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MULTIPLE SOLUTIONS FOR A CLASS OF KIRCHHOFF TYPE EQUATIONS WITH ZERO MASS AND HARDY-LITTLEWOOD-SOBOLEV CRITICAL NONLINEARITY

CHONGQING WEI, ANRAN LI*

School of Mathematical Sciences, Shanxi University, Taiyuan 030006, China

Abstract. In this paper, we study the multiplicity of solutions to the following Kirchhoff type equation with zero mass and Hardy-Littlewood-Sobolev critical nonlinearity

$$\begin{cases} -m(\int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u = \lambda K(x) f(u) + \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right) |u|^{2^*_\mu - 2} u, \quad x \in \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

where $N \geqslant 3$, $\lambda > 0$, $\mu \in (0, \min\{N,4\})$, $2^*_\mu = \frac{2N-\mu}{N-2}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, and m satisfies some local monotonicity conditions near zero. The nonlinearity f is odd in u and satisfies some classical superlinear and quasi-critical growth conditions. For any given $k \in \mathbb{N}$, k pairs of nontrivial solutions are obtained for λ large enough by a version of the symmetric mountain pass theorem and a version of the second concentration compactness principle.

Keywords. Hardy-Littlewood-Sobolev critical nonlinearity; Kirchhoff type equations; Variational method; Zero mass.

1. Introduction and Main Result

In this paper, we consider the following Kirchhoff type equation

$$\begin{cases} -m(\int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u = \lambda K(x) f(u) + \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right) |u|^{2^*_\mu - 2} u, \quad x \in \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geqslant 3$, $\lambda \in (0, +\infty)$, and $\mu \in (0, \min\{N,4\})$ are given parameters, $2^*_\mu = \frac{2N-\mu}{N-2}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality (see Lemma 2.1), and m, K and f satisfy the following assumptions:

- (m_0) $m \in C([0,+\infty),[0,+\infty))$ with $m(0) > 0$, and there exists $\sigma > 0$ such that m is increasing (or decreasing) in $[0,\sigma]$;
- (K_0) there exists $p \in [2, 2^*)$ such that $K \in L^\infty(\mathbb{R}^N) \cap L^{\frac{2^*}{2^*-p}}(\mathbb{R}^N)$, $K(x) > 0$, $a.e.\ x \in \mathbb{R}^N$, where $2^* = \frac{2N}{N-2}$ is the critical exponent of Sobolev embedding;

*Corresponding author.
E-mail address: weicq@sxu.edu.cn (C. Wei), anran0200@163.com (A. Li).
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(K_1) there exist $a_0, r > 0$, and $x_0 \in \mathbb{R}^N$ such that $K(x) \geq a_0$ for $a.e.$ $x \in B_r(x_0)$, where $B_r(x_0)$ is an open sphere in $\mathbb{R}^N$, which is centered at x_0 and with a radius of r;

(f_0) $f \in C(\mathbb{R}, \mathbb{R})$ is odd;

(f_1) there holds $\lim_{s\to 0} \frac{f(s)}{|s|^{2^*-2}s} = \lim_{|s|\to\infty} \frac{f(s)}{|s|^{2^*-2}s} = 0$;

(f_2) there exists $\nu \in (2, 2^*)$ such that $0 < \nu F(s) \leqslant sf(s)$, $s \neq 0$, where $F(s) = \int_0^s f(t)dt$.

Generally speaking, we call a nonlinear Schrödinger equation is with zero mass $-\Delta u + V(x)u = f(u)$, $x \in \mathbb{R}^N$, if $V = 0$ and f satisfies $f'(0) = 0$ (see [1, Section 5]). Problem (1.1) is a class of problem with zero mass, because under our assumptions the nonlinearity f can verify the condition that $f'(0) = 0$.

Our study is inspired by some works in recent years. On the one hand, various classes of Kirchhoff type equations have been under the spotlight of research for the past two decades. It was first proposed by Kirchhoff [2] with its origin in the theory of nonlinear vibration. In the case $m(t) = a + bt$ with $a, b > 0$, it is an extension of classical D'Alembert's wave equation for free vibrations of elastic strings. Since Lions in [3] proposed an abstract framework to this problem, Kirchhoff type equations have been widely studied in extensive literatures. We refer the readers to [4–23] and the references therein. Among them, the critical case were studied in [12, 15, 18–23]. Particularly, by truncating the nonlocal term, the following Kirchhoff type equation with critical growth was studied in [20]

$$\begin{cases} -m(\int_\Omega |\nabla u|^2 dx)\Delta u = \lambda f(x,u) + \mu |u|^{2^*-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where m is an increasing positive function in $[0, +\infty)$ and the nonlinearity f is odd in the second variable and enjoys some superlinear growth conditions. By applying a version of the symmetric mountain pass theorem and the second concentration compactness principle of Lions [24, 25], multiple solutions depending on μ and λ were obtained in [20].

On the other hand, the following Choquard equation

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.2}$$

has attracted much attentions due to its vast applications in physical models [26, 27]. The existence and uniqueness of positive solutions for (1.2) with $N = 3$, $V(x) \equiv 1$, $\alpha = 2$, and $p = 2$ was firstly obtained by Lieb in [26]. Later, Lions [28] obtained the existence and multiplicity results of normalized solutions on the same topic. Moroz and Schaftingen [29] studied the existence, asymptotic behavior, and symmetry of solutions for Choquard equations. Gao and Yang [30] studied the Brezis-Nirenberg type critical problems for nonlinear Choquard equations in bounded domains. Later, in [31], some existence and multiplicity results for Choquard equations with Hardy-Littlewood-Sobolev critical exponents in bounded domains were established. For related topics, we refer the readers to [32], a survey paper.

Recently, for the case $m(t) = a + bt^{\theta-1}$ and $\theta \in [1, 2^*_\mu)$, the following Kirchhoff type equation in the bounded domain with Hardy-Littlewood-Sobolev critical nonlinearity was studied in [33]

$$\begin{cases} -\left[a + b\left(\int_\Omega |\nabla u|^2 dx\right)^{\theta-1}\right]\Delta u = \lambda k(x)u + \left(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}dy\right)|u|^{2^*_\mu-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $0 < \mu < N$, and a, b, λ are positive real parameters. The function $k \in L^{\frac{2^*}{2^*-2}}(\Omega)$ is a nonnegative and continuous real valued function. By using the genus theory, introduced by Krasnoselskii, a variant of the mountain pass theorem for even functionals due to Rabinowitz [34], and a version of the second concentration compactness principle [35], the multiplicity of solutions for problem (1.3) was obtained. Motivated by [20] and [33], we studied the following Kirchhoff type equation in the bounded domain with Hardy-Littlewood-Sobolev critical nonlinearity in [36]

$$\begin{cases} -m(\int_\Omega |\nabla u|^2 dx)\Delta u = \lambda f(x,u) + \left(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right) |u|^{2^*_\mu - 2} u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

The assumptions on m and f in [36] were slightly weaker than those in [20]. The multiplicity of solutions was studied in [36] via a version of the symmetric mountain pass theorem and the truncation method.

More recently, multiplicity results for the following Choquard-Kirchhoff type equations on $\mathbb{R}^N$ with Hardy-Littlewood-Sobolev critical exponent were studied in [37]

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u = \alpha k(x)|u|^{q-2}u + \beta\left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right)|u|^{2^*_\mu-2}u,\ x \in \mathbb{R}^N,$$

where $a > 0, b \geqslant 0$, $N \geqslant 3$, α, β are positive real parameters, $k \in L^r(\mathbb{R}^N)$ with $r = \frac{2^*}{2^*-q}$ if $q \in (1, 2^*)$ and $r = \infty$ if $q \geqslant 2^*$. The multiplicity of solutions to the equation above was obtained by variational methods, depending on α, β, according to the different ranges of q.

As demonstrated in the literatures, various versions of concentration compactness principles play an important role in averting the lack of compactness. Particularly, on the whole space $\mathbb{R}^N$, the concentration compactness principle at infinity [38] provided some quantitative information about the loss of mass of a sequence at infinity. Base on [24, 25, 38], the authors in [35] established a version of the concentration compactness principle for Choquard type equations (see [35, Lemma 2.5]). According to it, we can find that the functional associated to some Choquard type equations with Hardy-Littlewood-Sobolev critical nonlinearity satisfies $(\mathrm{PS})_c$ condition for $c > 0$ small enough (see Proposition 2.2 for more details).

In this paper, under assumption (m_0), the equation (1.1) that we study can cover many kinds of Kirchhoff type equations. However, since we assume that $m(0) > 0$, degenerate Kirchhoff type problem ($m(0) = 0$) is not considered. Similar to [20, 36], the nonlinearity f just needs to satisfy the classic Ambrosetti-Rabinowitz condition instead of some 4-superlinear conditions.

The main working space in this paper is $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$, endowed with the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

$L^p(\mathbb{R}^N)$ is the usual p power Lebesgue integrable space, and we also denote by $|u|_p$ the norm of a function $u \in L^p(\mathbb{R}^N)$, for any $1 \leqslant p \leqslant \infty$.

Since, under our assumption (m_0), m only satisfies some local monotonicity conditions, we first deal with problem (1.1) by truncating m, which has been successfully used to study Kirchhoff type equations in [4, 12, 20, 36]. Condition (m_0) implies that there exists $\delta \in (0, \sigma)$ such that

$$\begin{cases} 0 < m(0) < m(\delta) < \frac{\nu}{2}m(0), & \text{if } m \text{ is increasing;} \\ 0 < m(\delta) < m(0) < \frac{\nu}{2}m(\delta), & \text{if } m \text{ is decreasing,} \end{cases}$$

where $\nu > 2$ is from (f_2). We set

$$m_\delta(t) = \begin{cases} m(t), & \text{if } 0 \leqslant t \leqslant \delta; \\ m(\delta), & \text{if } t > \delta. \end{cases}$$

It is easy to see that $m_\delta \in C([0,+\infty),(0,+\infty))$. Then, we consider the truncated problem

$$\begin{cases} -m_\delta(\|u\|^2)\Delta u = \lambda K(x)f(u) + \left(\displaystyle\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}dy\right)|u|^{2^*_\mu-2}u, \quad x \in \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.4}$$

As usual, the energy functional associated to problem (1.4) is given by

$$J_{\delta,\lambda}(u) = \frac{1}{2}M_\delta(\|u\|^2) - \lambda\int_{\mathbb{R}^N} K(x)F(u)dx - \frac{1}{2\cdot 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}dydx,$$

where $M_\delta(s) := \int_0^s m_\delta(t)dt$. By (f_0)–(f_2), one finds that $J_{\delta,\lambda}$ belongs to $C^1(D^{1,2}(\mathbb{R}^N),\mathbb{R})$. Its Fréchet derivative at u is given by

$$\begin{aligned} J'_{\delta,\lambda}(u)\phi =& m_\delta(\|u\|^2)\int_{\mathbb{R}^N}\nabla u\nabla\phi dx - \lambda\int_{\mathbb{R}^N}K(x)f(u)\phi(x)dx \\ &- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}|u(x)|^{2^*_\mu-2}u(x)\phi(x)dydx, \end{aligned}$$

for every $\phi \in D^{1,2}(\mathbb{R}^N)$. A weak solution to problem (1.4) is the critical point of $J_{\delta,\lambda}$. Moreover, by the definition of m_δ, if $u \in D^{1,2}(\mathbb{R}^N)$ is a weak solution to problem (1.4) and $\|u\| \leqslant \delta^{\frac{1}{2}}$, then $m_\delta(\|u\|^2) = m(\|u\|^2)$, which implies that u is also a weak solution to (1.1), the original problem. Hence, in order to obtain the weak solution of problem (1.1), we look for the critical point of $J_{\delta,\lambda}$ with the small norm. We show that this is true if the parameter λ is large enough.

Theorem 1.1. *Suppose that* (m_0), (K_0), (K_1), *and* (f_0)–(f_2) *hold. Then, for any given* $k \in \mathbb{N}$, *there exists* $\lambda_k^* > 0$ *such that problem* (1.1) *has at least k pairs of nontrivial weak solutions for every* $\lambda > \lambda_k^*$.

The rest of the paper is organized as follows. We give some preliminaries in Section 2. The proof of Theorem 1.1 is given in Section 3.

2. Preliminary

In this section, we provide some necessary preliminary results.

Lemma 2.1. (Hardy-Littlewood-Sobolev inequality, [39]) *Let* $p,q > 1$ *and* $\mu \in (0,N)$ *be such that* $\frac{1}{p}+\frac{1}{q}+\frac{\mu}{N} = 2$. *Then, for* $f \in L^p(\mathbb{R}^N)$ *and* $g \in L^q(\mathbb{R}^N)$, *we have*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)g(y)}{|x-y|^\mu}dxdy\right| \leqslant C(N,\mu,p)|f|_p\cdot|g|_q.$$

Definition 2.1. If E is a real Banach space and $J \in C^1(E,\mathbb{R})$, we say that J satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ ($(\mathrm{PS})_c$ condition for short) if every sequence $\{u_n\}_{n=1}^{\infty} \subset E$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ possesses a convergent subsequence. J satisfies the Palais-Smale condition ((PS) condition for short) if J satisfies $(\mathrm{PS})_c$ condition for every $c \in \mathbb{R}$.

The following version of the symmetric mountain pass theorem (see [40, 41]) is needed to prove our main result.

Lemma 2.2. [41] *Let $E = V \oplus W$ be a real Banach space with $\dim V < \infty$. Suppose that $J \in C^1(E,\mathbb{R})$ is an even functional with $J(0) = 0$ and*

- (J_1) *there exist $\rho, \alpha > 0$ such that $\inf_{u \in \partial B_\rho(0) \cap W} I(u) \geqslant \alpha$;*
- (J_2) *there exists a subspace $\hat{V} \subset E$ such that $\dim V < \dim \hat{V} < \infty$ and $\max_{u \in \hat{V}} I(u) \leqslant M$ for some $M > 0$;*
- (J_3) *J satisfies $(PS)_c$ condition for any $c \in (0,M)$ with M as in (J_2).*

Then J possesses at least $\dim \hat{V} - \dim V$ pairs of nontrivial critical points.

Proposition 2.1. *Suppose that f satisfies (f_0)–(f_2). Then every $(PS)_c$ sequence $\{u_n\}_{n=1}^{\infty}$ of $J_{\delta,\lambda}$ at positive level c is bounded in $D^{1,2}(\mathbb{R}^N)$.*

Proof. Condition (m_0) and the definition of m_δ imply that

$$\begin{cases} 0 < m(0) \leqslant m_\delta(s) \leqslant m(\delta) < \frac{\nu}{2} m(0), \text{ if } m \text{ is increasing;} \\ 0 < m(\delta) \leqslant m_\delta(s) \leqslant m(0) < \frac{\nu}{2} m(\delta), \text{ if } m \text{ is decreasing,} \end{cases}$$

for $s \in [0,+\infty)$. Hence, for every $(\mathrm{PS})_c$ sequence $\{u_n\}_{n=1}^{\infty}$ of $J_{\delta,\lambda}$, it follows from Definition 2.1, (f_2), and $\nu < 2^* < 2 \cdot 2^*_\mu$ that there exists $C > 0$ such that for n large enough

$$\begin{aligned} C + \|u_n\| &\geqslant J_{\delta,\lambda}(u_n) - \frac{1}{\nu} J'_{\delta,\lambda}(u_n)u_n \\ &\geqslant \left(\frac{\min\{m(0), m(\delta)\}}{2} - \frac{\max\{m(0), m(\delta)\}}{\nu} \right) \|u_n\|^2 \\ &= \begin{cases} (\frac{m(0)}{2} - \frac{m(\delta)}{\nu}) \|u_n\|^2, \text{ if } m \text{ is increasing;} \\ (\frac{m(\delta)}{2} - \frac{m(0)}{\nu}) \|u_n\|^2, \text{ if } m \text{ is decreasing.} \end{cases} \end{aligned}$$

Therefore, $\{u_n\}_{n=1}^{\infty}$ is bounded in $D^{1,2}(\mathbb{R}^N)$. □

Set

$$S = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : u \in D^{1,2}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} |u(x)|^{2^*} dx = 1 \right\},$$

$$S_{H,L} = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : u \in D^{1,2}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu} |u(x)|^{2^*_\mu}}{|x-y|^\mu} dy dx = 1 \right\}.$$

The following result (the complete proof was given in [35]) due to Lions [24, 25, 38] plays an important role in the proof that $J_{\delta,\lambda}$ satisfies $(\mathrm{PS})_c$ condition with $c > 0$ small enough.

Lemma 2.3. [35, Lemma 2.5] *Suppose that $\{u_n\}_{n=1}^{\infty} \subset D^{1,2}(\mathbb{R}^N)$ is such that $u_n \rightharpoonup u$ weakly in $D^{1,2}(\mathbb{R}^N)$ and $|\nabla u_n|^2 \rightharpoonup \omega$, $|u_n|^{2^*} \rightharpoonup \zeta$, and $(\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu} dy) |u_n|^{2^*_\mu} \rightharpoonup \upsilon$ weakly in the sense of measures, where ω, ζ, and υ are nonnegative and bounded measures on $\mathbb{R}^N$. Then there exist* *an at most countable index set I, which can be empty and a set of distinct points $\{x_i\}_{i\in I}\subset\mathbb{R}^N$ and three families of positive numbers $\{\omega_i\}_{i\in I}$, $\{\zeta_i\}_{i\in I}$, and $\{\upsilon_i\}_{i\in I}$ such that*

$$\omega\geqslant|\nabla u|^2+\sum_{i\in I}\omega_i\delta_{x_i},\ \zeta=|u_n|^{2^*}+\sum_{i\in I}\zeta_i\delta_{x_i},\upsilon=\left(\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}dy\right)|u_n|^{2_\mu^*}+\sum_{i\in I}\upsilon_i\delta_{x_i},$$

with $S\upsilon_i^{\frac{2}{2^*}}\leqslant\omega_i, S_{H,L}\upsilon_i^{\frac{1}{2_\mu^*}}\leqslant\omega_i$, and $\upsilon_i\leqslant C(N,\mu)\zeta_i^{\frac{2N-\mu}{N}}$ for $i\in I$. In particular, $\sum_{i\in I}\upsilon_i^{\frac{1}{2_\mu^*}}<\infty$, where δ_x is the Dirac-mass of mass 1 concentrated at $x\in\mathbb{R}^N$. Define

$$\omega_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{B_R^c(0)}|\nabla u_n|^2dx,\ \zeta_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{B_R^c(0)}|u_n|^{2^*}dx,$$
$$\upsilon_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{B_R^c(0)}\left(\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}dy\right)|u_n(x)|^{2_\mu^*}dx.$$

Then

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2dx=\omega_\infty+\int_{\mathbb{R}^N}d\omega,\ \limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}dx=\zeta_\infty+\int_{\mathbb{R}^N}d\zeta,$$
$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2_\mu^*}|u_n(x)|^{2_\mu^*}}{|x-y|^\mu}dydx=\upsilon_\infty+\int_{\mathbb{R}^N}d\upsilon.$$

Furthermore,

$$S\zeta_\infty^{\frac{2}{2^*}}\leqslant\omega_\infty,\ C(N,\mu)^{\frac{2N}{\mu-2N}}\upsilon_\infty^{\frac{2N}{2N-\mu}}\leqslant\zeta_\infty\left(\int_{\mathbb{R}^N}d\zeta+\zeta_\infty\right),\ S_{H,L}^2\upsilon_\infty^{\frac{2}{2_\mu^*}}\leqslant\omega_\infty\left(\int_{\mathbb{R}^N}d\omega+\omega_\infty\right).$$

Proposition 2.2. *Under the assumptions of Theorem 1.1, functional $J_{\delta,\lambda}$ satisfies (PS)$_c$ condition for any $c>0$ small enough.*

Proof. Let $\{u_n\}_{n=1}^\infty\subset D^{1,2}(\mathbb{R}^N)$ be such that

$$J'_{\delta,\lambda}(u_n)\to0\text{ and }J_{\delta,\lambda}(u_n)\to c<c^*:=\left(\frac{1}{\nu}-\frac{1}{2\cdot2_\mu^*}\right)(\min\{m(0),m(\delta)\}S_{H,L})^{\frac{2N-\mu}{N-\mu+2}}.$$

Proposition 2.1 indicates that $\{u_n\}_{n=1}^\infty$ is bounded in $D^{1,2}(\mathbb{R}^N)$. First of all, we can prove that the set I given by Lemma 2.3 is empty. Indeed, suppose by contradiction that there exists some $i_0\in I$ with $\upsilon_{i_0}>0$. For any $\varepsilon>0$, define $\phi_\varepsilon\in C_0^\infty(\mathbb{R}^N,[0,1])$ in the following way

$$\phi_\varepsilon(x)=1,\ x\in B(x_{i_0},\varepsilon),$$
$$\phi_\varepsilon(x)=0,\ x\in\mathbb{R}^N\setminus B(x_{i_0},2\varepsilon),$$
$$|\nabla\phi_\varepsilon(x)|\leqslant\frac{2}{\varepsilon},\ x\in\mathbb{R}^N.$$

It is easy to see that $\{u_n\phi_\varepsilon\}_{n=1}^\infty$ is also bounded in $D^{1,2}(\mathbb{R}^N)$. Thus

$$\lim_{n\to\infty}J'_{\delta,\lambda}(u_n)(u_n\phi_\varepsilon)=0.$$

That is,

$$\begin{aligned} o_n(1) =& m_\delta(\|u_n\|^2)\int_{\mathbb{R}^N}\nabla u_n\nabla(u_n\phi_\varepsilon)dx-\lambda\int_{\mathbb{R}^N}K(x)f(u_n)u_n\phi_\varepsilon dx \\ &-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}|u_n(x)|^{2^*_\mu}\phi_\varepsilon(x)}{|x-y|^\mu}dydx \\ =& m_\delta(\|u_n\|^2)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\phi_\varepsilon dx+\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\phi_\varepsilon dx\right) \\ &-\lambda\int_{\mathbb{R}^N}K(x)f(u_n)u_n\phi_\varepsilon dx-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}|u_n(x)|^{2^*_\mu}\phi_\varepsilon(x)}{|x-y|^\mu}dydx. \end{aligned} \tag{2.1}$$

By Hölder inequality, one has

$$\begin{aligned} \left|\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\phi_\varepsilon dx\right| &= \left|\int_{B(x_{i_0},2\varepsilon)}u_n\nabla u_n\nabla\phi_\varepsilon dx\right| \\ &\leqslant \left(\int_{B(x_{i_0},2\varepsilon)}|\nabla u_n|^2dx\right)^{\frac{1}{2}}\left(\int_{B(x_{i_0},2\varepsilon)}|u_n\nabla\phi_\varepsilon|^2dx\right)^{\frac{1}{2}} \\ &\leqslant C\left(\int_{B(x_{i_0},2\varepsilon)}|u_n\nabla\phi_\varepsilon|^2dx\right)^{\frac{1}{2}}. \end{aligned}$$

Since

$$\lim_{n\to\infty}\int_{B(x_{i_0},2\varepsilon)}|u_n\nabla\phi_\varepsilon|^2dx=\int_{B(x_{i_0},2\varepsilon)}|u\nabla\phi_\varepsilon|^2dx$$

and

$$\begin{aligned} \left(\int_{B(x_{i_0},2\varepsilon)}|u\nabla\phi_\varepsilon|^2dx\right)^{\frac{1}{2}} &\leqslant \left(\int_{B(x_{i_0},2\varepsilon)}|u|^{2^*}dx\right)^{\frac{1}{2^*}}\left(\int_{B(x_{i_0},2\varepsilon)}|\nabla\phi_\varepsilon|^Ndx\right)^{\frac{1}{N}} \\ &\leqslant C\left(\int_{B(x_{i_0},2\varepsilon)}|u|^{2^*}dx\right)^{\frac{1}{2^*}},\ \text{where } C>0 \text{ is independent of } \varepsilon, \end{aligned}$$

together with the facts that $\{u_n\}_{n=1}^\infty$ is bounded in $D^{1,2}(\mathbb{R}^N)$ and m_δ is continuous, we have

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}m_\delta(\|u_n\|^2)\left|\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\phi_\varepsilon dx\right|=0. \tag{2.2}$$

By the definitions of ϕ_ε and m_δ, we have

$$\begin{aligned} \liminf_{n\to\infty}m_\delta(\|u_n\|^2)\int_{\mathbb{R}^N}|\nabla u_n|^2\phi_\varepsilon dx &\geqslant \lim_{n\to\infty}\min\{m(0),m(\delta)\}\int_{B(x_{i_0},2\varepsilon)}|\nabla u_n|^2\phi_\varepsilon dx \\ &= \min\{m(0),m(\delta)\}\left(\int_{B(x_{i_0},2\varepsilon)}|\nabla u|^2\phi_\varepsilon dx+\left\langle\sum_{i\in I}\eta_i\delta_{x_i},\phi_\varepsilon\right\rangle\right) \\ &\geqslant \min\{m(0),m(\delta)\}\left(\int_{B(x_{i_0},2\varepsilon)}|\nabla u|^2\phi_\varepsilon dx+\omega_{i_0}\right). \end{aligned}$$

According to the absolute continuity of Lebesgue integral, we see that $\int_{B(x_{i_0},2\varepsilon)}|\nabla u|^2\phi_\varepsilon dx \to 0$ as $\varepsilon \to 0$. Then

$$\liminf_{\varepsilon\to 0}\liminf_{n\to\infty} m_\delta(\|u_n\|^2)\int_{\mathbb{R}^N}|\nabla u_n|^2\phi_\varepsilon dx \geqslant \min\{m(0),m(\delta)\}\omega_{i_0}. \tag{2.3}$$

By (f$_1$), for every $\varepsilon>0$, there exists $C_\varepsilon>0$ such that $|f(s)s|\leqslant \varepsilon|s|^{2^*}+C_\varepsilon|s|^p$, $s\in\mathbb{R}$, where $p\in[2,2^*)$ is from (K$_0$). Then

$$\begin{aligned}\left|\int_{\mathbb{R}^N}K(x)f(u_n)u_n\phi_\varepsilon dx\right| &\leqslant \int_{B(x_{i_0},2\varepsilon)}|K(x)f(u_n)u_n\phi_\varepsilon|dx\\ &\leqslant |K|_\infty\int_{B(x_{i_0},2\varepsilon)}(\varepsilon|u_n|^{2^*}+C_\varepsilon|u_n|^p)dx\\ &\leqslant C|K|_\infty\varepsilon + C_\varepsilon|K|_\infty\int_{B(x_{i_0},2\varepsilon)}|u_n|^p dx.\end{aligned}$$

Since the local compact embedding theorem implies that

$$\lim_{n\to\infty}\int_{B(x_{i_0},2\varepsilon)}|u_n|^p dx=\int_{B(x_{i_0},2\varepsilon)}|u|^p dx,$$

then

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N}K(x)f(u_n)u_n\phi_\varepsilon dx\right|\leqslant C|K|_\infty\varepsilon+C_\varepsilon|K|_\infty\int_{B(x_{i_0},2\varepsilon)}|u|^p dx.$$

Therefore, the arbitrariness of ε and the absolute continuity of Lebesgue integral lead to

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N}K(x)f(u_n)u_n\phi_\varepsilon dx\right|=0. \tag{2.4}$$

Finally, due to the facts that $(\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}dy)|u_n|^{2_\mu^*}\rightharpoonup \upsilon$ weakly in the sense of measures and $\sum\limits_{i\in I}\upsilon_i^{\frac{1}{2_\mu^*}}<\infty$, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2_\mu^*}|u_n(x)|^{2_\mu^*}\phi_\varepsilon(x)}{|x-y|^\mu}dydx\\ =&\lim_{\varepsilon\to 0}\left(\int_{B(x_{i_0},2\varepsilon)}|u|^{2^*}\phi_\varepsilon dx+\left\langle\sum_{i\in I}\upsilon_i\delta_{x_i},\phi_\varepsilon\right\rangle\right)\\ =&\upsilon_{i_0}.\end{aligned} \tag{2.5}$$

It follows from (2.1) that

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2_\mu^*}|u_n(x)|^{2_\mu^*}\phi_\varepsilon(x)}{|x-y|^\mu}dydx\\ =&m_\delta(\|u_n\|^2)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\phi_\varepsilon dx\right)-\lambda\int_{\mathbb{R}^N}K(x)f(u_n)u_n\phi_\varepsilon dx+o_n(1)\\ \geqslant&\min\{m(0),m(\delta)\}\int_{\mathbb{R}^N}|\nabla u_n|^2\phi_\varepsilon dx-m_\delta(\|u_n\|^2)\left|\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\phi_\varepsilon dx\right|\\ &-\lambda\left|\int_{\mathbb{R}^N}K(x)f(u_n)u_n\phi_\varepsilon dx\right|+o_n(1).\end{aligned}$$

Then

$$\begin{aligned}
&\liminf_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}|u_n(x)|^{2^*_\mu}\phi_\varepsilon(x)}{|x-y|^\mu}dydx\\
\geqslant&\liminf_{n\to\infty}\left[\min\{m(0),m(\delta)\}\int_{\mathbb{R}^N}|\nabla u_n|^2\phi_\varepsilon dx-m_\delta(\|u_n\|^2)\left|\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\phi_\varepsilon dx\right|\right.\\
&\left.-\lambda\left|\int_{\mathbb{R}^N}K(x)f(u_n)u_n\phi_\varepsilon dx\right|\right]\\
\geqslant&\liminf_{n\to\infty}\min\{m(0),m(\delta)\}\int_{\mathbb{R}^N}|\nabla u_n|^2\phi_\varepsilon dx-\limsup_{n\to\infty}m_\delta(\|u_n\|^2)\left|\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\phi_\varepsilon dx\right|\\
&-\limsup_{n\to\infty}\lambda\left|\int_{\mathbb{R}^N}K(x)f(u_n)u_n\phi_\varepsilon dx\right|.
\end{aligned}$$

By taking limits as $\varepsilon\to 0$ on both sides of the last inequality, it follows from (2.2)–(2.5) that

$$\upsilon_{i_0}\geqslant\min\{m(0),m(\delta)\}\omega_{i_0}.$$

Because of $S_{H,L}\upsilon_i^{\frac{1}{2^*_\mu}}\leqslant\omega_i$ for $i\in I$, one has

$$\upsilon_{i_0}\geqslant(\min\{m(0),m(\delta)\}S_{H,L})^{\frac{2N-\mu}{N-\mu+2}}. \tag{2.6}$$

It follows from $J_{\delta,\lambda}(u_n)\to c$ and $J'_{\delta,\lambda}(u_n)\to 0$ that

$$\begin{aligned}
c=&\lim_{n\to\infty}\left(J_{\delta,\lambda}(u_n)-\frac{1}{\nu}\langle J'_{\delta,\lambda}(u_n),u_n\rangle\right)\\
\geqslant&\liminf_{n\to\infty}\left(\frac{1}{\nu}-\frac{1}{2\cdot 2^*_\mu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}|u_n(x)|^{2^*_\mu}\phi_\varepsilon(x)}{|x-y|^\mu}dydx.
\end{aligned}$$

Then by taking limits as $\varepsilon\to 0$ on both sides of the inequality above, (2.6) leads to

$$c\geqslant\left(\frac{1}{\nu}-\frac{1}{2\cdot 2^*_\mu}\right)(\min\{m(0),m(\delta)\}S_{H,L})^{\frac{2N-\mu}{N-\mu+2}}.$$

This is a contradiction with the fact that $c<c^*$. Thus I is empty.

Next, in order to obtain that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}|u_n(x)|^{2^*_\mu}}{|x-y|^\mu}dydx=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}dydx,$$

it suffices to show that $\upsilon_\infty=0$. On the contrary, we assume that $\upsilon_\infty>0$. Let $\psi_R\in C^\infty(\mathbb{R}^N,[0,1])$ be a cut-off function such that

$$\psi_R(x)=0,\ |x|<R;\ \psi_R(x)=1,\ |x|\geqslant 2R,\ \text{and}\ |\nabla\psi_R|\leqslant\frac{2}{R}.$$

It is also easy to see that $\{u_n\psi_R\}_{n=1}^\infty$ is bounded in $D^{1,2}(\mathbb{R}^N)$. Then

$$\begin{aligned}
o_n(1)=&m_\delta(\|u_n\|^2)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\psi_R dx+\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\psi_R dx\right)\\
&-\lambda\int_{\mathbb{R}^N}K(x)f(u_n)u_n\psi_R dx-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}|u_n(x)|^{2^*_\mu}\psi_R(x)}{|x-y|^\mu}dydx.
\end{aligned}\tag{2.7}$$

By Hölder inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^N} u_n\nabla u_n\nabla\psi_R dx\right| &= \left|\int_{\{x:\ R\leqslant|x|\leqslant 2R\}} u_n\nabla u_n\nabla\psi_R dx\right|\\ &\leqslant \left(\int_{\{x:\ R\leqslant|x|\leqslant 2R\}}|\nabla u_n|^2dx\right)^{\frac{1}{2}}\left(\int_{\{x:\ R\leqslant|x|\leqslant 2R\}}|u_n\nabla\psi_R|^2dx\right)^{\frac{1}{2}}\\ &\leqslant C\left(\int_{\{x:\ R\leqslant|x|\leqslant 2R\}}|u_n\nabla\psi_R|^2dx\right)^{\frac{1}{2}}.\end{aligned}$$

Since

$$\lim_{n\to\infty}\int_{\{x:\ R\leqslant|x|\leqslant 2R\}}|u_n\nabla\psi_R|^2dx=\int_{\{x:\ R\leqslant|x|\leqslant 2R\}}|u\nabla\psi_R|^2dx,$$

$$\begin{aligned}\left(\int_{\{x:\ R\leqslant|x|\leqslant 2R\}}|u\nabla\psi_R|^2dx\right)^{\frac{1}{2}} &\leqslant \left(\int_{\{x:\ R\leqslant|x|\leqslant 2R\}}|u|^{2^*}dx\right)^{\frac{1}{2^*}}\left(\int_{\{x:\ R\leqslant|x|\leqslant 2R\}}|\nabla\psi_R|^Ndx\right)^{\frac{1}{N}}\\ &\leqslant C\left(\int_{\{x:\ R\leqslant|x|\leqslant 2R\}}|u|^{2^*}dx\right)^{\frac{1}{2^*}}\to 0,\end{aligned}$$

as $R\to\infty$, and the sequence $\{m_\delta(\|u_n\|^2)\}_{n=1}^\infty$ is bounded in $\mathbb{R}$, we arrive at

$$\lim_{R\to\infty}\limsup_{n\to\infty} m_\delta(\|u_n\|^2)\left|\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\psi_R dx\right|=0. \tag{2.8}$$

By the definition of ψ_R, we see that

$$\int_{\mathbb{R}^N}|\nabla u_n|^2\psi_R dx\geqslant\int_{B^c_{2R}(0)}|\nabla u_n|^2dx,$$

where $B^c_{2R}(0)=\{x\in\mathbb{R}^N:\ |x|\geqslant 2R\}$. Thus, the definition of m_δ leads to

$$\limsup_{n\to\infty} m_\delta(\|u_n\|^2)\int_{\mathbb{R}^N}|\nabla u_n|^2\psi_R dx\geqslant\limsup_{n\to\infty}\min\{m(0),m(\delta)\}\int_{B^c_{2R}(0)}|\nabla u_n|^2dx.$$

Therefore, the definition of ω_∞ implies that

$$\limsup_{R\to\infty}\limsup_{n\to\infty} m_\delta(\|u_n\|^2)\int_{\mathbb{R}^N}|\nabla u_n|^2\psi_R dx\geqslant\min\{m(0),m(\delta)\}\omega_\infty. \tag{2.9}$$

In view of (f_1), for any fixed $\varepsilon>0$, we see that there exists $C_\varepsilon>0$ such that

$$|f(s)s|\leqslant\varepsilon|s|^{2^*}+C_\varepsilon|s|^p,\ s\in\mathbb{R},$$

where $p\in[2,2^*)$ is from (K_0). It follows that

$$\begin{aligned}\left|\int_{\mathbb{R}^N}K(x)f(u_n)u_n\psi_R dx\right| &\leqslant\int_{B^c_R(0)}|K(x)f(u_n)u_n\psi_R|dx\\ &\leqslant|K|_\infty\varepsilon\int_{B^c_R(0)}|u_n|^{2^*}dx+C_\varepsilon\int_{B^c_R(0)}K(x)|u_n|^pdx.\end{aligned}$$

On account of $K\in L^{\frac{2^*}{2^*-p}}(\mathbb{R}^N)$ and $|u_n|^p\rightharpoonup|u|^p$ in $L^{\frac{2^*}{p}}(B^c_R(0))$, we conclude that

$$\lim_{n\to\infty}\int_{B^c_R(0)}K(x)|u_n|^pdx=\int_{B^c_R(0)}K(x)|u|^pdx.$$

Then

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N}K(x)f(u_n)u_n\psi_R dx\right|\leqslant C|K|_\infty\varepsilon+C_\varepsilon\int_{B_R^c(0)}K(x)|u|^p dx.$$

Since $\lim_{R\to\infty}\int_{B_R^c(0)}K(x)|u|^p dx=0$, for the above $\varepsilon>0$, we have that there exists $R_\varepsilon>0$ such that

$$C_\varepsilon\int_{B_R^c(0)}K(x)|u|^p dx<\varepsilon,\ R>R_\varepsilon.$$

Thus, the arbitrariness of ε implies that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N}K(x)f(u_n)u_n\psi_R dx\right|=0. \tag{2.10}$$

Then, by the definitions of ψ_R and υ_∞, we have

$$\begin{aligned}
&\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}|u_n(x)|^{2^*_\mu}\psi_R(x)}{|x-y|^\mu}dydx\\
=&\lim_{R\to\infty}\limsup_{n\to\infty}\int_{B_R^c(0)}\left(\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}dy\right)|u_n(x)|^{2^*_\mu}dx\\
=&\upsilon_\infty.
\end{aligned}\tag{2.11}$$

It follows from (2.7) that

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}|u_n(x)|^{2^*_\mu}\psi_R(x)}{|x-y|^\mu}dydx\\
\geqslant&\limsup_{n\to\infty}\Bigg[\min\{m(0),m(\delta)\}\int_{\mathbb{R}^N}|\nabla u_n|^2\psi_R dx-m_\delta(\|u_n\|^2)\left|\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\psi_R dx\right|\\
&-\lambda\left|\int_{\mathbb{R}^N}K(x)f(u_n)u_n\psi_R dx\right|\Bigg]\\
\geqslant&\limsup_{n\to\infty}\min\{m(0),m(\delta)\}\int_{\mathbb{R}^N}|\nabla u_n|^2\psi_R dx\\
&-\limsup_{n\to\infty}m_\delta(\|u_n\|^2)\left|\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\psi_R dx\right|-\limsup_{n\to\infty}\lambda\left|\int_{\mathbb{R}^N}K(x)f(u_n)u_n\psi_R dx\right|.
\end{aligned}$$

By taking limits as $R\to+\infty$ on both sides of the above inequality, (2.8)-(2.11) lead to

$$\upsilon_\infty\geqslant\min\{m(0),m(\delta)\}\omega_\infty. \tag{2.12}$$

For each

$$c\in\begin{cases}(0,\frac{\nu m(0)-2m(\delta)}{2\nu}\frac{S}{2}), & \text{if } m \text{ is increasing;}\\ (0,\frac{\nu m(\delta)-2m(0)}{2\nu}\frac{S}{2}), & \text{if } m \text{ is decreasing,}\end{cases}$$

similarly to the proof of Proposition 2.1, we can obtain that $|u_n|_{2^*} \leqslant 1$ for n large enough. Then Hardy-Littlewood-Sobolev inequality (Lemma 2.1) and Lemma 2.3 lead to

$$\begin{aligned}\upsilon_\infty &= \lim_{R\to\infty}\limsup_{n\to\infty}\int_{B_R^c(0)}\left(\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}dy\right)|u_n(x)|^{2_\mu^*}dx\\ &\leqslant \lim_{R\to\infty}\limsup_{n\to\infty}\left[C(N,\mu)|u_n|_{2^*}^{2_\mu^*}\left(\int_{B_R^c(0)}|u_n(x)|^{2^*}dx\right)^{\frac{2N-\mu}{2N}}\right]\\ &\leqslant C(N,\mu)\zeta_\infty^{\frac{2_\mu^*}{2^*}}.\end{aligned}\tag{2.13}$$

Since $S\zeta_\infty^{\frac{2}{2^*}} \leqslant \omega_\infty$ and $\mu < 4$, (2.12) and (2.13) imply that

$$\omega_\infty \geqslant \left(\frac{\min\{m(0),m(\delta)\}S^{\frac{2_\mu^*}{2}}}{C(N,\mu)}\right)^{\frac{2}{2_\mu^*-2}}.$$

Then, by choosing

$$c\in\begin{cases}(0,\min\{\frac{\nu m(0)-2m(\delta)}{2\nu}\frac{S}{2},\frac{\nu m(0)-2m(\delta)}{2\nu}(\frac{m(0)S^{\frac{2_\mu^*}{2}}}{C(N,\mu)})^{\frac{2}{2_\mu^*-2}}\}), & \text{if } m \text{ is increasing;}\\ (0,\min\{\frac{\nu m(\delta)-2m(0)}{2\nu}\frac{S}{2},\frac{\nu m(\delta)-2m(0)}{2\nu}(\frac{m(\delta)S^{\frac{2_\mu^*}{2}}}{C(N,\mu)})^{\frac{2}{2_\mu^*-2}}\}), & \text{if } m \text{ is decreasing,}\end{cases}$$

we find a contradiction respectively. In fact, for the case that m is increasing, since

$$\begin{aligned}c=\lim_{n\to\infty}\left(J_{\delta,\lambda}(u_n)-\frac{1}{\nu}\langle J'_{\delta,\lambda}(u_n),u_n\rangle\right) &\geqslant \limsup_{n\to\infty}\left(\frac{m(0)}{2}-\frac{m(\delta)}{\nu}\right)\int_{\mathbb{R}^N}|\nabla u_n|^2dx\\ &\geqslant \left(\frac{m(0)}{2}-\frac{m(\delta)}{\nu}\right)\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\psi_R dx,\end{aligned}$$

by taking limits as $R\to\infty$ on both sides of the above inequality, we see that

$$c \geqslant \left(\frac{m(0)}{2}-\frac{m(\delta)}{\nu}\right)\omega_\infty \geqslant \frac{\nu m(0)-2m(\delta)}{2\nu}\left(\frac{m(0)S^{\frac{2_\mu^*}{2}}}{C(N,\mu)}\right)^{\frac{2}{2_\mu^*-2}}.$$

Similarly, for the case that m is decreasing, we have

$$\begin{aligned}c &= \lim_{n\to\infty}\left(J_{\delta,\lambda}(u_n)-\frac{1}{\nu}\langle J'_{\delta,\lambda}(u_n),u_n\rangle\right)\\ &\geqslant \left(\frac{m(\delta)}{2}-\frac{m(0)}{\nu}\right)\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\psi_R dx\\ &= \left(\frac{m(\delta)}{2}-\frac{m(0)}{\nu}\right)\omega_\infty\\ &\geqslant \frac{\nu m(\delta)-2m(0)}{2\nu}\left(\frac{m(\delta)S^{\frac{2_\mu^*}{2}}}{C(N,\mu)}\right)^{\frac{2}{2_\mu^*-2}}.\end{aligned}$$

Therefore, it results that $\upsilon_\infty = 0$, for every

$$c \in \begin{cases} (0, \min\{\frac{\nu m(0)-2m(\delta)}{2\nu}\frac{S}{2}, \frac{\nu m(0)-2m(\delta)}{2\nu}(\frac{m(0)S^{\frac{2^*_\mu}{2}}}{C(N,\mu)})^{\frac{2}{2^*_\mu-2}}\}), & \text{if } m \text{ is increasing;} \\ (0, \min\{\frac{\nu m(\delta)-2m(0)}{2\nu}\frac{S}{2}, \frac{\nu m(\delta)-2m(0)}{2\nu}(\frac{m(\delta)S^{\frac{2^*_\mu}{2}}}{C(N,\mu)})^{\frac{2}{2^*_\mu-2}}\}), & \text{if } m \text{ is decreasing.} \end{cases}$$

Furthermore, from the above analyses, for every

$$c \in \begin{cases} (0, \min\{c^*, \frac{\nu m(0)-2m(\delta)}{2\nu}\frac{S}{2}, \frac{\nu m(0)-2m(\delta)}{2\nu}(\frac{m(0)S^{\frac{2^*_\mu}{2}}}{C(N,\mu)})^{\frac{2}{2^*_\mu-2}}\}), & \text{if } m \text{ is increasing;} \\ (0, \min\{c^*, \frac{\nu m(\delta)-2m(0)}{2\nu}\frac{S}{2}, \frac{\nu m(\delta)-2m(0)}{2\nu}(\frac{m(\delta)S^{\frac{2^*_\mu}{2}}}{C(N,\mu)})^{\frac{2}{2^*_\mu-2}}\}), & \text{if } m \text{ is decreasing,} \end{cases}$$

we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}|u_n(x)|^{2^*_\mu}}{|x-y|^\mu}dydx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}dydx. \tag{2.14}$$

Finally, we see that $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$. On the one hand, (f_1), (K_0), and the definition of the weak convergence in $L^{\frac{2^*}{p}}(\mathbb{R}^N)$ also imply that

$$\int_{\mathbb{R}^N}K(x)f(u_n)u_n dx \to \int_{\mathbb{R}^N}K(x)f(u)udx, \ \ n\to\infty.$$

Together with $J'_{\delta,\lambda}(u_n)u_n = o_n(1)$ and (2.14), we have

$$\lim_{n\to\infty}(m_\delta(\|u_n\|^2)\|u_n\|^2) = \lambda\int_{\mathbb{R}^N}K(x)f(u)udx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}dydx. \tag{2.15}$$

On the other hand, by $J'_{\delta,\lambda}(u_n)u = o_n(1)$ and the definition of the weak convergence, we can obtain

$$\lim_{n\to\infty}m_\delta(\|u_n\|^2)\|u\|^2 = \lambda\int_{\mathbb{R}^N}K(x)f(u)udx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}dydx. \tag{2.16}$$

(2.15) and (2.16) imply that

$$\lim_{n\to\infty}m_\delta(\|u_n\|^2)\lim_{n\to\infty}\|u_n\|^2 = \lim_{n\to\infty}(m_\delta(\|u_n\|^2)\|u_n\|^2) = \lim_{n\to\infty}m_\delta(\|u_n\|^2)\|u\|^2.$$

Since $m_\delta(t) \geqslant \min\{m(0), m(\delta)\} > 0$ for $t\in[0,+\infty)$, then $\lim_{n\to\infty}\|u_n\|^2 = \|u\|^2$. Therefore, it follows from $D^{1,2}(\mathbb{R}^N)$ is a Hilbert space that $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$. □

3. Proof of Theorem 1.1

In this section, we prove our main result. The main tool is Lemma 2.2 with $V = \{0\}$ and $W = D^{1,2}(\mathbb{R}^N)$. First of all, similar to [20], we verify that the functional $J_{\delta,\lambda}$ satisfies conditions (J_1) and (J_2).

Proposition 3.1. *Under the assumptions of Theorem 1.1,*

(1) *for each $\lambda > 0$, there exist $\rho_\lambda, \alpha_\lambda > 0$ such that* $\inf_{u\in\partial B_{\rho_\lambda(0)}}J_{\delta,\lambda}(u) \geqslant \alpha_\lambda$;

(2) *for any given $k \in \mathbb{N}$ and $M > 0$, there exists $\lambda_{k,M} > 0$ with the following property: for any $\lambda > \lambda_{k,M}$ one can find a subspace $V_k^\lambda \subset D^{1,2}(\mathbb{R}^N)$ with $\dim V_k^\lambda = k$ such that*

$$\max_{u \in V_k^\lambda} J_{\delta,\lambda}(u) < M.$$

Proof. (1) By (f_0) and (f_1), there exists $C > 0$ such that $F(s) \leqslant C|s|^{2^*}$, $s \in \mathbb{R}$. Hence, for $u \in D^{1,2}(\mathbb{R}^N)$,

$$\begin{aligned}
J_{\delta,\lambda}(u) &\geqslant \frac{\min\{m(0), m(\delta)\}}{2}\|u\|^2 - C|K|_\infty \lambda |u|_{2^*}^{2^*} - \frac{1}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^{2_\mu^*}|u(x)|^{2_\mu^*}}{|x-y|^\mu} dydx \\
&\geqslant \frac{\min\{m(0), m(\delta)\}}{2}\|u\|^2 - C_1\lambda\|u\|^{2^*} - C_2\|u\|^{2\cdot 2_\mu^*},
\end{aligned}$$

where $C_i, i = 1,2$ are positive constants independent of λ. Since $2 < 2^* < 2\cdot 2_\mu^*$, the first result can be easily obtained if we choose $\rho_\lambda > 0$ small enough.

(2) Let $\varphi \in C_0^\infty(B_1(0), \mathbb{R})$. We choose $\{x_1, \cdots, x_k\} \subset B_r(x_0)$ and $\tau > 0$ such that $B_\tau(x_i) \subset B_r(x_0)$ with $B_\tau(x_i) \cap B_\tau(x_j) = \emptyset$ if $i, j \in \{1, \cdots, k\}$ and $i \neq j$. For each $i \in \{1, \cdots, k\}$, we set $\varphi_i^\tau(x) := \varphi(\frac{x - x_i}{\tau})$, $x \in B_\tau(x_i)$. Then

$$A_\tau := \frac{\|\varphi_i^\tau\|^2}{|\varphi_i^\tau|_\nu^2} = \tau^{N-2-\frac{2N}{\nu}} \frac{\|\varphi\|^2}{|\varphi|_\nu^2}. \tag{3.1}$$

Since $\mathbb{R}^k$ is finite dimensional, there exists $d_1 = d_1(k, \nu) > 0$ such that

$$\sum_{i=1}^k |y_i|^\nu \geqslant d_1 \left(\sum_{i=1}^k |y_i|^2\right)^{\frac{\nu}{2}}, \ (y_1, y_2, \cdots, y_k) \in \mathbb{R}^k. \tag{3.2}$$

Hence, set $V_{k,\tau} := \mathrm{span}\{\varphi_1^\tau, \cdots, \varphi_k^\tau\}$. By (3.1) and (3.2), there holds

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^\nu dx &= \int_{\cup_{i=1}^k B_\tau(x_i)} \left|\sum_{i=1}^k \alpha_i\varphi_i^\tau\right|^\nu dx = \sum_{i=1}^k |\alpha_i\varphi_i^\tau|_\nu^\nu \\
&\geqslant d_1\left(\sum_{i=1}^k |\alpha_i\varphi_i^\tau|_\nu^2\right)^{\frac{\nu}{2}} \\
&= d_1\left(\sum_{i=1}^k A_\tau^{-1}\|\alpha_i\varphi_i^\tau\|^2\right)^{\frac{\nu}{2}} \\
&= d_2\tau^{-(N-2-\frac{2N}{\nu})\frac{\nu}{2}}\|u\|^\nu,
\end{aligned} \tag{3.3}$$

for every $u = \sum_{i=1}^k \alpha_i\varphi_i^\tau$, $\alpha_i \in \mathbb{R}$, where $d_2 = d_1|\varphi|_\nu^\nu\|\varphi\|^{-\nu}$. By (f_1) and (f_2), there exist $d_3, d_4 > 0$ such that $F(s) \geqslant d_3|s|^\nu - d_4$, $s \in \mathbb{R}$. On account of (3.3) and (K_1), we have

$$\begin{aligned}
J_{\delta,\lambda}(u) &\leqslant \frac{\max\{m(0), m(\delta)\}}{2}\|u\|^2 - \lambda a_0\sum_{i=1}^k\int_{B_\tau(x_i)} F(u)dx \\
&\leqslant \frac{\max\{m(0), m(\delta)\}}{2}\|u\|^2 - \lambda d_2 d_3 a_0\tau^{-(N-2-\frac{2N}{\nu})\frac{\nu}{2}}\|u\|^\nu + \lambda d_4 k a_0\tau^N\omega_N,
\end{aligned}$$

where ω_N is the volume of the unitary ball in $\mathbb{R}^N$. Setting $\gamma := N + \nu - \frac{N\nu}{2}$, $d_5 = a_0 d_2 d_3, d_6 = a_0 d_4 k \omega_N$, one has

$$J_{\delta,\lambda}(u) \leqslant \frac{\max\{m(0), m(\delta)\}}{2}\|u\|^2 - \lambda d_5 \tau^\gamma \|u\|^\nu + \lambda d_6 \tau^N,\ u \in V_{k,\tau}. \tag{3.4}$$

Since $\nu < 2^*$, we have that $0 < \gamma < N$. Then we can choose $\gamma_0 \in (\gamma, N)$ and set $\lambda = \tau^{-\gamma_0}$. We consider the function

$$h_\tau(t) := \frac{\max\{m(0), m(\delta)\}}{2} t^2 - d_5 \tau^{-\gamma_0+\gamma} t^\nu + d_6 \tau^{-\gamma_0+N},\ t > 0.$$

It obtains its maximum at

$$t_\tau = [\max\{m(0), m(\delta)\}(d_5 \nu)^{-1} \tau^{\gamma_0 - \gamma}]^{\frac{1}{\nu-2}}.$$

This fact and $\gamma_0 \in (\gamma, N)$ imply that $t_\tau \to 0,\ \tau \to 0^+$. Then $h_\tau(t_\tau) \to 0,\ \tau \to 0^+$. Thus, for any $M > 0$, there exists $\tau^* = \tau^*(k, \nu, N, \delta, M) > 0$ such that

$$h_\tau(t_\tau) = \max_{t \geqslant 0} h_\tau(t) \leqslant \frac{M}{2},\ \tau \in (0, \tau^*]. \tag{3.5}$$

By choosing $\lambda_{k,M} = (\tau^*)^{-\gamma_0}$, we set $V_k^\lambda := V_{k,\lambda^{-\frac{1}{\gamma_0}}}$ for every $\lambda \geqslant \lambda_{k,M}$. It is a subspace of $D^{1,2}(\mathbb{R}^N)$ and its dimension is k. Since $\lambda \geqslant \lambda_{k,M}$ implies that $\tau \leqslant \tau^*$, it follows from (3.4) and (3.5) that

$$J_{\delta,\lambda}(u) \leqslant \max_{t \geqslant 0} h_\tau(t) \leqslant \frac{M}{2} < M, \text{ for } u \in V_k^\lambda.$$

□

Proof of Theorem 1.1 It follows from Proposition 3.1 that the (J_1) and (J_2) in Lemma 2.2 hold. Condition (J_3) follows from Proposition 2.2 with

$$M_0 = \begin{cases} \min\{c^*, \frac{\nu m(0) - 2m(\delta)}{2\nu}\frac{S}{2}, \frac{\nu m(0) - 2m(\delta)}{2\nu}\left(\frac{m(0) S^{\frac{2^*_\mu}{2}}}{C(N,\mu)}\right)^{\frac{2}{2^*_\mu - 2}}, \frac{\nu m(0) - 2m(\delta)}{2\nu}\delta\}, & m \text{ is increasing;} \\ \min\{c^*, \frac{\nu m(\delta) - 2m(0)}{2\nu}\frac{S}{2}, \frac{\nu m(\delta) - 2m(0)}{2\nu}\left(\frac{m(\delta) S^{\frac{2^*_\mu}{2}}}{C(N,\mu)}\right)^{\frac{2}{2^*_\mu - 2}}, \frac{\nu m(\delta) - 2m(0)}{2\nu}\delta\}, & m \text{ is decreasing.} \end{cases}$$

Since (f_0) and (f_1) imply that $J_{\delta,\lambda}(0) = 0$ and $J_{\delta,\lambda}$ is even, Lemma 2.2 implies that the truncated problem (1.4) with $\lambda \geqslant \lambda_k^* := \lambda_{k,M_0}$ enjoys at least k pairs of nontrivial solutions for every $k \in \mathbb{N}$. Let $u \in D^{1,2}(\mathbb{R}^N)$ be one of these solutions. Since $J_{\delta,\lambda}(u) \leqslant M_0$, we find from (f_2) that

$$\frac{\nu m(0) - 2m(\delta)}{2\nu}\delta \geqslant M_0 \geqslant J_{\delta,\lambda}(u) - \frac{1}{\nu} J'_{\delta,\lambda}(u)u \geqslant \frac{\nu m(0) - 2m(\delta)}{2\nu}\|u\|^2, \text{ if } m \text{ is increasing;}$$
$$\frac{\nu m(\delta) - 2m(0)}{2\nu}\delta \geqslant M_0 \geqslant J_{\delta,\lambda}(u) - \frac{1}{\nu} J'_{\delta,\lambda}(u)u \geqslant \frac{\nu m(\delta) - 2m(0)}{2\nu}\|u\|^2, \text{ if } m \text{ is decreasing.}$$

Hence, $\|u\|^2 \leqslant \delta$ and it follows from the definition of m_δ that $m_\delta(\|u\|^2) = m(\|u\|^2)$, that is, u is also a weak solution to problem 1.1. The proof is completed. □

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REFERENCES

[1] H. Berestycki, P.L. Lions, Nonlinear scalar field equations. I Existence of a ground state, Arch. Ration. Mech. Anal. 82 (1983), 313-345.
[2] G. Kirchhoff, Vorlesungen über Mathematische Physik: Mechanik, Teubner, Leipzig, 1883.
[3] J.L. Lions, On some questions in boundary value problems of mathematical physics. Contemporary developments in continuum mechanics and partial differential equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977), North-Holland Math. Stud. 30 (1978), 284-346.
[4] C.O. Alves, F.J.S.A. Corrêa, T.F. Ma. Positive solutions for a quasilinear elliptic equation of Kirchhoff type, Comput. Math. Appl. 49 (2005), 85-93.
[5] K. Perera, Z.T. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, J. Differential Equations 221 (2006), 246-255.
[6] Z.T. Zhang, K. Perera, Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow, J. Math. Anal. Appl. 317 (2006), 456-463.
[7] A.M. Mao, Z.T. Zhang, Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition, Nonlinear Anal. 70 (2009), 1275-1287.
[8] X. Wu, Existence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhoff-type equations in $\mathbb{R}^N$, Nonlinear Anal. 12 (2011), 1278-1287.
[9] C.Y. Chen, Y.C. Kuo, T.F. Wu, The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions, J. Differential Equations 250 (2011), 1876-1908.
[10] Y.H. Li, F.Y. Li, J.P. Shi, Existence of a positive solution to Kirchhoff type problems without compactness conditions, J. Differential Equations 253 (2012), 2285-2294.
[11] X.M. He, W.M. Zou, Existence and concentration behavior of positive solutions for a Kirchhoff type equation in $\mathbb{R}^3$, J. Differential Equations 252 (2012), 1813-1834.
[12] G.M. Figueiredo, Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, J. Math. Anal. Appl. 401 (2013), 706-713.
[13] Z.P. Liang, F.Y. Li, J.P. Shi, Positive solutions to Kirchhoff type equations with nonlinearity having prescribed asymptotic behavior, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014), 155-167.
[14] G.B. Li, H.Y. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, J. Differential Equations 257 (2014), 566-600.
[15] D. Naimen, The critical problem of Kirchhoff type elliptic equations in dimension four, J. Differential Equations 257 (2014), 1168-1193.
[16] Z.J. Guo, Ground states for Kirchhoff equations without compact condition, J. Differential Equations 259 (2015), 2884-2902.
[17] A.R. Li, J.B. Su, Existence and multiplicity of solutions for Kirchhoff type equation with radial potentials in $\mathbb{R}^3$, Z. Angew. Math. Phys. 66 (2015), 3147-3158.
[18] E. Hebey, Multiplicity of solutions for critical Kirchhoff type equations, Comm. Partial Differential Equations 41 (2016), 913-924.
[19] L. Wang, K. Xie, B.L. Zhang, Existence and multiplicity of solutions for critical Kirchhoff-type p-Laplacian problems, J. Math. Anal. Appl. 458 (2018), 361-378.
[20] M.F. Furtado, L.D. de Oliveira, J.P.P. da Silva, Multiple solutions for a Kirchhoff equation with critical growth, Z. Angew. Math. Phys. 70 (2019), 11.
[21] F. Faraci, C. Farkas, On an open question of Ricceri concerning a Kirchhoff-type problem, Minimax Theory Appl. 4 (2019), 271-280.
[22] F. Faraci, C. Farkas, On a critical Kirchhoff-type problem, Nonlinear Anal. 192 (2020), 111679.
[23] F. Faraci, K. Silva, On the Brezis-Nirenberg problem for a Kirchhoff type equation in high dimension, Calc. Var. Partial Differential Equations 60 (2021), 22.
[24] P.L. Lions, The concentration-compactness principle in the calculus of variations, the limit case, I, Rev. Mat. Iberoam. 1 (1985), 145-201.
[25] P.L. Lions, The concentration-compactness principle in the calculus of variations, the limit case, II, Rev. Mat. Iberoam. 2 (1985), 45-121.

[26] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, Stud. Appl. Math. 57 (1977), 93-105.
[27] S. Pekar, Untersuchungen über Die Elektronentheorie Der Kristalle, Akademie Verlag, Berlin, 1954.
[28] P.L. Lions, The Choquard equation and related questions, Nonlinear Anal. 4 (1980), 1063-1072.
[29] V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, J. Funct. Anal. 265 (2013), 153-184.
[30] F.S. Gao, M.B. Yang, On the Brezis-Nirenberg type critical problem for nonlinear Choquard equation, Sci. China Math. 61 (2018), 1219-1242.
[31] F.S. Gao, M.B. Yang, On nonlocal Choquard equations with Hardy-Littlewood-Sobolev critical exponents, J. Math. Anal. Appl. 448 (2017), 1006-1041.
[32] V. Moroz, J. Van Schaftingen, A guide to the Choquard equation, J. Fixed Point Theory Appl. 19 (2017), 773-813.
[33] Y.Q. Song, S.Y. Shi, Existence and multiplicity of solutions for Kirchhoff equations with Hardy-Littlewood-Sobolev critical nonlinearity, Appl. Math. Lett. 92 (2019), 170-175.
[34] P.H. Rabinowitz, Minimax methods in critical point theory with applications to diferential equations, CBMS Reg. Conf. Series in Math., no. 65, American Mathematical Society, Providence, Rhode Island, 1986.
[35] F.S. Gao, E.D. da Silva, M.B. Yang, J.Z. Zhou, Existence of solutions for critical Choquard equations via the concentration compactness method, Proc. Roy. Soc. Edinburgh, Sect. A, Math. 150 (2020), 921-954.
[36] A.R. Li, P.T. Wang, C.Q. Wei, Multiplicity of solutions for a class of Kirchhoff type equations with Hardy-Littlewood-Sobolev critical nonlinearity, Appl. Math. Lett. 102 (2020), 106105.
[37] S.H. Liang, P. Pucci, B.L. Zhang, Multiple solutions for critical Choquard-Kirchhoff type equations, Adv. Nonlinear Anal. 10 (2021), 400-419.
[38] J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, Calc. Var. Partial Differential Equations 3 (1995), 493-512.
[39] E.H. Lieb, M. Loss, Analysis, in: Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, Rhode Island, 1997.
[40] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973), 349-381.
[41] E.A.B. Silva, M.S. Xavier, Multiplicity of solutions for quasilinear elliptic problems involving critical Sobolev exponents, Ann. Inst. H. Poincaré Anal. Non Linéaire 20 (2003), 341-358.