J. Nonlinear Var. Anal. 7 (2023), No. 6, pp. 985-994
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.7.2023.6.07

A NEW q-FRACTIONAL INTEGRAL OPERATOR AND ITS APPLICATIONS TO THE COEFFICIENT PROBLEM INVOLVING THE SECOND HANKEL DETERMINANT FOR q-STARLIKE AND q-CONVEX FUNCTIONS

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Abstract. In this article, we introduce and study a new q-fractional integral operator which essentially stems from a successive application of the Srivastava-Owa operator of fractional integration with the q-Ruscheweyh derivative operator. As an application of this new q-fractional integral operator, we investigate a coefficient problem involving the second Hankel determinant $H_2^2(f)$ for normalized analytic and univalent functions $f(z)$ belonging to some normalized families of q-starlike and q-convex functions in the open unit disk $\mathbb{U}$.

Keywords. Analytic functions; Hankel determinants; Quantum calculus; q-Starlike and q-Convex functions; Univalent functions.

1. Introduction, Definitions, and Preliminaries

Let $\mathscr{A}$ denote the class of functions f, which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and normalized by the following conditions:

$$f(0) = 0 \qquad \text{and} \qquad f'(0) - 1 = 0.$$

Thus, clearly, each function $f \in \mathscr{A}$ has a Taylor-Maclaurin series expansion of the form given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbb{U}). \tag{1.1}$$

In recent years, many researchers focused their attention on the connection between the fractional calculus (that is, the calculus of integrals and derivatives of arbitrary real or complex order) and the quantum (or q-) calculus in the area of Geometric Function Theory of Complex

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Received April 21, 2023; Accepted July 21, 2023.

Analysis, especially with the study univalent and multivalent functions. In fact, Srivastava's recent survey-cum-expository review article [1] has shown how important and potentially useful it is to study the concepts of the fractional calculus and the q-calculus in Geometric Function Theory of Complex Analysis. In [1], as well as in its predecessors [2] and [3], the interested reader can find a hybrid-type family of operators of Fractional Calculus with the kernel involving a unified form of various extensions and generalizations of the Mittag-Leffler and the Hurwitz-Lerch functions.

The celebrated operator of the Riemann-Liouville fractional integral has been successfully combined with such other powerful operators as (for example) the Ruscheweyh and Săalăgean derivative operators with a view to obtaining a beneficial function that can be used to introduce new families of operators (see, for details, [4–6]).

Fractional integral operators on the complex plane $\mathbb{C}$ were defined by Srivastava and Owa [7] as follows.

Definition 1.1. (see Srivastava and Owa [7]) The fractional integral of order α is defined, for a function $f(z)$, by

$$I_z^{\alpha} f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1}\, f(\zeta)\, \mathrm{d}\zeta \qquad (\alpha > 0), \tag{1.2}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{\sigma-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Recently, Aldweby and Darus [8] presented the q-Ruscheweyh derivative operator, which has subsequently been used widely to introduce and study several new classes of univalent functions (see, for example, [9–12]; see also the recent works [13–15]).

Definition 1.2. (see Aldweby and Darus [8]) The q-Ruscheweyh derivative operator is given by

$$R_q^{\tau} f(z) := z + \sum_{k=2}^{\infty} \frac{[k+\tau-1]_q!}{[\tau]_q!\,[k-1]_q!}\, a_k\, z^k \qquad (0 < q < 1;\ \tau > -1), \tag{1.3}$$

where the quantum (or q-) factorial $[\nu]_q!$ is defined for every non-negative integer ν by

$$[\nu]_q! = \begin{cases} [1]_q\,[2]_q\,[3]_q \cdots [\nu]_q & (\nu \in \mathbb{N} \setminus \{1\}) \\ 1 & (\nu = 1) \end{cases}$$

and, for the q-number $[\lambda]_q$,

$$[\lambda]_q = \begin{cases} \dfrac{1-q^{\lambda}}{1-q} & (\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}) \\ 0 & (\lambda = 0), \end{cases}$$

$\mathbb{N}$ being the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In this work, we introduce a presumably new q-fractional integral operator which is defined by successive application of the Srivastava-Owa operator and the q-Ruscheweyh derivative operator, which are given in Definition 1.1 and Definition 1.2, respectively. We also examine the second Hankel determinant of a class of normalized starlike and convex functions.

We recall that Noonan and Thomas [16] studied the following $\mathfrak{q}^{\text{th}}$ Hankel determinant $H_{\mathfrak{q}}^{n}(f)$ for a function $f(z)$ in the analytic function class $\mathscr{A}$, and given by (1.1), for $\mathfrak{q}, n \in \mathbb{N}$:

$$H_{\mathfrak{q}}^{n}(f) := \det \begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+\mathfrak{q}-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+\mathfrak{q}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+\mathfrak{q}-1} & a_{n+\mathfrak{q}} & \cdots & a_{n+2\mathfrak{q}-2} \end{pmatrix}.$$

Obviously, for $\mathfrak{q} = n = 2$, we have the second Hankel determinant defined by

$$H_2^2(f) := \det \begin{pmatrix} a_2 & a_3 \\ a_3 & a_4 \end{pmatrix} = \left|a_2 a_4 - a_3^2\right|.$$

In Geometric Function Theory of Complex Analysis, the finding of the upper bounds for the Hankel determinants for various subclasses of the normalized analytic function class $\mathscr{A}$ (such as the class $\mathscr{S}^*$ of starlike functions in $\mathbb{U}$ and the class $\mathscr{C}$ of convex functions in $\mathbb{U}$) continues to be an interesting problem (see, for details, the recent developments which are reported in [17–24]). Motivated by many of these recent developments, here we apply the approaches used by Janteng et al. (see [25] and [26]), and by Lupaş and Oros [5], to derive upper bounds for the second Hankel determinant $H_2^2(f)$ for functions f belonging to the following function classes $\mathscr{S}^*_{q,\alpha}$ and $\mathscr{C}_{q,\alpha}$:

$$\mathscr{S}^*_{q;\alpha,\tau} := \left\{ f : f \in \mathscr{A} \ \text{ and } \ \Re\left(\frac{z\mathfrak{D}_q\left(\mathfrak{I}_q^{\alpha,\tau} f(z)\right)}{\mathfrak{I}_q^{\alpha,\tau} f(z)} \right) > 0 \qquad (z \in \mathbb{U}) \right\} \tag{1.4}$$

and

$$\mathscr{C}_{q;\alpha,\tau} := \left\{ f : f \in \mathscr{A} \ \text{ and } \ \Re\left(1 + \frac{qz\mathfrak{D}_q^2\left(\mathfrak{I}_q^{\alpha,\tau} f(z)\right)}{\mathfrak{D}_q\left(\mathfrak{I}_q^{\alpha,\tau} f(z)\right)} \right) > 0 \qquad (z \in \mathbb{U}) \right\}, \tag{1.5}$$

respectively. Here, and in what follows, the aforementioned q-fractional integral operator $\mathfrak{I}_q^{\alpha,\tau}$ is given by Definition 1.3 below and $\mathfrak{D}_q$ denotes the q-derivative operator defined as follows:

$$\mathfrak{D}_q\big(f(z)\big) := \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\[2ex] f'(0) & (z = 0). \end{cases}$$

Clearly, when $q \to 1-$, the q-derivative $\mathfrak{D}_q\big(f(z)\big)$ reduces to the ordinary derivative $f'(z)$.

Definition 1.3. By successively applying the Srivastava-Owa operator I_z^{α}, defined by (1.2) and the q-Ruscheweyh derivative operator R_q^{τ}, defined by (1.3), we are led eventually to a presumably new q-fractional operator defined, for a function $f \in \mathscr{A}$ given by (1.1), as follows

$$\begin{aligned} \mathfrak{I}_q^{\alpha,\tau} f(z) &:= I_z^{\alpha}\, R_q^{\tau} f(z) \\ &= I_z^{\alpha}\, R_q^{\tau} \left(z + \sum_{k=2}^{\infty} a_k\, z^k \right) \\ &= z + \sum_{k=2}^{\infty} \frac{[k+\tau-1]_q!}{[\tau]_q!\,[k-1]_q!}\, \frac{\Gamma(k+1)\Gamma(\alpha+2)}{\Gamma(k+1+\alpha)}\, a_k\, z^k, \end{aligned}$$

which is easily verifiable.

The following lemmas will be needed in our present investigation.

Lemma 1.1. (see Duren [27] and [28]) *If the function $p(z)$ is in the class $\mathscr{P}$, for which*

$$\Re\big(p(z)\big) > 0 \qquad (z \in \mathbb{U})$$

and

$$p(z) = 1 + p_1 z + p_2 z + p_3 z^3 + \cdots \qquad (z \in \mathbb{U}),$$

then $|p_k| \leqq 2 \;\; (k \in \mathbb{N})$.

Lemma 1.2. (see Libera and Złotkiewicz [29]) *Let the function $p(z)$ be in the class $\mathscr{P}$, for which*

$$\Re\big(p(z)\big) > 0 \qquad (z \in \mathbb{U})$$

and

$$p(z) = 1 + p_1 z + p_2 z + p_3 z^3 + \cdots \qquad (z \in \mathbb{U}).$$

Then there exist $\zeta, z \in \mathbb{C}$ with $\max\{|\zeta|, |z|\} \leqq 1$ *such that*

$$p_2 = \frac{1}{2}\Big(p_1^2 + \zeta\left(4 - p_1^2\right)\Big)$$

and

$$p_3 = \frac{1}{4}\Big(p_1^3 + 2\left(4 - p_1^2\right)p_1\zeta - p_1\left(4 - p_1^2\right)\zeta^2 + 2\left(4 - p_1^2\right)\left(1 - |\zeta|^2\right)z\Big).$$

2. Applications to the Coefficient Problem for the Function Classes $\mathscr{S}^*_{q;\alpha,\tau}$ and $\mathscr{C}_{q;\alpha,\tau}$

In this section, we begin by stating our first main result as Theorem 2.1 below.

Theorem 2.1. *Let the function $f(z)$, given by (1.1), be in the class $\mathscr{S}^*_{q;\alpha,\tau}$. Then*

$$\left|a_2a_4 - a_3^2\right| \leqq \frac{4\big(([2]_q+1)([3]_q+1)-2\big)}{\hbar_2\hbar_4([2]_q-1)^2([3]_q-1)([4]_q-1)} + \frac{4([2]_q+1)^2}{\hbar_3^2([2]_q-1)^2([3]_q-1)^2},$$

where

$$\hbar_2 = [\tau+1]_q \, \frac{\Gamma(3)\,\Gamma(2+\alpha)}{\Gamma(3+\alpha)},$$

$$\hbar_3 = \frac{[\tau+2]_q[\tau+1]_q}{[2]_q!} \, \frac{\Gamma(4)\,\Gamma(2+\alpha)}{\Gamma(4+\alpha)}$$

and

$$\hbar_4 = \frac{[\tau+3]_q[\tau+2]_q[\tau+1]_q}{[3]_q!} \, \frac{\Gamma(5)\,\Gamma(2+\alpha)}{\Gamma(5+\alpha)}.$$

Proof. Since $f(z) \in \mathscr{S}^*_{q;\alpha,\tau}$, in light of the definition in equation (1.4), there exists a function $p(z)$ given by $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$ in $\mathscr{P}$ such that

$$z\mathfrak{D}_q\left(\mathfrak{I}_q^{\alpha,\tau} f(z)\right) = \left(\mathfrak{I}_q^{\alpha,\tau} f(z)\right) p(z) \qquad (\forall\, z \in \mathbb{U}).$$

This implies that

$$z\mathfrak{D}_q\left(z + \hbar_2a_2z^2 + \hbar_3a_3z^3 + \hbar_4a_4z^4 + \cdots\right) = \left(z + \hbar_2a_2z^2 + \hbar_3a_3z^3 + \hbar_4a_4z^4 + \cdots\right)$$
$$\left(1 + p_1z + p_2z^2 + p_3z^3 + \cdots\right),$$

which yields

$$\begin{aligned} z&+[2]_q\hbar_2 a_2 z^2+[3]_q\hbar_3 a_3 z^3+[4]_q\hbar_4 a_4 z^4+\cdots \\ &= z+(p_1+\hbar_2 a_2)\,z^2+(p_2+\hbar_2 p_1 a_2+\hbar_3 a_3)\,z^3+(p_3+\hbar_2 p_2 a_2+\hbar_3 p_1 a_3+\hbar_4 a_4)\,z^4+\cdots. \end{aligned}$$

Upon comparing the coefficients on both sides of this last equation, we are led to the following three equations:

$$a_2=\frac{p_1}{\hbar_2([2]_q-1)},$$

$$a_3=\frac{1}{\hbar_3([3]_q-1)}\left(p_2+\frac{p_1^2}{[2]_q-1}\right),$$

and

$$a_4=\frac{1}{\hbar_4([4]_q-1)}\left(p_3+\frac{p_2p_1}{[2]_q-1}+\frac{p_1}{[3]_q-1}\left(p_2+\frac{p_1^2}{[2]_q-1}\right)\right).$$

Thus, for the second Hankel determinant $H_2^2(f)$, we obtain

$$\begin{aligned} |H_2(2)| &:= \left|a_2a_4-a_3^2\right| \\ &= \left|\begin{array}{c} \dfrac{p_1}{\hbar_2\hbar_4([2]_q-1)([4]_q-1)}\left(p_3+\frac{p_2p_1}{[2]_q-1}+\dfrac{p_1}{[3]_q-1}\left(p_2+\frac{p_1^2}{[2]_q-1}\right)\right) \\ -\left(\frac{1}{\hbar_3([3]_q-1)}\left(p_2+\frac{p_1^2}{[2]_q-1}\right)\right)^2 \end{array}\right| \\ &= \left|\begin{array}{c} \dfrac{1}{\hbar_2\hbar_4([2]_q-1)([4]_q-1)}\left(p_1p_3+\frac{p_1^2p_2}{[2]_q-1}+\frac{p_1^2}{[3]_q-1}\left(p_2+\frac{p_1^2}{[2]_q-1}\right)\right) \\ -\dfrac{1}{\hbar_3^2([3]_q-1)^2}\left(p_2^2+\frac{2p_1^2p_2}{[2]_q-1}+\frac{p_1^4}{([2]_q-1)^2}\right) \end{array}\right|. \end{aligned}$$

Now, without any loss of generality, we can assume that $p_1 = \mathfrak{p}$ with $\mathfrak{p} \in [0,2]$. Then, by applying Lemma 1.1 and Lemma 1.2, we find that

$$\begin{aligned}
&\left|a_2a_4 - a_3^2\right| \\
&= \left| \begin{array}{c} \dfrac{1}{\hbar_2\hbar_4([2]_q-1)([4]_q-1)} \\ \cdot \left\{ \begin{array}{c} \dfrac{\mathfrak{p}}{4}\left(\mathfrak{p}^3 + 2\left(4-\mathfrak{p}^2\right)\mathfrak{p}\zeta - \mathfrak{p}\left(4-\mathfrak{p}^2\right)\zeta^2 + 2\left(4-\mathfrak{p}^2\right)\left(1-|\zeta|^2\right)z\right) \\ + \dfrac{\mathfrak{p}^2}{[2]_q-1}\left(\frac{1}{2}\left(\mathfrak{p}^2+\zeta\left(4-\mathfrak{p}^2\right)\right)\right) \\ + \dfrac{\mathfrak{p}^2}{[3]_q-1}\left(\frac{1}{2}\left(\mathfrak{p}^2+\zeta\left(4-\mathfrak{p}^2\right)\right) + \frac{\mathfrak{p}^2}{[2]_q-1}\right) \end{array} \right\} \\ -\dfrac{1}{\hbar_3^2([3]_q-1)^2}\left(\dfrac{1}{4}\left(\mathfrak{p}^2+\zeta\left(4-\mathfrak{p}^2\right)\right)^2 + \frac{2\mathfrak{p}^2}{[2]_q-1}\left(\frac{1}{2}\left(\mathfrak{p}^2+\left(4-\mathfrak{p}^2\right)\zeta\right)\right) + \frac{\mathfrak{p}^4}{([2]_q-1)^2}\right) \end{array} \right|.
\end{aligned}$$

By simplifying the right-hand side of the above equation and using the triangle inequality, we find for $r := |\zeta| \leqq 1$ that

$$\begin{aligned}
\left|a_2a_4 - a_3^2\right| \leqq\ & \frac{1}{4\hbar_2\hbar_4([2]_q-1)^2([3]_q-1)([4]_q-1)} \\
& \cdot \left\{ \begin{array}{c} \big(([2]_q+1)([3]_q+1)-2\big)p^4 \\ +([2]_q-1)([3]_q-1)\mathfrak{p}^2(4-\mathfrak{p}^2)r^2 \\ +2([2]_q-1)([3]_q-1)\mathfrak{p}\left(4-\mathfrak{p}^2\right)\left(1-r^2\right) \\ +\big(2([2]_q[3]_q-1)\big)\mathfrak{p}^2\left(4-\mathfrak{p}^2\right)r \end{array} \right\} \\
& + \frac{1}{4\hbar_3^2([2]_q-1)^2([3]_q-1)^2} \\
& \cdot \Big(\left([2]_q+1\right)^2\mathfrak{p}^4 + \left([2]_q-1\right)^2 r^2\left(4-\mathfrak{p}^2\right)^2 \\
& \qquad + 2\left([2]_q^2-1\right)r\mathfrak{p}^2(4-\mathfrak{p}^2)\Big).
\end{aligned}$$

If we denote the right-hand side of this last equation by $\Upsilon(\mathfrak{p},r)$ and make use of the derivative test to maximize the function $\Upsilon(\mathfrak{p},r)$ with

$$(\mathfrak{p},r) \in [0,2]\times[0,1],$$

we can eventually have

$$\begin{aligned}
\left|a_2a_4-a_3^2\right| &= \Upsilon(\mathfrak{p},r) \\
&\leqq \max_{0\leqq r\leqq 1;\, 0\leqq \mathfrak{p}\leqq 2}\{\Upsilon(\mathfrak{p},r)\} = \Upsilon(2,1) \\
&= \frac{1}{4\hbar_2\hbar_4([2]_q-1)^2([3]_q-1)([4]_q-1)}\Big(\big(2^4([2]_q+1)([3]_q+1)-2\big)\Big) \\
&\quad + \frac{1}{4\hbar_3^2([2]_q-1)^2([3]_q-1)^2}\Big(2^4([2]_q+1)^2\Big) \\
&= \frac{4\big(([2]_q+1)([3]_q+1)-2\big)}{\hbar_2\hbar_4([2]_q-1)^2([3]_q-1)([4]_q-1)} + \frac{4([2]_q+1)^2}{\hbar_3^2([2]_q-1)^2([3]_q-1)^2},
\end{aligned}$$

with evidently completes the proof of Theorem 2.1. □

We next turn to the coefficient problem involving the upper bounds for the second Hankel determinant $H_2^2(f)$ for the functions in the class $\mathscr{C}_q^{\alpha,\tau}$ defined by (1.5).

Theorem 2.2. *Let the function $f(z)$, given by (1.1), be in the class $\mathscr{C}_q^{\alpha,\tau}$. Then*

$$\begin{aligned}
\left|a_2a_4-a_3^2\right| \leqq{}& \frac{4([3]_q+1)([2]_q+1)}{([2]_q-1)^2[2]_q([3]_q-1)([4]_q-1)[4]_q\widetilde{\hbar}_2\widetilde{\hbar}_4} \\
&+ \frac{8([2]_q^2+1)}{([3]_q^2\widetilde{\hbar}_3-[3]_q\widetilde{\hbar}_3)^2([2]_q-1)^2},
\end{aligned}$$

where

$$\widetilde{\hbar}_2 = [\tau+1]_q\,\frac{\Gamma(3)\,\Gamma(2+\alpha)}{\Gamma(3+\alpha)},$$

$$\widetilde{\hbar}_3 = \frac{[\tau+2]_q[\tau+1]_q}{[2]_q!}\,\frac{\Gamma(4)\,\Gamma(2+\alpha)}{\Gamma(4+\alpha)},$$

and

$$\widetilde{\hbar}_4 = \frac{[\tau+3]_q[\tau+2]_q[\tau+1]_q}{[3]_q!}\,\frac{\Gamma(5)\,\Gamma(2+\alpha)}{\Gamma(5+\alpha)}.$$

Proof. First of all, the Product Rule for the q-derivative operator $\mathfrak{D}_q$ states that

$$\mathfrak{D}_q\big(f(z)\cdot g(z)\big) = \mathfrak{D}_q\big(f(z)\big)\cdot g(z) + f(qz)\cdot\mathfrak{D}_q\big(g(z)\big),$$

provided that each member exists. Thus, clearly, the following inequality in the definition (1.5):

$$\Re\left(1+\frac{qz\mathfrak{D}_q^2\left(\mathfrak{I}_q^{\alpha,\tau}f(z)\right)}{\mathfrak{D}_q\left(\mathfrak{I}_q^{\alpha,\tau}f(z)\right)}\right) > 0 \qquad (z\in\mathbb{U})$$

can be rewritten as follows

$$\Re\left(\frac{\mathfrak{D}_q\Big(z\mathfrak{D}_q\big(\mathfrak{I}_q^{\alpha,\tau}f(z)\big)\Big)}{\mathfrak{D}_q\left(\mathfrak{I}_q^{\alpha,\tau}f(z)\right)}\right) > 0 \qquad (z\in\mathbb{U}).$$

Let us now assume that $f(z) \in \mathscr{C}_q^{\alpha,\tau}$. Then, in view of the definition (1.5), there exists a function $p(z)$ given by $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ in $\mathscr{P}$ such that

$$1 + \frac{qz\mathfrak{D}_q^2\left(\mathfrak{I}_q^{\alpha,\tau} f(z)\right)}{\mathfrak{D}_q\left(\mathfrak{I}_q^{\alpha,\tau} f(z)\right)} = p(z) \qquad (\forall\, z \in \mathbb{U})$$

or, equivalently, that

$$\mathfrak{D}_q\left(z\mathfrak{D}_q\left(\mathfrak{I}_q^{\alpha,\tau} f(z)\right)\right) = \mathfrak{D}_q\left(\mathfrak{I}_q^{\alpha,\tau} f(z)\right) \cdot p(z) \qquad (\forall\, z \in \mathbb{U}). \tag{2.1}$$

If we set

$$\mathfrak{F}_{q;\alpha,\tau}(z) := \mathfrak{D}_q\left(\mathfrak{I}_q^{\alpha,\tau} f(z)\right) \qquad (z \in \mathbb{U}),$$

then the following extension of Alexander's theorem can be easily verified from the definitions (1.4) and (1.5):

$$\mathfrak{F}_{q;\alpha,\tau}(z) \in \mathscr{C}_q^{\alpha,\tau} \iff z\mathfrak{D}_\alpha\left(\mathfrak{F}_{q;\alpha,\tau}(z)\right).$$

We now write equation (2.1) as follows:

$$\begin{aligned}\mathfrak{D}_q&\left(z\mathfrak{D}_q\left(z + \widetilde{\hbar}_2 a_2 z^2 + \widetilde{\hbar}_3 a_3 z^3 + \widetilde{\hbar}_4 a_4 z^4 + \cdots\right)\right)\\ &= \mathfrak{D}_q\left(z + \widetilde{\hbar}_2 a_2 z^2 + \widetilde{\hbar}_3 a_3 z^3 + \widetilde{\hbar}_4 a_4 z^4 + \cdots\right) \cdot \left(1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots\right),\end{aligned}$$

which readily yields

$$\begin{aligned}1 &+ [2]_q^2 \widetilde{\hbar}_2 a_2 z + [3]_q^2 \widetilde{\hbar}_3 a_3 z^2 + [4]_q^2 \widetilde{\hbar}_4 a_4 z^3 + \cdots\\ &= 1 + \left(p_1 + [2]_q \widetilde{\hbar}_2 a_2\right) z + \left(p_2 + [2]_q \widetilde{\hbar}_2 p_1 a_2 + [3]_q \widetilde{\hbar}_3 a_3\right) z^2\\ &\quad + \left(p_3 + [2]_q \widetilde{\hbar}_2 p_2 a_2 + [3]_q \widetilde{\hbar}_3 p_1 a_3 + [4]_q \widetilde{\hbar}_4 a_4\right) z^3 + \cdots.\end{aligned} \tag{2.2}$$

If we compare the coefficients on both sides of this last equation (2.2), we are led to the following three equations:

$$a_2 = \frac{p_1}{[2]_q^2 \widetilde{\hbar}_2 - [2]_q \widetilde{\hbar}_2},$$

$$a_3 = \frac{1}{[3]_q^2 \widetilde{\hbar}_3 - [3]_q \widetilde{\hbar}_3}\left(p_2 + \frac{p_1^2}{[2]_q - 1}\right),$$

and

$$a_4 = \frac{1}{[4]_q^2 \widetilde{\hbar}_4 - [4]_q \widetilde{\hbar}_4}\left(p_3 + \frac{p_1 p_2}{[2]_q - 1} + \frac{p_1 p_2}{[3]_q - 1} + \frac{p_1^3}{([2]_q - 1)([3]_q - 1)}\right).$$

The remainder of the proof of Theorem 2.2 would run parallel to that of Theorem 2.2, which we have already presented fairly adequately. We, therefore, leave the details involved as an exercise for the interested reader. □

3. Concluding Remarks and Observations

In our present investigation, we introduced a presumably new q-fractional integral operator $\mathfrak{I}_q^{\alpha,\tau}$, which combines the q-Ruscheweyh derivative operator R_q^τ with the Srivastava-Owa fractional integral operator I_z^α. We then applied this q-fractional integral operator $\mathfrak{I}_q^{\alpha,\tau}$ in order to define two general function classes $\mathscr{S}^*_{q;\alpha,\tau}$ and $\mathscr{C}_q^{\alpha,\tau}$ that are analogous to the widely- and extensively-studied classes q-starlike and q-convext functions, respectively. Finally, for functions belonging to each of these newly-refined function classes $\mathscr{S}^*_{q;\alpha,\tau}$ and $\mathscr{C}_q^{\alpha,\tau}$, we derived upper bounds for the second Hankel determinant $H_2^2(f)$. Our findings in this paper could potentially lead to further researches and applications in related areas of mathematical sciences.

In conclusion, we remark that a large number of mostly amateurish-type researchers on these and other related topics continue to produce and publish obvious and inconsequential variations and straightforward translations of the known q-results in terms of the so-called $(\widetilde{p},q)$-calculus by unnecessarily forcing-in an obviously superfluous (or redundant) parameter $\widetilde{p}$ into the classical q-calculus and thereby falsely claiming "generalization" (see [1, p. 340] and [2, Section 5, pp. 1511–1512]; see also the recent survey-cum-expository review article [30, Section 6]). Such tendencies to produce and flood the literature with trivialities and amateurish-type researches should be discouraged by all means.

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