\alpha > 0$ and for any $\beta > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$(\beta B_Y - C_{\varepsilon_n}) \cap C_{\varepsilon_n} \subseteq \alpha B_Y + C_{\varepsilon_0}, \quad \forall n \geq n_0.$$

Proof. It follows from Lemma 4.6 that, for any $\beta > 0$, there exists $\theta > 0$ and $n_1 \in \mathbb{N}$ such that

$$(\beta B_Y - C_{\varepsilon_n}) \cap C_{\varepsilon_n} \subseteq \theta B_Y, \quad \forall n \geq n_1,$$

so

$$(\beta B_Y - C_{\varepsilon_n}) \cap C_{\varepsilon_n} \subseteq \theta B_Y \cap C_{\varepsilon_n}, \quad \forall n \geq n_1. \tag{4.4}$$

For any $\alpha > 0$ and for the above $\theta > 0$, we conclude from Lemma 3.4 that there exists $n_2 \in \mathbb{N}$ such that

$$(\theta B_Y) \cap C_{\varepsilon_n} \subseteq \alpha B_Y + C_{\varepsilon_0}, \quad \forall n \geq n_2. \tag{4.5}$$

Thanks to (4.4) and (4.5), for any $n \geq \max\{n_1, n_2\}$, one has $(\beta B_Y - C_{\varepsilon_n}) \cap C_{\varepsilon_n} \subseteq \alpha B_Y + C_{\varepsilon_0}$. This completes the proof. □

Theorem 4.1. *Let $\bar{B}$ be a bounded base of C. Assume that K is nonempty compact and F is C-u.s.c. and strictly quasi l-C-convexlike on K with nonempty C-compact values. Then*

$$W_l(F,K,C) = E_l(F,K,C) = \mathrm{cl}(H_l(F,K,C)).$$

Proof. It follows from Lemma 2.4 that $H_l(F,K,C) \subseteq W_l(F,K,C)$. By Lemma 4.5, we obtain that $W_l(F,K,C)$ is closed, so $\mathrm{cl}(H_l(F,K,C)) \subseteq W_l(F,K,C)$. Let $x_0 \in W_l(F,K,C)$, and let $\{\varepsilon_n\} \subseteq (0,\delta)$ with $\varepsilon_n \to 0$. We claim that, for any neighborhood W of $Q_l(x_0,C)$, there exists $n_0 \in \mathbb{N}$ such that

$$Q_l(x_0, C_{\varepsilon_n}) \subseteq W, \quad \forall n \geq n_0. \tag{4.6}$$

In fact, if (4.6) is not true, without loss of generality, we assume that there exists a neighborhood W_0 of $Q_l(x_0,C)$ such that $Q_l(x_0,C_{\varepsilon_n}) \not\subseteq W_0$ for any $n \in \mathbb{N}$. Then there exists $y_n \in Q_l(x_0,C_{\varepsilon_n})$ such that

$$y_n \notin W_0, \quad \forall n \in \mathbb{N}. \tag{4.7}$$

Since K is compact and $y_n \in K$, without loss of generality, we assume that $y_n \to y_0 \in K$.

Next, we prove that $F(x_0) \subseteq F(y_0)+C$, which implies $y_0 \in Q_l(x_0,C)$. Suppose that $F(x_0) \not\subseteq F(y_0)+C$. Then there exists $z_0 \in F(x_0)$ such that $z_0 \notin F(y_0)+C$. Since $F(y_0)$ is C-compact, we see that $F(y_0)$ is C-closed, so $F(y_0)+C$ is closed. Then there exists $\alpha_0 > 0$ such that

$$z_0 \notin F(y_0)+2\alpha_0 B_Y + C. \tag{4.8}$$

Due to $y_n \in Q_l(x_0,C_{\varepsilon_n})$, one has $F(y_n) \leq^l_{C_{\varepsilon_n}} F(x_0)$, so $F(x_0) \subseteq F(y_n)+C_{\varepsilon_n}$. This implies that there are $a_n \in F(y_n)$ and $c_n \in C_{\varepsilon_n}$ such that $z_0 = a_n + c_n$. Since F is C-u.s.c. at y_0, we have

$$a_n \in F(y_n) \subseteq F(y_0)+\alpha_0 B_Y + C \tag{4.9}$$

for n large enough. This means that there exist $v_n \in F(y_0)$, $w_n \in \alpha_0 B_Y$ and $h_n \in C$ such that $a_n = v_n + w_n + h_n$. Noting that $F(y_0)$ is C-compact, we obtain that $F(y_0)$ is C-bounded, so there exists $\varphi > 0$ such that $F(y_0) \subseteq \varphi B_Y + C$, which implies that there exist $b_n \in \varphi B_Y$ and $s_n \in C$ such that $v_n = b_n + s_n$. Thus,

$$z_0 = a_n + c_n = b_n + s_n + w_n + h_n + c_n. \tag{4.10}$$

Let $\beta = \|z_0\| + \varphi + \alpha_0$. It is clear that

$$z_0 - b_n - w_n - h_n - s_n \in \beta B_Y - C \subseteq \beta B_Y - C_{\varepsilon_n}.$$

Combining this with (4.10), we have

$$z_0 - b_n - w_n - h_n - s_n = c_n \in (\beta B_Y - C_{\varepsilon_n}) \cap C_{\varepsilon_n}. \tag{4.11}$$

For $\beta > 0$ and for $\alpha_0 > 0$, it follows from Lemma 4.7 that $(\beta B_Y - C_{\varepsilon_n}) \cap C_{\varepsilon_n} \subseteq \alpha_0 B_Y + C$ for n large enough. This together with (4.9) and (4.11) implies that

$$\begin{aligned} z_0 &\in b_n + w_n + h_n + s_n + \alpha_0 B_Y + C = a_n + \alpha_0 B_Y + C \\ &\subseteq F(y_0) + \alpha_0 B_Y + C + \alpha_0 B_Y + C \subseteq F(y_0) + 2\alpha_0 B_Y + C, \end{aligned}$$

which contradicts (4.8). Thus $y_0 \in Q_l(x_0,C)$. Noting that $y_n \to y_0 \in Q_l(x_0,C) \subseteq W_0$, we have $y_n \in W_0$ for n large enough, which contradicts (4.7). Therefore, we see that (4.6) holds. For any $y \in K$, noting that $F(y)$ is C-compact, it follows from Lemma 2.1 that $F(y)$ is C_{ε_n}-compact, so $F(y)$ is C_{ε_n}-closed. By Lemma 4.1, we obtain that $Q_l(x_0,C_{\varepsilon_n})$ is closed. Since K is compact and $Q_l(x_0,C_{\varepsilon_n}) \subseteq K$, we obtain that $Q_l(x_0,C_{\varepsilon_n})$ is compact. Due to Lemma 4.4, we have $W_l(F,Q_l(x_0,C_{\varepsilon_n}),C_{\varepsilon_n}) \neq \emptyset$. Let $x_n \in W_l(F,Q_l(x_0,C_{\varepsilon_n}),C_{\varepsilon_n})$. It is clear that $x_n \in Q_l(x_0,C_{\varepsilon_n})$. It follows from Lemma 4.3 that $Q_l(x_0,C) = \{x_0\}$. Combining this with (4.6), we have $x_n \to x_0$. By Remarks 2.6 and 2.8, we conclude

$$x_n \in W_l(F,Q_l(x_0,C_{\varepsilon_n}),C_{\varepsilon_n}) \subseteq W_l(F,K,C_{\varepsilon_n}) \subseteq H_l(F,K,C),$$

so $x_0 \in \mathrm{cl}(H_l(F,K,C))$. Therefore, we obtan $W_l(F,K,C) \subseteq \mathrm{cl}(H_l(F,K,C))$. In view of Lemma 4.2, one has

$$W_l(F,K,C) = E_l(F,K,C) = \mathrm{cl}(H_l(F,K,C)).$$

This completes the proof. □

5. Connectedness of the Weak Henig Proper Solution Set

In this section, we discuss connectedness and arcwise connectedness of weak Henig proper solution sets for set optimization problems. We define $\xi : K \times K \times [0,\delta) \to \mathbb{R} \cup \{-\infty,+\infty\}$ by

$$\xi(x,y,\varepsilon) = G_e(F(x),F(y),\varepsilon), \quad \forall (x,y,\varepsilon) \in K \times K \times [0,\delta).$$

Remark 5.1. Assume that $F(x)$ is C-bounded for any $x \in K$. Then we conclude from Remark 3.6 that $\xi(x,y,\varepsilon) \in \mathbb{R}$ for any $(x,y,\varepsilon) \in K \times K \times [0,\delta)$.

The set-valued mapping $H : K \times [0,\delta) \to K$ is defined by

$$\begin{aligned} H(x,\varepsilon) &= \{v \in K : G_e(F(y),F(x),\varepsilon) \geq G_e(F(v),F(x),\varepsilon),\ \forall y \in K\} \\ &= \{v \in K : \xi(y,x,\varepsilon) \geq \xi(v,x,\varepsilon),\ \forall y \in K\}, \quad \forall (x,\varepsilon) \in K \times [0,\delta). \end{aligned}$$

Lemma 5.1. [44] *Assume that A is a nonempty and connected subset of a topological space and $F : A \to 2^Y$ is an upper semicontinuous set-valued mapping with nonempty connected values. Then $F(A)$ is connected.*

The following lemma is well known.

Lemma 5.2. *Let Υ and Ω be two topological spaces. Assume that A is a nonempty and arcwise connected subset of Υ and $f : A \to \Omega$ is a continuous single-valued mapping. Then $f(A) := \bigcup_{x\in A} \{f(x)\}$ is arcwise connected.*

Lemma 5.3. *Assume that $F(x)$ is C-compact for any $x \in K$. Then*

$$H_l(F,K,C) = \bigcup_{(x,\varepsilon)\in K\times(0,\delta)} H(x,\varepsilon).$$

Proof. For any $x \in K$, since $F(x)$ is C-compact, it follows from Lemma 2.1 that $F(x)$ is C_ε-compact for any $\varepsilon \in (0,\delta)$. Similar to the proof of the Theorem 3.1 in [28], we can prove that

$$W_l(F,K,C_\varepsilon) = \bigcup_{x\in K} H(x,\varepsilon).$$

Combining this with Remark 2.6, we arrive at

$$H_l(F,K,C) = \bigcup_{\varepsilon\in(0,\delta)} \bigcup_{x\in K} H(x,\varepsilon) = \bigcup_{(x,\varepsilon)\in K\times(0,\delta)} H(x,\varepsilon).$$

This completes the proof. □

Similar to the proof of [28, Theorem 4.1 (i)], by Theorem 3.2, it is easy to obtain the following lemma.

Lemma 5.4. *Assume that K is nonempty and compact, and F is C-continuous with nonempty and C-compact values. Then H is u.s.c. on $K \times (0,\delta)$.*

Lemma 5.5. *Let K be a nonempty subset of X and $(x_0,\varepsilon_0) \in K \times (0,\delta)$. Assume that F is strictly quasi l-C-convexlike on K with nonempty C-compact values. Then $H(x_0,\varepsilon_0)$ is a singleton.*

Proof. It follows from [42, Lemma 3.2] that $\xi(\cdot, x_0, \varepsilon_0)$ is a strictly quasi convexlike function on K, i.e., for any $y_1, y_2 \in K$ with $y_1 \neq y_2$, there exist $y_3 \in K$ and $\lambda \in [0,1]$ such that

$$\xi(y_3, x_0, \varepsilon_0) < \lambda \xi(y_1, x_0, \varepsilon_0) + (1-\lambda)\xi(y_2, x_0, \varepsilon_0).$$

Thus, it is easy to see that $H(x_0, \varepsilon_0)$ is a singleton. This completes the proof. □

Theorem 5.1. *Let K be nonempty, convex and compact. Assume that F is C-continuous and naturally quasi C-convex on K with nonempty C-compact values. Then $H_l(F,K,C)$ is connected.*

Proof. Similar to the proof in [28, Lemmas 4.1 and 4.3], we can prove that $H(x,\varepsilon)$ is nonempty and convex for any $(x,\varepsilon) \in K \times (0,\delta)$. It follows from Lemma 5.4 that H is u.s.c. on $K \times (0,\delta)$. By Lemma 5.3, we have

$$H_l(F,K,C) = \bigcup_{(x,\varepsilon)\in K\times(0,\delta)} H(x,\varepsilon).$$

Therefore, we conclude from Lemma 5.1 that $H_l(F,K,C)$ is connected. This completes the proof. □

From Theorem 5.1 and Remark 2.7, we can obtain the following corollary.

Corollary 5.1. *Let K be nonempty, convex and compact. Assume that F is C-continuous and C-convex on K with nonempty C-compact values. Then $H_l(F,K,C)$ is connected.*

Next, we give an example to illustrate Theorem 5.1 and Corollary 5.1.

Example 5.1. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = [0,1]$ and $C = \mathbb{R}^2_+ = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$. The set-valued mapping $F_1 : X \to 2^Y$ is defined as follows

$$F_1(x) = \left((x-1)^2, \cos\left(x + \frac{\pi}{2}\right)\right) + B_Y, \quad x \in X.$$

It is clear that F_1 is C-continuous and C-convex on K with nonempty C-compact values. It follows from Corollary 5.1 that $H_l(F_1,K,C)$ is connected. The set-valued mapping $F_2 : X \to 2^Y$ is defined as follows

$$F_2(x) = \left(x^2, 1-x^2\right) + B_Y, \quad x \in X.$$

We can check that F_2 is C-continuous and naturally quasi C-convex on K with nonempty C-compact values. However, F_2 is not C-convex on K. We conclude from Theorem 5.1 that $H_l(F_2,K,C)$ is connected.

Theorem 5.2. *Let K be a nonempty, arcwise connected and compact subset of X. Assume that F is C-continuous and strictly quasi l-C-convexlike on K with nonempty C-compact values. Then $H_l(F,K,C)$ is arcwise connected.*

Proof. We conclude from Lemma 5.3 that

$$H_l(F,K,C) = \bigcup_{(x,\varepsilon)\in K\times(0,\delta)} H(x,\varepsilon).$$

Thanks to Lemmas 5.4 and 5.5, we obtain that $H : K \times (0,\delta) \to K$ is a continuous single-valued mapping. Noting that K is arcwise connected, we can see that $K \times (0,\delta)$ is arcwise connected. Therefore, it follows from Lemma 5.2 that $H_l(F,K,C)$ is arcwise connected. This completes the proof. □

Remark 5.2. Comparing Theorem 5.2 with [26, Theorem 3.1], we would like to point out the following facts.

(i) In Theorem 5.2, we do not need to require that K is convex. It is clear that convexity is a special case of arcwise connectedness. We weaken the convexity of K by assuming that K is arcwise connected.

(ii) For any $x \in K$, we replace compactness of $F(x)$ by C-compactness of $F(x)$, which is a weaker assumption.

(iii) It is clear that the class of C-continuous mappings is strictly larger than the class of continuous mappings. We weaken the continuity of F by assuming that F is C-continuous.

(iv) Obviously, if F is strictly naturally quasi C-convex on K, then F is strictly quasi C-convexlike on K. Thus, we weaken the assumption that F is strictly naturally quasi C-convex on K by assuming that F is strictly quasi C-convexlike on K.

(v) In Theorem 5.2, we derive arcwise connectedness of $H_l(F,K,C)$ by using a well known lemma (Lemma 5.2). It is worth noting that the proof methods in Theorem 5.2 are different from the ones in Theorem 3.1 of [26].

Finally, we give an example to illustrate Theorem 5.2.

Example 5.2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = [0,\pi]$ and $C = \mathbb{R}^2_+ = \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\right\}$. The set-valued mapping $F : X \to 2^Y$ is defined as follows

$$F(x) = \left(-2\sin x, x^2 - 6x + 3\right) + B_Y, \quad x \in X.$$

It is easy to see that $0 \notin W_l(F,K,C)$, and so $0 \notin H_l(F,K,C)$. This means that $H_l(F,K,C) \neq K$. We can check that all conditions of Theorem 5.2 are satisfied. Therefore, it follows from Theorem 5.2 that $H_l(F,K,C)$ is arcwise connected.

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