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GLOBAL SOLUTIONS TO NONLINEAR WAVE EQUATIONS ARISING FROM A VARIATIONAL PRINCIPLE

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Abstract. In this paper, we establish the global existence of weak solutions to the initial-boundary value and initial value problems for two classes of nonlinear wave equations which are the Euler-Lagrange equation of a variational principle. We use the method of energy-dependent coordinates to rewrite these equations as semilinear systems and resolve all singularities by introducing a new set of dependent and independent variables. The global weak solutions can be constructed by expressing the solutions of these semilinear systems in terms of the original variables.

Keywords. Existence; Energy-dependent coordinates; Nonlinear wave equations; Weak solutions.

1. Introduction

We are interested in the following variational principle [1,2]

$$\delta \int \left\{ A^{ij}_{\mu\nu}(\mathbf{u}) \frac{\partial u^\mu}{\partial x_i} \frac{\partial u^\nu}{\partial x_j} + B^i_\mu(\mathbf{u}) \frac{\partial u^\mu}{\partial x_i} + F(\mathbf{u}) \right\} \mathrm{d}\mathbf{x} = 0, \tag{1.1}$$

where the summation convention is employed. Here, $\mathbf{x} \in \mathbb{R}^{d+1}$ are the space-time independent variables and $\mathbf{u} : \mathbb{R}^{d+1} \to \mathbb{R}^n$ are the dependent variables, and the coefficients $A^{ij}_{\mu\nu} : \mathbb{R}^n \to \mathbb{R}$, B^i_μ, and F are smooth functions.

Let $n = 1$ and $d = 1$. Then the Euler-Lagrange equation for (1.1) reads that

$$\begin{aligned} &(2A^{11}u_t + (A^{12} + A^{21})u_x + B^1)_t + ((A^{12} + A^{21})u_t + 2A^{22}u_x + B^2)_x \\ &= \frac{\partial A^{11}}{\partial u} u_t^2 + \frac{\partial (A^{12} + A^{21})}{\partial u} u_t u_x + \frac{\partial A^{22}}{\partial u} u_x^2 + \frac{\partial B^1}{\partial u} u_t + \frac{\partial B^2}{\partial u} u_x + \frac{\partial F}{\partial u}. \end{aligned} \tag{1.2}$$

Part of the motivation for studying (1.1) and (1.2) comes from the theory of nematic and cholesteric liquid crystals. In nematic liquid crystals, the average orientation of molecules can be described macroscopically by the director field $\mathbf{n}(\mathbf{x}, t) \in \mathbb{S}^2$ at a spatial location $\mathbf{x}$ and time

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t. The potential energy density for the director field $\mathbf{n}$ is given by the well-known Oseen-Frank energy density

$$W(\mathbf{n},\nabla\mathbf{n})=\frac{1}{2}k_1(\nabla\cdot\mathbf{n})^2+\frac{1}{2}k_2(\mathbf{n}\cdot\nabla\times\mathbf{n})^2+\frac{1}{2}k_3|\mathbf{n}\times(\nabla\times\mathbf{n})|^2, \tag{1.3}$$

where k_1, k_2, and k_3 are the elastic constants of the material. By neglecting the viscosity effects for the rotational motion of the molecules, Hunter and Saxton [24] modeled the propagation of the orientation waves in the director field $\mathbf{n}$ by the least action principle

$$\delta\int\left(\frac{1}{2}\partial_t\mathbf{n}\cdot\partial_t\mathbf{n}-W(\mathbf{n},\nabla\mathbf{n})\right)\mathrm{d}\mathbf{x}\mathrm{d}t=0, \qquad \mathbf{n}\cdot\mathbf{n}=1. \tag{1.4}$$

For cholesteric liquid crystals, the Oseen-Frank potential energy density is expressed as the sum of an elastic and a chiral contribution (neglecting a constant factor)

$$W(\mathbf{n},\nabla\mathbf{n})=\frac{1}{2}k_1(\nabla\cdot\mathbf{n})^2+\frac{1}{2}k_2(\mathbf{n}\cdot\nabla\times\mathbf{n})^2+\frac{1}{2}k_3|\mathbf{n}\times(\nabla\times\mathbf{n})|^2+\lambda\mathbf{n}\cdot\nabla\times\mathbf{n}; \tag{1.5}$$

see, e.g., [17, 26, 27]. Here λ is a parameter representing the molecular chirality. Compared with the nematic, the Lagrangian density in the case of cholesteric liquid crystals includes linear terms as well as quadratic terms in derivatives of the director field $\mathbf{n}$. We note that the variational principles (1.4) with (1.3) and (1.5) are the special cases of (1.1). The term F in (1.1) can be regarded as the contribution from the external electrical or magnetic field.

One of the simplest model derived from (1.4) is the so-called variational wave equation

$$u_{tt}-c(u)[c(u)u_x]_x=0 \tag{1.6}$$

with a given smooth function $c(\cdot)$; see, e.g., [2, 24, 25] for the details on the derivation and physical background of the above equation. Many efforts have been made to study equation (1.6), such as the finite-time singularity of smooth solutions [11, 15, 16, 18, 23], the dissipative solutions [6, 28–30] and the conservative solutions [3, 5, 7, 19], and so on. In particular, Bressan and Zheng [7] established the global existence of conservative solutions to the Cauchy problem of (1.6) by introducing the method of energy-dependent coordinates. This method allows one to rewrite the equation as a semilinear system and resolve all singularities in a new set of variables related to the energy. We also refer the reader to Refs. [4, 8–11, 14, 20, 21, 31, 32] for the application of the method to the related systems.

Based on the energy-dependent coordinates method, the second author established in [22] the global existence of weak solutions to (1.2) under the assumptions that $A^{11}\cdot A^{22}(z)\neq 0$ for any $z\in\mathbb{R}$. If $F\equiv 0$, then the solution constructed above is conservative in the sense that the total energy represented by a Radon measure equals a constant. In the present paper, we consider (1.2) with two extreme cases, i.e., Case I: $A^{11}\equiv 0$ and Case II: $A^{22}\equiv 0$. For Case I, we let

$$(A^{ij})_{2\times 2}=\frac{1}{2}\begin{pmatrix}0 & \beta_1\\ \beta_2 & \gamma\end{pmatrix}(u),$$

and then (1.2) reduces to

$$\left(\frac{1}{2}\beta u_x+B^1\right)_t+\left(\frac{1}{2}\beta u_t+\gamma u_x+B^2\right)_x=\frac{1}{2}\beta_u u_t u_x+\frac{1}{2}\gamma_u u_x^2+B_u^1 u_t+B_u^2 u_x+F_u, \tag{1.7}$$

where and below $\beta =: \beta_1+\beta_2$. For Case II: $A^{22}\equiv 0$, we let

$$(A^{ij})_{2\times 2}=\frac{1}{2}\begin{pmatrix}\alpha & \beta_1\\ \beta_2 & 0\end{pmatrix}(u).$$

In this case, (1.2) can be reduced to

$$\left(\alpha u_t+\frac{1}{2}\beta u_x+B^1\right)_t+\left(\frac{1}{2}\beta u_t+B^2\right)_x=\frac{1}{2}\alpha_u u_t^2+\frac{1}{2}\beta_u u_t u_x+B^1_u u_t+B^2_u u_x+F_u. \tag{1.8}$$

Throughout the paper, we assume $\alpha,\gamma,\beta,B^1,B^2$, and F are smooth bounded functions of u.

For equation (1.7), we further assume

$$\gamma(0)\ge 0,\quad F'(0)=0,\quad \beta(z)\ge\beta_0>0,\ \forall\, z\in\mathbb{R} \tag{1.9}$$

for some constant β_0, and investigate its initial-boundary value problem on the region $(t,x)\in\mathbb{R}^+\times\mathbb{R}^+$ with the initial and boundary conditions

$$u(0,x)=u_0(x)\in H^1_{\text{loc}}(\mathbb{R}^+),\quad u(t,0)=0, \tag{1.10}$$

and the compatibility conditions

$$u_0(0)=u_0'(0)=0. \tag{1.11}$$

For equation (1.8), we further assume

$$\alpha(z)\ge\alpha_0>0,\quad \beta(z)\ge\beta_0>0,\ \forall\, z\in\mathbb{R} \tag{1.12}$$

for some constants α_0,β_0, and consider its initial value problem on the region $(t,x)\in\mathbb{R}^+\times\mathbb{R}$ with the initial data

$$u(0,x)=u_0(x)\in H^1(\mathbb{R}),\quad u_t(0,x)=u_1(x)\in L^2(\mathbb{R}). \tag{1.13}$$

The approach that we used here is the method of energy-dependent coordinates inspired by the work of Bressan and Zheng [7] to deal with (1.6). Based on the energy-dependent coordinates, Chen and Shen [12] considered a family of nonlinear wave equations with a parameter and established the global existence of weak solutions for their initial boundary value problems. Moreover, the global existence of Hölder continuous solutions for the Cauchy problem to the λ-family water wave equations was discussed by Chen, Shen, and Zhu [13]. The main idea of this method is to introduce a new set of variables depending on the energy, which allows one to rewrite the original equation as a semilinear system on these new variables. The global existence of solutions for the semilinear system can be established by deriving a priori estimates and a global weak solution to the original equation can be constructed by returning to the original variables.

Let us first give the definitions of weak solutions to the above two problems.

Definition 1.1. A function $u(t,x)$ with $(t,x)\in\mathbb{R}^+\times\mathbb{R}^+$ is a **weak solution** to initial-boundary problem (1.7) (1.10)-(1.11) if the following hold:
(i) The function $u(t,x)$ is locally Hölder continuous with exponent $1/2$.
(ii) The function $u(t,x)$ satisfies initial and boundary conditions (1.10) and (1.11) pointwise.

(iii) Equation (1.7) is satisfied in the distributional sense

$$\iint_{\mathbb{R}^+\times\mathbb{R}^+}\left\{\varphi_t\left(\frac{1}{2}\beta u_x+B^1\right)+\varphi_x\left(\frac{1}{2}\beta u_t+\gamma u_x+B^2\right)\right.$$
$$\left.+\varphi\left(\frac{1}{2}\beta_u u_t u_x+\frac{1}{2}\gamma_u u_x^2+B_u^1 u_t+B_u^2 u_x+F_u\right)\right\}\mathrm{d}x\mathrm{d}t=0 \tag{1.14}$$

for all test functions $\varphi\in C_c^1(\mathbb{R}^+\times\mathbb{R}^+)$.

Definition 1.2. A function $u(t,x)$ with $(t,x)\in\mathbb{R}^+\times\mathbb{R}$ is a **weak solution** to Cauchy problem (1.8) (1.13) if the following hold:
(i) Function u is locally Hölder continuous with exponent $1/2$. Function $t\mapsto u_t(t,\cdot)$ is continuous with values in $\mathbf{L}^\theta_{\mathrm{loc}}$ for all $1\le\theta<2$. Moreover, for any $T>0$, it satisfies the Lipschitz continuity property

$$\|u(t,\cdot)-u(s,\cdot)\|_{L^2_{\mathrm{loc}}}\le L|t-s|,\quad \forall\, t,s\in(0,T] \tag{1.15}$$

for some constant L depending on T with $L=O(\sqrt{T})$.
(ii) Function $u(t,x)$ takes on the initial condition in (1.13) pointwise, while its temporal derivative holds in $\mathbf{L}^\theta_{\mathrm{loc}}$ for $\theta\in[1,2)$.
(iii) Equation (1.8) is satisfied in the distributional sense

$$\iint_{\mathbb{R}^+\times\mathbb{R}}\left\{\varphi_t(\alpha u_t+\frac{1}{2}\beta u_x+B^1)+\varphi_x(\frac{1}{2}\beta u_t+B^2)\right.$$
$$\left.+\varphi\left(\frac{1}{2}\alpha_u u_t^2+\frac{1}{2}\beta_u u_t u_x+B_u^1 u_t+B_u^2 u_x+F_u\right)\right\}\mathrm{d}x\mathrm{d}t=0 \tag{1.16}$$

for all test functions $\varphi\in C_c^1(\mathbb{R}^+\times\mathbb{R})$.

The conclusions of this paper are as follows.

Theorem 1.1. *Let condition (1.9) hold. Then initial-boundary problem (1.7) (1.10)-(1.11) admits a weak solution under Definition 1.1.*

Theorem 1.2. *Let condition (1.12) hold. Then Cauchy problem (1.8) (1.13) exists a weak solution under Definition 1.2.*

After this introduction, we divide the paper into two main sections, one focusing on the proof of Theorem 1.1 and the other pertaining to prove Theorem 1.2.

2. Proof of Theorem 1.1

In this section, we consider initial-boundary value problem (1.7) (1.10)-(1.11) and prove Theorem 1.1. This section is split into three subsections. In Subsection 2.1, we introduce a new set of dependent and independent variables and derive an equivalent semilinear system of (1.7) for smooth solutions. In Subsection 2.2, we demonstrate the existence results for solutions to the equivalent semilinear system. Finally, we return the solution to the original variables and present the proof to Theorem 1.1 in Subsection 2.3.

2.1. **Equivalent system of (1.7).** Denote

$$R := u_x, \quad S := \beta u_t + \gamma u_x, \tag{2.1}$$

so that

$$u_t = \frac{S - \gamma R}{\beta}, \quad u_x = R.$$

Then (1.7) can be rewritten as

$$\begin{cases} \beta R_t + \gamma R_x = aR^2 - bRS + F_u, \\ S_x = -aR^2 + bRS + F_u, \\ \beta u_t + \gamma u_x = S, \end{cases} \tag{2.2}$$

where

$$a = \frac{\gamma\beta_u - \beta\gamma_u}{2\beta}, \quad b = \frac{\beta_u}{2\beta}.$$

System (2.2) is equivalent to equation (1.7) for smooth solutions if we supplement it with the restriction at $t = 0$ and $x = 0$: $u_x = R$, due to the following identity

$$\beta G_t + \gamma G_x = [2a(R + u_x) - 2bS]G$$

for $G = R - u_x$, which implies that $G \equiv 0$ for all $t > 0, x > 0$ if it vanishes at $t = 0$ and $x = 0$.

For convenience to deal with possibly unbounded value of R, we introduce new dependent variables

$$\ell := \frac{R}{1+R^2}, \quad h := \frac{1}{1+R^2}, \tag{2.3}$$

from which one easily checks that $\ell^2 + h^2 = h$. Then we have by a direct calculation

$$\beta h_t + \gamma h_x = -2\ell[a(1-h) - b\ell S + F_u h], \tag{2.4}$$

$$\beta \ell_t + \gamma \ell_x = (2h-1)[a(1-h) - b\ell S + F_u h], \tag{2.5}$$

$$hS_x = -a(1-h) + b\ell S + F_u h. \tag{2.6}$$

We define the characteristic passing through the point (t,x) as follows

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}\widetilde{x}(s;t,x) = \frac{\gamma}{\beta}(u(s;\widetilde{x}(s;t,x))), \\ \widetilde{x}|_{s=t} = x, \end{cases} \quad \text{or} \quad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}y}\widetilde{t}(y;t,x) = \frac{\beta}{\gamma}(u(\widetilde{t}(y;t,x),y)), \\ \widetilde{t}|_{y=x} = t. \end{cases}$$

Now we define the coordinate transformation $(t,x) \to (T,X)$, where $T = T(t,x) = t$ and

$$X = X(t,x) = \begin{cases} \int_0^{\widetilde{x}(0;t,x)} (1 + R^2(0,\xi))\, \mathrm{d}\xi, \\ \quad \text{when the characteristic passing (t,x) interacts t} = 0; \\ -\frac{\gamma}{\beta}(0)\widetilde{t}(0;t,x), \\ \quad \text{when the characteristic passing (t,x) interacts x} = 0. \end{cases}$$

Obviously, one has

$$\beta X_t + \gamma X_x = 0, \tag{2.7}$$

from which it turns out that

$$\begin{aligned} f_t &= f_T T_t + f_X X_t = f_T - \tfrac{\gamma}{\beta} X_x f_X, \\ f_x &= f_T T_x + f_X X_x = X_x f_X \end{aligned} \tag{2.8}$$

for any smooth function $f(t,x)$. Moreover, it follows from (2.7) and (2.3) that

$$\beta X_{tx}+\gamma X_{xx}=\frac{2a\ell}{h}X_x. \tag{2.9}$$

Now, we introduce the new variable

$$p=\frac{1+R^2}{X_x},$$

which suggests by (2.3) that

$$p=\frac{1}{hX_x},\quad \frac{1}{X_x}=ph. \tag{2.10}$$

Then we use (2.4) and (2.9) to obtain

$$\beta p_t+\gamma p_x=2p[-a\ell-b(1-h)S+F_u\ell]. \tag{2.11}$$

Summing up (2.4)-(2.6) and (2.11) and making use of (2.8) and (2.10), we then obtain a semilinear hyperbolic system with smooth coefficients for the variables h,ℓ,p,S,u in (T,X) coordinates as follows:

$$\begin{cases} \partial_T h=-\frac{2\ell}{\beta}[a(1-h)-b\ell S+F_u h],\\ \partial_T \ell=\frac{2h-1}{\beta}[a(1-h)-b\ell S+F_u h],\\ \partial_X S=p[-a(1-h)+b\ell S+F_u h],\\ \partial_T p=\frac{2}{\beta}p[-a\ell-b(1-h)S+F_u\ell],\\ \partial_T u=\frac{S}{\beta}\quad (\text{or } \partial_X u=p\ell). \end{cases} \tag{2.12}$$

Here we point out that we may use either u_T or u_X in (2.12) since there holds

$$\partial_T(\partial_X u)=\partial_T(p\ell)=\frac{p}{\beta}[a(h-1-bS\ell+F_u h)]=\partial_X\left(\frac{S}{\beta}\right)=\partial_X(\partial_T u).$$

We now consider the initial-boundary conditions of system (2.12) in the new coordinates (T,X), corresponding to (1.10)-(1.11) in the original coordinates (t,x). From the definition of the coordinate transformation $(t,x)\to(T,X)$, one sees that the lines $t=0$ with $x\geq 0$ and $x=0$ with $t\geq 0$ are, respectively, transformed to $T=0$ with $X\geq 0$ and the line $\Gamma_0: X=-\frac{\gamma}{\beta}(0)T$ with $T\geq 0$. The coordinate transformation maps the domain $\mathbb{R}^+\times\mathbb{R}^+$ into the set

$$\Omega:=\{(T,X);\ T\geq 0, X\geq -\frac{\gamma}{\beta}(0)T\}. \tag{2.13}$$

On the lines $T=0$ with $X\geq 0$ and Γ_0, we can thus assign the data of (h,ℓ,S,p,u) defined by

$$\begin{cases} h(0,X)=\frac{1}{1+(u_0')^2},\ \ell(0,X)=\frac{u_0'}{1+(u_0')^2},\ p(0,X)=1,\ S(0,X)=\overline{S},\ u(0,X)=u_0,\\ h(\Gamma_0)=1,\ \ell(\Gamma_0)=0,\ p(\Gamma_0)=1,\ S(\Gamma_0)=0,\ u(\Gamma_0)=0, \end{cases} \tag{2.14}$$

where $\overline{S}=\overline{S}(x)$ is defined by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\overline{S}(x)=\frac{\beta_u(u_0)}{2\beta(u_0)}u_0'\overline{S}(x)+\frac{\beta\gamma_u-\gamma\beta_u}{2\beta}(u_0)(u_0')^2+F_u(u_0),\\ \overline{S}(x)|_{x=0}=0. \end{cases}$$

2.2. **Solutions to system (2.12).** In this subsection, we prove the existence of a unique global solution to the system (2.12) with the initial-boundary data (2.14) in the coordinates (T,X).

We first establish a priori estimates for solutions to (2.12) in Ω. By a direct calculation, one obtains

$$\begin{aligned}\partial_T(h^2+\ell^2) = 2h\partial_T h + 2\ell\partial_T\ell &= \left(-\frac{4h\ell}{\beta}+\frac{2\ell(2h-1)}{\beta}\right)[a(1-h)-b\ell S+F_u h]\\ &= -\frac{2\ell}{\beta}[a(1-h)-b\ell S+F_u h] = \partial_T h,\end{aligned}$$

which together with (2.14) implies that $h^2+\ell^2=h$ for all $(T,X)\in\Omega$. It follows that

$$0\le h\le 1,\quad |\ell|\le\frac{1}{2}. \tag{2.15}$$

Now we estimate p and S. It follows directly from the equation of p in (2.12) and (2.14) that p is positive in Ω. In addition, one can easily check that there hold

$$\partial_T[\beta p(1-h)]-\partial_X(2F)=0,\quad \partial_T(ph)-\partial_X\left(\frac{\gamma}{\beta}\right)=0, \tag{2.16}$$

which means that the two differential forms

$$\beta p(1-h)\mathrm{d}X+2F\mathrm{d}T, \tag{2.17}$$

$$ph\mathrm{d}X+\frac{\gamma}{\beta}\mathrm{d}T \tag{2.18}$$

have zero integral along every closed curve contained in Ω. Then, for every $(T,X)\in\Omega$, we construct the closed curve $\mathscr{C}$ composed of the following four parts: the vertical segment with the endpoints $(0,X)$ and (T,X), the horizontal segment with the endpoints (T,X) and $(T,-\frac{\gamma}{\beta}(0)T)$, the line Γ_0 with the endpoints $(T,-\frac{\gamma}{\beta}(0)T)$ and $(0,0)$, and the line $T=0$ with the endpoints $(0,0)$ and $(0,X)$. See Fig. 1 for illustration. Integrating (2.17) along the closed curve $\mathscr{C}$ and employing the boundary data (2.14) give

$$\begin{aligned}\int_{-\frac{\gamma}{\beta}(0)T}^{X}\beta p(1-h)(T,X')\mathrm{d}X' &= \int_0^T 2F(u(T',X))\mathrm{d}T'+\int_0^X\beta p(1-h)(0,X')\mathrm{d}X'\\ &\quad -\int_0^T\left[-\frac{\gamma}{\beta}(0)\beta p(1-h)+2F(u)\right](T',-\frac{\gamma}{\beta}(0)T')\mathrm{d}T',\end{aligned}$$

from which one concludes that

$$\int_{-\frac{\gamma}{\beta}(0)T}^{X} p(T,X')\mathrm{d}X'\le\int_{-\frac{\gamma}{\beta}(0)T}^{X} ph(T,X')\mathrm{d}X'+C(T+|X|). \tag{2.19}$$

Throughout this paper, C denotes a positive constant, which may change from line to line.

On the other hand, we integrate (2.18) along the closed curve $\mathscr{C}$ to find

$$\begin{aligned}\int_{-\frac{\gamma}{\beta}(0)T}^{X} ph(T,X')\mathrm{d}X' &= \int_0^T\frac{\gamma}{\beta}(T',X)\mathrm{d}T'+\int_0^X ph(0,X')\mathrm{d}X'\\ &\quad -\int_0^T\left[-\frac{\gamma}{\beta}(0)ph+\frac{\gamma}{\beta}\right](T',-\frac{\gamma}{\beta}(0)T')\mathrm{d}T'\\ &\le C(T+|X|).\end{aligned}$$

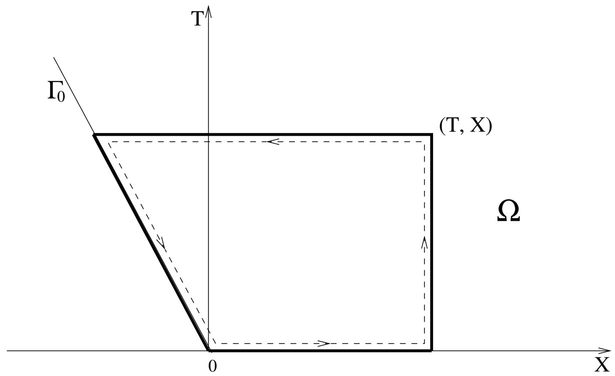

FIGURE 1. The Closed Curve $\mathscr{C}$.

Putting the above into (2.19) leads to

$$\int_{-\frac{\gamma}{\beta}(0)T}^{X} p(T,X')\mathrm{d}X' \le C(T+|X|). \tag{2.20}$$

Integrating the equation of S in (2.12) horizontally from $-\frac{\gamma}{\beta}(0)T$ to X and making use of (2.14) and (2.20), one sees that

$$\begin{aligned}
|S(T,X)| = &\Bigg|\exp\left\{\int_{-\frac{\gamma}{\beta}(0)T}^{X} bp\ell(T,X')\mathrm{d}X'\right\}\\
&\quad\times\int_{-\frac{\gamma}{\beta}(0)T}^{X} p[F_u h - a(1-h)]\exp\left\{-\int_{-\frac{\gamma}{\beta}(0)T}^{X'} bp\ell(T,X'')\mathrm{d}X''\right\}(T,X')\mathrm{d}X'\Bigg|\\
\le&\exp\left\{C\int_{-\frac{\gamma}{\beta}(0)T}^{X} p(T,X')\mathrm{d}X'\right\}\\
&\quad\times C\int_{-\frac{\gamma}{\beta}(0)T}^{X} p\exp\left\{C\int_{-\frac{\gamma}{\beta}(0)T}^{X} p(T,X'')\mathrm{d}X''\right\}(T,X')\mathrm{d}X'\\
\le&C(T+|X|)e^{C(T+|X|)}.
\end{aligned} \tag{2.21}$$

In view of (2.21), we obtain by integrating the equation of u in (2.12)

$$|u(T,X)| \le |u_0| + \int_0^T \left|\frac{S}{\beta}(T',X)\right|\mathrm{d}T' \le |u_0| + CT(T+|X|)e^{C(T+|X|)}. \tag{2.22}$$

To estimate the function p, we integrate the equation for p in (2.12) vertically and use (2.14) and (2.21) to see that

$$\begin{aligned}
p(T,X) = &\exp\left\{\int_{T_0}^{T}\frac{2}{\beta}[-a\ell - b(1-h)S + F_u\ell](T',X)\mathrm{d}T'\right\}\\
\le&\exp\left\{CT(1+T+|X|)e^{C(T+|X|)}\right\},
\end{aligned} \tag{2.23}$$

where

$$T_0 = \begin{cases} -\frac{\gamma}{\beta}(0)T, & X<0;\\ 0, & X\ge 0.\end{cases}$$

We notice that all right-hand side functions in system (2.12) are locally Lipschitz continuous, which leads to the local existence of solutions, which can be established straightforward by fixed point methods. Due to a priori estimates (2.15) and (2.21)-(2.23), it is easy to extend this local solution to entire domain Ω by applying the technique in Bressan and Zheng [7]. Thus we have the global existence theorem.

Theorem 2.1. *Let the assumptions in Theorem 1.1 hold. Then problem* (2.12) (2.14) *has a unique global solution defined for all* $(T,X)\in\Omega$*. Moreover, if there exists a sequence of smooth functions* $\{u_0^\nu\}_{\nu\geq 1}$ *satisfying*

$$u_0^\nu \to u_0, \quad (u_0^\nu)_x \to (u_0)_x,$$

uniformly on any bounded subset of $\mathbb{R}^+$*, then it has the following convergence properties:*

$$(u^\nu, h^\nu, \ell^\nu, p^\nu) \to (u,h,\ell,p),$$

uniformly on bounded subsets of Ω*.*

2.3. **Solutions to equation** (1.7)**.** This subsection is devoted to returning the solution constructed above to original variables (t,x). Since initial data $(u_0)_x$ is assumed only to be in L^2, we see that, on any bounded subset of Ω,
– h, ℓ, and p are Lipschitz continuous w.r.t. T, measurable w.r.t. X,
– S is Lipschitz continuous w.r.t. X, measurable w.r.t. T,
– u is Lipschitz continuous w.r.t. both T and X,
– u, h, ℓ, S, and p have finite L^∞ norm and $p>0$.

In order to define u as functions of the original variables (t,x), we need the inverse functions $T=T(t,x)$, $X=X(t,x)$. Since $T(t,x)=t$, then it suffices to construct the function $x=x(T,X)$. Thanks to (2.8), it suggests that

$$x_T = \frac{\gamma}{\beta}, \quad x_X = ph, \tag{2.24}$$

which along with (2.16) gives $x_{TX}=x_{XT}$, which indicates that we may integrate one of the equations in (2.24) to obtain the function $x=x(T,X)$. Then, for any $(t,x)\in\mathbb{R}^+\times\mathbb{R}^+$, we define $u(t,x)=u(T,x(T,X))$. We here note that the map $(T,X)\mapsto x$ may not be one-to-one, which, however, does not cause any real difficulty due to the following assertion: for any fixed (t,x), the values of u do not depend on the choice of X. Suppose it holds. For each given point (t^*,x^*), we can then choose an arbitrary X^* satisfying $x(t^*,X^*)=x^*$, and define $u(t^*,x^*):=u(t^*,X^*)$. We next demonstrate the above assertion. Assume (t^*,X_1) and (t^*,X_2) with $X_1<X_2$ are two distinct points Ω such that $x(t^*,X_1)=x(t^*,X_2)=x^*$. On account of (2.24), we have $h(t^*,X)=0$ and then $\ell(t^*,X)=0$ for $X\in[X_1,X_2]$. Thus it follows from (2.12) that

$$u(t^*,X_2)-u(t^*,X_1)=\int_{X_1}^{X_2} u_X(t^*,X)\mathrm{d}X=\int_{X_1}^{X_2} p\ell(t^*,X)\mathrm{d}X=0.$$

We now prove that the function $u(t,x)$, constructed as above, is Hölder continuous on bounded sets. In fact, for any fixed time t, we obtain

$$\int_{x_1}^{x_2} u_x^2\mathrm{d}x=\int_{X_1}^{X_2}(u_X X_x)^2\cdot X_x\mathrm{d}X=\int_{X_1}^{X_2} p(1-h)\mathrm{d}X<\infty$$

for any bounded interval $[x_1,x_2]\subset\mathbb{R}^+$, which combined with the fact $\beta u_t+\gamma u_x=S\in L^\infty_{\mathrm{loc}}$ yields that $u=u(t,x)$ is locally Hölder continuous with exponent $1/2$. Furthermore, the function R at

(2.1) is square integrable on bounded subsets of the t-x plane. From the identity

$$u_x = u_X X_x = \frac{\ell}{h} = R,$$

one sees that R is indeed the same as recovered from (2.3).

Next, we check that function $u = u(t,x)$ satisfies (1.14). For any test function $\varphi \in C_c^1(\mathbb{R}^+ \times \mathbb{R}^+)$, we directly compute by (2.8)

$$\begin{aligned}
&\iint_{\mathbb{R}^+\times\mathbb{R}^+} \left\{\varphi_t\left(\frac{1}{2}\beta u_x + B^1\right) + \varphi_x\left(\frac{1}{2}\beta u_t + \gamma u_x + B^2\right)\right\}\mathrm{d}x\mathrm{d}t \\
=& \iint_{\Omega} \left\{\left(\varphi_T - \frac{\gamma}{\beta ph}\varphi_X\right)\left(\frac{1}{2}\beta\frac{\ell}{h} + B^1\right) + \frac{\varphi_X}{ph}\left(\frac{1}{2}(S - \gamma\frac{\ell}{h}) + \gamma\frac{\ell}{h} + B^2\right)\right\}\cdot ph\mathrm{d}X\mathrm{d}T \\
=& \iint_{\Omega} \left\{\varphi_T\left(\frac{1}{2}\beta p\ell + phB^1\right) + \varphi_X\left(\frac{1}{2}S - \frac{\gamma}{\beta}B^1 + B^2\right)\right\}\mathrm{d}X\mathrm{d}T \\
=& -\iint_{\Omega} \varphi\left\{\left(\frac{1}{2}\beta p\ell + phB^1\right)_T + \left(\frac{1}{2}S - \frac{\gamma}{\beta}B^1 + B^2\right)_X\right\}\mathrm{d}X\mathrm{d}T.
\end{aligned}\tag{2.25}$$

A straightforward computation gives

$$\begin{aligned}
&\left(\frac{1}{2}\beta p\ell + phB^1\right)_T + \left(\frac{1}{2}S - \frac{\gamma}{\beta}B^1 + B^2\right)_X \\
=& \frac{1}{2}[(\beta p\ell)_T + S_X] + phB_u^1 u_T + \left(B_u^2 - \frac{\gamma}{\beta}B_u^1\right)u_X \\
=& ph\cdot\left\{-a\left(\frac{\ell}{h}\right)^2 + bS\frac{\ell}{h} + F_u + \frac{1}{\beta}B_u^1 S + \left(B_u^2 - \frac{\gamma}{\beta}B_u^1\right)\frac{\ell}{h}\right\}.
\end{aligned}\tag{2.26}$$

Inserting (2.26) into (2.25) leads to

$$\begin{aligned}
&\iint_{\mathbb{R}^+\times\mathbb{R}^+} \left\{\varphi_t\left(\frac{1}{2}\beta u_x + B^1\right) + \varphi_x\left(\frac{1}{2}\beta u_t + \gamma u_x + B^2\right)\right\}\mathrm{d}x\mathrm{d}t \\
=& -\iint_{\mathbb{R}^+\times\mathbb{R}^+} \varphi\left\{-aR^2 + bSR + F_u + \frac{1}{\beta}B_u^1 S + \left(B_u^2 - \frac{\gamma}{\beta}B_u^1\right)R\right\}\mathrm{d}x\mathrm{d}t \\
=& -\iint_{\mathbb{R}^+\times\mathbb{R}^+} \varphi\left\{-\frac{\gamma\beta_u - \beta\gamma_u}{2\beta}u_x^2 + \frac{\beta_u}{2\beta}(\beta u_t + \gamma u_x)u_x + F_u\right. \\
&\qquad\left. + \frac{1}{\beta}B_u^1(\beta u_t + \gamma u_x) + \left(B_u^2 - \frac{\gamma}{\beta}B_u^1\right)u_x\right\}\mathrm{d}x\mathrm{d}t \\
=& -\iint_{\mathbb{R}^+\times\mathbb{R}^+} \varphi\left\{\frac{1}{2}\beta_u u_t u_x + \frac{1}{2}\gamma_u u_x^2 + B_u^1 u_t + B_u^2 u_x + F_u\right\}\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

which concludes (1.14). The proof of Theorem 1.1 is completed.

3. Proof of Theorem 1.2

This section is devoted to providing the proof to Theorem 1.2. We divide the section into four subsections. Following Section 2, we introduce new variables and derive an equivalent semilinear system of (1.8) for smooth solutions in Subsection 3.1 and establish its existence results in Subsection 3.2. Weak solutions of (1.8) are constructed and the proof of Theorem 1.2 is completed in Subsection 3.3.

3.1. **Equivalent system of (1.8).** We denote

$$\bar{R} := u_t, \quad \bar{S} := \alpha u_t + \beta u_x, \tag{3.1}$$

so that

$$u_t = \bar{R}, \quad u_x = \frac{\bar{S} - \alpha\bar{R}}{\beta}.$$

Then (1.8) can be rewritten as

$$\begin{cases} \alpha\bar{R}_t + \beta\bar{R}_x = d\bar{R}^2 - b\bar{R}\bar{S} + F_u, \\ \bar{S}_t = -d\bar{R}^2 + b\bar{R}\bar{S} + F_u, \\ \alpha u_t + \beta u_x = \bar{S}, \end{cases} \tag{3.2}$$

where

$$d = \frac{\alpha\beta_u - \beta\alpha_u}{2\beta}, \;\; b = \frac{\beta_u}{2\beta}.$$

For smooth solutions, system (3.2) is equivalent to equation (1.8) if it supplements the restriction $u_t = \bar{R}$ at time zero since there holds

$$\alpha(\bar{R} - u_t)_t + \beta(\bar{R} - u_t)_x = [2d(\bar{R} + u_t) - 2b\bar{S}](\bar{R} - u_t).$$

Introduce the new dependent variables

$$m_1 = \frac{\bar{R}}{1+\bar{R}^2}, \quad g_1 = \frac{1}{1+\bar{R}^2}, \quad m_2 = \frac{\bar{S}}{1+\bar{S}^2}, \quad g_2 = \frac{1}{1+\bar{S}^2}, \tag{3.3}$$

which gives $m_1^2 + g_1^2 = g_1$ and $m_2^2 + g_2^2 = g_2$. Then it follows from (3.2) that

$$\begin{cases} g_2(\alpha g_{1t} + \beta g_{1x}) = -2m_1\{d(1-g_1)g_2 - bm_1m_2 + F_u g_1 g_2\}, \\ g_2(\alpha m_{1t} + \beta m_{1x}) = (2g_1 - 1)\{d(1-g_1)g_2 - bm_1m_2 + F_u g_1 g_2\}, \\ g_1 g_{2t} = -2m_2\{-d(1-g_1)g_2 + bm_1m_2 + F_u g_1 g_2\}, \\ g_1 m_{2t} = (2g_2 - 1)\{-d(1-g_1)g_2 + bm_1m_2 + F_u g_1 g_2\}. \end{cases} \tag{3.4}$$

Define the two characteristics passing through the point (t,x) as follows:

$$x_-(s;t,x) = x, \qquad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}x_+(s;t,x) = \frac{\beta}{\alpha}(u(s, x_+(s;t,x))), \\ x_+|_{s=t} = x. \end{cases}$$

Then we define the coordinate transformation $(t,x) \to (Y,Z)$, where

$$Z = Z(t,x) = \int_0^x (1 + \bar{S}^2(0,\xi))\mathrm{d}\xi, \quad Y = Y(t,x) = \int_{x_+(0;t,x)}^0 (1 + \bar{R}^2(0,\xi))\mathrm{d}\xi,$$

which implies that $Z_t = 0$ and $\alpha Y_t + \beta Y_x = 0$, from which one finds that

$$\alpha Y_{tx} + \beta Y_{xx} = -u_x(\alpha_u Y_t + \beta_u Y_x) = 2d\frac{g_1 m_2 - \alpha m_1 g_2}{\alpha g_1 g_2}(-Y_x), \tag{3.5}$$

and

$$\begin{aligned} f_t &= f_Z Z_t + f_Y Y_t = -\tfrac{\beta}{\alpha} Y_x f_Y, \\ f_x &= f_Z Z_x + f_Y Y_x \end{aligned} \tag{3.6}$$

for any smooth function $f(t,x)$.

We next introduce the new variable

$$q_1 = \frac{1+\bar{R}^2}{-Y_x}, \quad q_2 = \frac{1+\bar{S}^2}{Z_x},$$

from which and (3.3) we have

$$\frac{1}{-Y_x} = q_1 g_1, \quad \frac{1}{Z_x} = q_2 g_2.$$

By (3.4) and (3.5), it obtains that

$$\begin{aligned} g_2(\alpha q_{1t} + \beta q_{1x}) &= 2q_1\{-dmg_2 - b(1-g_1)m_2 + \tfrac{d}{\alpha}m_2 + F_u m_1 g_2\}, \\ g_1 q_{2t} &= 2q_2\{-d(1-g_1)m_2 + bm_1(1-g_2) + F_u g_1 m_2\}. \end{aligned} \tag{3.7}$$

Combining (3.4) and (3.7) and employing (3.6), we obtain a semilinear hyperbolic system with smooth coefficients for the variables $(g_1, g_2, m_1, m_2, q_1, q_2, u)$ in (Y, Z) coordinates as follows:

$$\begin{cases} \partial_Z g_1 = -\frac{2m_1 q_2}{\beta}\{d(1-g_1)g_2 - bm_1 m_2 + F_u g_1 g_2\}, \\ \partial_Z m_1 = \frac{q_2(2g_1-1)}{\beta}\{d(1-g_1)g_2 - bm_1 m_2 + F_u g_1 g_2\}, \\ \partial_Z q_1 = \frac{2q_1 q_2}{\beta}\{-dm_1 g_2 - b(1-g_1)m_2 + \frac{d}{\alpha}m_2 + F_u m_1 g_2\}, \\ \partial_Y g_2 = -\frac{2\alpha q_1 m_2}{\beta}\{-d(1-g_1)g_2 + bm_1 m_2 + F_u g_1 g_2\}, \\ \partial_Y m_2 = \frac{\alpha q_1(2g_2-1)}{\beta}\{-d(1-g_1)g_2 + bm_1 m_2 + F_u g_1 g_2\}, \\ \partial_Y q_2 = \frac{2\alpha q_1 q_2}{\beta}\{-d(1-g_1)m_2 + bm_1(1-g_2) + F_u g_1 m_2\}, \\ \partial_Z u = \frac{q_2 m_2}{\beta}, \quad (\text{or } \partial_Y u = \frac{\alpha}{\beta}q_1 m_1). \end{cases} \tag{3.8}$$

Here we have used the following equality

$$\partial_Y\left(\frac{q_2 m_2}{\beta}\right) = \frac{\alpha q_1 q_2}{\beta^2}\left\{-d(1-g_1)g_2 - bm_1 m_2 + F_u g_1 g_2\right\} = \partial_Z\left(\frac{\alpha}{\beta}q_1 m_1\right).$$

We next consider the boundary conditions of system (3.8) in the coordinates (Y, Z), corresponding to (1.13) in original coordinates (t, x). It is easy to know by (1.13) that

$$\bar{R}(0,x) = u_1(x) \in L^2, \quad \bar{S}(0,x) = \alpha(u_0(x))u_1(x) + \beta(u_0(x))u_0'(x) \in L^2,$$

which mean that the two functions

$$Z = Z(x) = \int_0^x (1+\bar{S}^2(0,\xi))\mathrm{d}\xi, \quad Y = Y(x) = \int_x^0 (1+\bar{R}^2(0,\xi))\mathrm{d}\xi \tag{3.9}$$

are well defined and absolutely continuous. Furthermore, Z is strictly increasing while Y is strictly decreasing. Hence the function $Y = \psi(Z)$ defined by (3.9) is continuous and strictly decreasing and satisfies

$$|Z + \psi(Z)| \le \int_{\mathbb{R}} (\bar{S}^2(0,\xi) + \bar{R}^2(0,\xi))\mathrm{d}\xi =: \mathscr{E}_0 < \infty.$$

It is obvious that the coordinate transformation maps the line $t = 0$ and the domain $[0,\infty)\times\mathbb{R}$ in (t,x) plane into the curve $\Gamma_1 : Y = \psi(Z)$, and the set $\Omega^+ := \{(Z,Y);\ Y \ge \psi(Z)\}$ in the (X,Y) plane, respectively. Along curve Γ_1, we can thus assign the boundary data defined by

$$\begin{cases} g_1(\Gamma_1) = \frac{1}{1+\bar{R}^2(0,x)},\ m_1(\Gamma_1) = \bar{R}(0,x)g_1(\Gamma_1),\ q_1(\Gamma_1) = 1, \\ g_2(\Gamma_1) = \frac{1}{1+\bar{S}^2(0,x)},\ m_2(\Gamma_1) = \bar{S}(0,x)g_2(\Gamma_1),\ q_2(\Gamma_1) = 1, \end{cases} \quad u(\Gamma_1) = u_0(x). \tag{3.10}$$

It is easily checked that $g_i^2+m_i^2=g_i$ $(i=1,2)$ on Γ_1.

3.2. **Solutions to system (3.8).** We establish the existence of a unique global solution to system (3.8) with boundary data (3.10) in coordinates (Z,Y) in this subsection.

Here we present, without derivation, some identities from system (3.8), which are outlined in the following:

$$\partial_Z(g_1^2+m_1^2-g_1)=0, \quad \partial_Y(g_2^2+m_2^2-g_2)=0, \tag{3.11}$$

$$\partial_Y(q_2g_2)=\frac{2\alpha b}{\beta}q_1q_2m_1(g_2-g_2^2-m_2^2), \tag{3.12}$$

$$\partial_Z(\frac{\alpha}{\beta}q_1g_1)-\partial_Y(\frac{\alpha}{\beta}q_2g_2)=0, \tag{3.13}$$

$$\partial_Z(\alpha q_1(1-g_1))-\partial_Y(2Fq_2g_2)=\frac{4\alpha b}{\beta}Fq_1q_2m_1(g_2^2+m_2^2-g_2). \tag{3.14}$$

According to (3.11) and boundary conditions (3.10), one finds

$$g_1^2+m_1^2=g_1, \quad g_2^2+m_2^2=g_2, \quad \forall\,(Z,Y)\in\Omega^+, \tag{3.15}$$

from which and (3.12) and (3.14), we see that

$$\partial_Y(q_2g_2)=0, \tag{3.16}$$

$$\partial_Z(\alpha q_1(1-g_1))-\partial_Y(2Fq_2g_2)=0. \tag{3.17}$$

We now derive a priori estimates for solutions to the semilinear hyperbolic system (3.8) in Ω^+. Obviously, it turns out by (3.15) that

$$0\le g_1\le 1, \quad 0\le g_2\le 1, \quad |m_1|\le\frac{1}{2}, \quad |m_2|\le\frac{1}{2}. \tag{3.18}$$

To estimate q_1 and q_2, we first see from (3.8) and initial condition $q_1(\Gamma_1)=q_2(\Gamma_1)=1$ that q_1 and q_2 are positive in Ω^+. On the other hand, by (3.13) and (3.17), the two differential forms

$$\frac{\alpha}{\beta}q_1g_1\mathrm{d}Y+\frac{\alpha}{\beta}q_2g_2\,\mathrm{d}Z, \tag{3.19}$$

$$\alpha q_1(1-g_1)\mathrm{d}Y+2Fq_2g_2\mathrm{d}Z \tag{3.20}$$

have zero integral along every closed curve contained in Ω^+. Then, for every $(Z,Y)\in\Omega^+$, we construct the closed curve $\mathscr{S}$ composed of the following three parts: the vertical segment with endpoints $(Z,\psi(Z))$ and (Z,Y), the horizontal segment with endpoints (Z,Y) and $(\psi^{-1}(Y),Y)$, and the boundary curve Γ_1 with endpoints $(\psi^{-1}(Y),Y)$ and $(Z,\psi(Z))$. Here ψ^{-1} denotes the inverse of ψ. See Fig 2 for illustration. We integrate (3.20) along closed curve $\mathscr{S}$ and use boundary data (2.13) to obtain

$$\begin{aligned}&\int_{\psi(Z)}^{Y}\alpha q_1(1-g_1)\mathrm{d}Y\\ =&\int_{\psi^{-1}(Y)}^{Z}2Fq_2g_2\mathrm{d}Z+\int_{\psi(Z)}^{Y}\alpha q_1(1-g_1)(\psi^{-1}(Y),Y)\mathrm{d}Y-\int_{\psi^{-1}(Y)}^{Z}2Fq_2g_2(Z,\psi(Z))\mathrm{d}Z,\end{aligned}$$

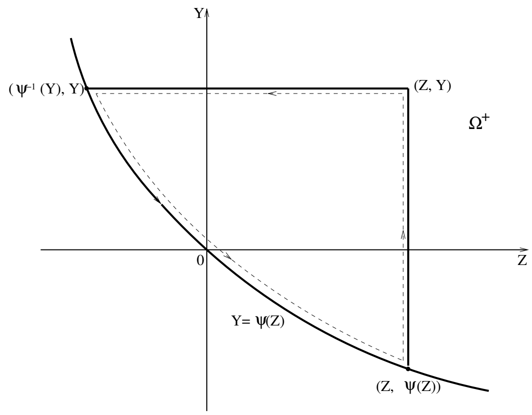

FIGURE 2. The closed curve $\mathscr{S}$.

from which one has

$$\begin{aligned}\int_{\psi(Z)}^{Y} q_1 \mathrm{d}Y &\le C\left\{\int_{\psi(Z)}^{Y} q_1 g_1 \mathrm{d}Y + \int_{\psi^{-1}(Y)}^{Z} q_2 g_2 \mathrm{d}Z + Y - \psi(Z) + Z - \psi^{-1}(Y)\right\}\\ &\le C\left\{\int_{\psi(Z)}^{Y} q_1 g_1 \mathrm{d}Y + \int_{\psi^{-1}(Y)}^{Z} q_2 g_2 \mathrm{d}Z + Y + Z + \mathscr{E}_0\right\}. \end{aligned} \tag{3.21}$$

Here and below, C denotes a positive constant. Making use of (3.16) leads to

$$\int_{\psi^{-1}(Y)}^{Z} q_2 g_2 \mathrm{d}Z = \int_{\psi^{-1}(Y)}^{Z} q_2 g_2 (Z, \psi(Z)) \mathrm{d}Z \le Z - \psi^{-1}(Y). \tag{3.22}$$

In addition, integrating (3.19) along the closed curve $\mathscr{S}$ yields

$$\begin{aligned}&\int_{\psi(Z)}^{Y} \frac{\alpha}{\beta} q_1 g_1 \mathrm{d}Y\\ &= \int_{\psi^{-1}(Y)}^{Z} \frac{\alpha}{\beta} q_2 g_2 \mathrm{d}Z + \int_{\psi(Z)}^{Y} \frac{\alpha}{\beta} q_1 g_1 (\psi^{-1}(Y), Y) \mathrm{d}Y - \int_{\psi^{-1}(Y)}^{Z} \frac{\alpha}{\beta} q_2 g_2 (Z, \psi(Z)) \mathrm{d}Z,\end{aligned}$$

which together with (3.22) and (1.12) gives

$$\int_{\psi(Z)}^{Y} q_1 g_1 \mathrm{d}Y \le C(Z + Y + \mathscr{E}_0). \tag{3.23}$$

Inserting (3.22) and (3.23) into (3.21) arrives at

$$\int_{\psi(Z)}^{Y} q_1 \mathrm{d}Y \le C(Y + Z + \mathscr{E}_0). \tag{3.24}$$

We now integrate the equation for q_2 in (3.8) and use (3.24) to obtain

$$q_2(Z,Y) \le \exp\left\{C \int_{\psi(Z)}^{Y} q_1 \mathrm{d}Y\right\} \le \exp\{C(Y + Z + \mathscr{E}_0)\}. \tag{3.25}$$

In view of the equation for q_1, we obtain

$$q_1(Z,Y) \le \exp\left\{C \int_{\psi^{-1}(Y)}^{Z} q_2 \mathrm{d}Z\right\} \le \exp\left\{C(Y + Z + \mathscr{E}_0) \exp\{C(Y + Z + \mathscr{E}_0)\}\right\}. \tag{3.26}$$

Applying (3.25) and (3.8) deduces the estimate of $u(Z,Y)$. Since all right-hand side functions in system (3.8) are locally Lipschitz continuous, then the local existence of solutions follows directly from the fixed point method. Based on a priori estimates (3.18), (3.25), and (3.26), this local solution can be easily extended to the entire domain Ω^+ by using the technique in Bressan and Zheng [7]. Thus we have the following global existence theorem

Theorem 3.1. *Let the assumptions in Theorem 1.2 hold. Then problem (3.8) (3.10) has a unique global solution defined for all $(Z,Y)\in\Omega^+$. Moreover, if there exists a sequence of smooth functions $(u_0^\nu,u_1^\nu)_{\nu\geq 1}$ satisfying*

$$u_0^\nu\to u_0,\quad (u_0^\nu)_x\to(u_0)_x,\quad u_1^\nu\to u_1,$$

uniformly on any bounded subset of $\mathbb{R}$, then it has the following convergence properties:

$$(u^\nu,g_1^\nu,g_2^\nu,m_1^\nu,m_2^\nu,q_1^\nu,q_2^\nu)\to(u,g_1,g_2,m_1,m_2,q_1,q_2),$$

uniformly on bounded subsets of Ω^+.

3.3. **Solutions to equation (1.8).** In this subsection, we construct a weak solution of (1.8) by returning the function $u(Z,Y)$ to the original variables (t,x). Since the initial data $(u_0)_x$ and u_1 are assumed only to be in L^2, we see that, on bounded subsets of the Z-Y plane,
– g_1, m_1, and q_1 are Lipschitz continuous w.r.t. Z, measurable w.r.t. Y,
– g_2,m_2 and q_2 are Lipschitz continuous w.r.t. Y, measurable w.r.t. Z,
– u is Lipschitz continuous w.r.t. both Z and Y,
– u, g_1,g_2, m_1,m_2, q_1, and q_2 have finite L^∞ norm and $q_1>0,q_2>0$.

To return the solution u to original variables (t,x), we need the inverse functions $Z=Z(x)$ and $Y=Y(t,x)$. We may integrate $x_Z=q_2g_2,t_Z=\alpha q_2g_2/\beta$, or $x_Y=0,t_Y=\alpha q_1g_1/\beta$ to obtain $x=x(Z),t=t(Z,Y)$ by the facts $x_{ZY}=x_{YZ}$ and $t_{ZY}=t_{YZ}$. The map $(Z,Y)\mapsto t$ may not be one-to-one, however, we have the following assertion: for any fixed (t,x), the values of u do not depend on the choice of Y. Since if it holds, for each given point (t^*,x^*), we can solve a unique Z^* from $x^*=x(Z)$ and choose an arbitrary Y^* satisfying $t(Z^*,Y^*)=t^*$, and then define $u(t^*,x^*):=u(Z^*,Y^*)$. To prove the above assertion, we assume (Z^*,Y_1) and (Z^*,Y_2) with $Y_1<Y_2$ are two distinct points in the region Ω^+ such that $t(Z^*,Y_1)=t(Z^*,Y_2)=t^*$. By the equation $t_Y=\alpha q_1g_1/\beta$, we have $g_1(Z^*,Y)=0$ and then $m_1(Z^*,Y)=0$ for $Y\in[Y_1,Y_2]$. Thus it suggests that

$$u(Z^*,Y_2)-u(Z^*,Y_1)=\int_{Y_1}^{Y_2}u_Y(Z^*,Y)\mathrm{d}Y=\int_{Y_1}^{Y_2}\frac{\alpha}{\beta}q_1m_1(Z^*,Y)\mathrm{d}Y=0,$$

which concludes the proof of the assertion.

We next show that the function $u(t,x)$, constructed as above, is Hölder continuous on bounded sets. In fact, integrating along any forward characteristic $t\mapsto x^+(t)$ and noting $Y=const.$ on this kind of characteristics achieves

$$\int_0^\tau(\alpha u_t+\beta u_x)^2\mathrm{d}t=\int_{Z_0}^{Z_\tau}(\beta u_Z Z_x)^2 t_Z\mathrm{d}Z=\int_{Z_0}^{Z_\tau}\frac{\alpha}{\beta}q_2(1-g_2)\mathrm{d}X\leq C_\tau \tag{3.27}$$

for some constant C_τ depending only on τ. Similarly, one has

$$\int_0^\tau u_t^2\mathrm{d}t=\int_{Y_0}^{Y_\tau}\frac{\alpha}{\beta}q_1(1-g_1)\mathrm{d}Y\leq C_\tau,$$

which along with (3.27) indicates that $u=u(t,x)$ is Hölder continuous with exponent $1/2$. Moreover, it turns out that $\bar{R}$ and $\bar{S}$ at (3.1) are square integrable on bounded subsets of the t-x plane and these two functions are indeed the same as recovered from (3.3) by the following identities

$$\begin{aligned}
u_t &= \frac{\beta}{\alpha}(-Y_x)u_Y = \frac{\beta}{\alpha}\cdot\frac{1}{q_1g_1}\cdot\frac{\alpha}{\beta}q_1m_1 = \frac{m_1}{g_1} = \bar{R},\\
\alpha u_t + \beta u_x &= \beta u_Z Z_x = \beta\cdot\frac{q_2m_2}{\beta}\cdot\frac{1}{q_2g_2} = \frac{m_2}{g_2} = \bar{S}.
\end{aligned}$$

To check (1.16), for any test functions $\varphi\in C_c^1(\mathbb{R}^+\times\mathbb{R})$, we use (3.6) to compute

$$\begin{aligned}
&\iint_{\mathbb{R}^+\times\mathbb{R}}\Big\{\varphi_t(\alpha u_t+\frac{1}{2}\beta u_x+B^1)+\varphi_x(\frac{1}{2}\beta u_t+B^2)\Big\}\,\mathrm{d}x\mathrm{d}t\\
=&\iint_{\Omega^+}\Big\{-\frac{\beta}{\alpha}Y_x\varphi_Y\Big(\alpha\frac{m_1}{g_1}+\frac{m_2g_1-\alpha m_1g_2}{2g_1g_2}+B^1\Big)\\
&\qquad+(\varphi_ZZ_x+\varphi_YY_x)\Big(\frac{\beta m_1}{2g_1}+B^2\Big)\Big\}\cdot\frac{\alpha}{\beta}q_1q_2g_1g_2\,\mathrm{d}Z\mathrm{d}Y\\
=&\iint_{\Omega^+}\Big\{\varphi_Z\frac{\alpha}{\beta}q_1(\beta m_1+B^2g_1)+\varphi_Yq_2\Big(\frac{m_2}{2}+B^1g_2-\frac{\alpha}{\beta}B^2g_2\Big)\Big\}\,\mathrm{d}Z\mathrm{d}Y\\
=&-\iint_{\Omega^+}\varphi\Big\{\Big[\frac{\alpha q_1}{\beta}(\beta m_1+B^2g_1)\Big]_Z+\Big[q_2\Big(\frac{m_2}{2}+B^1g_2-\frac{\alpha}{\beta}B^2g_2\Big)\Big]_Y\Big\}\,\mathrm{d}Z\mathrm{d}Y.
\end{aligned}\tag{3.28}$$

By a direct calculation, we deduce

$$\begin{aligned}
&\Big[\frac{\alpha q_1}{\beta}(\beta m_1+B^2g_1)\Big]_Z+\Big[q_2\Big(\frac{m_2}{2}+B^1g_2-\frac{\alpha}{\beta}B^2g_2\Big)\Big]_Y\\
=&\frac{\alpha}{\beta}q_1q_2\Big\{-dg_2(1-g_1)+bm_1m_2+B_u^1m_1g_2+B_u^2\frac{m_2g_1-\alpha m_1g_2}{\beta}+F_ug_1g_2\Big\}\\
=&\frac{\alpha}{\beta}q_1q_2g_1g_2\Big\{-d\frac{1-g_1}{g_1}+b\frac{m_1}{g_1}\frac{m_2}{g_2}+B_u^1\frac{m_1}{g_1}+B_u^2\frac{m_2g_1-\alpha m_1g_2}{\beta g_1g_2}+F_u\Big\}.
\end{aligned}$$

Putting the above into (3.28) yields

$$\begin{aligned}
&\iint_{\mathbb{R}^+\times\mathbb{R}}\Big\{\varphi_t(\alpha u_t+\frac{1}{2}\beta u_x+B^1)+\varphi_x(\frac{1}{2}\beta u_t+B^2)\Big\}\,\mathrm{d}x\mathrm{d}t\\
=&-\iint_{\mathbb{R}^+\times\mathbb{R}}\varphi\Big\{-d\frac{1-g_1}{g_1}+b\frac{m_1}{g_1}\frac{m_2}{g_2}+B_u^1\frac{m_1}{g_1}+B_u^2\frac{m_2g_1-\alpha m_1g_2}{\beta g_1g_2}+F_u\Big\}\,\mathrm{d}x\mathrm{d}t\\
=&-\iint_{\mathbb{R}^+\times\mathbb{R}}\varphi\Big\{-d\bar{R}^2+b\bar{R}\bar{S}+B_u^1\bar{R}+B_u^2\frac{\bar{S}-\alpha\bar{R}}{\beta}+F_u\Big\}\,\mathrm{d}x\mathrm{d}t\\
=&-\iint_{\mathbb{R}^+\times\mathbb{R}}\varphi\Big\{\frac{1}{2}\alpha_uu_t^2+\frac{1}{2}\beta_uu_tu_x+B_u^1u_t+B_u^2u_x+F_u\Big\}\,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

which finishes the proof of (1.16).

Finally, we complete the proof of Theorem 1.2 by proving (1.15) and the continuity of $t \mapsto u_t(t,\cdot)$. For $M > 0$ and $t > 0$, we first prove

$$\mathscr{E}_M(t) := \int_{-M}^{M} u_t^2(t,x)\,\mathrm{d}x \le \widehat{C}(\mathscr{E}_0 + M + t) \tag{3.29}$$

for some positive constant $\widehat{C}$ independent of M and t. To show it, we fix $\tau > 0$ and denote $\Gamma_\tau := \{(Z,Y):\ t(Z,Y) = \tau\}$. Let A_1 and A_2 on Γ_τ be any two corresponding points of the points $(\tau,-M)$ and (τ,M) in t-x plane, respectively. Then we draw the horizontal and vertical lines from A_1 and A_2 up to Γ_1 at points A_4 and A_3, respectively. We consider the region D bounded by Γ_1, Γ_τ, A_1A_4, and A_2A_3, integrate (3.20) along its boundary and use (3.16) to find that

$$\begin{aligned}\int_{A_1A_2} -\alpha q_1(1-g_1)\mathrm{d}Y =& \int_{A_1A_2} 2Fq_2g_2\mathrm{d}Z + \int_{A_2A_3} \alpha q_1(1-g_1)\mathrm{d}Y \\ &+ \int_{A_3A_4} \alpha q_1(1-g_1)\mathrm{d}Y + 2Fq_2g_2\mathrm{d}Z + \int_{A_4A_1} 2Fq_2g_2\mathrm{d}Z \\ \le& C(\mathscr{E}_0 + M + \tau).\end{aligned}$$

Here we have used the fact that the length of the segment corresponding to A_4A_3 in the initial line $t = 0$ is less or equal to $(2M + \max\{\alpha/\beta\}\tau)$. Therefore, we have

$$\int_{A_1A_2} -q_1(1-g_1)\mathrm{d}Y \le \widehat{C}(\mathscr{E}_0 + M + \tau) \tag{3.30}$$

for some positive constant $\widehat{C}$ independent of M and t. On the other hand, it follows that

$$\int_{-M}^{M} u_t^2(\tau,x)\mathrm{d}x = \int_{A_1A_2\cap\{g_1\neq 0\}} -q_1(1-g_1)\mathrm{d}Y,$$

which together with (3.30) concludes (3.29). Now for any $t,s \in \mathbb{R}^+$, we see that

$$\begin{aligned}\|u(t,x)-u(s,x)\|_{L^2([-M,M])} =& \left\|(t-s)\int_0^1 u_t(s+\xi(t-s),x)\mathrm{d}\xi\right\|_{L^2([-M,M])} \\ \le& |t-s|\int_0^1 \|u_t(s+\xi(t-s),x)\|_{L^2([-M,M])}\mathrm{d}\xi \\ \le& \sqrt{\widehat{C}(\mathscr{E}_0 + M + t + s)}|t-s|,\end{aligned} \tag{3.31}$$

which leads to (1.15).

In order to verify the continuity of the function $t \mapsto u_t(t,\cdot)$ in $L^\theta([-M,M])$ $(1 \le \theta < 2)$, we first consider the arguments for smooth initial data with compact support, in which, $u = u(Z,Y)$ remains smooth on Ω^+. For a fixed time τ and any fixed $M > 0$, we assert that $\frac{\mathrm{d}}{\mathrm{d}t}u(t,\cdot)|_{t=\tau} = u_t(\tau,\cdot)$ in interval $[-M,M]$, where

$$u_t(\tau,x) := u_Z Z_t + u_Y Y_t = \frac{m_1}{g_1},$$

which defines the value of $u_t(\tau,\cdot)$ at almost every point $x \in [-M,M]$ by (3.29). Consider the curve segment A_1A_2 as before. For any $\theta \in [1,2)$, let $\sigma := 2/(2-\theta)$ be the conjugate exponent of $2/\theta$ and denote $\widetilde{M} := \widehat{C}(\mathscr{E}_0 + M + 2\tau)$. Given any $\varepsilon > 0$, it is obvious that there exist finitely many disjoint intervals $[a_i, b_i] \subset [-M, M]$, $i = 1, 2, \cdots, N$, such that

$$g_1(P) < \frac{2\varepsilon}{(\widetilde{M}+1)^\sigma} \tag{3.32}$$

for every point $P = (Z(x_P, \tau), Y(x_P, \tau))$ and

$$g_1(Q) > \frac{\varepsilon}{(\widetilde{M}+1)^\sigma}$$

for every point $Q = (Z(x_Q, \tau), Y(x_Q, \tau))$, where $x_P \in \bigcup_{i=1}^{N} [a_i, b_i] =: J$ and $x_Q \in [-M, M] \setminus J =: J'$. We notice that $u = u(t, x)$ is smooth in a neighborhood of $\{\tau\} \times J'$ by the construction of J'. Making use of Minkowski's inequality gives

$$\begin{aligned}
&\lim_{\rho \to 0} \frac{1}{\rho} \left[\int_{-M}^{M} |u(\tau+\rho, x) - u(\tau, x) - \rho u_t(\tau, x)|^\theta \mathrm{d}x \right]^{\frac{1}{\theta}} \\
\le &\lim_{\rho \to 0} \frac{1}{\rho} \left[\int_J |u(\tau+\rho, x) - u(\tau, x)|^\theta \mathrm{d}x \right]^{\frac{1}{\theta}} + \left[\int_J |u_t(\tau, x)|^\theta \mathrm{d}x \right]^{\frac{1}{\theta}}.
\end{aligned} \tag{3.33}$$

By (3.32) and (3.31), one can estimate the measure of the "bad" set J

$$\begin{aligned}
\mathrm{meas}(J) = \int_J \mathrm{d}x = &\sum_{i=1}^{N} \int_{(Z_{a_i}, Y_{a_i})}^{(Z_{b_i}, Y_{b_i})} q_2 g_2 \mathrm{d}Z = \sum_{i=1}^{N} \int_{(Z_{a_i}, Y_{a_i})}^{(Z_{b_i}, Y_{b_i})} -q_1 g_1 \mathrm{d}Y \\
\le &\frac{\frac{2\varepsilon}{(\widetilde{M}+1)^\sigma}}{1 - \frac{2\varepsilon}{(\widetilde{M}+1)^\sigma}} \sum_{i=1}^{N} \int_{(Z_{a_i}, Y_{a_i})}^{(Z_{b_i}, Y_{b_i})} -q_1 (1 - g_1) \mathrm{d}Y \\
\le &\frac{2\varepsilon}{(1-2\varepsilon)(\widetilde{M}+1)^\sigma} \int_{A_1 A_2} -q_1 (1 - g_1) \mathrm{d}Y \\
\le &\frac{2\widetilde{M}\varepsilon}{(1-2\varepsilon)(\widetilde{M}+1)^\sigma},
\end{aligned}$$

where $(Z_{a_i}, Y_{a_i}) = (Z(a_i, \tau), Y(a_i, \tau))$ and $(Z_{b_i}, Y_{b_i}) = (Z(b_i, \tau), Y(b_i, \tau))$. Applying Hölder's inequality and recalling (3.31) gives

$$\begin{aligned}
\int_J |u(\tau+\rho, x) - u(\tau, x)|^\theta \mathrm{d}x \le &\mathrm{meas}(J)^{\frac{1}{\sigma}} \left(\int_J |u(\tau+\rho, x) - u(\tau, x)|^2 \mathrm{d}x \right)^{\frac{\theta}{2}} \\
\le &\left(\frac{2\widetilde{M}\varepsilon}{(1-2\varepsilon)(\widetilde{M}+1)^\sigma} \right)^{\frac{1}{\sigma}} \|u(\tau+\rho, \cdot) - u(\tau, \cdot)\|_{L^2([-M,M])}^\theta \\
\le &\left(\frac{2\widetilde{M}\varepsilon}{(1-2\varepsilon)(\widetilde{M}+1)^\sigma} \right)^{\frac{1}{\sigma}} (\widetilde{M} + \widehat{C}\rho)^{\frac{\theta}{2}} \rho^\theta,
\end{aligned}$$

from which one has

$$\limsup_{\rho\to 0}\frac{1}{\rho}\left(\int_J |u(\tau+\rho,x)-u(\tau,x)|^\theta \mathrm{d}x\right)^{\frac{1}{\theta}}$$

$$\le\sqrt{\widetilde{M}}\left(\frac{2\widetilde{M}\varepsilon}{(1-2\varepsilon)(\widetilde{M}+1)^\sigma}\right)^{\frac{1}{\sigma\theta}}\le\left(\frac{2\varepsilon}{1-2\varepsilon}\right)^{\frac{1}{\sigma\theta}}. \tag{3.34}$$

Moreover, we have

$$\left(\int_J |u_t(\tau,x)|^\theta \mathrm{d}x\right)^{\frac{1}{\theta}}\le \mathrm{meas}(J)^{\frac{1}{\sigma\theta}}\left(\int_J |u_t(\tau,x)|^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

$$\le\left(\frac{2\widetilde{M}\varepsilon}{(1-2\varepsilon)(\widetilde{M}+1)^\sigma}\right)^{\frac{1}{\sigma\theta}}\|u_t(\tau,\cdot)\|_{L^2([-M,M])}$$

$$\le\sqrt{2\widetilde{M}}\left(\frac{2\widetilde{M}\varepsilon}{(1-2\varepsilon)(\widetilde{M}+1)^\sigma}\right)^{\frac{1}{\sigma\theta}}\le\sqrt{2}\left(\frac{2\varepsilon}{1-2\varepsilon}\right)^{\frac{1}{\sigma\theta}}. \tag{3.35}$$

Combining with (3.33), (3.34), and (3.35), it suggests that

$$\lim_{\rho\to 0}\frac{1}{\rho}\left(\int_{-M}^{M} |u(\tau+\rho,x)-u(\tau,x)-\rho u_t(\tau,x)|^\theta \mathrm{d}x\right)^{\frac{1}{\sigma}}=0.$$

The continuity of $t\mapsto u_t(t,\cdot)$ can be obtained by the same method. In fact, it is easily seen that

$$\limsup_{\rho\to 0}\int_{-M}^{M} |u_t(\tau+\rho,x)-u_t(\tau,x)|^\theta \mathrm{d}x\le \limsup_{\rho\to 0}\int_J |u_t(\tau+\rho,x)-u_t(\tau,x)|^\theta \mathrm{d}x$$

$$\le \limsup_{\rho\to 0}\mathrm{meas}(J)^{\frac{1}{\sigma}}\left(\int_J |u_t(\tau+\rho,x)-u_t(\tau,x)|^2 \mathrm{d}x\right)^{\frac{\theta}{2}}$$

$$\le \limsup_{\rho\to 0}\left(\frac{2\widetilde{M}\varepsilon}{(1-2\varepsilon)(\widetilde{M}+1)^\sigma}\right)^{\frac{1}{\sigma}}\left(2\|u_t(\tau+\rho,x)\|^2_{L^2([-M,M])}+2\|u_t(\tau,x)\|^2_{L^2([-M,M])}\right)^{\frac{\theta}{2}}$$

$$\le\left(\frac{2\widetilde{M}\varepsilon}{(1-2\varepsilon)(\widetilde{M}+1)^\sigma}\right)^{\frac{1}{\sigma}}(8\widetilde{M})^{\frac{\theta}{2}}\le 8^{\frac{\theta}{2}}\left(\frac{2\varepsilon}{1-2\varepsilon}\right)^{\frac{1}{\sigma}},$$

which completes the proof by the arbitrariness of ε.

To extend the result to general initial data $(u_0), u_1\in L^2$, we let $\{(u_0^\nu)_x\},\{u_1^\nu\}\in C_c^\infty$ be a sequence of smooth initial data such that $u_0^\nu\to u_0$ uniformly, $(u_0^\nu)_x\to(u_0)_x$ almost everywhere and in L^2, $u_1^\nu\to u_1$ almost everywhere and in L^2. The proof is concluded by Theorem 3.1.

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