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OPTIMAL CONTROL OF A PHASE FIELD TUMOR GROWTH MODEL WITH CHEMOTAXIS AND ACTIVE TRANSPORT

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Abstract. This paper is concerned with a distributed optimal control problem for a phase field model describing tumor growth with chemotaxis and active transport. First, comparing with the results in [H. Garcke, K.F. Lam, Well-posedness of a Cahn-Hilliard system modelling tumour growth with chemotaxis and active transport, European J. Appl. Math. 28 (2017), 284–316], we prove the existence of solutions for such a system with more general potential, the regularity of solutions and the continuous dependence of initial data as well as control variable with respect to a strong topology. It is worth pointing out that the potentials cover the case of classical quartic double-well potential, which is the standard approximation for the physical relevant logarithmic potential. Furthermore, the existence of an optimal control is proved by monotonicity arguments and compactness theorems. Beyond that, by overcoming some difficulties in mathematical analysis and calculation, especially in the proof of the Fréchet differentiability of the control-to-state operator, we derive the corresponding first-order necessary conditions of optimality in terms of the adjoint variables and the usual variational inequality.

Keywords. Active transport; Chemotaxis; optimal control; Tumor growth; Well-posedness.

1. INTRODUCTION

In the recent several years, there has been a great development in phase field models for tumor growth. Based on the continuum mixture theory, many models involving transport and reaction terms have been derived to describe the evolution of a tumor colony surrounded by healthy tissues, which experience biological mechanisms, such as proliferation via nutrient consumption, apoptosis, chemotaxis, and active transport of specific chemical species. For the case of a young tumor, before the development of quiescent cells, the phase field models often consist of a Cahn-Hilliard equation coupled with a reaction-diffusion equation for the nutrient [10, 22, 24, 25, 34]. One may also consider the effects of fluid flow into the evolution of the tumor, leading to the development of Cahn-Hilliard-Darcy systems [2, 11, 12, 14, 22, 28, 42].

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In this paper, we mainly investigate the following model for tumor growth originally proposed in [22]:

$$\begin{cases} \partial_t \varphi = \Delta\mu + (\lambda_p \sigma - \lambda_a - u)h(\varphi), & (x,t) \in Q, \\ \mu = -\Delta\varphi + F'(\varphi) - \chi_\varphi \sigma, & (x,t) \in Q, \\ \partial_t \sigma = \chi_\sigma \Delta\sigma - \chi_\varphi \Delta\varphi - \lambda_c \sigma h(\varphi), & (x,t) \in Q, \\ \partial_\nu \varphi = \partial_\nu \mu = 0,\ \chi_\sigma \partial_\nu \sigma = K(\sigma^* - \sigma), & (x,t) \in \Sigma, \\ \varphi(x,0) = \varphi_0(x),\ \ \sigma(x,0) = \sigma_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^3$ is an open and bounded domain with smooth boundary Γ. We denote by ∂_ν the outward normal derivative on Γ. Let $T > 0$ be a fixed final time and

$$Q := \Omega \times (0,T),\ \Sigma := \Gamma \times (0,T).$$

φ represents the local concentration of tumor cells, where $\varphi = \pm 1$, with $\varphi = 1$ denoting the tumor phase and $\varphi = -1$ representing the healthy phase. μ, u, and σ denote a chemical potential associated to φ, the concentration of cytotoxic drugs, and the concentration of an unspecified chemical species acting as nutrient for the tumor cells, respectively. Moreover, $F(\varphi)$ is a potential function describing the phase separations. The typical potentials are the regular potential, the logarithmic potential, and the double obstacle potential are defined as:

$$F_{pol}(r) := \frac{1}{4}(r^2-1)^2,\ r \in \mathbb{R},$$

$$F_{log}(r) := \begin{cases} \frac{\theta}{2}[(1+r)\ln(1+r) + (1-r)\ln(1-r)] - \frac{\theta_0}{2} r^2, & r \in (-1,1),\ 0 < \theta < \theta_0, \\ \theta \ln 2 - \frac{\theta_0}{2}, & r \in \{-1,1\}, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$F_{dob}(r) := \begin{cases} c(1-r^2) & r \in [-1,1], \\ +\infty & \text{otherwise.} \end{cases} c > 0.$$

The given function $h(\varphi)$ is an interpolation function such that $h(-1) = 0$ and $h(1) = 1$, and σ^* is the nutrient supply on the boundary Γ. The parameters $\lambda_a, \lambda_c, \lambda_p$, and K denote tumor apoptosis rate, nutrient consumption rate, the constant tumor proliferation rate, and nutrient supply rate, respectively. $\chi_\sigma \geq 0$ represents the diffusivity of the nutrient, $\chi_\varphi \geq 0$ denotes the parameter for transport mechanisms such as chemotaxis and active uptake.

Now, let us briefly describe the role of the occurring terms from a modeling viewpoint and more details can be found in [19,22]. The terms $\lambda_p \sigma h(\varphi)$ and $\lambda_a h(\varphi)$ stand for the proliferation of tumor cells and the apoptosis of tumor cells, whereas $\lambda_c \sigma h(\varphi)$ indicates the consumption of the nutrient only by the tumor cells. The term $-uh(\varphi)$ in the first equation of problem (1.1) models the elimination of tumor cells by cytotoxic drugs and the function u will act as our control. In addition, let us point out that the contributions $\chi_\sigma \Delta\sigma$ and $\chi_\varphi \Delta\varphi$ model pure chemotaxis, namely, the movement of tumor cells towards regions of high nutrients, and the active transport that describes the movement of the nutrient towards the tumor cells.

To the best of our knowledge, there are four ways of treatments for cancer including surgery, immunotherapy (strengthening the immune system), radiotherapy (using radiation to kill cancer cells), and chemotherapy (using drugs to kill cancer cells). The latter three treatments are typically conducted in cycles to shrink the tumor into a more manageable size for which surgery can be applied. The so-called a cycle is a period of treatment followed by a (longer) period of rest, so that the patient's body can build new healthy cells. Moreover, further therapeutic treatments may be necessary to destroy the cancer cells that may remain after the surgery. Hence, in this paper, we consider the following optimal distribution control problem involving a cancer treatment with cytotoxic drugs.

(**CP**) Minimize the cost functional

$$\begin{aligned}\mathscr{J}((\varphi,\mu,\sigma);u) =& \frac{\alpha_0}{2}\int_Q |\varphi(x,t)-\varphi_Q(x,t)|^2\,dxdt + \frac{\alpha_1}{2}\int_\Omega |\varphi(x,T)-\varphi_\Omega(x)|^2\,dx \\ &+ \frac{\beta_0}{2}\int_Q |\sigma(x,t)-\sigma_Q(x,t)|^2\,dxdt + \frac{\beta_1}{2}\int_\Omega |\sigma(x,T)-\sigma_\Omega(x)|^2\,dx \\ &+ \frac{\beta_2}{2}\int_Q |u(x,t)|^2\,dxdt\end{aligned}$$

subject to the condition that (φ,μ,σ) solves the state system (1.1) with a control

$$u \in \mathscr{U}_{ad} = \{u \in L^2(Q) : u_{min} \le u \le u_{max},\ a.e.(x,t) \in Q\}. \tag{1.2}$$

Here, α_0, α_1, β_0, β_1, and β_2 are some fixed constants that do not vanish simultaneously, φ_Q, σ_Q, and φ_Ω, σ_Ω indicate some target functions, and u_{min}, u_{max} denote some prescribed functions.

In the past several decades, there have been many recent contributions regarding the well-posedness, asymptotic behavior, and optimal control for phase field tumor growth models; see, for example, [3–9,15,18–21,23,32–34,36–39] where the velocity contributions were neglected, and [12,13,16,17,27–31,43] considering the velocity contributions. In this paper, we focus on the case that velocity contributions are neglected. We now compare (1.1) with the other models for tumour growth studied in the literature. Assuming different linear phenomenological laws for chemical reactions, the authors in [25] (see also [26,34]) introduced the following different thermodynamically consistent model

$$\begin{cases}\alpha\partial_t\mu + \partial_t\varphi = \Delta\mu + p(\varphi)(\chi_\sigma\sigma + \chi_\varphi(1-\varphi)-\mu), & (x,t)\in Q,\\ \mu = \beta\partial_t\varphi - \Delta\varphi + F'(\varphi) - \chi_\varphi\sigma, & (x,t)\in Q,\\ \partial_t\sigma = \chi_\sigma\Delta\sigma - \chi_\varphi\Delta\varphi - p(\varphi)(\chi_\sigma\sigma + \chi_\varphi(1-\varphi)-\mu), & (x,t)\in Q,\\ \partial_\nu\varphi = \partial_\nu\mu = 0,\ \chi_\sigma\partial_\nu\sigma = K(\sigma^*-\sigma), & (x,t)\in\Sigma,\\ \varphi(x,0)=\varphi_0(x),\ \ \mu(x,0)=\mu_0(x),\ \ \sigma(x,0)=\sigma_0(x), & x\in\Omega.\end{cases} \tag{1.3}$$

When $\alpha=\beta=0$, it is not difficult to verify that system (1.3) possesses a Lyapunov-type energy functional which means the standard a priori estimates can be obtained even in the case that F has polynomial growth of order 6. Hence, the authors in [3,6,23] analysed the well-posedness, long-time behavior, and optimal control problem for system (1.3) with regular potentials having polynomial growth of order 6. When $\alpha,\beta>0$, we call these extension models of the case $\alpha=\beta=0$. Thanks to the regularizing effect of the artificial relaxation terms $\alpha\partial_t\mu$ and $\beta\partial_t\varphi$, a uniform separation principle for phase variable can be verified, which is a key property to handle singular potential. Hence, the well-posedness, vanishing viscosity limits, and optimal control problems of these extension models with more general potentials including singular potentials were established in [4,5,8,9,36–38].

For system (1.1), neglecting the effects of chemotaxis and active transport, i.e., $\chi_\varphi=0$, the authors in [20,32] considered the well-posedness, long time behavior (in terms of global attractors), and optimal control of treatment time and cytotoxic drug with more general regular potentials having at least cubic and at most exponential growth at infinity. While $\chi_\varphi > 0$, the extension models of problem (1.1) with general potentials were considered in [7, 39]. In particular, the authors in [7] studied a distributed optimal control problem with two control variables. The sparse optimal control, i.e, the cost functional contains a non-differentiable but convex contribution like L^1-norm, was established in [39]. For system (1.1), the authors [19] established the well-posedness and its quasi-static limits of system (1.1) with at most quadratic potentials. Specifically, in the case of the non-constant mobility, they obtained the existence of weak solutions by a Faedo-Galerkin scheme along with *a priori* estimates wherein they had to restrict the regular potentials with at most quadratic growth in order to estimate the mean of μ via the Poincaré inequality. Furthermore, they showed continuous dependence on initial data under the additional assumptions concluding the mobilities are constants. Later on, to improve the restriction of growth order of potential, they replaced the Neumann boundary conditions in system (1.1) with non-zero Dirichlet boundary conditions [18]. They proved the existence of weak solutions with the regular potentials having arbitrary polynomial growth, and verified the regularity and continuous dependence on initial data with the regular potentials with polynomial growth of order up to six in three dimensions.

In this paper, comparing with the results in [19], under the assumptions of more general potentials (see Assumptions (A_2) in Section 2), we improve the well-posedness result in [19], and further study the optimal distributed control problem of system (1.1). Due to the presence of chemotaxis and active transport terms, the nonlinear function F with growth order greater than two is indeed the key challenge in the analysis. Here let us emphasize the following differences and give some comments on the problem discussed in [19]:

(i) When the mobilities are constants, we prove the existence of weak solutions for the system (1.1) with the Neumann boundary conditions and the nonlinear function F under assumption $(A4)$ which includes the classical quartic double-well potential, the standard approximation for the physical relevant logarithmic potential. In this case, to estimate the free energy functional, we have to first establish the a priori estimate of σ in $L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$ by choosing suitable testing functions.

(ii) We first establish the regularity of the solution component φ via the regularity theory of elliptic equations and the bootstrapping method. Then, we prove the stability estimates on both initial data and control parameter in the space that the solution belongs to.

(iii) We investigate the optimal control problem of system (1.1). In particular, our stability estimates are allowed to prove the Fréchet differentiability (in suitable spaces) of the control-to-state mapping, which can be used to study the optimal control problem of cost functional involved the L^2-norm of $\nabla\varphi$. For the sake of simplicity, we here only focus on the case of cost functional with L^2-norm of φ, but the scheme proposed in this paper can be applied to the cost functional with the H^1-norm of φ as well.

(iv) It is possible to extend the results in this paper to the optimal time control problem, namely, the terminal time in the cost functional is free, which penalises long treatment time. In addition, it also can be used to deal with the sparse optimal control problem with a cost functional containing a non-differentiable term like L^1-norm describing the sparsity effects. These works will be illustrated in the forthcoming papers.

The remaining part of this paper is organized as follows: In Section 2, we state some useful lemmas and basic assumptions. The well-posedness result for system (1.1) is addressed in

Section 3. Section 4 states the optimal control problem (**CP**), the Fréchet differentiability of the control-to-state operator, as well as the first-order necessary optimality conditions for problem (**CP**).

2. Preliminaries and Assumptions

For simplification, we denote L^p the standard Lebesgue space $L^p(\Omega)$ equipped with the norms $\|\cdot\|_{L^p}$ and $W^{k,p}$ the Sobolev spaces $W^{k,p}(\Omega)$ equipped with the norms $\|\cdot\|_{W^{k,p}}$ for any $p\in[1,\infty]$ and $k>0$. In the case that $p=2$, we use $H^k:=W^{k,2}$ and the norm $\|\cdot\|_{H^k}$. By Fubini's theorem, we use the notation $L^p(Q)$ to denote the space $L^p(\Omega\times(0,T))$ for $1\le p<\infty$. Moreover, the dual space of a Banach space X is denoted by X^*, and the duality pairing between X and X^* is written by $\langle\cdot,\cdot\rangle_X$, whereas, the L^2-inner product is written by $(\cdot,\cdot)$. In the rest of this paper, unless otherwise stated, we shall denote by C a generic positive constant which may change from line to line.

2.1. Useful preliminaries.

Throughout this paper, we make repeated use of the following Gagliardo-Nirenberg interpolation inequality in dimension d (see [1]): let $\Omega\subset\mathbb{R}^d$ be a bounded domain with C^m boundary, $f\in W^{m,r}\cap L^q$ with $1\le q,r\le\infty$. For any integer j with $0\le j<m$, suppose that there exists $\alpha\in[0,1]$ such that

$$\frac{1}{p}=\frac{j}{d}+\left(\frac{1}{r}-\frac{m}{d}\right)\alpha+\frac{1-\alpha}{q},\quad \frac{j}{m}\le\alpha\le 1.$$

If $r\in(1,\infty)$ and $m-j-\frac{d}{r}$ is a non-negative integer, we in addition assume $\alpha\neq 1$. Under these assumptions, there exists a positive constant C depending only on Ω,m,j,q,r and α, such that

$$\|D^j f\|_{L^p}\le C\|f\|_{W^{m,r}}^{\alpha}\|f\|_{L^q}^{1-\alpha}.$$

Let us also recall the Poincaré inequality and a weak version of dominated convergence theorem.

Lemma 2.1. [40] *Let $\Omega\subset\mathbb{R}^d$ with $d\ge1$ be a bounded domain with Lipschitz boundary. Then there exists a positive constant C_1 depending only on Ω and d, such that*

$$\|u\|_{L^2}\le C_1\left(\|\nabla u\|_{L^2}+\|u\|_{L^2(\Gamma)}\right),$$
$$\|u-M(u)\|_{L^2}\le C_1\|\nabla u\|_{L^2},$$

for any $u\in H^1$, where the operator $M:L^1\to\mathbb{R}$ is defined by

$$M(f)=\frac{1}{|\Omega|}\int_\Omega f(x)\,dx$$

for any $f\in L^1$.

Lemma 2.2. [35] *Let $\Omega\subset\mathbb{R}^d$ with $d\ge1$ be a bounded domain and $\{g_n\}\subset L^q$, where $1<q<\infty$ is given. Assume that there exist a positive constant L independent of n and $g\in L^q$, such that $\|g_n\|_{L^q}\le L$ for any natural number n. Moreover, $g_n\to g$ almost everywhere in Ω as $n\to\infty$. Then $g_n\to g$ weakly in L^q as $n\to\infty$.*

2.2. **Assumptions.** Now, we make the following assumptions on the data of the state system.

(A_1) λ_p, λ_a, λ_c, and χ_φ are fixed non-negative constants, and χ_σ, and K are fixed positive constants.

(A_2) The initial and boundary data satisfy

$$\varphi_0 \in H^1,\ \ \sigma_0 \in L^2,\ \ \sigma^* \in L^2(\Sigma).$$

(A_3) For any $s \in \mathbb{R}$, the functions $h(s)$, $h'(s)$ are Lipschitz continuous with Lipschitz constant L_h, and there exists a positive constant h_0, such that

$$0 \leq h(t) \leq h_0,\ \ \forall t \in \mathbb{R}.$$

(A_4) The potential function $F \in \mathscr{C}^3(\mathbb{R})$ satisfies that there exist exponent $p \in (2,6)$ and some positive constants $L_i > 0 (i = 1,2,3,4)$ such that

$$L_1|s|^{p-2} - L_2 \leq F''(s) \leq L_3(1+|s|^{p-2}),\ \forall s \in \mathbb{R},$$
$$|F'''(s) - F'''(r)| \leq L_4(1+|s|^2+|r|^2)|s-r|,\ \forall s,r \in \mathbb{R}.$$

Remark 2.1. Assumption (A_4) is weaker than the one supposed in [19]. Obviously, the double well potential $F(s) = \frac{1}{4}(s^2-1)^2$ satisfies assumption (A_4). For convenience, we can easily deduce from assumption (A_4) that there exist constants $L_i > 0$ $(i = 5,6,7)$ such that

$$L_5|s|^p - L_6 \leq F(s) \leq L_7(1+|s|^p),\ \ \forall\, s \in \mathbb{R}. \tag{2.1}$$

Remark 2.2. Notice that, here we just assume the initial data satisfies assumption (A_2). If we ask the initial data with higher regularity, e.g., $\varphi_0 \in H^2$, the component φ of the solution (φ,μ,σ) will belong to $L^\infty(0,T;H^2(\Omega)$ as in [7], which implies that $\varphi \in L^\infty(0,T;L^\infty(\Omega)$ by the Sobolev embedding theorem $H^2(\Omega) \subset L^\infty(\Omega)$ for $\Omega \subset \mathbf{R}^3$. To this point, $F''(\varphi(t))$ is bounded and Lipschitz continuous in φ for $t \geq 0$, such that the growth condition of F'' and the locally Lipschitz continuity of F''' in assumption (A_4) can be removed.

Moreover, for the optimal control problem under investigation, we make the following general assumptions:

(A_5) α_0, α_1, β_0, β_1, and β_2 are non-negative but not all zero.

(A_6) $\varphi_Q \in L^2(0,T;H^1)$, $\sigma_Q \in L^2(Q)$, $\varphi_\Omega \in H^1$, $\sigma_\Omega \in L^2$, $u_{min}, u_{max} \in L^\infty(Q)$ with $u_{min} \leq u_{max}$ for almost everywhere $(x,t) \in Q$.

(A_7) $\mathscr{U}$ is nonempty open and bounded subset of $L^2(Q)$ containing $\mathscr{U}_{ad}$, and there exists a constant $R > 0$, such that $\|u\|_{L^\infty(Q)} \leq R$ for all $u \in \mathscr{U}$.

Remark 2.3. From inequality (1.2), we know that the set of admissible controls $\mathscr{U}_{ad}$ is a nonempty, closed, bounded, and convex subset in $L^2(Q)$. Thus, $\mathscr{U}_{ad}$ is weakly closed, i.e., it contains its all the weak limits.

3. The Well-Posedness of the State System

In this section, we state our results about the existence of a weak solution to (1.1) by a suitable Galerkin approximation scheme and give the continuous dependence on both the initial data and control parameter.

3.1. **The existence of weak solutions.** We are now in a position to present the result regarding the existence of weak solutions to the state system (1.1) whose proof relies on a suitable Galerkin procedure introduced in [40] and suitable a priori high-order energy estimates. In what follows, in order to estimate the free energy functional, we have to establish the a priori estimate of σ in $L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$.

Theorem 3.1. *Suppose that hypotheses (A_1)-(A_4) are satisfied. For any $T>0$, there exists at least one global weak solution (φ,μ,σ) to problem (1.1) with initial data (φ_0,σ_0) on $[0,T]$, such that*

$$\begin{aligned}
&\varphi\in L^\infty(0,T;H^1)\cap L^2(0,T;H^3)\cap H^1(0,T;(H^1)^*),\\
&\mu\in L^2(0,T;H^1),\\
&\sigma\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1)\cap H^1(0,T;(H^1)^*).
\end{aligned}$$

Moreover, the triple (φ,μ,σ) satisfies

$$\begin{aligned}
&\langle\partial_t\varphi,\psi\rangle_{H^1}+(\nabla\mu,\nabla\psi)=((\lambda_p\sigma-\lambda_a-u)h(\varphi),\psi),\\
&(\nabla\varphi,\nabla\eta)+\big(F'(\varphi)-\chi_\varphi\sigma,\eta\big)=(\mu,\eta),\\
&\langle\partial_t\sigma,\phi\rangle_{H^1}+\big(\chi_\sigma\nabla\sigma-\chi_\varphi\nabla\varphi,\nabla\phi\big)+\lambda_c\,(\sigma h(\varphi),\phi)+K\,(\sigma-\sigma^*,\phi)_{L^2(\Gamma)}=0
\end{aligned}$$

and

$$\varphi(0)=\varphi_0,\ \ (\sigma(0),\phi)=(\sigma_0,\phi)$$

for any ψ, η, $\phi\in H^1$ and a.e. $t\in(0,T)$.

Proof. We consider the eigenvalue problem $-\Delta\omega=\lambda\omega$ subject to the homogeneous Neumann boundary condition $\partial_\nu\omega=0$. From the standard spectral theory, we know that there exists two sequences $\{\lambda_n\}_{n=1}^\infty$ and $\{\omega_n\}_{n=1}^\infty$, such that, for every $n\geq 1$, $\lambda_n\geq 0$ is an eigenvalue and $\omega_n\neq 0$ is the corresponding eigenfunction, the sequence $\{\lambda_n\}$ is nondecreasing, tending to infinity as $n\to+\infty$, and the sequence $\{\omega_n\}$ forms orthonormal basis of L^2 and is also an orthogonal basis of H^1. We notice that $\lambda=0$ is an eigenvalue. If $\lambda_1=0$, we choose $\omega_1=1$. For any $n\geq 1$, we introduce a finite-dimensional space $W_n=span\{\omega_1,...,\omega_n\}\subset H^1$ and let P_n be the orthogonal projection from L^2 to W_n.

Then we seek for the approximate solution $(\phi^n(t),\mu^n(t),\sigma^n(t))$ in the form

$$\phi^n(t)=\sum_{i=1}^n\alpha_i(t)\omega_i,\ \mu^n(t)=\sum_{i=1}^n\beta_i(t)\omega_i,\ \sigma^n(t)=\sum_{i=1}^n\gamma_i(t)\omega_i,$$

by solving the following problem

$$\begin{cases}
(\partial_t\varphi^n,\psi)=-(\nabla\mu^n,\nabla\psi)+((\lambda_p\sigma^n-\lambda_a-u)h(\varphi^n),\psi),\\
(\mu^n,\eta)=(\nabla\varphi^n,\nabla\eta)+(F'(\varphi^n),\eta)-\chi_\varphi(\sigma^n,\eta),\\
(\partial_t\sigma^n,\phi)=\big(\chi_\varphi\nabla\varphi^n-\chi_\sigma\nabla\sigma^n,\nabla\phi\big)-(\lambda_c\sigma^nh(\varphi^n),\phi)-K\int_\Gamma(\sigma^n-\sigma^*)\phi\,dS,\\
\alpha_i(0)=(\varphi_0,\omega_i),\ \ \gamma_i(0)=(\sigma_0,\omega_i),\ \ i=1,2,\cdots,n
\end{cases}\tag{3.1}$$

for any ψ, η, $\phi\in W_n$.

Thanks to the continuity of $F'(\cdot)$ and $h(\cdot)$, there exists a local solution $(\varphi^n,\mu^n,\sigma^n)$ of the approximating problem (3.1). In what follows, we show that $(\varphi^n,\mu^n,\sigma^n)$ are uniformly bounded with respect to n in suitable function spaces. Below, the symbol $\mathscr{K}(T)$ denotes a generic positive constant that independent of the parameter n and may vary from line by line.

First estimate. Letting $\psi=\varphi^n$, $\eta=-\Delta\varphi^n$, and $\phi=\sigma^n$ in system (3.1), we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\varphi^n\|_{L^2}^2+\|\sigma^n\|_{L^2}^2\right)+\|\Delta\varphi^n\|_{L^2}^2+\int_\Omega F''(\varphi^n)|\nabla\varphi^n|^2\,dx\\&+\chi_\sigma\int_\Omega|\nabla\sigma^n|^2\,dx+\lambda_c\int_\Omega h(\varphi^n)|\sigma^n|^2\,dx+K\int_\Gamma|\sigma^n|^2\,dS\\=&\int_\Omega(\lambda_p\sigma^n-\lambda_a-u)h(\varphi^n)\varphi^n\,dx-2\chi_\varphi\int_\Omega\sigma^n\Delta\varphi^n\,dx+K\int_\Gamma\sigma^n\sigma^*\,dS.\end{aligned}$$

Along with the Hölder's inequality, Young's inequality, and assumption (A_4), we derive that, for any $t\in[0,T]$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\varphi^n\|_{L^2}^2+\|\sigma^n\|_{L^2}^2\right)+\|\Delta\varphi^n\|_{L^2}^2+\chi_\sigma\|\nabla\sigma^n\|_{L^2}^2+K\|\sigma^n\|_{L^2(\Gamma)}^2\\\le&\lambda_p h_0\|\sigma^n\|_{L^2}\|\varphi^n\|_{L^2}+\lambda_a h_0|\Omega|^{\frac{1}{2}}\|\varphi^n\|_{L^2}+h_0\|u\|_{L^2}\|\varphi^n\|_{L^2}+2\chi_\varphi\|\sigma^n\|_{L^2}\|\Delta\varphi^n\|_{L^2}\\&+K\|\sigma^n\|_{L^2(\Gamma)}\|\sigma^*\|_{L^2(\Gamma)}+L_2\|\nabla\varphi^n\|_{L^2}^2.\end{aligned}$$

It follows from the interpolation inequality and the boundary trace imbedding theorem that

$$\begin{aligned}&\frac{d}{dt}\left(\|\varphi^n\|_{L^2}^2+\|\sigma^n\|_{L^2}^2\right)+\|\Delta\varphi^n\|_{L^2}^2+\chi_\sigma\|\nabla\sigma^n\|_{L^2}^2+K\|\sigma^n\|_{L^2(\Gamma)}^2\\\le&C\left(1+\|\varphi^n\|_{L^2}^2+\|\sigma^n\|_{L^2}^2+\|\sigma^*\|_{L^2(\Gamma)}^2+\|u\|_{L^2}^2\right).\end{aligned}$$

Therefore, a Gronwall argument yields that, for any $t\in(0,T]$,

$$\begin{aligned}&\|\varphi^n(t)\|_{L^2}^2+\|\sigma^n(t)\|_{L^2}^2+\int_0^t\|\Delta\varphi^n(s)\|_{L^2}^2+\chi_\sigma\|\nabla\sigma^n(s)\|_{L^2}^2+K\|\sigma^n(s)\|_{L^2(\Gamma)}^2\,ds\\\le&\left(\|\varphi_0\|_{L^2}^2+\|\sigma_0\|_{L^2}^2+CT+C\int_0^T\|\sigma^*(s)\|_{L^2(\Gamma)}^2+\|u(s)\|_{L^2}^2\,ds\right)e^{M_1T}\\=:&\mathscr{K}_1(T).\end{aligned}\tag{3.2}$$

Second estimate. Let $\eta=1$ in the second equality of equality (3.1). From assumption (A_4), inequality (2.1), and Hölder's inequality, we find that

$$\left|\int_\Omega\mu^n\,dx\right|\le C\left(\|\sigma^n\|_{L^2}+1+\|\varphi^n\|_{L^p}^{p-1}\right)\le C\left(\|\sigma^n\|_{L^2}+1+\int_\Omega F(\varphi^n)\,dx\right).\tag{3.3}$$

Moreover, let $\psi=\mu^n+\chi_\varphi\sigma^n$ and $\eta=\partial_t\varphi^n$ in (3.1). By Hölder's inequality, we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\nabla\varphi^n\|_{L^2}^2+\int_\Omega F(\varphi^n)\,dx\right)+\|\nabla\mu^n\|_{L^2}^2\\=&-\chi_\varphi\int_\Omega\nabla\mu^n\cdot\nabla\sigma^n\,dx+\int_\Omega(\lambda_p\sigma^n-\lambda_a-u)h(\varphi^n)(\mu^n+\chi_\varphi\sigma^n)\,dx\\\le&\chi_\varphi\|\nabla\mu^n\|_{L^2}\|\nabla\sigma^n\|_{L^2}+(\lambda_p\|\sigma^n\|_{L^2}+\lambda_a|\Omega|^{\frac{1}{2}}+\|u\|_{L^2})h_0(\|\mu^n\|_{L^2}+\chi_\varphi\|\sigma^n\|_{L^2}).\end{aligned}$$

Therefore, we conclude from Young's inequality, Poincaré's inequality, and inequality (3.3) that

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla\varphi^n\|_{L^2}^2+2\int_\Omega F(\varphi^n)\,dx\right)+\|\nabla\mu^n\|_{L^2}^2\\ \le& C(\|\sigma^n\|_{L^2}^2+\|u\|_{L^2}^2+1)\int_\Omega F(\varphi^n)\,dx+C(1+\|\sigma^n\|_{L^2}^2+\|u\|_{L^2}^2+\|\nabla\sigma^n\|_{L^2}^2).\end{aligned}$$

Using the classical Gronwall inequality and inequalities (2.1) and (3.2) yields

$$\|\nabla\varphi^n(t)\|_{L^2}^2+2\int_\Omega F(\varphi^n(x,t))\,dx+\int_0^t\|\nabla\mu^n(s)\|_{L^2}^2\,ds\le C\left(\|\varphi_0\|_{H^1}^p+\mathscr{K}_1(T)\right)e^{C\mathscr{K}_1(T)},$$

which implies

$$\begin{aligned}&\|\varphi^n\|_{L^\infty(0,T;H^1)}+\|F(\varphi^n)\|_{L^\infty(0,T;L^1)}+\|\sigma^n\|_{L^\infty(0,T;L^2)\cap L^2(0,T;H^1)}+\|\mu^n\|_{L^2(0,T;H^1)}\\ \le&\mathscr{K}(T).\end{aligned} \tag{3.4}$$

This uniform estimate (3.4) ensures that we can extend $(\varphi^n,\mu^n,\sigma^n)$ to the full time interval $[0,T]$ for all $n\in\mathbb{N}$.

Third estimate. Now, we establish the higher order estimates for φ^n via regularity theory of elliptic equations and the bootstrapping method (see for example [17, 18]).

Denote by $g=\mu^n+\chi_\varphi\sigma^n+\varphi^n-F'(\varphi^n)$. Then φ^n satisfies the following equation

$$\begin{cases}-\Delta\varphi^n+\varphi^n=g, & x\in\Omega,\\ \partial_\nu\varphi^n=0, & x\in\Gamma.\end{cases} \tag{3.5}$$

From inequality (3.4), we know that $\mu^n+\chi_\varphi\sigma^n+\varphi^n\in L^2(0,T;H^1)$. Therefore, it suffices to prove that $F'(\varphi^n)\in L^2(0,T;H^1)$. To do this, we define a sequence of numbers $\{\ell_j\}_{j\in\mathbb{Z}^+}$ by $\ell_1=1$ and for any $j\ge1$,

$$\ell_{j+1}=\frac{6\ell_j}{6-(6-p)\ell_j}. \tag{3.6}$$

Define $f(x)=\frac{6x}{6-(6-p)x}$. Then f is strictly increasing and positive in $[1,\frac{6}{6-p})$. Hence, $\ell_j<\ell_{j+1}$ for any $j\ge1$ and $\ell_j>1$ for any $j\ge2$. We derive from the assumption (A_4) that

$$\left(\int_\Omega|F'(\varphi^n)|^{\ell_1}\,dx\right)^{\frac{2}{\ell_1}}\le C(1+\|\varphi^n\|_{L^{(p-1)\ell_1}}^{2(p-1)})\le C(1+\|\varphi^n\|_{H^1}^{2(p-1)}).$$

Since $\varphi^n\in L^\infty(0,T;H^1)$, we obtain $F'(\varphi^n)\in L^2(0,T;L^{\ell_1})$. Furthermore, using the Sobolev embedding theorem $L^2(0,T;H^1)\subset L^2(0,T;L^{\ell_1})$, we know $g\in L^2(0,T;L^{\ell_1})$. Therefore, we conclude from the regularity theory of the elliptic equation of second order that $\varphi^n\in L^2(0,T;W^{2,\ell_1})$. From the Gagliardo-Nirenberge inequality, one can see that

$$\|\varphi^n\|_{L^{(p-1)\ell_{j+1}}}^{2(p-1)}\le C\|\varphi^n\|_{W^{2,\ell_j}}^2\|\varphi^n\|_{H^1}^{2(p-2)}$$

for any $\varphi^n \in W^{2,\ell_j} \cap H^1$. Therefore, for each $j \geq 1$, we have

$$\begin{aligned}\int_0^T \left(\int_\Omega |F'(\varphi^n)|^{\ell_{j+1}}\, dx\right)^{\frac{2}{\ell_{j+1}}} dt \leq & C\int_0^T \left(\int_\Omega (1+|\varphi^n|^{(p-1)\ell_{j+1}})\, dx\right)^{\frac{2}{\ell_{j+1}}} dt \\ \leq & C\int_0^T (1+\|\varphi^n\|^{2(p-1)}_{L^{(p-1)\ell_{j+1}}})\, dt \\ \leq & C(1+\|\varphi^n\|^2_{L^2(0,T;W^{2,\ell_j})}\|\varphi^n\|^{2(p-2)}_{L^\infty(0,T;H^1)}).\end{aligned}$$

i.e., $F'(\varphi^n) \in L^2(0,T;L^{\ell_{j+1}})$, which further implies that $\varphi^n \in L^2(0,T;W^{2,\ell_{j+1}})$. Since $\frac{6}{7-p} < \frac{6}{6-p}$, we infer that $\ell_j \geq \frac{6}{7-p}$ after a finite number of steps. At this point, the bootstrapping procedure will be stopped and $\ell_{j+1} \geq 6$. Therefore, we obtain $\varphi^n \in L^2(0,T;W^{2,6})$.

In what follows, we will prove that $F'(\varphi^n) \in L^2(0,T;H^1)$. With the combination of the Gagliardo-Nirenberge inequality and $\varphi^n \in L^2(0,T;W^{2,6})$, we obtain

$$\begin{aligned}\int_0^T \int_\Omega |F'(\varphi^n)|^2\, dxdt \leq & C\left(1+\int_0^T \|\varphi^n\|^{2(p-1)}_{L^{2(p-1)}}\, dt\right) \\ \leq & C\left(1+\|\varphi^n\|^2_{L^2(0,T;W^{2,\frac{6}{9-p}})}\|\varphi^n\|^{2(p-2)}_{L^\infty(0,T;H^1)}\right) \\ \leq & C\left(1+\|\varphi^n\|^2_{L^2(0,T;W^{2,6})}\|\varphi^n\|^{2(p-2)}_{L^\infty(0,T;H^1)}\right)\end{aligned}$$

and

$$\begin{aligned}\int_0^T \int_\Omega |F''(\varphi^n)|^2|\nabla\varphi^n(x,t)|^2\, dxdt \leq & C\left(1+\int_0^T \int_\Omega |\varphi^n|^{2(p-2)}|\nabla\varphi^n(x,t)|^2\, dt\right) \\ \leq & C\left(1+\|\varphi^n\|^{2(p-2)}_{L^{2(p-2)}(0,T;L^\infty)}\|\varphi^n\|^2_{L^\infty(0,T;H^1)}\right) \\ \leq & C\left(1+\|\varphi^n\|^{\frac{p-2}{2}}_{L^2(0,T;W^{2,6})}\|\varphi^n\|^{\frac{3p-2}{2}}_{L^\infty(0,T;H^1)}\right),\end{aligned}$$

i.e., $F'(\varphi^n) \in L^2(0,T;H^1)$. Then by the regularity theory of the system (3.5), we obtain that

$$\|\varphi^n\|_{H^3} \leq C(\|\mu^n\|_{H^1} + \|F'(\varphi^n)\|_{H^1} + \|\sigma^n\|_{H^1} + \|\varphi^n\|_{L^2}),$$

which together with inequality (3.4) yields

$$\|\varphi^n\|_{L^2(0,T;H^3)} + \|F'(\varphi^n)\|_{L^2(0,T;H^1)} \leq \mathscr{K}(T). \tag{3.7}$$

Estimates on time derivatives. From the first and third equations of (3.1), we find

$$\begin{aligned}&\|\partial_t\varphi^n\|_{(H^1)^*} \leq \|\nabla\mu^n\|_{L^2} + \lambda_p h_0\|\sigma^n\|_{L^2} + \lambda_a h_0|\Omega|^{\frac{1}{2}} + h_0\|u\|_{L^2},\\ &\|\partial_t\sigma^n\|_{(H^1)^*} \leq \chi_\sigma\|\nabla\sigma^n\|_{L^2} + \chi_\varphi\|\nabla\varphi^n\|_{L^2} + \lambda_c h_0\|\sigma^n\|_{L^2} + K\|\sigma^n\|_{L^2(\Gamma)} + K\|\sigma^*\|_{L^2(\Gamma)}.\end{aligned}$$

Recalling the assumption (A_4) and inequality(3.4), we have

$$\begin{aligned}&\int_0^T \|\partial_t\sigma^n(t)\|^2_{(H^1)^*}\, dt + \int_0^T \|\partial_t\varphi^n(t)\|^2_{(H^1)^*}\, dt + \int_0^T \|F'(\varphi^n(t))\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}}\, dt \\ &+ \int_0^T \|\mu^n(t)\|^2_{L^2}\, dt \leq \mathscr{K}(T).\end{aligned} \tag{3.8}$$

In view of (3.4), (3.7), and (3.8), we can obtain the existence of a weak solution for problem (1.1) by the standard (weak-) compactness results and monotone arguments as in [19]. □

3.2. **Continuous dependence on initial data and control parameter.** Now, we state the stability property of weak solutions (i.e., continuous dependence) with respect to the initial data and the control parameter u in a strong topology. It not only implies the uniqueness of the weak solution (1.1), but also helps to verify the Fréchet differentiability of the control-to-state mapping.

Theorem 3.2. *Assume that (A_1)-(A_4) hold. Let $(\varphi_1,\mu_1,\sigma_1)$, $(\varphi_2,\mu_2,\sigma_2)$ be two weak solutions to problem (1.1) with the initial data $(\varphi_{01},\sigma_{01})$, $(\varphi_{02},\sigma_{02})$ and the functions u_1, u_2, respectively, and denote by $(\varphi,\mu,\sigma,u)=(\varphi_1-\varphi_2,\mu_1-\mu_2,\sigma_1-\sigma_2,u_1-u_2)$. Then there exists a constant $K_2>0$ depending only on some physical parameters and the initial data $(\varphi_{01},\sigma_{01})$, $(\varphi_{02},\sigma_{02})$ of problem (1.1), such that*

$$
\begin{aligned}
&\|\varphi(t)\|^2_{L^\infty(0,T;H^1)\cap L^2(0,T;H^3)}+\|\sigma(t)\|^2_{L^\infty(0,T;L^2)\cap L^2(0,T;H^1)}\\
\leq &K_2\left(\|\varphi_{01}-\varphi_{02}\|^2_{H^1}+\|\sigma_{01}-\sigma_{02}\|^2_{L^2}+\|u\|^2_{L^2(Q)}\right),
\end{aligned}
$$

which implies that the weak solution to problem (1.1) is unique.

Proof. It is easy to verify that (φ,μ,σ,u) satisfies the following system

$$
\begin{cases}
\langle\partial_t\varphi,\psi\rangle+(\nabla\mu,\nabla\psi)=((\lambda_p\sigma-u)h(\varphi_1),\psi)+((\lambda_p\sigma_2-\lambda_a-u_2)(h(\varphi_1)-h(\varphi_2)),\psi),\\
(\mu,\eta)=(\nabla\varphi,\nabla\eta)+\left(F'(\varphi_1)-F'(\varphi_2)-\chi_\varphi\sigma,\eta\right),\\
\langle\partial_t\sigma,\phi\rangle+\left(\chi_\sigma\nabla\sigma-\chi_\varphi\nabla\varphi,\nabla\phi\right)=-\lambda_c\left(\sigma h(\varphi_1),\phi\right)-\lambda_c\left(\sigma_2(h(\varphi_1)-h(\varphi_2)),\phi\right)\\
-K\left(\sigma-\sigma^*,\phi\right)_{L^2(\Gamma)},\\
\varphi(x,0)=\varphi_0(x),\ \ \sigma(x,0)=\sigma_0(x)
\end{cases}
$$

for any $\psi,\eta,\phi\in H^1$ and a.e. $t\in(0,T)$. Let $\psi=2\varphi-2\Delta\varphi$, $\eta=-\Delta\varphi$, and $\phi=2\delta\sigma$ in the above system, where the constant $\delta\geq 1$ will be specified later. Then

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\varphi\|^2_{H^1}+\delta\|\sigma\|^2_{L^2}\right)+2\|\Delta\varphi\|^2_{H^1}+2\delta\chi_\sigma\|\nabla\sigma\|^2_{L^2}+2\delta K\|\sigma\|^2_{L^2(\Gamma)}+2\lambda_c\delta\int_\Omega h(\varphi_1)|\sigma|^2\,dx\\
=&-2(1+\delta)\chi_\varphi\int_\Omega\sigma\Delta\varphi\,dx-2\delta\lambda_c\int_\Omega\sigma_2(h(\varphi_1)-h(\varphi_2))\sigma\,dx-2\chi_\varphi\int_\Omega\nabla\sigma\cdot\nabla\Delta\varphi\,dx\\
&+2\int_\Omega(\lambda_p\sigma-u)h(\varphi_1)(\varphi-\Delta\varphi)\,dx+2\int_\Omega(\lambda_p\sigma_2-\lambda_a-u_2)(h(\varphi_1)-h(\varphi_2))(\varphi-\Delta\varphi)\,dx\\
&+2\int_\Omega(F'(\varphi_1)-F'(\varphi_2))\Delta\varphi\,dx+2\int_\Omega(F''(\varphi_1)-F''(\varphi_2))\nabla\varphi_1\cdot\nabla\Delta\varphi\,dx\\
&+2\int_\Omega F''(\varphi_2)\nabla\varphi\cdot\nabla\Delta\varphi\,dx\\
=:&\sum_{j=1}^{8}J_j. \qquad (3.9)
\end{aligned}
$$

Now, we estimate each term of the right hand side of (3.9) one by one.

Applying (A_3)-(A_4), Hölder's inequality, and Poincaré inequality, one can derive

$$
\begin{aligned}
|J_1| \le& 2(1+\delta)\chi_\varphi\|\sigma\|_{L^2}\|\Delta\varphi\|_{L^2},\\
|J_2| \le& 2\delta\lambda_c L_h\|\sigma_2\|_{L^4}\|\sigma\|_{L^2}\|\varphi\|_{L^4} \le C\delta\|\sigma_2\|_{H^1}\|\sigma\|_{L^2}\|\varphi\|_{H^1},\\
|J_3| \le& 2\chi_\varphi\|\nabla\sigma\|_{L^2}\|\nabla\Delta\varphi\|_{L^2},\\
|J_4| \le& h_0(\lambda_p\|\sigma\|_{L^2}+\|u\|_{L^2})(\|\varphi\|_{L^2}+\|\Delta\varphi\|_{L^2}),\\
|J_5| \le& C(1+\|\sigma_2\|_{L^2}+\|u_2\|_{L^2})\|\varphi\|_{H^1}^2 + C(1+\|\sigma_2\|_{L^2}+\|u_2\|_{L^2})\|\varphi\|_{L^6}\|\Delta\varphi\|_{L^3}\\
\le& C(1+\|\sigma_2\|_{L^2}+\|u_2\|_{L^2})\|\varphi\|_{H^1}^2 + C(1+\|\sigma_2\|_{H^1}+\|u_2\|_{L^2})\|\varphi\|_{H^1}\|\Delta\varphi\|_{H^1}.
\end{aligned}
$$

Using Gagliardo-Nirenburg inequality yields

$$
\begin{aligned}
|J_6| =& \left|\int_\Omega\int_0^1 F''(r\varphi_1+(1-r)\varphi_2)\,dr\,\varphi\Delta\varphi\,dx\right|\\
\le& C(1+\|\varphi_1\|_{L^6}^4+\|\varphi_2\|_{L^6}^4)\|\varphi\|_{L^6}\|\Delta\varphi\|_{L^6}\\
\le& C(1+\|\varphi_1\|_{H^1}^4+\|\varphi_2\|_{H^1}^4)\|\varphi\|_{H^1}\|\Delta\varphi\|_{H^1},\\
|J_7| =& \left|\int_\Omega\int_0^1 F'''(r\varphi_1+(1-r)\varphi_2)\,dr\,\varphi\nabla\varphi_1\cdot\nabla\Delta\varphi\,dx\right|\\
\le& C(1+\|\varphi_1\|_{L^{12}}^3+\|\varphi_2\|_{L^{12}}^3)\|\varphi\|_{L^6}\|\nabla\varphi_1\|_{L^{12}}\|\nabla\Delta\varphi\|_{L^2}\\
\le& C(1+\|\varphi_1\|_{H^1}^{\frac{21}{8}}\|\varphi_1\|_{H^3}^{\frac{3}{8}}+\|\varphi_2\|_{H^1}^{\frac{21}{8}}\|\varphi_2\|_{H^3}^{\frac{3}{8}})\|\varphi_1\|_{H^1}^{\frac{3}{8}}\|\|\varphi_1\|_{H^3}^{\frac{5}{8}}\|\varphi\|_{H^1}\|\Delta\varphi\|_{H^1}\\
\le& C(1+\|\varphi_2\|_{H^1}^{\frac{7}{2}}\|\varphi_2\|_{H^3}^{\frac{1}{2}})(\|\varphi\|_{H^1}+\|\varphi\|_{H^1}^{\frac{1}{2}}\|\Delta\varphi\|_{H^1}^{\frac{1}{2}})\|\Delta\varphi\|_{H^1},\\
|J_8| \le& \|F''(\varphi_2)\|_{L^3}\|\nabla\varphi\|_{L^6}\|\nabla\Delta\varphi\|_{L^2}\\
\le& C(1+\|\varphi_2\|_{L^{12}}^4)(\|\varphi\|_{H^1}+\|\varphi\|_{H^1}^{\frac{1}{2}}\|\nabla\Delta\varphi\|_{L^2}^{\frac{1}{2}})\|\Delta\varphi\|_{H^1}.
\end{aligned}
$$

Therefore, we conclude that there exists a positive constant $\mathscr{K}$ independent of δ, such that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\varphi\|_{H^1}^2+\delta\|\sigma\|_{L^2}^2\right)+\|\nabla\Delta\varphi\|_{L^2}^2+\|\Delta\varphi\|_{L^2}^2+\delta\chi_\sigma\|\nabla\sigma\|_{L^2}^2+\delta K\|\sigma\|_{L^2(\Gamma)}^2\\
\le& \mathscr{K}L_\delta(t)\left(\|\varphi\|_{H^1}^2+\delta\|\sigma\|_{L^2}^2\right)+\mathscr{K}\|u\|_{L^2}^2+\mathscr{K}\chi_\sigma\|\nabla\sigma\|_{L^2}^2,
\end{aligned}
$$

where

$$
L_\delta(t)=\delta\left(1+\delta\|\sigma_2\|_{H^1}^2+(1+\|\varphi_1\|_{H^1}^6+\|\varphi_2\|_{H^1}^{14})(1+\|\varphi_1\|_{H^3}^2+\|\varphi_2\|_{H^3}^2)+\|u_2\|_{L^2}^2\right).
$$

Letting $\delta=\mathscr{K}+1$, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\varphi\|_{H^1}^2+(\mathscr{K}+1)\|\sigma\|_{L^2}^2\right)+\|\nabla\Delta\varphi\|_{L^2}^2+\|\Delta\varphi\|_{L^2}^2+\chi_\sigma\|\nabla\sigma\|_{L^2}^2+K\|\sigma\|_{L^2(\Gamma)}^2\\
\le& \mathscr{K}L_{\mathscr{K}+1}(t)\left(\|\varphi\|_{H^1}^2+(\mathscr{K}+1)\|\sigma\|_{L^2}^2\right)+\mathscr{K}\|u\|_{L^2}^2,
\end{aligned}
$$

where $L_{\mathscr{K}+1}(t)\in L^1(0,T)$. Hence, applying the classical Gronwall inequality, we obtain

$$
\begin{aligned}
&\|\varphi(t)\|_{H^1}^2+(\mathscr{K}+1)\|\sigma(t)\|_{L^2}^2+\int_0^t\left(\|\nabla\Delta\varphi(s)\|_{L^2}^2+\|\Delta\varphi(s)\|_{L^2}^2+\chi_\sigma\|\nabla\sigma(s)\|_{L^2}^2\right)ds\\
\le& e^{\mathscr{K}\int_0^t L_{\mathscr{K}+1}(s)\,ds}\left(\|\varphi_0\|_{H^1}^2+(\mathscr{K}+1)\|\sigma_0\|_{L^2}^2+\mathscr{K}\|u(s)\|_{L^2(Q)}^2\right). \qquad (3.10)
\end{aligned}
$$

Therefore, we infer from inequality (3.10) that the weak solution to problem (1.1) is unique. □

From Theorem 3.2, we infer that the control-to-state mapping $S : u \to (\varphi, \mu, \sigma)$ is well-defined and a locally Lipschitz continuous mapping from $L^2(Q)$ into $\mathscr{Y} = \mathscr{X}_2 \times L^2(0,T;H^1) \times \mathscr{X}_1$, where $\mathscr{X}_2 = H^1(0,T;(H^1)^*) \cap L^2(0,T;H^3)$ and $\mathscr{X}_1 = H^1(0,T;(H^1)^*) \cap L^2(0,T;H^1)$.

4. The Optimal Control Problem

In this section, we are in a position to consider the optimal control problem (**CP**) under assumptions (A_1)-(A_7). The main results are the existence of an optimal control and the corresponding first-order necessary conditions for optimality.

4.1. **The existence of an optimal control.**

For optimal control problem **(CP)**, we next demonstrate the following existence result.

Theorem 4.1. *Assume that* (A_1)-(A_7) *are in force. Then optimal control problem* **(CP)** *admits at least one solution.*

Proof. The proof makes use of the direct method from the calculus of variations. In fact, the cost functional is non-negative, convex, and weakly lower semi-continuous. To this end, let $\{u_n\}_{n=1}^{\infty} \subset \mathscr{U}_{ad}$ be a minimizing sequence for problem (**CP**), and let $(\varphi_n, \mu_n, \sigma_n) = S(u_n)$ for each $n \in \mathbb{N}$. From Theorem 3.1 and assumption (A_4), we conclude that there exists

$$
\begin{aligned}
& u \in L^2(Q), \\
& \varphi \in L^\infty(0,T;H^1) \cap L^2(0,T;H^3) \cap H^1(0,T;(H^1)^*), \\
& \mu \in L^2(0,T;H^1), \\
& \sigma \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1) \cap H^1(0,T;(H^1)^*), \\
& \chi \in L^{\frac{p}{p-1}}(0,T;L^{\frac{p}{p-1}}),
\end{aligned}
$$

such that we can extract a subsequence $\{(u_{n_j}, \varphi_{n_j}, \mu_{n_j}, \sigma_{n_j}, F'(\phi_{n_j}))\}_{j=1}^{\infty}$ of $\{(u_n, \varphi_n, \mu_n, \sigma_n, F'(\phi_n))\}_{n=1}^{\infty}$ satisfying

$$
\begin{aligned}
& u_{n_j} \rightharpoonup u \textit{ weakly in } L^2(Q), \\
& \varphi_{n_j} \rightharpoonup \varphi \textit{ weakly in } L^2(0,T;H^3) \cap H^1(0,T;(H^1)^*), \\
& \mu_{n_j} \rightharpoonup \mu \textit{ weakly in } L^2(0,T;H^1), \\
& \sigma_{n_j} \rightharpoonup \sigma \textit{ weakly in } L^2(0,T;H^1) \cap H^1(0,T;(H^1)^*), \\
& F'(\phi_{n_j}) \rightharpoonup \chi \textit{ weakly in } L^{\frac{p}{p-1}}(0,T;L^{\frac{p}{p-1}}).
\end{aligned}
$$

Since $\mathscr{U}_{ad}$ is weakly closed, we obtain $u \in \mathscr{U}_{ad}$. Using the similar proof of Theorem 3.1, it is easy to prove that (φ, μ, σ) is the weak solution to the problem (1.1) with the control u, i.e., $(\varphi, \mu, \sigma) = S(u)$, which implies that $(u, (\varphi, \mu, \sigma))$ is admissible for problem (**CP**). It then follows from the weakly lower sequential semicontinuity of cost functional $\mathscr{J}$ that u is in fact an optimal control for problem (**CP**). □

4.2. **First-order necessary optimality conditions.**

In what follows, we firstly derive the first-order necessary optimality conditions for **(CP)** in form. It follows from the quadratic form of $\mathscr{J}$ and the chain rule that the Fréchet derivative of the reduced cost functional

$$J(u) = \mathscr{J}(S(u), u)$$

at every $\bar{u} \in \mathscr{U}$ is

$$DJ(\bar{u}) = D_{(\varphi,\mu,\sigma)}\mathscr{J}(S(\bar{u}),\bar{u}) \circ DS(\bar{u}) + D_u\mathscr{J}(S(\bar{u}),\bar{u}).$$

Thanks to the convexity of $\mathscr{U}_{ad}$, we deduce the formal first-order necessary condition for optimality characterized by the following variational inequality:

$$DJ(\bar{u})(u-\bar{u}) \geq 0, \ \forall \ u \in \mathscr{U}_{ad} \tag{4.1}$$

for any minimizer $\bar{u}$ of $\mathscr{J}$ in $\mathscr{U}_{ad}$.

4.2.1. *Well-posedness of linearized system.* To verify the above inference, we have to prove the Fréchet differentiability of the reduced cost functional J. Thanks to the Fréchet differentiability of $\mathscr{J}$, it suffices to prove that $\mathscr{S}$ is Fréchet differentiable. In fact, assume that $\bar{u} \in \mathscr{U}_{ad}$ and $(\bar{\varphi}, \bar{\mu}, \bar{\sigma})$ stands for its associated state. It is necessary to establish the well-posedness of the following linearized system:

$$\begin{cases} \partial_t \rho = \Delta\eta + (\lambda_p\theta - g)h(\bar{\varphi}) + h'(\bar{\varphi})\rho(\lambda_p\bar{\sigma} - \lambda_a - \bar{u}), \\ \eta = -\Delta\rho + F''(\bar{\varphi})\rho - \chi_\varphi\theta, \\ \partial_t\theta = \chi_\sigma\Delta\theta - \chi_\varphi\Delta\rho - \lambda_c h(\bar{\varphi})\theta - \lambda_c h'(\bar{\varphi})\rho\bar{\sigma}, \\ \partial_\nu\rho = \partial_\nu\eta = 0, \ \chi_\sigma\partial_\nu\theta + K\theta = 0, \\ \rho(x,0) = 0, \ \ \theta(x,0) = 0 \end{cases} \tag{4.2}$$

for any fixed $g \in L^2(Q)$.

Theorem 4.2. *Assume that $g \in L^2(Q)$. Then there exists a unique weak solution (ρ,η,θ) to problem (4.2) with the control term g on $[0,T]$. Furthermore, there exists a constant $K_3 > 0$ depending only on R, T, L_5, L_7, $|\Omega|$, and the initial data (φ_0,σ_0) of problem (1.1), such that*

$$\|\rho\|^2_{\mathscr{X}_2} + \|\eta\|^2_{L^2(0,T;H^1)} + \|\theta\|^2_{\mathscr{X}_1} \leq K_3\|g\|^2_{L^2(Q)}. \tag{4.3}$$

Proof. It is similar with the proof of Theorem 3.1 to seek for the following approximate solution $(\rho^n(t), \eta^n(t), \theta^n(t))$ in the form

$$\rho^n(t) = \sum_{i=1}^n \alpha_i(t)\omega_i, \ \eta^n(t) = \sum_{i=1}^n \beta_i(t)\omega_i, \ \theta^n(t) = \sum_{i=1}^n \gamma_i(t)\omega_i,$$

by solving the following problem

$$\begin{cases} \langle\partial_t\rho^n, \psi\rangle + (\nabla\eta^n, \nabla\psi) = (h(\bar{\varphi})(\lambda_p\theta^n - g) + h'(\bar{\varphi})\rho^n(\lambda_p\bar{\sigma} - \lambda_a - \bar{u}), \psi), \\ (\eta^n, \xi) = (\nabla\rho^n, \nabla\xi) + (F''(\bar{\varphi})\rho^n, \xi) - \left(\chi_\varphi\theta^n, \xi\right), \\ \langle\partial_t\theta^n, \phi\rangle + \chi_\sigma(\nabla\theta^n, \nabla\phi) = \chi_\varphi(\nabla\rho^n, \nabla\phi) - \lambda_c(h(\bar{\varphi})\theta^n + h'(\bar{\varphi})\rho^n\bar{\sigma}, \phi) \\ -K(\theta^n, \phi)_{L^2(\Gamma)}, \\ \alpha_i(0) = 0, \ \ \gamma_i(0) = 0, \ \ i = 1, 2, \cdots, n \end{cases} \tag{4.4}$$

for any ψ, ξ, $\phi \in W_n$. Now, we establish some a priori estimates for $(\rho^n(t), \eta^n(t), \theta^n(t))$.

First estimate. Taking $\psi=\rho^n$, $\xi=-\Delta\rho^n$, and $\phi=\theta^n$ in system (4.4), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\rho^n\|_{L^2}^2+\|\theta^n\|_{L^2}^2\right)+\|\Delta\rho^n\|_{L^2}^2+\chi_\sigma\|\nabla\theta^n\|_{L^2}^2+K\|\theta^n\|_{L^2(\Gamma)}^2+\lambda_c\int_\Omega h(\bar\varphi)|\theta^n|^2\,dx\\
=&-2\chi_\varphi\int_\Omega\theta^n\Delta\rho^n\,dx+\int_\Omega F''(\bar\varphi)\rho^n\Delta\rho^n+\int_\Omega h'(\bar\varphi)\rho^n(\lambda_p\bar\sigma-\lambda_a-\bar u)\rho^n\,dx\\
&-\lambda_c\int_\Omega h'(\bar\varphi)\rho^n\bar\sigma\theta^n\,dx+\int_\Omega h(\bar\varphi)(\lambda_p\theta^n-g)\rho^n\,dx\\
=:&\sum_{i=1}^5 J_i.
\end{aligned}
$$

From Hölder's inequality and Gagliardo-Nirenberge inequality, we deduce

$$
\begin{aligned}
|J_1|\le&2\chi_\varphi\|\theta^n\|_{L^2}\|\Delta\rho^n\|_{L^2},\\
|J_2|\le&\|F''(\bar\varphi)\|_{L^3}\|\rho^n\|_{L^6}\|\Delta\rho^n\|_{L^2}\le(1+\|\bar\varphi\|_{H^1}^{\frac{7}{2}}\|\bar\varphi\|_{H^3}^{\frac{1}{2}})(\|\rho^n\|_{L^2}+\|\rho^n\|_{L^2}^{\frac{1}{2}}\|\Delta\rho^n\|_{L^2}^{\frac{1}{2}})\|\Delta\rho^n\|_{L^2},\\
|J_3|\le&L_h\|\rho^n\|_{L^4}^2\|\lambda_p\bar\sigma-\lambda_a-\bar u\|_{L^2},\\
|J_4|\le&L_h\lambda_c\|\rho^n\|_{L^6}\|\bar\sigma\|_{L^3}\|\theta^n\|_{L^2}\le L_h\lambda_c(\|\rho^n\|_{L^2}+\|\rho^n\|_{L^2}^{\frac{1}{2}}\|\Delta\rho^n\|_{L^2}^{\frac{1}{2}})\|\bar\sigma\|_{H^1}\|\theta^n\|_{L^2},\\
|J_5|\le&h_0(\lambda_p\|\theta^n\|_{L^2}+\|g\|_{L^2})\|\rho^n\|_{L^2}\le h_0(\lambda_p\|\theta^n\|_{L^2}+\|g\|_{L^2})\|\rho^n\|_{L^2}.
\end{aligned}
$$

In view of Young's inequality, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\rho^n\|_{L^2}^2+\|\theta^n\|_{L^2}^2\right)+\|\Delta\rho^n\|_{L^2}^2+\chi_\sigma\|\nabla\theta^n\|_{L^2}^2+K\|\theta^n\|_{L^2(\Gamma)}^2\\
\le&C(1+\|\bar\sigma\|_{H^1}^2+\|\bar u\|_{L^2}^2+\|\bar\varphi\|_{H^1}^{14}\|\bar\varphi\|_{H^3}^2)(\|\rho^n\|_{L^2}^2+\|\theta^n\|_{L^2}^2)+C\|g\|_{L^2}^2.
\end{aligned}
$$

We derive from the classical Gronwall inequality that, for any $t\in[0,T]$,

$$
\begin{aligned}
&\|\rho^n(t)\|_{L^2}^2+\|\theta^n(t)\|_{L^2}^2+\int_0^t\left(\|\Delta\rho^n(s)\|_{L^2}^2+\chi_\sigma\|\nabla\theta^n(s)\|_{L^2}^2\right)ds\\
\le&C\|g\|_{L^2(Q)}^2\exp\left(\int_0^T\mathscr{L}(r)\,dr\right),
\end{aligned}
\tag{4.5}
$$

where

$$\mathscr{L}(t)=\mathscr{K}_1(1+\|\bar\sigma\|_{H^1}^2+\|\bar u\|_{L^2}^2+\|\bar\varphi\|_{H^1}^{14}\|\bar\varphi\|_{H^3}^2)\in L^1(0,T).$$

Second estimate. Let $\psi=-\Delta\rho^n$ in the first equation of problem (4.4). Accounting for Hölder's inequality, it is easy to see that

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla\rho^n\|_{L^2}^2+2\|\nabla\Delta\rho^n\|_{L^2}^2\\
\le&2\lambda_p h_0\left(\|\theta^n\|_{L^2}+\|g\|_{L^2}\right)\|\Delta\rho^n\|_{L^2}+2L_h\|\rho^n\|_{L^6}\|\lambda_p\bar\sigma-\lambda_a-\bar u\|_{L^2}\|\Delta\rho^n\|_{L^3}\\
&+2(\|F''(\bar\varphi)\|_{L^3}\|\nabla\rho^n\|_{L^6}+\|F'''(\bar\varphi)\|_{L^4}\|\nabla\bar\varphi\|_{L^{12}}\|\rho^n\|_{L^6})\|\nabla\Delta\rho^n\|_{L^2}\\
&+2\chi_\varphi\|\nabla\theta^n\|_{L^2}\|\nabla\Delta\rho^n\|_{L^2}.
\end{aligned}
$$

Young's inequality along with Gagliardo-Nirenberge inequality leads us to

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla\rho^n\|_{L^2}^2+\|\nabla\Delta\rho^n\|_{L^2}^2+\|\nabla\rho^n\|_{L^2}^2\\
\le&C(1+\|\bar\sigma\|_{L^2}^2\|\bar u\|_{L^2}^2+\|F'''(\bar\varphi)\|_{L^4}^2\|\nabla\bar\varphi\|_{L^{12}}^2+\|F''(\bar\varphi)\|_{L^3}^4)\|\rho^n\|_{H^1}^2+C(\|\theta^n\|_{L^2}^2+\|g\|_{L^2}^2)
\end{aligned}
$$

and

$$\|F'''(\bar{\varphi})\|_{L^4}^2\|\nabla\bar{\varphi}\|_{L^{12}}^2 \le C(1+\|\bar{\varphi}\|_{H^1}^{\frac{21}{4}}\|\bar{\varphi}\|_{H^3}^{\frac{3}{4}})\|\bar{\varphi}\|_{H^1}^{\frac{3}{4}}\|\bar{\varphi}\|_{H^3}^{\frac{5}{4}} \le (1+\|\bar{\varphi}\|_{H^1}^6)\|\bar{\varphi}\|_{H^3}^2,$$
$$\|F''(\bar{\varphi})\|_{L^3}^4 \le C(1+\|\varphi\|_{H^1}^{14}\|\varphi\|_{H^3}^2).$$

At this point, we collect the above estimates, and apply the classical Gronwall's inequality, to conclude that there exists a positive constant $D_2(T)$ such that, for any $t\in[0,T]$,

$$\|\nabla\rho^n(t)\|_{L^2}^2+\int_0^t\|\nabla\Delta\rho^n(s)\|_{L^2}^2\,ds+\int_0^t\|\nabla\rho^n(s)\|_{L^2}^2\,ds \le D_2(T)\|g\|_{L^2(Q)}^2. \tag{4.6}$$

Inserting $\xi=-\Delta\eta^n$, we arrive at

$$\begin{aligned}\|\nabla\eta^n\|_{L^2}^2 =& -\int_\Omega\left(\nabla\Delta\rho^n\cdot\nabla\eta^n-\nabla(F''(\bar{\varphi})\rho^n)\cdot\nabla\eta^n+\chi_\varphi\nabla\theta^n\cdot\nabla\eta^n\right)dx\\ \le&(\|F''(\bar{\varphi})\|_{L^3}\|\nabla\rho^n\|_{L^6}+\|F'''(\bar{\varphi})\|_{L^4}\|\nabla\bar{\varphi}\|_{L^{12}}\|\rho^n\|_{L^6}+\|\nabla\Delta\rho^n\|_{L^2})\|\nabla\eta^n\|_{L^2}\\ &+\chi_\varphi\|\nabla\theta^n\|_{L^2}\|\nabla\eta^n\|_{L^2},\end{aligned}$$

which implies that

$$\begin{aligned}\|\nabla\eta^n\|_{L^2}^2 \le& C\left(1+\|F''(\bar{\varphi})\|_{L^3}^4+\|F'''(\bar{\varphi})\|_{L^4}^2\|\nabla\bar{\varphi}\|_{L^{12}}^2\right)\|\rho^n\|_{H^1}^2\\ &+C\left(\|\nabla\Delta\rho^n\|_{L^2}^2+\|\nabla\theta^n\|_{L^2}^2\right).\end{aligned}$$

Furthermore, letting $\xi=1$, one obtains that

$$\left|\int_\Omega\eta^n(x,t)\,dx\right| \le \chi_\varphi\|\theta^n\|_{L^2}|\Omega|^{\frac{1}{2}}+\|F''(\bar{\varphi})\|_{L^3}\|\rho^n\|_{L^{\frac{3}{2}}}.$$

In view of Lemma 2.1, one sees that

$$\begin{aligned}\|\eta^n(t)\|_{L^2} \le& \|\eta^n-M(\eta^n)\|_{L^2}+|M(\eta^n)||\Omega|^{\frac{1}{2}}\\ \le& C\left(\|\nabla\eta^n\|_{L^2}+\|\theta^n\|_{L^2}+\|F''(\bar{\varphi})\|_{L^3}\|\rho^n\|_{H^1}\right).\end{aligned}$$

We infer from the above inequalities and Theorem 3.1 that there exists a positive constant $D_3(T)$ such that

$$\int_0^t\|\eta^n(s)\|_{H^1}^2\,ds \le D_3(T)\|g\|_{L^2(Q)}^2. \tag{4.7}$$

Estimates on time derivatives. From the first and third equations of problem (4.4), we find

$$\|\partial_t\rho^n\|_{(H^1)^*} \le \lambda_p h_0\|\theta^n\|_{L^2}+h_0\|g\|_{L^2}+\|\nabla\eta^n\|_{L^2}+L_h\|\lambda_p\bar{\sigma}-\lambda_a-\bar{u}\|_{L^2}\|\rho^n\|_{L^3}$$

and

$$\begin{aligned}\|\partial_t\theta^n\|_{(H^1)^*} \le& \chi_\sigma\|\nabla\theta^n\|_{L^2}+\chi_\varphi\|\nabla\rho^n\|_{L^2}+\lambda_c h_0\|\theta^n\|_{L^2}+\lambda_c L_h\|\bar{\sigma}\|_{L^3}\|\rho^n\|_{L^2}\\ &+K\|\theta^n\|_{L^2(\Gamma)}.\end{aligned}$$

Consequently, there exists a positive constant $D_4(T)$, such that

$$\int_0^T\|\partial_t\rho^n(t)\|_{(H^1)^*}^2\,dt+\int_0^T\|\partial_t\theta^n(t)\|_{(H^1)^*}^2\,dt \le D_4(T)\|g\|_{L^2(Q)}^2. \tag{4.8}$$

Therefore, we can obtain the existence of a weak solution for problem (4.2) by combining the above inequalities and compactness theorem. Moreover, we conclude from inequalities (4.5)-(4.8) and the lower semi-continuity of norms that inequality (4.3) is valid.

In what follows, we demonstrate the uniqueness of weak solutions to problem (4.2). Suppose that (ρ_1,η_1,θ_1) and (ρ_2,η_2,θ_2), respectively, are two weak solutions to problem (4.2). Let $(\rho,\eta,\theta)=(\rho_1-\rho_2,\eta_1-\eta_2,\theta_1-\theta_2)$. Then (ρ,η,θ) satisfies the following equations

$$\begin{cases}\partial_t\rho=\Delta\eta+\lambda_p h(\bar{\varphi})\theta+h'(\bar{\varphi})\rho(\lambda_p\bar{\sigma}-\lambda_a-\bar{u}),\\ \eta=-\Delta\rho+F''(\bar{\varphi})\rho-\chi_\varphi\theta,\\ \partial_t\theta=\chi_\sigma\Delta\theta-\chi_\varphi\Delta\rho-\lambda_c h(\bar{\varphi})\theta-\lambda_c h'(\bar{\varphi})\rho\bar{\sigma},\\ \partial_\nu\rho=\partial_\nu\eta=0,\ \chi_\sigma\partial_\nu\theta+K\theta=0,\\ \rho(x,0)=0,\ \ \theta(x,0)=0.\end{cases}$$

Repeating the proof of inequality (4.5) with $g\equiv 0$, we obtain

$$(\rho_1(x,t),\eta_1(x,t),\theta_1(x,t))=(\rho_2(x,t),\eta_2(x,t),\theta_2(x,t))$$

for almost everywhere $(x,t)\in Q$. □

4.2.2. *Fréchet differentiability.* Based on Theorem 3.2, we are in a position to verify the Fréchet differentiability of operator S.

Theorem 4.3. *Assume that (A_1)-(A_7) are satisfied. Then the following two conclusions hold:*

(i) *For any $\bar{u}\in\mathscr{U}_{ad}$, the control-to-state mapping S is Fréchet differentiable in $\mathscr{U}$ as a mapping from $L^\infty(Q)$ into $\mathscr{Y}$. Moreover, the Fréchet derivative $DS(\bar{u})\in\mathscr{L}(L^\infty(Q),\mathscr{Y})$ is defined as follows: for any $g\in L^\infty(Q)$, $DS(\bar{u})g=(\rho,\eta,\theta)$, where (ρ,η,θ) is the unique weak solution to the linearized system (4.2) associated with g.*

(ii) *The mapping $DS:\mathscr{U}\to\mathscr{L}(L^\infty(Q),\mathscr{Y})$ is Lipschitz continuous on $\mathscr{U}$ in the following sense: there is a constant $K_4>0$ such that, for any $u,\bar{u}\in\mathscr{U}$ and any $g\in L^\infty(Q)$,*

$$\|DS(u)g-DS(\bar{u})g\|_{\mathscr{Y}}\le K_4\|u-\bar{u}\|_{L^2(Q)}\|g\|_{L^\infty(Q)}. \tag{4.9}$$

Proof. For any fixed $\bar{u}\in\mathscr{U}$, let $(\bar{\varphi},\bar{\mu},\bar{\sigma})=S(\bar{u})\in\mathscr{Y}$ be the associated solution to system (1.1). Since $\mathscr{U}$ is an open subset of $L^\infty(Q)$, there exists some $\delta>0$ such that, for any $g\in L^\infty(Q)$ with $\|g\|_{L^\infty(Q)}\le\delta$, $\bar{u}+g\in\mathscr{U}$. Let $(\varphi^g,\mu^g,\sigma^g)$ be the unique weak solution to the state system (1.1) with $\bar{u}+g$ and (ρ,η,θ) be the unique weak solution to the linearized system (4.2) associated with g.

For convenience, denote by $\varphi(t):=\varphi^g(t)-\bar{\varphi}(t)$, $\mu(t):=\mu^g(t)-\bar{\mu}(t)$, $\sigma(t):=\sigma^g(t)-\bar{\sigma}(t)$, $\psi(t):=\varphi(t)-\rho(t)$, $\chi(t):=\mu(t)-\eta(t)$, $\xi(t):=\sigma(t)-\theta(t)$, $\mathscr{F}(t)=F'(\varphi^g)-F'(\bar{\varphi})-F''(\bar{\varphi})\rho$, and $\mathscr{H}(t):=h(\varphi^g(t))-h(\bar{\varphi})(t)-h'(\bar{\varphi}(t))\rho(t)$ for any $t\ge 0$. Then (ψ,χ,ξ) satisfies the following equations:

$$\begin{cases}\langle\partial_t\psi,\Psi\rangle+(\nabla\chi,\nabla\Psi)=(\lambda_p h(\bar{\varphi})\xi,\Psi)+(\mathscr{H}(\lambda_p\bar{\sigma}-\lambda_a-\bar{u}),\Psi)\\ +((\lambda_p\sigma-g)(h(\varphi^g)-h(\bar{\varphi})),\Psi),\\ (\chi,\Theta)+(\nabla\psi,\nabla\Theta)=\left(\mathscr{F}-\chi_\varphi\xi,\Theta\right),\\ \langle\partial_t\xi,\Phi\rangle+\left(\chi_\sigma\nabla\xi-\chi_\varphi\nabla\psi,\nabla\Phi\right)=-(\lambda_c h(\bar{\varphi})\xi,\Phi)-\lambda_c(\mathscr{H}\bar{\sigma},\Phi)\\ -\lambda_c((h(\varphi^g)-h(\bar{\varphi}))\sigma,\Phi)-K(\xi,\Phi)_{L^2(\Gamma)},\\ \psi(x,0)=0,\ \ \xi(x,0)=0\end{cases} \tag{4.10}$$

for any $\Psi,\Theta,\Phi\in H^1$, and a.e. $t\in(0,T)$.

First estimates. Inserting $\Psi=\psi$, $\Theta=-\Delta\phi$, and $\Phi=\xi$ in system (4.10), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\psi\|_{L^2}^2+\|\xi\|_{L^2}^2\right)+\|\Delta\psi\|_{L^2}^2+\chi_\sigma\|\nabla\xi\|_{L^2}^2+K\|\xi\|_{L^2(\Gamma)}^2+\lambda_c\int_\Omega h(\bar\varphi)|\xi|^2\,dx\\ =&\int_\Omega \mathscr{F}\Delta\psi\,dx-\lambda_c\int_\Omega \mathscr{H}\bar\sigma\xi\,dx+\lambda_p\int_\Omega h(\bar\varphi)\xi\psi\,dx+\int_\Omega\mathscr{H}(\lambda_p\bar\sigma-\lambda_a-\bar u)\psi\,dx\\ &+\int_\Omega(\lambda_p\sigma-g)(h(\varphi^g)-h(\bar\varphi))\psi\,dx-\lambda_c\int_\Omega(h(\varphi^g)-h(\bar\varphi))\sigma\xi\,dx\\ &-2\chi_\varphi\int_\Omega\Delta\psi\xi\,dx. \end{aligned}\tag{4.11}$$

Now, we estimate each term of the right hand side of inequality (4.11) one by one. Thanks to

$$f(\varphi^g)-f(\bar\varphi)-f'(\bar\varphi)\rho=\left(\int_0^1(f'(r\varphi^g+(1-r)\bar\varphi)-f'(\bar\varphi))dr\right)\varphi+f'(\bar\varphi)\psi \tag{4.12}$$

for any function $f\in\mathscr{C}^1(\mathbb{R})$, Hölder inequality, Gagliardo-Nirenberge inequality, and assumption (A_4), we infer that

$$\begin{aligned}\left|\int_\Omega\mathscr{F}\Delta\varphi\,dx\right|\le&C(1+\|\varphi^g\|_{L^{18}}^3+\|\bar\varphi\|_{L^{18}}^3)\|\varphi\|_{L^6}^2\|\Delta\psi\|_{L^2}+\|F''(\bar\varphi)\|_{L^3}\|\psi\|_{L^6}\|\Delta\psi\|_{L^2}\\ \le&C(1+\|\varphi^g\|_{H^1}^{\frac{5}{2}}\|\varphi^g\|_{H^3}^{\frac{1}{2}}+\|\bar\varphi\|_{H^1}^{\frac{5}{2}}\|\bar\varphi\|_{H^3}^{\frac{1}{2}})\|\varphi\|_{H^1}^2\|\Delta\psi\|_{L^2}\\ &+C(1+\|\bar\varphi\|_{H^1}^{\frac{7}{2}}\|\bar\varphi\|_{H^3}^{\frac{1}{2}})(\|\psi\|_{L^2}+\|\psi\|_{L^2}^{\frac{1}{2}}\|\Delta\psi\|_{L^2}^{\frac{1}{2}})\|\Delta\psi\|_{L^2}\end{aligned}\tag{4.13}$$

and

$$\begin{aligned}\left|\int_\Omega\mathscr{H}\bar\sigma\xi\,dx\right|\le&L_h\|\varphi\|_{L^6}^2\|\bar\sigma\|_{L^2}\|\xi\|_{L^6}+L_h\|\psi\|_{L^2}\|\bar\sigma\|_{L^3}\|\xi\|_{L^6}\\ \le&C\left(\|\varphi\|_{H^1}^2\|\bar\sigma\|_{L^2}\|\xi\|_{H^1}+\|\psi\|_{L^2}\|\bar\sigma\|_{H^1}\|\xi\|_{H^1}\right).\end{aligned}\tag{4.14}$$

Similarly, we can prove that

$$\begin{aligned}&\left|\int_\Omega\mathscr{H}(\lambda_p\bar\sigma-\lambda_a-\bar u)\psi\,dx\right|\\ \le&C\|\varphi\|_{H^1}^2(1+\|\bar\sigma\|_{L^2}+\|\bar u\|_{L^2})(\|\psi\|_{L^2}+\|\psi\|_{L^2}^{\frac{1}{2}}\|\Delta\psi\|_{L^2}^{\frac{1}{2}})\\ &+C(1+\|\bar\sigma\|_{L^2}+\|\bar u\|_{L^2})(\|\psi\|_{L^2}^2+\|\psi\|_{L^2}^{\frac{5}{4}}\|\Delta\psi\|_{L^2}^{\frac{3}{4}}).\end{aligned}\tag{4.15}$$

In addition, we obtain

$$\begin{aligned}&\left|\int_\Omega(\lambda_p\sigma-g)(h(\varphi^g)-h(\bar\varphi))\psi\,dx\right|\\ \le&(\|\sigma\|_{L^2}+\|g\|_{L^2})\|\varphi\|_{L^3}\|\psi\|_{L^6}\\ \le&(\|\sigma\|_{L^2}+\|g\|_{L^2})\|\varphi\|_{H^1}(\|\psi\|_{L^2}+\|\psi\|_{L^2}^{\frac{1}{2}}\|\Delta\psi\|_{L^2}^{\frac{1}{2}})\end{aligned}\tag{4.16}$$

and

$$\left|\int_\Omega(h(\varphi^g)-h(\bar\varphi))\sigma\xi\,dx\right|\le C\|\varphi\|_{H^1}\|\sigma\|_{L^2}(\|\xi\|_{L^2}+\|\xi\|_{L^2}^{\frac{1}{2}}\|\nabla\xi\|_{L^2}^{\frac{1}{2}}).\tag{4.17}$$

It follows from Young's inequality, Hölder's inequality, and inequalities (4.11), (4.13)-(4.17) that

$$\begin{aligned}&\frac{d}{dt}\left(\|\psi\|_{L^2}^2+\|\xi\|_{L^2}^2\right)+\|\Delta\psi\|_{L^2}^2+\chi_\sigma\|\nabla\xi\|_{L^2}^2+K\|\xi\|_{L^2(\Gamma)}^2\\ \le& H_1(t)(\|\psi\|_{L^2}^2+\|\xi\|_{L^2}^2)+H_2(t),\end{aligned}$$

where

$$\begin{aligned}H_1(t)=&C(1+\|\bar\varphi\|_{H^1}^{14}\|\bar\varphi\|_{H^3}^2+\|\bar\sigma\|_{H^1}^2+\|\bar u\|_{L^2}^2),\\ H_2(t)=&C(1+\|\varphi^g\|_{H^1}^5\|\varphi^g\|_{H^3}+\|\bar\varphi\|_{H^1}^5\|\bar\varphi\|_{H^3}+\|\bar\sigma\|_{L^2}^2+\|\bar u\|_{L^2}^2)(\|\varphi\|_{H^1}^4+\|\sigma\|_{L^2}^4)\\ &+\|g\|_{L^2}^4.\end{aligned}$$

From the classical Gronwall inequality, Theorem 3.1, and Theorem 3.2, we deduce that there exists a positive constant $D_5(T)$ such that, for any $t\in[0,T]$,

$$\begin{aligned}&\|\psi(t)\|_{L^2}^2+\|\xi(t)\|_{L^2}^2+\chi_\sigma\int_0^t\|\nabla\xi(s)\|_{L^2}^2\,ds+\int_0^t\|\psi(s)\|_{H^2}^2\,ds+K\int_0^t\|\xi(s)\|_{L^2(\Gamma)}^2\,ds\\ \le& D_5(T)\|g\|_{L^2(Q)}^4.\end{aligned}\tag{4.18}$$

Second estimates. Letting $\Psi=-\Delta\psi$ in the first equation of system (4.10), one sees that

$$\begin{aligned}&\frac{d}{dt}\|\nabla\psi\|_{L^2}^2+\|\nabla\Delta\psi\|_{L^2}^2+2\|\nabla\chi\|_{L^2}^2\\ =&-2\int_\Omega\nabla\left(\mathscr{F}-\chi_\varphi\xi\right)\cdot(\nabla\Delta\psi+\nabla\chi)\,dx-2\int_\Omega\mathscr{H}(\lambda_p\bar\sigma-\lambda_a-\bar u)\Delta\psi\,dx\\ &-2\lambda_p\int_\Omega h(\bar\varphi)\xi\Delta\psi\,dx-2\int_\Omega(\lambda_p\sigma-g)(h(\varphi^g)-h(\bar\varphi))\Delta\psi\,dx\end{aligned}$$

With the help of Hölder's inequality and inequality (4.12), we obtain

$$\begin{aligned}&\frac{d}{dt}\|\nabla\psi\|_{L^2}^2+\|\nabla\Delta\psi\|_{L^2}^2+\|\nabla\chi\|_{L^2}^2\\ \le&\left(\|\nabla\mathscr{F}\|_{L^2}+\chi_\varphi\|\nabla\xi\|_{L^2}\right)\left(\|\nabla\Delta\psi\|_{L^2}+\|\nabla\chi\|_{L^2}\right)+2L_h(\|\sigma\|_{L^2}+\|g\|_{L^2})\|\varphi\|_{L^6}\|\Delta\psi\|_{L^3}\\ &+C(\|\varphi\|_{L^6}^2+\|\psi\|_{L^6})\|\lambda_p\bar\sigma-\lambda_a-\bar u\|_{L^2}\|\Delta\psi\|_{L^3}+2\lambda_p h_0\|\xi\|_{L^2}\|\Delta\psi\|_{L^2}.\end{aligned}\tag{4.19}$$

Next, we estimate the term $\|\nabla\mathscr{F}\|_{L^2}$ by inequality (4.12) and Hölder's inequality.

$$\begin{aligned}\|\nabla\mathscr{F}\|_{L^2}=&\|\nabla(F'(\varphi^g)-F'(\bar\varphi)-F''(\bar\varphi)\rho)\|_{L^2}\\ \le&\|(F''(\varphi^g)-F''(\bar\varphi)-F'''(\bar\varphi)\rho)\nabla\varphi^g\|_{L^2}+\|F''(\bar\varphi)\nabla\psi\|_{L^2}+\|F'''(\bar\varphi)\rho\nabla\varphi\|_{L^2}\\ \le&C(1+\|\varphi^g\|_{L^{24}}^2+\|\bar\varphi\|_{L^{24}}^2)\|\varphi\|_{L^6}^2\|\nabla\varphi^g\|_{L^{12}}+\|F'''(\bar\varphi)\|_{L^6}\|\nabla\varphi^g\|_{L^6}\|\psi\|_{L^6}\\ &+\|F''(\bar\varphi)\|_{L^3}\|\nabla\psi\|_{L^6}+\|F'''(\bar\varphi)\|_{L^6}\|\nabla\varphi\|_{L^6}\|\rho\|_{L^6}.\end{aligned}\tag{4.20}$$

Concluding from the classical Gronwall inequality, Theorem 3.1, Theorem 3.2, Theorem 4.2, inequalities (4.18)-(4.20), and Gagliardo-Nirenberge inequality, we know that there exists a positive constant $D_6(T)$ such that, for any $t\in[0,T]$,

$$\|\nabla\psi(t)\|_{L^2}^2+\int_0^t\|\nabla\Delta\psi(s)\|_{L^2}^2\,ds+\int_0^t\|\nabla\chi(s)\|_{L^2}^2\,ds\le D_6(T)\|g\|_{L^2(Q)}^4.\tag{4.21}$$

Estimates on time derivatives. From the first and third equations of system (4.10), we find

$$\|\partial_t \psi\|_{(H^1)^*} \le \|\nabla\chi\|_{L^2} + \lambda_p h_0 \|\xi\|_{L^2} + C\|\varphi\|_{L^6}^2 \|\lambda_p\bar{\sigma} - \lambda_a - \bar{u}\|_{L^2} + L_h\|\psi\|_{L^6}\|\lambda_p\bar{\sigma} - \lambda_a - \bar{u}\|_{L^2} + C(\|\sigma\|_{L^2} + \|g\|_{L^2})\|\varphi\|_{H^1} \tag{4.22}$$

and

$$\|\partial_t \xi\|_{(H^1)^*} \le \chi_\sigma\|\nabla\xi\|_{L^2} + \chi_\varphi\|\nabla\psi\|_{L^2} + \lambda_c h_0\|\xi\|_{L^2} + C\|\varphi\|_{L^6}^2\|\bar{\sigma}\|_{L^6} + L_h\lambda_c\|\psi\|_{L^6}\|\bar{\sigma}\|_{L^3} + \lambda_c L_h\|\varphi\|_{L^3}\|\sigma\|_{L^2} + K\|\xi\|_{L^2(\Gamma)}. \tag{4.23}$$

With the help of Theorem 3.1, Theorem 3.2, and inequalities (4.18), (4.21)-(4.23), we infer that there exists a generic positive constant $D_7(T)$ such that, for any $t \in [0,T]$,

$$\int_0^t \|\partial_t \psi(s)\|_{(H^1)^*}^2\, ds + \int_0^t \|\xi_t(s)\|_{(H^1)^*}^2\, ds \le D_7(T)\|g\|_{L^2(Q)}^4. \tag{4.24}$$

Hence, we obtain the validity of (i).

Now, we prove the validity of (ii). For any fixed $\bar{u} \in \mathscr{U}$, let $k \in L^\infty(Q)$ satisfy $\bar{u}+k \in \mathscr{U}$. Denote by $(\varphi^k,\mu^k,\sigma^k) = S(\bar{u}+k)$, $(\bar{\varphi},\bar{\mu},\bar{\sigma}) = S(\bar{u})$ and $(\varphi,\mu,\sigma) = (\varphi^k-\bar{\varphi},\mu^k-\bar{\mu},\sigma^k-\bar{\sigma}) = S(\bar{u}+k)-S(\bar{u})$. For any fixed $g \in L^2(Q)$, we write $(\rho^g,\eta^g,\theta^g) = DS(\bar{u}+k)g$, $(\bar{\rho},\bar{\eta},\bar{\theta}) = DS(\bar{u})g$, and $(\rho,\eta,\theta) = (\rho^g-\bar{\rho},\eta^g-\bar{\eta},\theta^g-\bar{\theta}) = DS(\bar{u}+k)g - DS(\bar{u})g$. Then, (ρ,η,θ) satisfies the following system:

$$\begin{cases} \langle\partial_t\rho,\Psi\rangle + (\nabla\eta,\nabla\Psi) = \big((h(\varphi^k)-h(\bar{\varphi}))(\lambda_p\theta^g - g) + \lambda_p h(\bar{\varphi})\theta,\Psi\big) \\ +\big(h'(\bar{\varphi})\bar{\rho}(\lambda_p\sigma - k) + h'(\bar{\varphi})\rho(\lambda_p\sigma^k - \lambda_a - \bar{u} - k),\Psi\big) \\ +\big((h'(\varphi^k)-h'(\bar{\varphi}))\rho^g(\lambda_p\sigma^k - \lambda_a - \bar{u} - k),\Psi\big), \\ (\eta,\Theta) + (\nabla\rho,\nabla\Theta) = \big((F''(\varphi^k)-F''(\bar{\varphi}))\rho^g + F''(\bar{\varphi})\rho - \chi_\varphi\theta,\Theta\big), \\ \langle\partial_t\theta,\Phi\rangle + \big(\chi_\sigma\nabla\theta - \chi_\varphi\nabla\rho,\nabla\Phi\big) = -\lambda_c\left((h(\varphi^k)-h(\bar{\varphi}))\theta^g + h(\bar{\varphi})\theta,\Phi\right) \\ -\lambda_c\left(h'(\bar{\varphi})\rho\sigma^k + (h'(\varphi^k)-h'(\bar{\varphi}))\rho^g\sigma^k + h'(\bar{\varphi})\bar{\rho}\sigma,\Phi\right) - K(\theta,\Phi)_{L^2(\Gamma)}, \\ \rho(x,0) = 0, \ \ \theta(x,0) = 0 \end{cases} \tag{4.25}$$

for any $\Psi,\Theta,\Phi \in H^1$ and a.e. $t \in (0,T)$.

Here, we do not provide the details of the proof of inequality (4.9) since the process is quite similar to the proof of (i) by establishing some a priori estimates of the solutions to system (4.25). □

4.2.3. *Variational inequality.* By the above results, we can first reduce the formal first-order necessary condition (4.1) as follows.

Theorem 4.4. *Let* (A_1)-(A_7) *hold. If* $\bar{u} \in \mathscr{U}_{ad}$ *is an optimal control for the optimal control problem* (CP) *with associated state* $(\bar{\varphi},\bar{\mu},\bar{\sigma}) = S(\bar{u}) \in \mathscr{Y}$, *then, for any* $u \in \mathscr{U}_{ad}$,

$$\alpha_0\int_Q(\bar{\varphi}(x,t)-\varphi_Q)\rho(x,t)\,dxdt + \alpha_1\int_\Omega(\bar{\varphi}(x,T)-\varphi_\Omega)\rho(x,T)\,dx + \beta_0\int_Q(\bar{\sigma}(x,t)-\sigma_Q)\theta(x,t)\,dxdt + \beta_1\int_\Omega(\bar{\sigma}(x,T)-\sigma_\Omega)\theta(x,T)\,dx + \beta_2\int_Q\bar{u}(x,t)(u(x,t)-\bar{u}(x,t))\,dxdt \ge 0, \tag{4.26}$$

where $(\rho,\eta,\theta) = DS(\bar{u})(u-\bar{u}) \in \mathscr{Y}$ *is the unique weak solution to the linearized system* (4.2) *with* $g = u - \bar{u}$.

4.2.4. *Well-posedness of the adjoint system.* In order to simplify variational inequality (4.26) by eliminating the linearized variables ρ, θ, we should first introduce the following (formal) version of the adjoint system to (4.2) which can be obtained by the formal Lagrangian method described, e.g., in [41]:

$$\begin{cases} \partial_t \psi = -\Delta\chi - (\lambda_p \bar{\sigma} - \lambda_a - \bar{u})h'(\bar{\varphi})\psi + F''(\bar{\varphi})\chi + \chi_\varphi \Delta\zeta + \lambda_c h'(\bar{\varphi})\zeta\bar{\sigma} \\ -\alpha_0(\bar{\varphi}(x,t) - \varphi_Q), \\ \chi = -\Delta\psi, \\ \partial_t \zeta = -\chi_\sigma \Delta\zeta - \lambda_p h(\bar{\varphi})\psi - \chi_\varphi \chi + \lambda_c h(\bar{\varphi})\zeta - \beta_0(\bar{\sigma}(x,t) - \sigma_Q), \\ \partial_\nu \psi = \partial_\nu \chi = 0, \ \chi_\sigma \partial_\nu \zeta + K\zeta = 0, \\ \psi(x,T) = \alpha_1(\bar{\varphi}(x,T) - \varphi_\Omega), \\ \zeta(x,T) = \beta_1(\bar{\sigma}(x,T) - \sigma_\Omega). \end{cases} \tag{4.27}$$

Moreover, we have the following well-posedness result.

Theorem 4.5. *Let* (A_1)-(A_7) *hold. If* $\bar{u} \in \mathscr{U}_{ad}$ *is an optimal control for the optimal control problem* (CP) *with the associated state* $(\bar{\varphi}, \bar{\mu}, \bar{\sigma}) = S(\bar{u}) \in \mathscr{Y}$, *then the adjoint state system* (4.27) *has a unique solution* $(\psi, \chi, \zeta) \in \mathscr{Y}$.

Proof. A similar result on the existence, uniqueness, and regularity was proved in Theorem 3.1. Here, we do not provide the details to prove the well-posedness of the adjoint system (4.27). □

4.2.5. *The simplified variational inequality.* At the end of this section, we would like to employ the adjoint variables to eliminate the linearized variables ρ, θ from variational inequality (4.26).

Theorem 4.6. *Under the same assumptions of Theorem 4.5, we have*

$$\int_Q (\beta_2 \bar{u} - h(\bar{\varphi})\psi)(u - \bar{u})\, dx\, dt \geq 0 \tag{4.28}$$

for any $u \in \mathscr{U}_{ad}$.

Proof. Denote by (ρ, η, θ) the weak solution of (4.2) with $g = u - \bar{u}$ for any $u \in \mathscr{U}_{ad}$. Let (ψ, χ, ζ) be the weak solution to (4.27). Multiplying the first, the second, and the third equations of (4.2) with $g = u - \bar{u}$ by ψ, χ, and ζ, respectively, and integrating by parts, we derive that

$$\begin{aligned} &\int_Q \partial_t \rho \psi \, dxdt + \int_Q \nabla\eta \cdot \nabla\psi \, dxdt - \int_Q h(\bar{\varphi})(\lambda_p \theta - (u - \bar{u}))\psi \, dxdt + \int_Q F''(\bar{\varphi})\rho\chi \, dxdt \\ &- \int_Q h'(\bar{\varphi})\rho(\lambda_p \bar{\sigma} - \lambda_a - \bar{u})\psi \, dxdt - \int_Q \eta\chi \, dxdt + \int_Q \nabla\rho \cdot \nabla\chi \, dxdt - \chi_\varphi \int_Q \theta\chi \, dxdt \\ &+ \int_Q \partial_t \theta \zeta \, dxdt + \chi_\sigma \int_Q \nabla\theta \cdot \nabla\zeta \, dxdt - \chi_\varphi \int_Q \nabla\rho \cdot \nabla\zeta \, dxdt + \lambda_c \int_Q h(\bar{\varphi})\theta\zeta \, dxdt \\ &+ \lambda_c \int_Q h'(\bar{\varphi})\rho\bar{\sigma}\zeta \, dxdt + K \int_\Sigma \theta\zeta \, dSdt = 0. \end{aligned} \tag{4.29}$$

Meanwhile, multiplying the first, the second, and the third equation of (4.27) by ρ, η, and θ, respectively, and integrating by parts, we find

$$
\begin{aligned}
&-\int_Q \partial_t \psi \rho \, dxdt + \int_Q \nabla\chi \cdot \nabla\rho \, dxdt - \int_Q (\lambda_p \bar{\sigma} - \lambda_a - \bar{u}) h'(\bar{\varphi}) \psi \rho \, dxdt \\
&+ \int_Q F''(\bar{\varphi}) \chi \rho \, dxdt - \chi_\varphi \int_Q \nabla\zeta \cdot \nabla\rho \, dxdt + \lambda_c \int_Q h'(\bar{\varphi}) \zeta \bar{\sigma} \rho \, dxdt - \int_Q \chi \eta \, dxdt \\
&+ \int_Q \nabla\psi \cdot \nabla\eta \, dxdt - \int_Q \partial_t \zeta \theta \, dxdt + \chi_\sigma \int_Q \nabla\zeta \cdot \nabla\theta \, dxdt + K \int_\Sigma \zeta \theta \, dSdt \\
&- \lambda_p \int_Q h(\bar{\varphi}) \psi \theta \, dxdt - \chi_\varphi \int_Q \chi \theta \, dxdt + \lambda_c \int_Q h(\bar{\varphi}) \zeta \theta \, dxdt \\
=&\beta_0 \int_Q (\bar{\sigma}(x,t) - \sigma_Q) \theta \, dxdt + \alpha_0 \int_Q (\bar{\varphi}(x,t) - \varphi_Q) \rho \, dxdt. \qquad (4.30)
\end{aligned}
$$

Adding (4.29) to (4.30), one sees that, for any $u \in \mathscr{U}_{ad}$,

$$
\begin{aligned}
&\alpha_0 \int_Q (\bar{\varphi}(x,t) - \varphi_Q) \rho(x,t) \, dxdt + \alpha_1 \int_\Omega (\bar{\varphi}(x,T) - \varphi_\Omega) \rho(x,T) \, dx \\
&+ \beta_0 \int_Q (\bar{\sigma}(x,t) - \sigma_Q) \theta(x,t) \, dxdt + \beta_1 \int_\Omega (\bar{\sigma}(x,T) - \sigma_\Omega) \theta(x,T) \, dx \\
=& - \int_Q h(\bar{\varphi})(u - \bar{u}) \psi \, dxdt. \qquad (4.31)
\end{aligned}
$$

Therefore, inserting inequality (4.31) into inequality (4.26), we obtain

$$
\int_Q (\beta_2 \bar{u} - h(\bar{\varphi})\psi)(u - \bar{u}) \, dxdt \geq 0
$$

for any $u \in \mathscr{U}_{ad}$. □

According to inequality (4.28), we immediately conclude the following results as in [41].

Corollary 4.1. *Since $\mathscr{U}_{ad}$ is a nonempty, closed, and convex subset of $L^2(Q)$, from the standard arguments and inequality (4.28), we obtain the following conclusions:*

(i) *If $\beta_2 > 0$, the optimal control $\bar{u}$ is nothing but the $L^2(Q)$ orthogonal projection of $\frac{h(\bar{\varphi})\psi}{\beta_2}$ onto $\mathscr{U}_{ad}$. In other words,*

$$
\bar{u}(x,t) = \mathscr{P}\left(\frac{h(\bar{\varphi})\psi}{\beta_2}\right),
$$

where $\mathscr{P}$ is the orthogonal projector in $L^\infty(Q)$ onto $\mathscr{U}_{ad}$. Applying the standard arguments, it follows from this projection property that the pointwise condition

$$
\bar{u}(x,t) = \max\{u_{min}, \min\{\frac{h(\bar{\varphi})\psi}{\beta_2}, u_{max}\}\}, \quad \text{for a.e. } (x,t) \in Q.
$$

(ii) *If $\beta_2 = 0$, then*

$$
\bar{u}(x,t) = \begin{cases} u_{min}, & \text{if } h(\bar{\varphi})\psi > 0, \\ u_{max}, & \text{if } h(\bar{\varphi})\psi < 0. \end{cases}
$$

Based on the stability estimates of higher order both on the initial data and control parameter established in Theorem 3.2, we are eager to give the following remarks at the end of this paper.

Remark 4.1. In Theorem 4.3, we have demonstrated the Fréchet differentiability of the control-to-state mapping from $L^\infty(Q)$ to $\mathscr{Y}$. If the cost functional is given by

$$\begin{aligned}\mathscr{J}((\varphi,\mu,\sigma);u) =& \frac{\alpha_0}{2}\int_Q |\nabla\varphi(x,t)-\nabla\varphi_Q(x,t)|^2\,dxdt + \frac{\beta_0}{2}\int_Q |\sigma(x,t)-\sigma_Q(x,t)|^2\,dxdt \\ &+ \frac{\alpha_1}{2}\int_\Omega |\nabla\varphi(x,T)-\nabla\varphi_\Omega(x)|^2\,dx + \frac{\beta_1}{2}\int_\Omega |\sigma(x,T)-\sigma_\Omega(x)|^2\,dx \\ &+ \frac{\beta_2}{2}\int_Q |u(x,t)|^2\,dxdt,\end{aligned}$$

then, under the same assumptions in Theorem 4.6 but $\varphi_\Omega \in H^3$, by the similar proof of Theorem 4.1 and Theorem 4.4, the conclusion of Theorem 4.6 still holds true for the component ψ of the solution to the problem (4.27) with $(-\alpha_0(\bar\varphi(x,t)-\varphi_Q), \alpha_1(\bar\varphi(x,T)-\varphi_\Omega))$ replacing by $(\alpha_0\Delta(\bar\varphi(x,t)-\varphi_Q), -\alpha_1\Delta(\bar\varphi(x,T)-\varphi_\Omega))$.

Remark 4.2. Inspired by the idea of the so-called "two-norm discrepancy" (see [41]), we will be able to prove the twice Fréchet differentiability of the control-to-state mapping S by some tedious calculations. Furthermore, we can establish the second-order sufficient optimality conditions of the optimal control problem (CP) by some similar arguments as in [41].

5. Conclusion

In this work, we studied a phase field model for tumour growth with chemotaxis and active transport. In contrast to other models for tumour growth studied in the literatures, the model presented here admits an energy equality with non-dissipative right-hand sides and allows for some realistic source terms. We presented some theoretical results including the well-posedness of the system and the corresponding optimal distributed control problem under more general growth assumptions on the potential F than the ones in [19]. We pointed out that F can be the classical quartic double-well potential, which is the standard approximation for the physical relevant logarithmic potentials. Therefore, our results can play a role in dealing with singular potentials for phase field models for tumour growth with chemotaxis and active transport.

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