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GROUND STATE SOLUTIONS FOR FRACTIONAL CHOQUARD-SCHRÖDINGER-POISSON SYSTEM WITH CRITICAL GROWTH

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Abstract. In this paper, we study the following fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = (I_\mu * F(u))f(u) + |u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $0 < s,t < 1,\ 2(s+t) > 3, \mu \in (s+t,3),\ s \in [\frac{3}{4},1)$, and $2_s^* = \frac{6}{3-2s}$ is the fractional critical Sobolev exponent. By using a monotonicity argument and the global compactness lemma, we obtain the existence of a ground state solution for this system.

Keywords. Critical growth; Choquard equation; Fractional Schrödinger-Poisson system; Ground state solution; Pohožaev identity.

1. INTRODUCTION

In this paper, we study the following fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = (I_\mu * F(u))f(u) + |u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $0 < s,t < 1,\ 2(s+t) > 3,\ \mu \in (s+t,3),\ s \in [\frac{3}{4},1),\ F(t) = \int_0^t f(\tau)d\tau,\ 2_s^* = \frac{6}{3-2s}$ is the fractional critical Sobolev exponent, and $(-\Delta)^s$ is the fractional Laplacian operator defined as

$$(-\Delta)^s u(x) = C_s P.V. \int_{\mathbb{R}^3} \frac{u(x)-u(y)}{|x-y|^{3+2s}} dy, \quad x \in \mathbb{R}^3,$$

where $C_s = \left(\int_{\mathbb{R}^3} \frac{1-cos\zeta_1}{|\zeta|^{3+2s}} d\zeta\right)^{-1}$ is a suitable normalization constant (see [1]), *P.V.* is a commonly used abbreviation for the principal value sense, and $I_\mu : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ is the Riesz potential

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defined by

$$I_\mu(x) = \frac{\Gamma(\frac{3-\mu}{2})}{\Gamma(\frac{\mu}{2})\pi^{\frac{3}{2}}2^\mu |x|^{3-\mu}}, \quad x \in \mathbb{R}^3 \backslash \{0\},$$

where Γ is the Gamma function, and V and f satisfy the following assumptions:

(V_1) $V \in \mathscr{C}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $2sV(x) + (\nabla V(x), x) \geq 0$ for any $x \in \mathbb{R}^3$;

(V_2) $V(x) \leq \liminf_{|y| \to +\infty} V(y) = V_\infty \in \mathbb{R}^+$ for all $x \in \mathbb{R}^3$ and the inequality is strict in a subset of positive Lebesgue measure;

(V_3) there exits a constant $\alpha_0 > 0$ such that

$$\alpha_0 = \inf_{u \in H^s(\mathbb{R}^3) \backslash \{0\}} \frac{\int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{s}{2}} u|^2 + V(x)|u|^2 \right) dx}{\int_{\mathbb{R}^3} |u|^2 dx} > 0;$$

(f_1) $f \in \mathscr{C}(\mathbb{R}, \mathbb{R}),\ f(\tau) \equiv 0$ for all $\tau \in (-\infty, 0)$, and there exits $C_0 > 0$ and $1 + \frac{\mu}{3} < q < 2^*_{\mu,s}$ such that, for every $t \in \mathbb{R}$,

$$|f(t)| \leq C_0(|t|^{\frac{\mu}{3}} + |t|^{q-1}),$$

where $2^*_{\mu,s} = \frac{\mu+3}{3-2s}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality;

(f_2) $\lim_{\tau \to 0^+} \frac{f(\tau)}{\tau^{\mu/3}} = \lim_{\tau \to +\infty} \frac{f(\tau)}{\tau^{2^*_{\mu,s}-1}} = 0$;

(f_3) $[2(s+t)f(\tau)\tau - (3+\mu)F(\tau)]/\tau^{(6+\mu)/2(s+t)}$ is non-decreasing on $(0, +\infty)$, where $F(\tau) = \int_0^\tau f(u)du$;

(f_4) there exist $\nu > 0$ and $p \in (2, 2^*_{\mu,s})$ such that $f(\tau) \geq \nu \tau^{p-1}$ for any $\tau \geq 0$.

Remark 1.1. An exemple of V satisfying (V_1) and (V_2) can be found in [2, Remark 1.3].

When $s = t = 1$, the Schrödinger-Poisson type problem

$$\begin{cases} -\Delta u + V(x)u + \phi u = (I_\mu * f(u))F(u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

has been widely studied in recent years, where $V \in \mathscr{C}(\mathbb{R}^3, [0, +\infty))$ and $f \in \mathscr{C}(\mathbb{R}, \mathbb{R})$; see [3, 4, 5] and the references therein.

When $\phi(x) = 0$, system (1.2) reduces to the Choquard equation

$$-\Delta u + V(x)u = (I_\mu * f(u))F(u), \quad u \in H^1(\mathbb{R}^3). \tag{1.3}$$

For the case that $\mu = 2,\ p = 2,\ V(x) \equiv 1$, and $f(u) = u$, (1.3) is known as the Choquard-Pekar equation or the stationary Hartree equation, which was first introduced by Pekar [6] to describe the quantum mechanics of static polarons. Later, Choquard rediscovered it as an approximation of Hartree-Fock's theory of single-component plasma [7] on the modeling of an electron trapped in its own hole. For more details and applications, one refers to [8, 9, 10, 11, 12, 13, 14]. For the case that $V \equiv 1$ and $f(u) = |u|^{p-2}u$, Moroz and Van Schaftingen[9] obtained a solution to (1.3) when $1 + \frac{\mu}{3} < p < 3 + \mu$, where $1 + \frac{\mu}{3}$ and $3 + \mu$ are the lower and upper critical exponents. Afterwards, these results were extended to the case of general functions F, almost optimal in the sense of Berestyski-Lions or the more general potential V; see [15, 16, 17, 18]

When $V \equiv 1$, Xie, Chen, and Wu [19] studied the following Schrödinger-Poisson system with Hartree-type nonlinearity

$$\begin{cases} -\Delta u + u + \lambda \phi u = (I_\mu * |u|^p)|u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0,\ 0 < \mu < 3$, I_μ is the Riesz potential, and $\frac{3+\mu}{3} < p < 3+\mu$. Using the Pohožaev type identity and the filtration of Nehari manifold, they obtained the existence of positive ground state solutions.

In recent years, researchers paid much more attention to the existence of positive solutions, ground state solutions, multiple solutions, nodal solutions, and semi-classical states of fractional Schrödinger-Poisson systems such as

$$\begin{cases} (-\Delta)^s u + V(x)u + K(x)\phi|u|^{q-2}u = (I_\mu * |u|^p)|u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = K(x)|u|^q, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\frac{3+\mu}{3} < p < \frac{3+\mu}{3-2s},\ 1 < q < p$. For instance, Teng and Agarwal [20] established the existence of non-negative ground state solution and also discussed the nonexistence of ground states to (1.4) with subcritical Choquard nonlinearity. Che, Su, and Chen [21] obtained the existence of ground state solutions for fractional Choquard equations with competing potentials. The authors in [22, 23] recently studied quasilinear versions of the Choquard equation with the p-Laplace operator, and obtained the existence and nonexistence of positive solutions for the quasilinear elliptic inequalities and systems with nonlocal terms.

Inspired by the results above, when $K(x) \equiv 1$ and $q = 2$, we study the general convolution term to (1.4). To the best of our knowledge, there is no result for the existence of non negative least energy solutions for the fractional Choquard-Schrödinger-Poisson systems (1.1) with critical growth. Next, we list our main result.

Theorem 1.1. *Assume that $(V_1)-(V_3)$ and $(f_1)-(f_4)$ hold and $2(s+t) > 3,\ \mu \in (s+t,3)$, and $s \in [\frac{3}{4},1)$.*

(i) If $p \in (2, 2_s^* - 1]$, then there exists $\nu_1 > 0$ such that, for $\nu > \nu_1$, system (1.1) has a nonnegative ground state solution.

(ii) If $p \in (2_s^* - 1, 2_{\mu,s}^*)$, then, for any $\nu > 0$, system (1.1) has a nonnegative ground state solution.

Since f does not satisfy the Ambrosetti-Rabinowitz condition and $V(x)$ is not a constant, it is difficult to obtain the boundedness of (PS) sequences. In order to overcome this difficulty, we use a subtle approach developed by Jeanjean [24].

Lemma 1.1. *Let X be a Banach space and $\Lambda \subset \mathbb{R}^+$ an interval. Consider a family of $\mathscr{C}^1$ functional φ_λ on X with the form $\varphi_\lambda(u) = A(u) - \lambda B(u)$ for all $\lambda \in \Lambda$, where $B(u) \geq 0$ for all $u \in X$, such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to +\infty$. If there exists $v_1,\ v_2 \in X$ such that*

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi_\lambda(\gamma(t)) > \max\{\varphi_\lambda(v_1),\ \varphi_\lambda(v_2)\}, \quad \forall \lambda \in \Lambda$$

where $\Gamma = \{\gamma \in \mathscr{C}([0,1],X) : \gamma(0) = v_1,\ \gamma(1) = v_2\}$, then, for almost every $\lambda \in \Lambda$, there exists a sequence $\{v_n\} \subset X$ such that

(i) $\{v_n\}$ is bounded;

(ii) $\varphi_\lambda(v_n) \to c_\lambda$;
(iii) $\varphi'_\lambda(v_n) \to 0$ *in the dual* X' *of* X.
Moreover, $\lambda \mapsto c_\lambda$ *is left continuous.*

We point out that, for applying Lemma 1.1 to system (1.1), the corresponding limit problem plays an important role. Therefore, we consider the limit problem

$$\begin{cases} (-\Delta)^s u + V_\infty u + \phi u = (I_\mu * F(u))f(u) + |u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.5}$$

We obtain the following result.

Theorem 1.2. *Assume that* $(V_1)-(V_3)$ *and* $(f_1)-(f_4)$ *hold and* $2(s+t) > 3,\ \mu \in (s+t,3)$, *and* $s \in [\frac{3}{4},1)$.

(i) If $p \in (2, 2_s^*-1]$, *then there exists* $\nu_1 > 0$ *such that for* $\nu > \nu_1$, *system (1.5) has a nonnegative ground state solution.*

(ii) If $p \in (2_s^*-1, 2_{\mu,s}^*)$, *then, for any* $\nu > 0$, *system (1.5) has a nonnegative ground state solution.*

In this paper, we use the following notations:
- $L^p(\mathbb{R}^3), 1 \le p \le +\infty$,denotes a Lebesgue space, and the norm in $L^p(\mathbb{R}^3)$ is denoted by $\|\cdot\|_{L^p(\mathbb{R}^3)}$.
- For any $x \in \mathbb{R}^3$ and $R > 0$, $B_R(x) := \{y \in \mathbb{R}^3 : |y-x| < R\}$;
- C and C_i denote (possible different) any positive constant.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries results. In Section 3, we investigate the limit problem. Section 4 is devoted to the existence of ground state solutions to system (1.1).

2. Preliminaries

First, we introduce some notations and some necessary properties of the local term ϕu for studying Eq. (1.1). Define the homogeneous fractional Sobolev space $\mathscr{D}^{s,2}(\mathbb{R}^3)$ as

$$\mathscr{D}^{s,2}(\mathbb{R}^3) := \left\{ u \in L^{2_s^*}(\mathbb{R}^3) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{3+2s}{2}}} \in L^2\left(\mathbb{R}^3 \times \mathbb{R}^3\right) \right\}$$

which is the completion of $\mathscr{C}_0^\infty(\mathbb{R}^3)$ with the norm

$$\|u\|_{\mathscr{D}^{s,2}(\mathbb{R}^3)} := \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^{\frac{1}{2}}.$$

The fractional Sobolev space $H^s(\mathbb{R}^3)$ is defined by

$$H^s(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{3+2s}{2}}} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \right\}$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{s}{2}} u|^2 + u^2 \right) dx \right)^{\frac{1}{2}}.$$

It follows from [25, Proposition 3.4, Proposition3.6] that

$$\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx=\frac{1}{C(s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}dxdy.$$

It is well known that the embedding $\mathscr{D}^{s,2}(\mathbb{R}^3)\hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ is continuous and the best constant S_s can be defined as

$$S_s=\inf_{u\in\mathscr{D}^{s,2}(\mathbb{R}^3)\setminus\{0\}}\frac{\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx}{(\int_{\mathbb{R}^3}|u(x)|^{2_s^*}dx)^{\frac{2}{2_s^*}}}. \tag{2.1}$$

From $(V_2)-(V_3)$, we consider the work space $H^s(\mathbb{R}^3)$ with the norm

$$\|u\|:=\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx+\int_{\mathbb{R}^3}V(x)u^2dx\right)^{\frac{1}{2}},$$

which is equivalent to $\|\cdot\|_{H^s(\mathbb{R}^3)}$.

For $u\in H^s(\mathbb{R}^3)$, define the linear function as follows $\mathscr{L}_u(v)$ in $\mathscr{D}^{t,2}(\mathbb{R}^3)$ by

$$\mathscr{L}_u(v)=\int_{\mathbb{R}^3}u^2vdx.$$

From Hölder's inequality, Sobolev embedding theorem, and (2.1), we obtain

$$\begin{aligned}|\mathscr{L}_u(v)|&\le\left(\int_{\mathbb{R}^3}|u(x)|^{\frac{12}{3+2t}}\right)^{\frac{3+2t}{6}}\left(\int_{\mathbb{R}^3}|v(x)|^{2_t^*}\right)^{\frac{1}{2_t^*}}\\&\le S_t^{-\frac{1}{2}}\left(\int_{\mathbb{R}^3}|u(x)|^{\frac{12}{3+2t}}\right)^{\frac{3+2t}{6}}\|u\|_{\mathscr{D}^{t,2}(\mathbb{R}^3)}\le CS_t^{-\frac{1}{2}}\|u\|^2\|u\|_{\mathscr{D}^{t,2}(\mathbb{R}^3)}.\end{aligned} \tag{2.2}$$

It follows from the Lax-Milgram theorem that there exists a unique $\phi_u^t\in\mathscr{D}^{t,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3}(-\Delta)^{\frac{t}{2}}\phi_u^t(-\Delta)^{\frac{t}{2}}vdx=\int_{\mathbb{R}^3}u^2vdx,\quad\forall v\in\mathscr{D}^{t,2}(\mathbb{R}^3), \tag{2.3}$$

which implies that ϕ_u^t is a weak solution to $(-\Delta)^t\phi=u^2, x\in\mathbb{R}^3$, and the representation formula holds

$$\phi_u^t(x)=c_t\int_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|^{3-2t}}dy, \tag{2.4}$$

which is called t-Riesz potential, Here, $c_t>0$ is a constant. It is easy to see that $\phi_u^t\ge 0$ for all $x\in\mathbb{R}^3$. Combining (2.3) and (2.2), we obtain

$$\|\phi_u^t\|^2_{\mathscr{D}^{t,2}(\mathbb{R}^3)}=\int_{\mathbb{R}^3}\phi_u^tu^2dx\le CS_t^{-\frac{1}{2}}\|u\|^2\|\phi_u^t\|_{\mathscr{D}^{t,2}(\mathbb{R}^3)}, \tag{2.5}$$

which implies that $\|\phi_u^t\|_{\mathscr{D}^{t,2}(\mathbb{R}^3)}\le CS_t^{-\frac{1}{2}}\|u\|^2$. This together with (2.5) obtains

$$\int_{\mathbb{R}^3}\phi_u^tu^2dx\le CS_t^{-\frac{1}{2}}\|u\|^2\|\phi_u^t\|_{\mathscr{D}^{t,2}(\mathbb{R}^3)}\le C\|u\|^4.$$

Substituting ϕ_u^t into the first equation of (1.1), we obtain the next fractional Schrödinger equation

$$(-\Delta)^su+V(x)u+\phi_u^tu=(I_\mu*F(u))f(u)+|u|^{2_s^*-2}u,\quad x\in\mathbb{R}^3. \tag{2.6}$$

In order to investigate the non-local term $\int_{\mathbb{R}^3}(I_\mu*F(u))F(u)dx$, we recall the following Hardy-Littlewood-Sobolev inequality [26].

Lemma 2.1. *Let $\mu \in (0,3)$, $p,\ q > 1$ and $1 \le r < \alpha < +\infty$ be such that*

$$\frac{1}{p}+\frac{1}{q}=1+\frac{\mu}{3}, \qquad \frac{1}{r}-\frac{1}{\alpha}=\frac{\mu}{3}.$$

(i) For any $f \in L^p(\mathbb{R}^3)$ and $g \in L^q(\mathbb{R}^3)$, one has

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f(x)g(y)}{|x-y|^{3-\mu}}dxdy\right| \le C(\mu,p)\|f\|_{L^p(\mathbb{R}^3)}\|g\|_{L^q(\mathbb{R}^3)}.$$

(ii) For any $f \in L^r(\mathbb{R}^3)$ one has

$$\left\|\frac{1}{|\cdot|^{3-\mu}}*f\right\|_{L^\alpha(\mathbb{R}^3)} \le C(\mu,r)\|f\|_{L^r(\mathbb{R}^3)}.$$

Remark 2.1. From (f_1), Lemma 2.1, Sobolev embedding theorem, and (2.4), we can conclude that

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(I_\mu * F(u))F(u)dx\right| &\le C\|F(u)\|^2_{L^{\frac{6}{3+\mu}}(\mathbb{R}^3)}\\ &\le C\left[\int\left(|u|^{\frac{3+\mu}{3}}+|u|^q\right)^{\frac{6}{3+\mu}}\right]^{\frac{3+\mu}{3}}\\ &\le C\left(\|u\|^{\frac{2(3+\mu)}{3}}_{L^2(\mathbb{R}^3)}+\|u\|^{2q}_{L^{\frac{6q}{3+\mu}}(\mathbb{R}^3)}\right).\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}(I_\mu * F(u))f(u)vdx\right|\\ &\le C(\mu)\left(\int_{\mathbb{R}^3}|F(u)|^{\frac{6}{3+\mu}}dx\right)^{\frac{3+\mu}{6}}\left(\int_{\mathbb{R}^3}|f(u)v|^{\frac{6}{3+\mu}}dx\right)^{\frac{3+\mu}{6}}\\ &\le C\left(\|u\|^{\frac{3+\mu}{3}}_{L^2(\mathbb{R}^3)}+\|u\|^q_{L^{\frac{6q}{3+\mu}}(\mathbb{R}^3)}\right)\left(\|u\|^{\frac{\mu}{3}}_{L^2(\mathbb{R}^3)}\|v\|_{L^2(\mathbb{R}^3)}+\|u\|^{q-1}_{L^{\frac{6q}{3+\mu}}(\mathbb{R}^3)}\|v\|_{L^{\frac{6q}{3+\mu}}(\mathbb{R}^3)}\right),\end{aligned} \tag{2.8}$$

and

$$\int_{\mathbb{R}^3}\phi_v^t v^2dx = c_t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v(x)|^2|v(y)|^2}{|x-y|^{3-2t}}dxdy \le C\|v\|^4_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)}, \quad v \in L^{\frac{12}{3+2t}}(\mathbb{R}^3). \tag{2.9}$$

Hence, the energy functional $\Phi : H^s(\mathbb{R}^3) \to \mathbb{R}$ associated with (2.6)

$$\Phi(u)=\frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u|^2+V(x)u^2)dx+\frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2dx-\frac{1}{2}\int_{\mathbb{R}^3}(I_\mu * F(u))F(u)dx-\frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx$$

is well defined on $H^s(\mathbb{R}^3)$ and $\Phi \in \mathscr{C}^1(H^s(\mathbb{R}^3),\mathbb{R})$. Moreover, for any $v \in H^s(\mathbb{R}^3)$,

$$\langle\Phi'(u),v\rangle = \int_{\mathbb{R}^3}\left[(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v+V(x)uv+\phi_u^t uv-(I_\mu * F(u))f(u)v-|u|^{2_s^*-1}uv\right]dx.$$

It is clear that the critical points of Φ are the solutions to problem (2.6). Obviously, (u,ϕ_u^t) is a solution to system (1.1) if u is a solution to problem (2.6).

Define the Nehari-Pohožaev manifold (see [5]) by

$$\mathscr{M} := \{u \in H^s(\mathbb{R}^3)\setminus\{0\} : G(u)=0\},$$

where

$$G(u) := \frac{4s+2t-3}{2}\|u\|^2_{\mathscr{D}^{s,2}(\mathbb{R}^3)} + \frac{1}{2}\int_{\mathbb{R}^3}[(2s+2t-3)V(x)-(\nabla V(x),x)]u^2dx + \frac{4s+2t-3}{4}\int_{\mathbb{R}^3}\phi_u^t u^2 dx$$
$$+\frac{1}{2}\int_{\mathbb{R}^3}(I_\mu * F(u))[(3+\mu)F(u)-(2s+2t)f(u)u]dx - \frac{(s+t)2_s^*-3}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx.$$

Combining [2, Lemma 2.6], [15, Lemma 5.1], [16, Lemma 2.2], and [24], we can obtain the next Brezis-Lieb type lemma.

Lemma 2.2. *Assume that* $2s+2t>3$. *If* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$ *and* $u_n \to u$ *a.e. in* $\mathbb{R}^3$, *then*

$$\Phi(u_n) = \Phi(u)+\Phi(u_n-u)+o(1), \quad G(u_n) = G(u)+G(u_n-u)+o_n(1), \tag{2.10}$$

$$\Phi'(u_n) = \Phi'(u)+\Phi'(u_n-u)+o_n(1), \tag{2.11}$$

and

$$\langle\Phi'(u_n),u_n\rangle = \langle\Phi'(u),u\rangle + \langle\Phi'(u_n-u),u_n-u\rangle + o_n(1).$$

Now, we recall a fractional version of Lions vanishing Lemma [18].

Lemma 2.3. *Assume that* $\{u_n\}$ *is bounded in* $H^s(\mathbb{R}^3)$ *and*

$$\lim_{n\to+\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)}|u_n|^2dx = 0,$$

for some $R>0$. *Then* $u_n \to 0$ *in* $L^r(\mathbb{R}^3)$ *for all* $r\in(2,2_s^*)$.

The following general mini-max principle [27, proposition 2.8] is crucial in proving the existence of nontrivial solutions, which is a powerful variant of [28, Theorem 2.8].

Lemma 2.4. *Let X be a Banach space. Let* D_0 *be a closed subspace of the metric space D, and let* $\Gamma_0 \subset \mathscr{C}(D_0,X)$. *Define* $\widetilde{\Gamma} := \{\gamma\in\mathscr{C}(D,X) : \gamma|_{D_0}\in\Gamma_0\}$. *If* $\varphi\in\mathscr{C}^1(X,\mathbb{R})$ *verifies*

$$b := \sup_{\gamma_0\in\Gamma_0}\sup_{u\in D_0}\varphi(\gamma_0(u)) < c := \inf_{\gamma\in\widetilde{\Gamma}}\sup_{u\in D}\varphi(\gamma(u)) < \infty,$$

then, for every $\sigma\in(0,(c-b)/2)$, $\delta>0$ *and* $\gamma\in\widetilde{\Gamma}$ *satisfying* $\sup_D \varphi\circ\gamma \le c+\sigma$, *there exists* $u\in X$ *such that*
(a) $c-2\sigma \le \varphi(u) \le c+2\sigma$;
(b) $\operatorname{dist}(u,\gamma(D)) \le 2\delta$;
(c) $\|\varphi'(u)\|_{X'} \le 8\sigma/\delta$.

3. The Limit Problem

At the beginning of this section, we establish some key inequalities.

Lemma 3.1. *Assume that* (f_1) *and* (f_3) *hold. Then, for all* $\theta>0$ *and* $\tau\in\mathbb{R}$,

$$g(\theta,\tau) := \frac{3}{\theta^{\frac{3+\mu}{2}}}F(\theta^{s+t}\tau) + \left(1-\theta^{\frac{3}{2}}\right)[2(s+t)f(\tau)\tau-(3+\mu)F(\tau)] - 3F(\tau) \ge 0.$$

Proof. Clearly, $g(\theta,0)\geq 0$ for $\theta>0$. From (f_3), for $\tau\neq 0$, we obtain that

$$\begin{aligned}&\frac{d}{d\theta}g(\theta,\tau)\\&=\frac{3\theta^{\frac{1}{2}}|\tau|^{\frac{6+\mu}{2(s+t)}}}{2}\left[\frac{2(s+t)f(\theta^{s+t}\tau)\theta^{s+t}\tau-(3+\mu)F(\theta^{s+t}\tau)}{|\theta^{s+t}\tau|^{\frac{6+\mu}{2(s+t)}}}-\frac{2(s+t)f(\tau)\tau-(3+\mu)F(\tau)}{|\tau|^{\frac{6+\mu}{2(s+t)}}}\right]\\&\begin{cases}\geq 0, & \theta\geq 1,\\ <0, & 0<\theta<1,\end{cases}\end{aligned}$$

which implies that $g(\theta,\tau)\geq g(1,\tau)=0$ for all $\theta\in(0,+\infty)$ and $\tau\in(-\infty,0)\cup(0,+\infty)$. □

Lemma 3.2. *Assume that* $(f_1)-(f_3)$ *hold. Then*

$$\frac{F(\theta)}{\theta^{(8s+4t+\mu-6)/2(s+t)}}\text{ is nondecreasing on }(0,+\infty).$$

Proof. On account of (f_2), Lemma 3.1, and $2(s+t)>3$, we obtain that

$$\lim_{\theta\to 0}g(\theta,\tau)=2(s+t)f(\tau)\tau-(6+\mu)F(\tau)\geq 0. \tag{3.1}$$

Note that

$$\begin{aligned}&\frac{d}{d\theta}\left(\frac{F(\theta)}{\theta^{(8s+4t+\mu-6)/2(s+t)}}\right)\\&=\frac{1}{2(s+t)\theta^{(10s+6t+\mu-6)/2(s+t)}}[2(s+t)f(\theta)\theta-(8s+4t+\mu-6)F(\theta)]\geq 0.\end{aligned}$$

This completest the proof. □

Lemma 3.3. *Assume that* $(f_1)-(f_3)$ *hold. Then*

$$\begin{aligned}&h(\theta,u)\\&:=\int_{\mathbb{R}^3}\bigg\{\frac{4s+2t-3}{\theta^{3+\mu}}(I_\mu*F(\theta^{s+t}u))F(\theta^{s+t}u)\\&\quad+(1-\theta^{4s+2t-3})(I_\mu*F(u))[(2s+2t)f(u)u-(3+\mu)F(u)]-(4s+2t-3)(I_\mu*F(u))F(u)\bigg\}dx\\&\geq 0,\quad\forall\theta>0,\ u\in H^s(\mathbb{R}^3).\end{aligned}$$

Proof. It follows from (f_1) and Lemma 3.2 that

$$I_\mu*\left(\frac{F(\theta^{s+t}u)}{|\theta|^{(8s+4t+\mu-6)/2}}\right)-I_\mu*F(u)\begin{cases}\geq 0, & \theta\geq 1,\\ \leq 0, & 0<\theta<1.\end{cases} \tag{3.2}$$

From (f_1), (f_3), and (3.2), we obtain that

$$
\begin{aligned}
&\frac{d}{d\theta}h(\theta,u)\\
&=\int_{\mathbb{R}^3}\Bigg\{\frac{2(4s+2t-3)(s+t)}{\theta^{3+\mu}}(I_\mu * F(\theta^{s+t}u))f(\theta^{s+t}u)\theta^{s+t-1}u\\
&\quad-\frac{(4s+2t-3)(3+\mu)}{\theta^{\mu+4}}(I_\mu * F(\theta^{s+t}u))F(\theta^{s+t}u)\\
&\quad-(4s+2t-3)\theta^{4s+2t-4}(I_\mu * F(u))[(2s+2t)f(u)u-(3+\mu)F(u)]\Bigg\}dx\\
&=(4s+2t-3)\theta^{4s+2t-4}\int_{\mathbb{R}^3}|u|^{(6+\mu)/2(s+t)}\Bigg\{\left(I_\mu *\left(\frac{F(\theta^{s+t}u)}{|\theta|^{(8s+4t+\mu-6)/2}}\right)\right)\\
&\quad\frac{2(s+t)f(\theta^{s+t}u)\theta^{s+t}u-(3+\mu)F(\theta^{s+t}u)}{|\theta^{s+t}u|^{(6+\mu)/2(s+t)}}-(I_\mu * F(u))\frac{(2s+2t)f(u)u-(3+\mu)F(u)}{|u|^{(6+\mu)/2(s+t)}}\Bigg\}dx\\
&\begin{cases}\geq 0, & \theta\geq 1,\\ \leq 0, & 0<\theta<1,\end{cases}
\end{aligned}
$$

which yields $h(\theta,u)\geq h(1,u)=0$ for all $\theta>0$ and $u\in H^s(\mathbb{R}^3)$. □

Noting that $\lim_{|x|\to\infty}V(x)=V_\infty$, we consider the limit problem

$$
\begin{cases}(-\Delta)^s u+V_\infty u+\phi_u^t u=(I_\mu * F(u))f(u)+|u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3,\\ u\in H^s(\mathbb{R}^3),\ u>0, & \text{in } \mathbb{R}^3,\end{cases} \tag{3.3}
$$

whose energy functional is defined by

$$
\begin{aligned}
&\Phi_\infty(u)\\
&=\frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u|^2+V_\infty u^2)dx+\frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2dx-\frac{1}{2}\int_{\mathbb{R}^3}(I_\mu * F(u))F(u)dx-\frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx.
\end{aligned} \tag{3.4}
$$

Set $u_\theta=\theta^{s+t}u(\theta x)$. By direct calculation, we deduce that

$$
\begin{aligned}
\gamma(\theta)&=\Phi_\infty(u_\theta)\\
&=\frac{\theta^{4s+2t-3}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx+\frac{\theta^{2s+2t-3}}{2}\int_{\mathbb{R}^3}V_\infty u^2dx+\frac{\theta^{4s+2t-3}}{4}\int_{\mathbb{R}^3}\phi_u^t u^2dx\\
&\quad-\frac{1}{2\theta^{3+\mu}}\int_{\mathbb{R}^3}(I_\mu * F(\theta^{s+t}u))F(\theta^{s+t}u)dx-\frac{\theta^{(s+t)2_s^*}-3}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx
\end{aligned} \tag{3.5}
$$

which implies that $\Phi_\infty(u_\theta)\to-\infty$ as $\theta\to+\infty$. We obtain the following lemma.

Lemma 3.4. Φ_∞ *is not bounded from below.*

Define $\mathscr{M}_\infty = \{u \in H^s(\mathbb{R}^3)\backslash\{0\} : G_\infty(u) = 0\}$, where

$$\begin{aligned}
&G_\infty(u)\\
&= \frac{4s+2t-3}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx + \frac{2s+2t-3}{2}\int_{\mathbb{R}^3} V_\infty u^2 dx + \frac{4s+2t-3}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx\\
&\quad + \frac{3+\mu}{2}\int_{\mathbb{R}^N} (I_\mu * F(u))F(u)dx - (s+t)\int_{\mathbb{R}^3} (I_\mu * F(u))f(u)udx - \frac{4s+2t-3}{2}\int_{\mathbb{R}^3} |u|^{2_s^*}dx\\
&= \frac{d\Phi_\infty(u_\theta)}{d\theta}|_{\theta=1}.
\end{aligned} \tag{3.6}$$

Set

$$\xi(\theta) := \frac{1-\theta^{2s+2t-3}}{2} - \frac{(2s+2t-3)(1-\theta^{4s+2t-3})}{2(4s+2t-3)}$$

and

$$\zeta(\theta) := \frac{1-\theta^{4s+2t-3}}{2} - \frac{1-\theta^{(s+t)2_s^*-3}}{2_s^*}, \quad \forall\theta > 0.$$

It is easy to check that

$$\xi(\theta) > 0, \quad \zeta(\theta) > 0, \quad \theta \in (0,1)\cup(1,+\infty). \tag{3.7}$$

Lemma 3.5. *For any $u \in H^s(\mathbb{R}^3)$ and $\theta > 0$, the following inequality holds*

$$\Phi_\infty(u) \geq \Phi_\infty(u_\theta) + \frac{1-\theta^{4s+2t-3}}{4s+2t-3}G_\infty(u) + \xi(\theta)\int_{\mathbb{R}^3} V_\infty u^2 dx + \zeta(\theta)\int_{\mathbb{R}^3} |u|^{2_s^*}dx.$$

Proof. From (3.4)-(3.7) and Lemma 3.3, we obtain that

$$\begin{aligned}
&\Phi_\infty(u) - \Phi_\infty(u_\theta) - \frac{1-\theta^{4s+2t-3}}{4s+2t-3}G_\infty(u)\\
&= \xi(\theta)\int_{\mathbb{R}^3} V_\infty u^2 dx + \frac{1}{2(4s+2t-3)}\int_{\mathbb{R}^3}\Bigg\{\frac{4s+2t-3}{\theta^{3+\mu}}(I_\mu * F(\theta^{s+t}u))F(\theta^{s+t}u)\\
&\quad + (1-\theta^{4s+2t-3})(I_\mu * F(u))[2(s+t)f(u)u - (3+\mu)F(u)] - (4s+2t-3)(I_\mu * F(u))F(u)\Bigg\}dx\\
&\quad + \zeta(\theta)\int_{\mathbb{R}^3} |u|^{2_s^*}dx\\
&\geq 0,
\end{aligned}$$

which yields the conclusion. $\square$

Lemma 3.6. *Assume that $2s+2t > 3$. For any $u \in H^s(\mathbb{R}^3)\backslash\{0\}$, there exists a unique $\theta_0 > 0$ such that $u_{\theta_0} \in \mathscr{M}_\infty$. Moreover, $\Phi_\infty(u_{\theta_0}) = \max_{\theta\geq 0}\Phi_\infty(u_\theta)$.*

Proof. Letting $u \in H^s(\mathbb{R}^3)\backslash\{0\}$ be fixed, we observe that

$$\gamma'(\theta) = 0 \Leftrightarrow \theta\gamma'(\theta) = 0 \Leftrightarrow u_\theta \in \mathscr{M}_\infty, \quad \text{for } \theta > 0.$$

By $(V_2),(f_2)$, (2.7), and $q > 1+\frac{\mu}{3} > 1+\frac{\mu}{2(s+t)}$, we obtain $\lim_{\theta\to 0^+}\gamma(\theta) = 0$, $\gamma(\theta) > 0$ for $\theta > 0$ small and $\gamma(\theta) < 0$ for θ large. Hence, $\max_{\theta>0}\gamma(\theta)$ is achieved at $\theta = \theta_0(u) > 0$ such that $\gamma'(\theta_0) = 0$ and $u_{\theta_0} \in \mathscr{M}_\infty$.

Next, we prove that θ_0 is unique for any $u \in H^s(\mathbb{R}^3)\backslash\{0\}$. Suppose on the contrary that there exist $\theta_1, \theta_2 > 0$ such that $\gamma'(\theta_1) = \gamma'(\theta_2) = 0$. It follows from $G_\infty(u_{\theta_1}) = G_\infty(u_{\theta_2}) = 0$ and Lemma 3.5 that

$$\begin{aligned}
&\Phi_\infty(\theta_1^{s+t}u(\theta_1 x)) \\
&\geq \Phi_\infty(\theta_2^{s+t}u(\theta_2 x)) + \frac{\theta_1^{4s+2t-3} - \theta_2^{4s+2t-3}}{(4s+2t-3)\theta_1^{4s+2t-3}} G_\infty(\theta_1^{s+t}u(\theta_1 x)) \\
&\quad + \xi(\theta_2/\theta_1)\theta_1^{2s+2t-3}\int_{\mathbb{R}^3} V_\infty u^2 dx + \zeta(\theta_2/\theta_1)\theta_1^{\frac{(4s+2t-3)3}{3-2s}}\int_{\mathbb{R}^3}|u|^{2_s^*}dx \\
&\geq \Phi_\infty(\theta_2^{s+t}u(\theta_2 x)) + \xi(\theta_2/\theta_1)\theta_1^{2s+2t-3}\int_{\mathbb{R}^3} V_\infty u^2 dx + \zeta(\theta_2/\theta_1)\theta_1^{\frac{(4s+2t-3)3}{3-2s}}\int_{\mathbb{R}^3}|u|^{2_s^*}dx
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\Phi_\infty(\theta_2^{s+t}u(\theta_2 x)) \\
&\geq \Phi_\infty(\theta_1^{s+t}u(\theta_1 x)) + \frac{\theta_2^{4s+2t-3} - \theta_1^{4s+2t-3}}{(4s+2t-3)\theta_2^{4s+2t-3}} G_\infty(\theta_2^{s+t}u(\theta_2 x)) \\
&\quad + \xi(\theta_1/\theta_2)\theta_2^{2s+2t-3}\int_{\mathbb{R}^3} V_\infty u^2 dx + \zeta(\theta_1/\theta_2)\theta_2^{\frac{(4s+2t-3)3}{3-2s}}\int_{\mathbb{R}^3}|u|^{2_s^*}dx \\
&\geq \Phi_\infty(\theta_1^{s+t}u(\theta_1 x)) + \xi(\theta_1/\theta_2)\theta_2^{2s+2t-3}\int_{\mathbb{R}^3} V_\infty u^2 dx + \zeta(\theta_1/\theta_2)\theta_2^{\frac{(4s+2t-3)3}{3-2s}}\int_{\mathbb{R}^3}|u|^{2_s^*}dx.
\end{aligned} \tag{3.9}$$

Therefore, from (3.8) and (3.9), we obtain $\theta_1 = \theta_2$. That is, θ_0 is unique for any $u \in H^s(\mathbb{R}^3)$. □

Lemma 3.7. *The manifold $\mathscr{M}_\infty$ satisfies the following properties:*
(1) there exists $\rho > 0$ such that $\|u\| \geq \rho$, $\forall u \in \mathscr{M}_\infty$;
(2) $m_\infty = \inf_{u\in\mathscr{M}_\infty}\Phi_\infty(u) > 0$.

Proof. (1) From (2.7)-(2.8) and the Sobolev embedding theorem, we obtain that

$$\begin{aligned}
&\frac{2s+2t-3}{2}\|u\|^2 \\
&\leq \frac{4s+2t-3}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx + \frac{2s+2t-3}{2}\int_{\mathbb{R}^3}V_\infty u^2dx + \frac{4s+2t-3}{4}\int_{\mathbb{R}^3}\phi_u^t u^2 dx \\
&= \frac{1}{2}\int_{\mathbb{R}^3}(I_\mu * F(u))[2(s+t)f(u)u - (3+\mu)F(u)]dx + \frac{4s+2t-3}{2}\int_{\mathbb{R}^3}|u|^{2_s^*}dx \\
&\leq C(\|u\|^{2+\frac{2\mu}{3}} + \|u\|^{2q} + \|u\|^{2_s^*}),
\end{aligned}$$

which implies

$$\|u\| \geq \rho, \quad \forall u \in \mathscr{M}_\infty. \tag{3.10}$$

(2) Let $\{u_n\} \subset \mathscr{M}_\infty$ be such that $\Phi_\infty(u_n) \to m_\infty$. There exist two possible scenarios:
(i) $\inf_{n\in\mathbb{N}}\|u_n\|_{L^2(\mathbb{R}^3)} > 0$; or
(ii) $\inf_{n\in\mathbb{N}}\|u_n\|_{L^2(\mathbb{R}^3)} = 0$.

Case (i) $\inf_{n\in\mathbb{N}}\|u_n\|_{L^2(\mathbb{R}^3)}=\rho_1>0$. It follows from (3.4), (3.6), and (3.1) that

$$\begin{aligned} m_\infty=\Phi_\infty(u_n)&=\Phi_\infty(u_n)-\frac{1}{4s+2t-3}G_\infty(u_n)\\ &=\frac{3-2t}{4s+2t-3}\int_{\mathbb{R}^3}V_\infty u_n^2dx\\ &\quad+\frac{1}{2(4s+2t-3)}\int_{\mathbb{R}^3}(I_\mu*F(u_n))[2(s+t)f(u_n)u_n-(4s+2t+\mu)F(u_n)]dx\\ &\quad+\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\\ &\geq\frac{(3-2t)V_\infty}{4s+2t-3}\rho_1^2. \end{aligned}\tag{3.11}$$

Case (ii) $\inf_{n\in\mathbb{N}}\|u_n\|_{L^2(\mathbb{R}^3)}=0$. According to (3.10), passing to a sub-sequence, we obtain

$$\|u_n\|_{L^2(\mathbb{R}^3)}\to 0,\quad \|u_n\|_{\mathscr{D}^{s,2}(\mathbb{R}^3)}\geq\frac{\rho}{2}.\tag{3.12}$$

It follows from $(f_1)-(f_2)$ that, for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|F(\tau)|\leq C_\varepsilon|\tau|^{1+\frac{\mu}{3}}+\varepsilon|\tau|^{2^*_{\mu,s}},\quad \forall\tau\in\mathbb{R}.\tag{3.13}$$

From (2.1), (2.7), and (3.13), we deduce that

$$\int_{\mathbb{R}^3}(I_\mu*F(u))F(u)\leq CC_\varepsilon\|u\|_{L^2(\mathbb{R}^3)}^{2+\frac{2\mu}{3}}+C\varepsilon S_s^{-\frac{2_s^*}{2}}\|u\|_{\mathscr{D}^{s,2}(\mathbb{R}^3)}^{\frac{2_s^*(3+\mu)}{3}}.\tag{3.14}$$

Let

$$\theta_n=\left[(2_s^*)^{\frac{3-2s}{2s}}S_s^{\frac{3}{2s}}\|u_n\|^{-2}_{\mathscr{D}^{s,2}(\mathbb{R}^3)}\right]^{\frac{1}{4s+2t-3}}.$$

Then, due to (3.12), $\{\theta_n\}$ is bounded. Applying Lemma 3.5, (3.5), (3.12), and (3.14), we deduce that

$$\begin{aligned} m_\infty+o_n(1)&=\Phi_\infty(u_n)\geq\Phi_\infty((u_n)_{\theta_n})\\ &=\frac{\theta_n^{4s+2t-3}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2dx+\frac{\theta_n^{2s+2t-3}}{2}\int_{\mathbb{R}^3}V_\infty u_n^2dx+\frac{\theta_n^{4s+2t-3}}{4}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2dx\\ &\quad-\frac{1}{2\theta_n^{3+\mu}}\int_{\mathbb{R}^3}(I_\mu*F(\theta_n^{s+t}u_n))F(\theta_n^{s+t}u_n)dx-\frac{\theta_n^{\frac{3(4s+2t-3)}{3-2s}}}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\\ &\geq\frac{\theta_n^{4s+2t-3}}{2}\|u_n\|^2_{\mathscr{D}^{s,2}(\mathbb{R}^3)}-CC_\varepsilon\left(\theta_n^{2s+2t-3}\|u_n\|^2_{L^2(\mathbb{R}^3)}\right)^{\frac{3+\mu}{3}}\\ &\quad-C\varepsilon S_s^{-\frac{2_s^*}{2}}\left(\theta_n^{4s+2t-3}\|u_n\|^2_{\mathscr{D}^{s,2}(\mathbb{R}^3)}\right)^{\frac{3+\mu}{3-2s}}-\frac{S_s^{-\frac{2_s^*}{2}}}{2_s^*}\left(\theta_n^{4s+2t-3}\|u_n\|^2_{\mathscr{D}^{s,2}(\mathbb{R}^3)}\right)^{\frac{3}{3-2s}}\\ &=\frac{\theta_n^{4s+2t-3}\|u_n\|^2_{\mathscr{D}^{s,2}(\mathbb{R}^3)}}{4}\left[2-\frac{S_s^{-\frac{2_s^*}{2}}}{2_s^*}\left(\theta_n^{4s+2t-3}\|u_n\|^2_{\mathscr{D}^{s,2}(\mathbb{R}^3)}\right)^{\frac{2s}{3-2s}}\right]+o_n(1)\\ &=\frac{1}{4}(2_s^*)^{\frac{3-2s}{2s}}S_s^{\frac{3}{2s}}+o_n(1). \end{aligned}$$

It follows from Case (i) and Case (ii) that $m_\infty=\inf_{u\in\mathscr{M}_\infty}\Phi_\infty(u)>0$. □

By Lemmas 3.6 and 3.7, we easily derive the following results.

Lemma 3.8. *The following equality holds* $m_\infty = \bar{c}_\infty := \inf_{u\neq 0} \max_{\theta>0} \Phi_\infty(u_\theta)$.

Lemma 3.9. *Every critical point of* $\Phi_\infty|_{\mathscr{M}_\infty}$ *is a critical point of* Φ_∞.

Proof. The proof is similar to that of [2, Lemma 3.2]. Thus the proof is omitted here. □

Inspired by Jeanjean [24], we use Lemma 2.4 to obtain a Cerami sequence for the functional Φ_∞ with $G_\infty(u_n) \to 0$, where Φ_∞, $G_\infty(u_n)$ are given in (3.4) and (3.6), respectively.

Lemma 3.10. *There exists a sequence* $\{u_n\} \subset H^s(\mathbb{R}^3)$ *such that*

$$\Phi_\infty(u_n) \to c_\infty > 0, \quad \|\Phi_\infty'\|(1+\|u_n\|) \to 0 \text{ and } G_\infty(u_n) \to 0, \tag{3.15}$$

where

$$c_\infty := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\infty(\gamma(t)), \quad \Gamma := \left\{\gamma \in \mathscr{C}([0,1], H^s(\mathbb{R}^3)) : \gamma(0) = 0, \Phi_\infty(\gamma(1)) < 0\right\}.$$

Proof. For $u \in H^s(\mathbb{R}^3)\backslash\{0\}$, By (f_2), one has $\Phi_\infty(\tau u) \to -\infty$ as $\tau \to \infty$. By the standard arguments, one sees that $\Gamma \neq \emptyset$ and $c_\infty < \infty$. Furthermore, it is easy to check that there exit $\rho_0, \alpha_0 > 0$ such that $\Phi_\infty(u) \geq 0$ for all u with $\|u\| \leq \rho_0$ and $\Phi_\infty(u) \geq \alpha_0$ for all u with $\|u\| = \rho_0$. Together with the definition of Γ, we obtain $\|\gamma(1)\| > \rho_0$. From the continuity of $\gamma(t\tau)$ and the intermediate value theorem, there exists $\tau_\gamma \in (0,1)$ such that $\|\gamma(\tau_\gamma)\| = \rho_0$. Hence, we obtain

$$\sup_{\tau\in[0,1]} \Phi_\infty(\gamma(\tau)) \geq \Phi_\infty(\gamma(\tau_\gamma)) \geq \alpha_0 > 0,$$

which implies

$$\infty > c_\infty = \inf_{\gamma\in\Gamma} \sup_{\tau\in[0,1]} \Phi_\infty(\gamma(\tau)) \geq \alpha_0 > 0.$$

Define the continuous map

$$\eta : \mathbb{R}\times H^s(\mathbb{R}^3) \to H^s(\mathbb{R}^3), \quad \eta(\tau,v)(x) = e^{(s+t)\tau} v(e^\tau x), \quad \text{for } \tau\in\mathbb{R},\ v\in H^s(\mathbb{R}^3) \text{ and } x\in\mathbb{R}^3,$$

where $\mathbb{R}\times H^s(\mathbb{R}^3)$ is the Banach space with the product norm $\|(\tau,v)\| := (|\tau|^2+\|v\|^2)^{\frac{1}{2}}$. We define the following auxiliary functional:

$$\begin{aligned}
\widetilde{\Phi}_\infty(\tau,v) &= \Phi_\infty(\eta(\tau,v)) \\
&= \frac{1}{2}\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}}\eta(\tau,v)|^2 + V_\infty|\eta(\tau,v)|^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{\eta(\tau,v)}|\eta(\tau,v)|^2 dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^3} (I_\mu * F(\eta(\tau,v)))F(\eta(\tau,v))dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |\eta(\tau,v)|^{2_s^*}dx \\
&= \frac{e^{(4s+2t-3)\tau}}{2}\|v\|^2_{\mathscr{D}^{s,2}(\mathbb{R}^3)} + \frac{e^{(2s+2t-3)\tau}}{2}V_\infty\|v\|^2_{L^2(\mathbb{R}^3)} + \frac{e^{(4s+2t-3)\tau}}{2}\int_{\mathbb{R}^3}\phi_v^t v^2 dx \\
&\quad - \frac{e^{-(3+\mu)\tau}}{2}\int_{\mathbb{R}^3}(I_\mu * F(e^{(s+t)\tau}v))F(e^{(s+t)\tau}v)dx - \frac{e^{[2_s^*(s+t)-3]\tau}}{2_s^*}\|v\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^3)}.
\end{aligned}$$

Moreover, by direct calculations, we obtain $\widetilde{\Phi}_\infty \in \mathscr{C}^1(\mathbb{R}\times H^s(\mathbb{R}^3), \mathbb{R})$, and

$$\partial_\tau \widetilde{\Phi}_\infty(\tau,v) = G_\infty(\eta(\tau,v)), \quad \partial_v \widetilde{\Phi}_\infty(\tau,v)w = \Phi_\infty'(\eta(\tau,v))\eta(\tau,w)$$

for all $\tau \in \mathbb{R}$ and $v, w \in H^s(\mathbb{R}^3)$. We define the mini-max value $\tilde{c}_\infty$ for $\widetilde{\Phi}$ by

$$\tilde{c}_\infty = \inf_{\tilde{\gamma}\in\tilde{\Gamma}} \max_{\tau\in[0,1]} \widetilde{\Phi}_\infty(\tilde{\gamma}(\tau)),$$

where $\widetilde{\Gamma} = \{\tilde{\gamma} \in \mathscr{C}([0,1], \mathbb{R}\times H^s(\mathbb{R}^3)) : \tilde{\gamma}(0) = (0,0), \widetilde{\Phi}_\infty(\tilde{\gamma}(1)) < 0\}$. Since $\Gamma = \{\eta\circ\tilde{\gamma} : \tilde{\gamma} \in \widetilde{\Gamma}\}$, we deduce that $c_\infty = \tilde{c}_\infty$. By the definition of c_∞, there exists $\gamma_n \in \Gamma$ such that, for any $n \in \mathbb{N}$,

$$\max_{t\in[0,1]} \widetilde{\Phi}_\infty(0, \gamma_n(t)) = \max_{t\in[0,1]} \Phi_\infty(\gamma_n(t)) \le c_\infty + \frac{1}{n^2}.$$

Applying Lemma 2.4 to $\widetilde{\Phi}_\infty$, and setting $D = [0,1]$, $D_0 = \{0,1\}$, $B = \mathbb{R}\times H^s(\mathbb{R}^3)$, $\sigma = \frac{1}{n^2}$, $\delta = \frac{1}{n}$ and $\tilde{\gamma}_n(\tau) = (0, \gamma_n(\tau))$, we conclude that there exist $(\tau_n, v_n) \in \mathbb{R}\times H^s(\mathbb{R}^3)$ such that

$$\widetilde{\Phi}_\infty(\tau_n, v_n) \to c_\infty,$$

$$\|\widetilde{\Phi}'_\infty(\tau_n, v_n)\|(1 + \|(\tau_n, v_n)\|) \to 0, \tag{3.16}$$

$$\mathrm{dist}((\tau_n, v_n), \{0\}\times\gamma_n([0,1])) \to 0, \tag{3.17}$$

as $n \to \infty$. Thus, (3.17) implies $\tau_n \to 0$. It is easy to see that, for all $(\tau, w) \in \mathbb{R}\times H^s(\mathbb{R}^3)$,

$$\langle \widetilde{\Phi}'_\infty(\tau_n, v_n), (\iota, w)\rangle = \langle \Phi'_\infty(\eta(\tau_n, v_n)), \eta(\tau_n, w)\rangle + G_\infty(\eta(\tau_n, v_n))\iota. \tag{3.18}$$

Set $u_n = \eta(\tau_n, v_n)$. If we take $\iota = 1$ and $w = 0$ in (3.18), we obtain $G_\infty(u_n) \to 0$ as $n \to \infty$. For each $v \in H^s(\mathbb{R}^3)$, let $\iota = 0$ and $w_n = e^{-(s+t)\tau_n} v(e^{-\tau_n}x)$ in (3.18). We deduce from (3.16)-(3.17) that

$$\left|\langle \Phi'_\infty(u_n), v\rangle\right|(1 + \|u_n\|) = \left|\langle \Phi'_\infty(\eta(\tau_n, v_n)), \eta(\tau_n, w_n)\rangle\right|(1 + \|u_n\|) = o(1)\|w_n\|$$

as $n \to \infty$. Therefore, (3.15) holds. □

Lemma 3.11. *$c_\infty < \frac{s}{3} S_s^{\frac{3}{2s}}$, where S_s is given in (2.1).*

Proof. Let $\chi(x) \in \mathscr{C}_0^\infty(\mathbb{R}^3)$ be a cut-off function such that $0 \le \chi(x) \le 1$ in $\mathbb{R}^3$, $\chi \equiv 1$ in $B_1(0)$, and $\chi \equiv 0$ in $\mathbb{R}^3 \setminus B_2(0)$. It is known that S_s is achieved by

$$\tilde{u} := \kappa(\sigma^2 + |x - x_0|^2)^{-\frac{3-2s}{2}}$$

for any $\kappa \in \mathbb{R}$, $\sigma > 0$, and $x_0 \in \mathbb{R}^3$. Then, taking $x_0 = 0$, we define

$$u_\varepsilon(x) := \chi(x) U_\varepsilon(x), \quad x \in \mathbb{R}^3,$$

where

$$U_\varepsilon(x) = \varepsilon^{-\frac{(3-2s)}{2}} u^*(x/\varepsilon), \quad u^*(x) = \frac{\tilde{u}\left(x/S_s^{1/(2s)}\right)}{\|\tilde{u}\|_{L^{2_s^*}(\mathbb{R}^3)}}.$$

As in [29, 30], we obtain that

$$\mathscr{A}_\varepsilon := \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx \le S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) \tag{3.19}$$

and

$$\mathscr{B}_\varepsilon := \int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*} dx = S_s^{\frac{3}{2s}} + O(\varepsilon^3). \tag{3.20}$$

By a simple calculation, we observe

$$\mathscr{D}_\varepsilon := \int_{\mathbb{R}^3} |u_\varepsilon|^r dx = \begin{cases} O(\varepsilon^{\frac{3(2-r)+2sr}{2}}), & r > \frac{3}{3-2s}, \\ O(\varepsilon^{\frac{3(2-r)+2sr}{2}} |\log \varepsilon|), & r = \frac{3}{3-2s}, \\ O(\varepsilon^{\frac{(3-2s)r}{2}}), & r < \frac{3}{3-2s}. \end{cases} \tag{3.21}$$

Since $\sup_{\tau \geq 0} \Phi_\infty(\tau u_\varepsilon) = \Phi_\infty(\tau_\varepsilon u_\varepsilon) \geq \alpha_0 > 0$, we see that there exists $T_1 > 0$ such that $\tau_\varepsilon > T_1$. Moreover, we infer from $\Phi_\infty(\tau u_\varepsilon) \to -\infty$ as $\tau \to \infty$ that there exists $T_2 > 0$ such that $\tau_\varepsilon < T_2$. Then $T_1 < \tau_\varepsilon < T_2$. Note that

$$\begin{aligned} \Phi_\infty(\tau_\varepsilon u_\varepsilon) \leq & \frac{\tau_\varepsilon^2}{2} \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 + V_\infty |u_\varepsilon|^2) dx + \frac{\tau_\varepsilon^4}{4} \int_{\mathbb{R}^3} \phi_{u_\varepsilon}^t |u_\varepsilon|^2 dx \\ & - \frac{\tau_\varepsilon^{2p} \nu^2}{2p^2} \int_{\mathbb{R}^3} (I_\mu * |u_\varepsilon|^p) |u_\varepsilon|^p dx - \frac{\tau_\varepsilon^{2_s^*}}{2_s^*} \int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*} dx \\ = & \frac{\tau_\varepsilon^2}{2} \mathscr{A}_\varepsilon + \frac{\tau_\varepsilon^2}{2} V_\infty \int_{\mathbb{R}^3} |u_\varepsilon|^2 dx + \frac{\tau_\varepsilon^4}{4} \left(\int_{\mathbb{R}^3} |u_\varepsilon|^{\frac{12}{3+2t}} dx \right)^{\frac{3+2t}{3}} \\ & - \frac{\tau_\varepsilon^{2p}}{2p^2} \nu^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_\varepsilon(x)|^p |u_\varepsilon(y)|^p}{|x-y|^{3-\mu}} dx dy - \frac{\tau_\varepsilon^{2_s^*}}{2_s^*} \mathscr{B}_\varepsilon. \end{aligned} \tag{3.22}$$

Define

$$J_\varepsilon(\tau) := \frac{\tau^2}{2} \mathscr{A}_\varepsilon - \frac{\tau^{2_s^*}}{2_s^*} \mathscr{B}_\varepsilon.$$

According to (3.19)-(3.20), it is easy to verify that

$$\sup_{\tau \geq 0} J_\varepsilon(\tau) \leq \frac{s}{3} S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}). \tag{3.23}$$

By (3.22), (3.23), (3.24), and (3.25), we discuss two different situations.

Case 1. $s > \frac{3}{4}$, which yields $\frac{3}{3-2s} > 2$.

$$\Phi_\infty(\tau_\varepsilon u_\varepsilon) \leq \frac{s}{3} S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) + C \left(\int_{\mathbb{R}^3} |u_\varepsilon|^{\frac{12}{3+2t}} dx \right)^{\frac{3+2t}{3}} - C\nu^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_\varepsilon(x)|^p |u_\varepsilon(y)|^p}{|x-y|^{3-\mu}} dx dy.$$

Case 2. $s = \frac{3}{4}$, which yields $\frac{3}{3-2s} = 2$.

$$\Phi_\infty(\tau_\varepsilon u_\varepsilon) \leq \frac{s}{3} S_s^{\frac{3}{2s}} + O(\varepsilon^{2s} |\log \varepsilon|) + C \left(\int_{\mathbb{R}^3} |u_\varepsilon|^{\frac{12}{3+2t}} dx \right)^{\frac{3+2t}{3}} - C\nu^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_\varepsilon(x)|^p |u_\varepsilon(y)|^p}{|x-y|^{3-\mu}} dx dy.$$

Case 1. $s > \frac{3}{4}$. Furthermore, (2.9) and (3.21) yield

$$\lim_{\varepsilon \to 0^+} \frac{\left(\int_{\mathbb{R}^3} |u_\varepsilon|^{\frac{12}{3+2t}} dx \right)^{\frac{3+2t}{3}}}{\varepsilon^{3-2s}} = \begin{cases} \lim_{\varepsilon \to 0^+} \frac{O(\varepsilon^{2t+4s-3})}{\varepsilon^{3-2s}} = 0, & \frac{12}{3+2t} > \frac{3}{3-2s}, \\ \lim_{\varepsilon \to 0^+} \frac{O(\varepsilon^{2t+4s-3} |\log \varepsilon|^{\frac{3+2t}{3}})}{\varepsilon^{3-2s}} = 0, & \frac{12}{3+2t} = \frac{3}{3-2s}, \\ \lim_{\varepsilon \to 0^+} \frac{O(\varepsilon^{2(3-2s)})}{\varepsilon^{3-2s}} = 0, & \frac{12}{3+2t} < \frac{3}{3-2s}. \end{cases} \tag{3.24}$$

By virtue of (2.7) and (3.21), for any $p \in (2, 2_s^*)$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_\varepsilon(x)|^p |u_\varepsilon(y)|^p}{|x-y|^{3-\mu}} dxdy \\
&\geq C \int_{B_1(\frac{x_0}{\varepsilon})}\int_{B_1(\frac{x_0}{\varepsilon})} |U_\varepsilon(x)|^p |U_\varepsilon(y)|^p dxdy \\
&= C\left(\varepsilon^{3-\frac{(3-2s)p}{2}} \int_0^{\frac{1}{\varepsilon S_s^{1/(2s)}}} \frac{r^2}{(\sigma^2+r^2)^{\frac{(3-2s)p}{2}}} dr\right)^2 \\
&= \begin{cases} O(\varepsilon^{6-(3-2s)p}), & p > \frac{3}{3-2s}, \\ O(\varepsilon^{6-(3-2s)p} |\log \varepsilon|^2), & p = \frac{3}{3-2s}, \\ O(\varepsilon^{(3-2s)p}), & p < \frac{3}{3-2s}. \end{cases}
\end{aligned} \tag{3.25}$$

If $2_{\mu,s}^* > p > 2_s^* - 1$, then $0 < 6-(3-2s)p < 3-2s$, which implies that, for any fixed $\nu^2 > 0$, $c_\infty < \frac{s}{3} S_s^{\frac{3}{2s}}$ for $\varepsilon > 0$ small. If $2 < p \leq 2_s^* - 1$ and $\nu^2 \geq \varepsilon^{(\frac{1}{2}-p)(3-2s)}$, we also obtain $c_\infty < \frac{s}{3} S_s^{\frac{3}{2s}}$.

Case 2. $s = \frac{3}{4}$. According to $\frac{12}{3+2t} > 2 = \frac{3}{3-2s}$ and $2t + 2s > 3$, we obtain that

$$\lim_{\varepsilon \to 0^+} \frac{\left(\int_{\mathbb{R}^3} |u_\varepsilon|^{\frac{12}{3+2t}} dx\right)^{\frac{3+2t}{3}}}{\varepsilon^{2s} |\log \varepsilon|} \leq \lim_{\varepsilon \to 0^+} \frac{O(\varepsilon^{2t+4s-3})}{\varepsilon^{2s} |\log \varepsilon|} = 0.$$

Since $p > 2 = \frac{3}{3-2s}$, arguing as Case 1, we also obtain if $2_{\mu,s}^* > p > 2_s^* - 1$, then, for any fixed $\nu^2 > 0$, $c_\infty < \frac{s}{3} S_s^{\frac{3}{2s}}$ for $\varepsilon > 0$ small. If $2 < p \leq 2_s^* - 1$ and $\nu^2 \geq \varepsilon^{(3-2s)p-6}$, we also obtain $c_\infty < \frac{s}{3} S_s^{\frac{3}{2s}}$. □

Lemma 3.12. *The following equality holds*

$$\inf_{\gamma \in \Gamma} \sup_{\tau \in [0,1]} \Phi_\infty(\gamma(\tau)) = c_\infty = \bar{c}_\infty = \inf_{u \neq 0} \max_{\theta > 0} \Phi_\infty(u_\theta).$$

Proof. From Lemma 3.4, we can see that $\Phi_\infty(u_\theta) < 0$ for $u \in \mathbb{R} \times H^s(\mathbb{R}^3) \setminus \{0\}$ and θ large enough. This implies $c_\infty \leq \bar{c}_\infty$. Then, we show $c_\infty \geq \bar{c}_\infty$. We claim that, for any $\gamma \in \Gamma$, $\gamma([0,1] \cap \mathscr{M}_\infty) \neq \emptyset$. Indeed, from (3.11), we obtain that, for any $\gamma \in \Gamma$,

$$G_\infty(\gamma(1)) \leq (4s+2t-3)\Phi_\infty(\gamma(1)) - (3-2t)V_\infty \rho_1^2 < 0.$$

For any $u \in H^s(\mathbb{R}^3) \setminus \{0\}$, from (3.6), (2.7) -(2.8), and the Sobolev embedding theorem, we deduce that

$$\begin{aligned}
G_\infty(u) &= \frac{4s+2t-3}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{2s+2t-3}{2} \int_{\mathbb{R}^3} V_\infty u^2 dx + \frac{4s+2t-3}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^3} (I_\mu * F(u))[2(s+t)f(u)u - (3+\mu)F(u)] dx - \frac{4s+2t-3}{2} \int_{\mathbb{R}^3} |u|^{2_s^*} dx \\
&\geq \frac{2s+2t-3}{2} \|u\|^2 - C\|u\|^{2+\frac{2\mu}{3}} - C\|u\|^{2q} - C\|u\|^{2_s^*},
\end{aligned}$$

which yields that there exists $\rho_2 \in (0, \|\gamma(1)\|)$ and $\sigma_2 > 0$ such that $G_\infty(u) \geq \sigma_2$ for $\|u\| = \rho_2$. This implies that there exists $\tau_0 \in (0,1)$ such that $G_\infty(\gamma(\tau_0)) \geq \sigma_2$. Therefore, the curve $\gamma \in \Gamma$ must cross $\mathscr{M}_\infty$, which indicates $c_\infty \geq m_\infty$. Together with Lemma 3.8, we obtain $c_\infty \geq \bar{c}_\infty$. □

Proof of Theorem 1.2. By Lemma 3.10, there exists a sequence $\{u_n\} \subset H^s(\mathbb{R}^3)$ satisfying (3.15). It follows from (3.11) and (3.1) that

$$c_\infty = \Phi_\infty(u_n) = \Phi_\infty(u_n) - \frac{1}{4s+2t-3} G_\infty(u_n) \geq \frac{3-2t}{4s+2t-3} \int_{\mathbb{R}^3} V_\infty u_n^2 dx + \left(\frac{1}{2} - \frac{1}{2_s^*}\right) \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx.$$

Combining with the Hölder inequality, we deduce that $\{u_n\}$ is bounded in $L^r(\mathbb{R}^3)$ for $r \in [2, 2_s^*]$. Then, by $G_\infty(u_n) \to 0$, (2.7), and (2.8), we can see that

$$\begin{aligned}
&\frac{4s+2t-3}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}^3} (I_\mu * F(u_n))[2(s+t)f(u_n)u_n - (3+\mu)F(u_n)]dx + \frac{4s+2t-3}{2} \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \\
&\leq C(\|u_n\|_{L^2(\mathbb{R}^3)}^{2+\frac{2\mu_n}{3}} + \|u_n\|_{L^{\frac{6q}{3+\mu}}(\mathbb{R}^3)}^{2q} + \|u_n\|_{L^{2_s^*}(\mathbb{R}^3)}^{2_s^*}) \leq C.
\end{aligned}$$

This implies that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. Now, we claim that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^2 dx > 0.$$

If it does not occur, then it follows from Lemma 2.3 that $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for $r \in (2, 2_s^*)$. Hence, we obtain that

$$o_n(1) = \langle \Phi'_\infty(u_n), u_n \rangle = \|u_n\|_{\mathscr{D}^{s,2}(\mathbb{R}^3)}^2 + V_\infty \|u_n\|_{L^2(\mathbb{R}^3)}^2 - \|u_n\|_{L^{2_s^*}(\mathbb{R}^3)}^{2_s^*} + o_n(1),$$

as $n \to \infty$. Since $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$ and $c_\infty > 0$, we may assume that up to a sub-sequence, as $n \to \infty$, for some $l > 0$,

$$\|u_n\|_{\mathscr{D}^{s,2}(\mathbb{R}^3)}^2 + V_\infty \|u_n\|_{L^2(\mathbb{R}^3)}^2 \to l, \quad \|u_n\|_{L^{2_s^*}(\mathbb{R}^3)}^{2_s^*} \to l. \tag{3.26}$$

In view of (3.26) and (2.1), we obtain that

$$l = \lim_{n\to\infty} \left(\|u_n\|_{\mathscr{D}^{s,2}(\mathbb{R}^3)}^2 + V_\infty \|u_n\|_{L^2(\mathbb{R}^3)}^2 \right) \geq \lim_{n\to\infty} \|u_n\|_{\mathscr{D}^{s,2}(\mathbb{R}^3)}^2 \geq S_s \|u_n\|_{L^{2_s^*}(\mathbb{R}^3)}^2 = S_s l^{\frac{2}{2_s^*}},$$

which implies

$$l \geq S_s^{\frac{3}{2s}}. \tag{3.27}$$

Combining the fact

$$c_\infty + o_n(1) = \Phi_\infty(u_n) = \frac{1}{2} \left(\|u_n\|_{\mathscr{D}^{s,2}(\mathbb{R}^3)}^2 + V_\infty \|u_n\|_{L^2(\mathbb{R}^3)}^2 \right) - \frac{1}{2_s^*} \|u_n\|_{L^{2_s^*}(\mathbb{R}^3)}^{2_s^*} + o_n(1) = \frac{s}{3} l + o_n(1)$$

and (3.27), we observe that $c_\infty \geq \frac{s}{3} S_s^{\frac{3}{2s}}$, which contradicts Lemma 3.11. Hence, there exists $\delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that $\int_{B_1(y_n)} |u_n|^2 dx > \delta$. Let $v_n(x) = u_n(x + y_n)$. Then, as $n \to \infty$,

$$\Phi_\infty(v_n) \to c_\infty, \quad \Phi'_\infty(v_n) \to 0, \quad G_\infty(v_n) \to 0$$

and $\int_{B_1(0)} |v_n|^2 dx > \delta$. Hence, passing to a sub-sequence, there exists $v \in H^s(\mathbb{R}^3)$ such that

$$\begin{cases} v_n \rightharpoonup v & \text{in } H^s(\mathbb{R}^3), \\ v_n \to v & \text{in } L^r_{loc}(\mathbb{R}^3) \text{ for } r \in [1, 2_s^*), \\ v_n \to v & \text{a.e. in } \mathbb{R}^3. \end{cases}$$

By using standard arguments, we obtain that $\Phi_\infty'(v)=0$ and $\Phi_\infty(v)\geq \bar{c}_\infty$. Therefore, v is a nontrivial solution to (3.3). In view of Lemma 3.12, (3.1), and Fatou's Lemma, we obtain that

$$\begin{aligned}
\bar{c}_\infty = c_\infty &= \lim_{n\to\infty}\Phi_\infty(v_n)=\lim_{n\to\infty}\left[\Phi_\infty(v_n)-\frac{1}{4s+2t-3}G_\infty(v_n)\right]\\
&=\lim_{n\to\infty}\left\{\frac{(3-2t)V_\infty\|v_n\|_{L^2(\mathbb{R}^3)}}{4s+2t-3}\right.\\
&\quad+\frac{1}{2(4s+2t-3)}\int_{\mathbb{R}^3}(I_\mu*F(v_n))[2(s+t)f(v_n)v_n-(4s+2t+\mu)F(v_n)]dx\\
&\quad\left.+\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|v_n|^{2_s^*}dx\right\}\\
&\geq\frac{(3-2t)V_\infty}{4s+2t-3}\int_{\mathbb{R}^3}v^2dx\\
&\quad+\frac{1}{2(4s+2t-3)}\int_{\mathbb{R}^3}(I_\mu*F(v))[2(s+t)f(v)v-(4s+2t+\mu)F(v)]dx\\
&\quad+\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|v|^{2_s^*}dx\\
&=\Phi_\infty(v)-\frac{1}{4s+2t-3}G_\infty(v)\geq\bar{c}_\infty,
\end{aligned}$$

which implies that $\Phi_\infty(v)=\bar{c}_\infty=m_\infty$ by recalling Lemma 3.8. □

4. Existence of Ground State Solutions to (1.1)

In this section, our aim is to find ground state solution to (1.1), whose potential is not a constant. In order to use Jeanjean's monotonicity argument, i.e. Lemma 1.1, for $\lambda\in\Lambda=[\frac{1}{2},1]$, we introduce a family of functionals $\Phi_\lambda: H^s(\mathbb{R}^3)\to\mathbb{R}$ defined by

$$\begin{aligned}
&\Phi_\lambda(u)\\
&=\frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u|^2+V(x)u^2)dx+\frac{1}{4}\int_{\mathbb{R}^3}\phi_u^tu^2dx-\frac{\lambda}{2}\left[\int_{\mathbb{R}^3}(I_\mu*F(u))F(u)dx+\frac{2}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx\right].
\end{aligned}$$

Set

$$\begin{aligned}
G_\lambda(u)&=\frac{4s+2t-3}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx+\frac{4s+2t-3}{4}\int_{\mathbb{R}^3}\phi_u^tu^2dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^3}[(2s+2t-3)V(x)-(\nabla V(x),x)]u^2dx\\
&\quad+\frac{\lambda}{2}\left\{\int_{\mathbb{R}^3}(I_\mu*F(u))[(3+\mu)F(u)-2(s+t)f(u)u]dx-(4s+2t-3)\int_{\mathbb{R}^3}|u|^{2_s^*}dx\right\}.
\end{aligned}\tag{4.1}$$

Let us set $\Phi_\lambda(u)=A(u)-\lambda B(u)$, where

$$A(u)=\frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u|^2+V(x)u^2)dx+\frac{1}{4}\int_{\mathbb{R}^3}\phi_u^tu^2dx\to+\infty,$$

as $\|u\| \to +\infty$ and

$$B(u) = \frac{1}{2}\int_{\mathbb{R}^3}(I_\mu * F(u))F(u)dx + \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx \geq 0.$$

We can easily verify all the conditions of Lemma 1.1.

Lemma 4.1. *The functional Φ_λ satisfies the following properties:*
(i) there exists $e > 0$ such that $\Phi_\lambda(e) < 0$ for all $\lambda \in \Lambda$.
(ii) $c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi_\lambda(\gamma(t)) > \max\{\Phi_\lambda(0), \Phi_\lambda(e)\}$ for all $\lambda \in \Lambda$, where

$$\Gamma = \{\gamma \in \mathscr{C}([0,1], H^s(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = e\}.$$

Proof. (i) For $u \in H^s(\mathbb{R}^3)\backslash\{0\}$ fixed and $\lambda \in \Lambda$, we define

$$\begin{aligned}\Phi_\lambda^\infty(u) = &\frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u|^2 + V_\infty u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2 dx \\ &- \lambda\left[\frac{1}{2}\int_{\mathbb{R}^3}(I_\mu * F(u))F(u)dx + \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx\right].\end{aligned} \tag{4.2}$$

Thus, from (V_2), we obtain $\Phi_\lambda(u) \leq \Phi_\lambda^\infty(u)$. Set $u_\theta = \theta^{(s+t)}u(\theta x)$ for all $\theta > 0$. It follows from Lemma 3.4 that $\Phi_\lambda^\infty(u_\theta) \to -\infty$ as $\theta \to +\infty$. Take $e = u_\theta$ for θ large enough, (i) follows immediately.

(ii) From (2.7) and the Sobolev embedding theorem, we obtain that

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u\|^2 - C(\|u\|^{2+\frac{2\mu}{3}} + \|u\|^{2q} + \|u\|^{2_s^*}),$$

which implies that there exist $\alpha > 0$ and $\rho > 0$ such that

$$\Phi_\lambda(u) \geq \alpha > 0, \quad \forall\ \|u\| = \rho, \text{ for any } \lambda \in \Lambda.$$

Thus, for any $\lambda \in \Lambda$, there exists $t_0 \in (0,1)$ such that $\|\gamma(t_0)\| = \rho$ and

$$\max_{t\in[0,1]}\Phi_\lambda(\gamma(t)) \geq \Phi_\lambda(\gamma(t_0)) \geq \alpha > \max\{\Phi_\lambda(0), \Phi_\lambda(e)\},$$

which yields $c_\lambda > 0$. □

Combining Lemma 1.1 and Lemma 4.1, for almost all $\lambda \in \Lambda$, there exists a bounded sequence $\{u_n\} \subset H^s(\mathbb{R}^3)$ such that

$$\Phi_\lambda(u_n) \to c_\lambda, \quad \Phi_\lambda'(u_n) \to 0.$$

To prove that the above sequence $\{u_n\}$ satisfies the (PS) condition, we consider the following limit problem

$$\begin{cases}(-\Delta)^s u + V_\infty u + \phi_u^t u = \lambda(I_\mu * F(u))f(u) + \lambda|u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3, \\ u \in H^s(\mathbb{R}^3),\ u > 0, & \text{in } \mathbb{R}^3.\end{cases} \tag{4.3}$$

By Theorem 1.2, system (4.3) admits a ground state solution $u_\lambda^\infty \in H^s(\mathbb{R}^3)$, i.e. for any $\lambda \in \Lambda$, there exists $u_\lambda^\infty \in \mathscr{M}_\lambda^\infty$ such that

$$(\Phi_\lambda^\infty)'(u_\lambda^\infty) = 0, \quad \Phi_\lambda^\infty(u_\lambda^\infty) = m_\lambda^\infty = \inf_{u\in\in\mathscr{M}_\lambda^\infty}\Phi_\lambda^\infty(u),$$

where $\Phi_\lambda^\infty(u)$ defined in (4.2), $\mathscr{M}_\lambda^\infty = \{u \in H^s(\mathbb{R}^3)\setminus\{0\} : G_\lambda^\infty(u) = 0\}$, and

$$G_\lambda^\infty(u) = \frac{4s+2t-3}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx + \frac{2(s+t)-3}{2}\int_{\mathbb{R}^3}V_\infty u^2dx + \frac{4s+2t-3}{4}\int_{\mathbb{R}^3}\phi_u^t u^2dx$$
$$+\frac{\lambda}{2}\int_{\mathbb{R}^3}(I_\mu * F(u))[(3+\mu)F(u)dx - 2(s+t)f(u)u]dx - \frac{(4s+2t-3)\lambda}{2}\int_{\mathbb{R}^3}|u|^{2_s^*}dx.$$

Lemma 4.2. *For any $\lambda \in \Lambda$ fixed, $c_\lambda < m_\lambda^\infty$.*

Proof. Let u_λ^∞ be the minimizer of m_λ^∞. From Lemma 3.8, Lemma 4.1, and (V_2), choosing θ large enough, we conclude that for any $\lambda \in \Lambda$,

$$c_\lambda \leq \max_{\theta>0}\Phi_\lambda(\theta^{s+t}u_\lambda^\infty(\theta x)) < \max_{\theta>0}\Phi_\lambda^\infty(\theta^{s+t}u_\lambda^\infty(\theta x)) = \Phi_\lambda^\infty(u_\lambda^\infty) = m_\lambda^\infty.$$

This completes the proof. □

Lemma 4.3. *Let $\{u_n\}$ be a bounded $(PS)_{c_\lambda}$ sequence for the functional Φ_λ. Then there exist sub-sequence of $\{u_n\}$, still denoted $\{u_n\}$ and integer $k \in \mathbb{N}\cup\{0\}$, sequence $\{y_n^j\} \subset \mathbb{R}^3$, $w^j \in H^s(\mathbb{R}^3)$ for $1 \leq j \leq k$ such that*

(i) $u_n \rightharpoonup u_\lambda$ with $\Phi_\lambda'(u_\lambda) = 0$;

(ii) $y_n^j \to +\infty$ and $|y_n^i - y_n^j| \to +\infty$ for $i \neq j$;

(iii) $w^i \neq 0$ and $(\Phi_\lambda^\infty)'(w^i) = 0$ for $1 \leq i \leq k$;

(iv) $\|u_n - u_\lambda - \sum_{j=1}^k w^j(\cdot - y_n^k)\| \to 0$;

(v) $\Phi_\lambda(u_n) \to \Phi_\lambda(u_\lambda) + \sum_{j=1}^k \Phi_\lambda^\infty(w^j)$.

In addition, we agree that in the case $k = 0$ the above hold without w^j and y_n^j.

Proof. Since $\{u_n\} \subset H^s(\mathbb{R}^3)$ is a bounded sequence satisfying

$$\Phi_\lambda(u_n) \to c_\lambda > 0, \quad \Phi_\lambda'(u_n) \to 0,$$

one sees that there exists $u_\lambda \in H^s(\mathbb{R}^3)\setminus\{0\}$ satisfying

$$\begin{cases} u_n \rightharpoonup u_\lambda & \text{in } H^s(\mathbb{R}^3), \\ u_n \to u_\lambda & \text{in } L^r_{loc}(\mathbb{R}^3) \text{ for } r \in [1, 2_s^*), \\ u_n \to u_\lambda & \text{a.e. in } \mathbb{R}^3. \end{cases}$$

Moreover, using standard arguments, one can prove that $\Phi_\lambda'(u_\lambda) = 0$, so $G_\lambda(u_\lambda) = 0$. One deduces from (4.1), (V_1), and (3.1) that

$$\begin{aligned} &\Phi_\lambda(u_\lambda) \\ &= \Phi_\lambda(u_\lambda) - \frac{1}{4s+2t-3}G_\lambda(u_\lambda) \\ &= \frac{1}{2(4s+2t-3)}\int_{\mathbb{R}^3}[2sV(x) + (\nabla V(x), x)]u_\lambda^2dx \\ &\quad + \frac{\lambda}{2(4s+2t-3)}\int_{\mathbb{R}^3}(I_\mu * F(u_\lambda))[2(s+t)f(u_\lambda)u_\lambda - (4s+2t+\mu)F(u_\lambda)]dx + \frac{s\lambda}{3}\int_{\mathbb{R}^3}|u_\lambda|^{2_s^*}dx \\ &\geq 0. \end{aligned} \tag{4.4}$$

Setting $v_n^1 = u_n - u_\lambda$, we have $v_n^1 \rightharpoonup 0$. We next show that one of the following conclusions of v_n^1 is true:

Case 1: $v_n^1 \to 0$ in $H^s(\mathbb{R}^3)$, or
Case 2: there exists $R_0 > 0, \delta > 0$ and a sequence $\{y_n^1\} \in \mathbb{R}^3$ such that

$$\liminf_{n\to\infty} \int_{B_{R_0}(y_n^1)} |v_n^1|^2 dx \geq \delta > 0. \tag{4.5}$$

Indeed, assume that **Case 2** does not occur. Therefore, for any $R > 0$, we obtain that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y_n^1)} |v_n^1|^2 dx = 0.$$

Thus, Lemma 2.3 implies that $v_n^1 \to 0$ in $L^r(\mathbb{R}^3),\ r \in (2, 2_s^*)$. It follows from (2.7) and (2.9) that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} (I_\mu * F(v_n^1))F(v_n^1)dx = 0 \tag{4.6}$$

and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} \phi_{v_n^1}^t (v_n^1)^2 dx = 0 \tag{4.7}$$

Moreover, we infer from (2.10), (2.11), and (4.4) that

$$\lim_{n\to\infty} \Phi_\lambda(v_n^1) = \lim_{n\to\infty} \Phi_\lambda(u_n) - \Phi_\lambda(u_\lambda) \leq c_\lambda \tag{4.8}$$

and

$$\lim_{n\to\infty} (\Phi_\lambda)'(v_n^1) = \lim_{n\to\infty} (\Phi_\lambda)'(u_n) - (\Phi_\lambda)'(u_\lambda) = 0. \tag{4.9}$$

It follows from (4.6), (4.7), and (4.9) that

$$0 = \lim_{n\to\infty} \langle(\Phi_\lambda)'(v_n^1), v_n^1\rangle = \lim_{n\to\infty} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n^1|^2 dx + \int_{\mathbb{R}^3} V(x)(v_n^1)^2 dx - \lambda \int_{\mathbb{R}^3} |v_n^1|^{2_s^*} dx \right).$$

Since $\{v_n^1\}$ is bounded in $H^s(\mathbb{R}^3)$, then we may suppose that up to a sub-sequence, as $n \to \infty$,

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n^1|^2 dx + \int_{\mathbb{R}^N} V(x)(v_n^1)^2 dx \to l, \quad \lambda \int_{\mathbb{R}^3} |v_n^1|^{2_s^*} dx \to l \tag{4.10}$$

for some $l \geq 0$. If $l > 0$, in view of (2.1), we obtain that

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n^1|^2 dx + \int_{\mathbb{R}^3} V(x)(v_n^1)^2 dx \geq \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n^1|^2 dx \geq S_s \|v_n^1\|_{L^{2_s^*}(\mathbb{R}^3)}^2.$$

This together with (4.10) gives that $l \geq S_s^{\frac{3}{2s}} \lambda^{-\frac{3-2s}{2s}}$ for all $\lambda \in \Lambda$.

On the other hand, (4.8) implies that

$$\begin{aligned} c_\lambda \geq \lim_{n\to\infty} \Phi_\lambda(v_n^1) &= \lim_{n\to\infty} \left[\frac{1}{2} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n^1|^2 dx + \int_{\mathbb{R}^3} V(x)(v_n^1)^2 dx \right) - \frac{\lambda}{2_s^*} \int_{\mathbb{R}^3} |v_n^1|^{2_s^*} dx \right] \\ &\geq \frac{s}{3} S_s^{\frac{3}{2s}} \lambda^{-\frac{3-2s}{2s}}. \end{aligned} \tag{4.11}$$

By using similar argument as in Lemma 3.11, Lemma 3.8, and Lemma 3.12, we show that

$$m_\lambda^\infty < \frac{s}{3} S_s^{\frac{3}{2s}} \lambda^{-\frac{3-2s}{2s}}.$$

Combining with (4.11) and Lemma 4.2, we obtain

$$\frac{s}{3} S_s^{\frac{3}{2s}} \lambda^{-\frac{3-2s}{2s}} \leq c_\lambda < m_\lambda^\infty < \frac{s}{3} S_s^{\frac{3}{2s}} \lambda^{-\frac{3-2s}{2s}}, \quad \text{for all } \lambda \in \Lambda.$$

which is a contradiction. Thus, $l=0$. It follows from (4.10) that $\|v_n\|\to 0$, that is, $u_n\to u$ in $H^s(\mathbb{R}^3)$ and Lemma 4.3 holds with $k=0$ if **Case 2** does not occur.

In the following, we assume that **Case 2** is true, that is, (4.5) holds. Then, up to a sub-sequence, we obtain

$$|y_n^1|\to+\infty,\quad v_n^1(\cdot+y_n^1)\rightharpoonup w^1\neq 0,\quad (\Phi_\lambda^\infty)'w^1=0.$$

Indeed, let us consider $\widetilde{v_n^1}:=v_n^1(\cdot+y_n^1)$. Note that $\{v_n^1\}$ is bounded. Then together with (4.5), we deduce that $\widetilde{v_n^1}\rightharpoonup w^1\neq 0$. Therefore, it follows from $v_n^1\rightharpoonup 0$ in $H^s(\mathbb{R}^3)$ that $\{y_n^1\}$ is unbounded, up to a sub-sequence, $|y_n^1|\to+\infty$. Now we prove $(\Phi_\lambda^\infty)'(w^1)=0$. It suffices to prove that $\langle(\Phi_\lambda^\infty)'(\widetilde{v_n^1}),\varphi\rangle\to 0$ for any $\varphi\in\mathscr{C}_0^\infty(\mathbb{R}^3)$. According to (4.9), we obtain

$$|\langle\Phi_\lambda'(u_n),\varphi\rangle-\langle\Phi_\lambda'(u_\lambda),\varphi\rangle-\langle\Phi_\lambda'(v_n^1),\varphi\rangle|\le o_n(1)\|\varphi\|,$$

which implies $|\langle\Phi_\lambda'(v_n^1),\varphi\rangle|=o_n(1)\|\varphi\|$. Note that

$$\begin{aligned}
&\langle\Phi_\lambda'(v_n^1),\varphi(\cdot-y_n^1)\rangle\\
&=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(v_n^1(x)-v_n^1(y))(\varphi(x-y_n^1)-\varphi(y-y_n^1))}{|x-y|^{3+2s}}dxdy+\int_{\mathbb{R}^3}V(x)v_n^1(x)\varphi(x-y_n^1)dx\\
&\quad+\int_{\mathbb{R}^3}\phi_{v_n^1}^t v_n^1(x)\varphi(x-y_n^1)dx\\
&-\lambda\int_{\mathbb{R}^3}(I_\mu*F(v_n^1))f(v_n^1)\varphi(x-y_n^1)dx-\lambda\int_{\mathbb{R}^3}|v_n^1|^{2_s^*-2}v_n^1(x)\varphi(x-y_n^1)dx\\
&=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\widetilde{v_n^1}(x)-\widetilde{v_n^1}(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}}dxdy+\int_{\mathbb{R}^3}V(x+y_n^1)\widetilde{v_n^1}(x)\varphi(x)dx\\
&\quad+\int_{\mathbb{R}^3}\phi_{\widetilde{v_n^1}}^t\widetilde{v_n^1}(x)\varphi(x)dx-\lambda\int_{\mathbb{R}^3}(I_\mu*F(\widetilde{v_n^1}))f(\widetilde{v_n^1})\varphi(x)dx-\lambda\int_{\mathbb{R}^3}|\widetilde{v_n^1}|^{2_s^*-2}\widetilde{v_n^1}(x)\varphi(x)dx\\
&=o_n(1)\|\varphi\|.
\end{aligned}\tag{4.12}$$

Since $|y_n^1|\to+\infty$ and $\varphi\in\mathscr{C}_0^\infty(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3}[V(x+y_n^1)-V_\infty]\widetilde{v_n^1}(x)\varphi(x)dx\to 0.\tag{4.13}$$

Combining (4.12) and (4.13), we deduce that for any $\varphi\in\mathscr{C}_0^\infty(\mathbb{R}^3)$, $\langle(\Phi_\lambda^\infty)'(\widetilde{v_n^1}),\varphi\rangle\to 0$. By (V_2) and $u_n\to u_\lambda$ in $L^2_{loc}(\mathbb{R}^3)$, we can see

$$\int_{\mathbb{R}^3}(V(x)-V_\infty)(v_n^1)^2dx\to 0.\tag{4.14}$$

It follows immediately from (4.8) and (4.14) that

$$\Phi_\lambda(v_n^1)\to c_\lambda-\Phi_\lambda(u_\lambda),\quad \Phi_\lambda(u_n)-\Phi_\lambda(u_\lambda)-\Phi_\lambda^\infty(v_n^1)\to 0.\tag{4.15}$$

Set $v_n^2(\cdot) := v_n^1(\cdot) - w^1(\cdot - y_n^1)$, then $v_n^2 \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$. Noting that $\widetilde{v_n^1} \rightharpoonup w^1 \neq 0$, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^3} V(x)|v_n^2|^2 dx &= \int_{\mathbb{R}^3} V(x)|v_n^1|^2 dx + \int_{\mathbb{R}^3} V(x+y_n^1)|w^1(x)|^2 dx \\ &\quad - 2\int_{\mathbb{R}^3} V(x+y_n^1) v_n^1(x+y_n^1) w^1(x) dx \\ &= \int_{\mathbb{R}^3} V(x)|u_n|^2 dx - \int_{\mathbb{R}^3} V(x)|u_\lambda|^2 dx - \int_{\mathbb{R}^3} V_\infty |w^1|^2 dx + o_n(1).\end{aligned} \tag{4.16}$$

From the Brezis-Lieb Lemma, [2, Lemma 2.6] and [17, Lemma 2.9], we conclude that

$$\begin{aligned}\|v_n^2\|^2_{\mathscr{D}^{s,2}(\mathbb{R}^3)} &= \|u_n - u_\lambda - w^1(\cdot - y_n^1)\|^2_{\mathscr{D}^{s,2}(\mathbb{R}^3)} = \|u_n - u_\lambda\|^2_{\mathscr{D}^{s,2}(\mathbb{R}^3)} - \|w^1(\cdot - y_n^1)\|^2_{\mathscr{D}^{s,2}(\mathbb{R}^3)} + o_n(1) \\ &= \|u_n\|^2_{\mathscr{D}^{s,2}(\mathbb{R}^3)} - \|u_\lambda\|^2_{\mathscr{D}^{s,2}(\mathbb{R}^3)} - \|w^1(\cdot - y_n^1)\|^2_{\mathscr{D}^{s,2}(\mathbb{R}^3)} + o_n(1),\end{aligned} \tag{4.17}$$

$$\begin{aligned}\|v_n^2\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^3)} &= \|u_n - u_\lambda - w^1(\cdot - y_n^1)\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^3)} = \|u_n - u_\lambda\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^3)} - \|w^1(\cdot - y_n^1)\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^3)} + o_n(1) \\ &= \|u_n\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^3)} - \|u_\lambda\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^3)} - \|w^1\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^3)} + o_n(1),\end{aligned} \tag{4.18}$$

$$\begin{aligned}&\int_{\mathbb{R}^3} \phi^t_{v_n^2}(v_n^2)^2 dx = \int_{\mathbb{R}^3} \phi^t_{v_n^1}(v_n^1)^2 dx - \int_{\mathbb{R}^3} \phi^t_{w^1(x-y_n^1)}(w^1(x-y_n^1))^2 dx + o_n(1) \\ &= \int_{\mathbb{R}^3} \phi^t_{u_n}(u_n)^2 dx - \int_{\mathbb{R}^3} \phi^t_{u_\lambda}(u_\lambda)^2 dx - \int_{\mathbb{R}^3} \phi^t_{w^1}(w^1)^2 dx + o_n(1),\end{aligned} \tag{4.19}$$

$$\begin{aligned}&\int_{\mathbb{R}^3} \phi^t_{v_n^2} v_n^2 \varphi dx = \int_{\mathbb{R}^3} \phi^t_{v_n^1} v_n^1 \varphi dx - \int_{\mathbb{R}^3} \phi^t_{w^1(x-y_n^1)}(w^1(x-y_n^1))\varphi dx + o_n(1) \\ &= \int_{\mathbb{R}^3} \phi^t_{u_n} u_n \varphi dx - \int_{\mathbb{R}^3} \phi^t_{u_\lambda} u_\lambda \varphi dx - \int_{\mathbb{R}^3} \phi^t_{w^1} w^1 \varphi dx + o_n(1), \quad \forall \varphi \in (H^s(\mathbb{R}^3))',\end{aligned} \tag{4.20}$$

$$\begin{aligned}&\int_{\mathbb{R}^3} (I_\mu * F(v_n^2))F(v_n^2) dx \\ &= \int_{\mathbb{R}^3} (I_\mu * F(v_n^1))F(v_n^1) dx - \int_{\mathbb{R}^3} (I_\mu * F(w^1(x-y_n^1)))F(w^1(x-y_n^1)) dx + o_n(1) \\ &= \int_{\mathbb{R}^3} (I_\mu * F(u_n))F(u_n) dx - \int_{\mathbb{R}^3} (I_\mu * F(u_\lambda))F(u_\lambda) dx - \int_{\mathbb{R}^3} (I_\mu * F(w^1))F(w^1) dx + o_n(1),\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^3} (I_\mu * F(v_n^2)) f(v_n^2)\varphi dx \\ &= \int_{\mathbb{R}^3} (I_\mu * F(v_n^1)) f(v_n^1)\varphi dx - \int_{\mathbb{R}^3} (I_\mu * F(w^1(x-y_n^1))) f(w^1(x-y_n^1))\varphi dx + o_n(1) \\ &= \int_{\mathbb{R}^3} (I_\mu * F(u_n)) f(u_n)\varphi dx - \int_{\mathbb{R}^3} (I_\mu * F(u_\lambda)) f(u_\lambda)\varphi dx - \int_{\mathbb{R}^3} (I_\mu * F(w^1)) f(w^1)\varphi dx + o_n(1).\end{aligned} \tag{4.22}$$

By (4.16)-(4.22), we see that

$$\begin{cases}\Phi_\lambda(v_n^2) = \Phi_\lambda(u_n) - \Phi_\lambda(u_\lambda) - \Phi_\lambda^\infty(w^1) + o_n(1), \\ \Phi_\lambda^\infty(v_n^2) = \Phi_\lambda(v_n^1) - \Phi_\lambda^\infty(w^1) + o_n(1) \\ \langle \Phi_\lambda'(v_n^2), \varphi \rangle = \langle \Phi_\lambda'(u_n), \varphi \rangle - \langle \Phi_\lambda'(u_\lambda), \varphi \rangle - \langle (\Phi_\lambda^\infty)'(w^1), \varphi \rangle + o_n(1) = o_n(1).\end{cases}$$

Therefore, together with (4.15), we obtain

$$\Phi_\lambda(u_n) = \Phi_\lambda(u_\lambda) + \Phi_\lambda^\infty(v_n^1) + o_n(1) = \Phi_\lambda(u_\lambda) + \Phi_\lambda^\infty(w^1) + \Phi_\lambda^\infty(v_n^2) + o_n(1).$$

It follows from (4.4) and Lemma 4.2 that $\Phi_\lambda^\infty(v_n^2) = c_\lambda - \Phi_\lambda(u_\lambda) - \Phi_\lambda^\infty(w^1) \le c_\lambda$. Note that one of **Case 1** and **Case 2** is true for v_n^2. If **Case 1** holds, then Lemma 4.3 holds with $k=1$. If **Case 2** occurs, we repeat the above arguments. By iterating this process we have sequences of $\{y_n^j\} \subset \mathbb{R}^3$ such that $|y_n^j| \to +\infty$, $|y_n^j - y_n^i| \to +\infty$ for $i \ne j$ and $v_n^j = v_n^{j-1} - w^{j-1}(\cdot - y_n^{j-1})$ with $j \ge 2$ satisfying

$$v_n^j \rightharpoonup 0 \text{ in } H^s(\mathbb{R}^3), \quad (\Phi_\lambda^\infty)'(w^j) = 0$$

and

$$\begin{cases} \|u_n\|^2 - \|u_\lambda\|^2 - \sum_{j=1}^k \|w^j(\cdot - y_n^j)\|^2 = \|u_n - u_\lambda - \sum_{j=1}^k w^j(\cdot - y_n^j)\| + o_n(1), \\ \Phi_\lambda(u_n) - \Phi_\lambda(u_\lambda) - \sum_{j=1}^{k-1} \Phi_\lambda^\infty(w^j) - \Phi_\lambda^\infty(v_n^k) = o_n(1). \end{cases} \tag{4.23}$$

In view of $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$, (4.23) yields that the iteration stops at some k. That is, $v_n^{k+1} \to 0$ in $H^s(\mathbb{R}^3)$. From (4.23), it is easy to check that (iv) and (v) are true. The proof is complete. □

Lemma 4.4. *For almost every $\lambda \in \Lambda$, let $\{u_n\}$ be a bounded $(PS)_{c_\lambda}$ sequence of Φ_λ. Then up to a sub-sequence, $\{u_n\}$ converges to a nontrivial $u_\lambda \in H^s(\mathbb{R}^3) \setminus \{0\}$ such that*

$$\Phi_\lambda(u_\lambda) = c_\lambda, \quad (\Phi_\lambda)'(u_\lambda) = 0.$$

Proof. From Lemma 4.3, up to a sub-sequence, there exist $u_\lambda \in H^s(\mathbb{R}^3)$, nontrivial critical points w^j, $j = 1, \dots, k$ of Φ_λ^∞, $k \in \mathbb{N} \cup \{0\}$ and $\{y_n^j\} \subset \mathbb{R}^3$ with $|y_n^j| \to +\infty$, $1 \le j \le k$ such that

$$\Phi_\lambda'(u_\lambda) = 0, \quad u_n \rightharpoonup u_\lambda, \quad \Phi_\lambda(u_n) \to \Phi_\lambda(u_\lambda) + \sum_{j=1}^k \Phi_\lambda^\infty(w^k).$$

Together with (4.4), we infer that, if $k \ne 0$,

$$c_\lambda = \lim_{n\to\infty} \Phi_\lambda(u_n) = \Phi_\lambda(u_\lambda) + \sum_{j=1}^k \Phi_\lambda^\infty(w^k) \ge m_\lambda^\infty,$$

which contradicts Lemma 4.2. Therefore, this lemma follows. □

Proof of Theorem 1.1. Taking a sequence $\{\lambda_n\} \subset \Lambda$ satisfying $\lambda_n \to 1$, one sees from Lemma 4.1 that there exists a sequence of nontrivial critical points u_{λ_n} (which we denote it in the following by $\{u_n\}$) for Φ_{λ_n} and $\Phi_{\lambda_n}(u_n) = c_{\lambda_n}$. Now, we prove that $\{u_n\}$ is bounded. It follows from (3.1) and $\mu > s + t$ that for every $\tau \in \mathbb{R}$,

$$f(\tau)\tau - 2F(\tau) > f(\tau)\tau - \frac{6+\mu}{2(s+t)} F(\tau) \ge 0.$$

Combining $\langle \Phi_{\lambda_n}'(u_n), u_n \rangle = 0$ and $3 \le 4s$ we infer that

$$\begin{aligned} c_\delta \ge c_{\lambda_n} &= \Phi_{\lambda_n}(u_n) - \frac{1}{4}\langle \Phi_{\lambda_n}'(u_n), u_n \rangle \\ &= \frac{1}{4}\|u_n\|^2 + \frac{\lambda_n}{4}\int_{\mathbb{R}^3} (I_\mu * F(u_n))[f(u_n)u_n - 2F(u_n)]dx + \left(\frac{s}{3} - \frac{1}{4}\right)\lambda_n \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \\ &\ge \frac{1}{4}\|u_n\|^2, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. Therefore by Lemma 1.1, we obtain that

$$\begin{aligned}\lim_{n\to\infty}\Phi(u_n) &= \lim_{n\to\infty}\left\{\Phi_{\lambda_n}(u_n)+\frac{\lambda_n-1}{2}\left[\int_{\mathbb{R}^3}(I_\mu*F(u_n))F(u_n))dx+\frac{2}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\right]\right\}\\ &= \lim_{n\to\infty}c_{\lambda_n}=c_1,\end{aligned}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\langle\Phi'(u_n),\varphi\rangle\\ &=\lim_{n\to\infty}\left\{\langle\Phi'_{\lambda_n}(u_n),\varphi\rangle+(\lambda_n-1)\left[\int_{\mathbb{R}^3}(I_\mu*F(u_n))f(u_n)\varphi dx+\int_{\mathbb{R}^3}|u_n|^{2_s^*-2}u_n\varphi dx\right]\right\}\\ &=0,\end{aligned}$$

which implies that $\{u_n\}$ is a bounded $(PS)_{c_1}$ sequence for Φ. Therefore, by Lemma 4.4, there exists a nontrivial critical point $u_0\in$ for Φ with $\Phi(u_0)=c_1$.

At last, we prove the existence of a ground state solution to Eq. (1.1). Set

$$m=\inf\{\Phi(u):u\neq 0,\ \Phi'(u)=0\}.$$

It is easy to see that $0\le m\le\Phi(u_0)=c_1<+\infty$. For any u satisfying $\Phi'(u)=0$, by standard arguments, we see that $\|u\|\ge\rho$ for some $\rho>0$. While, it follows from (V_1), $G(u)=0$ and (3.1) that

$$\begin{aligned}\Phi(u)&=\Phi(u)-\frac{1}{4s+2t-3}G(u)\\ &=\frac{1}{2(4s+2t-3)}\int_{\mathbb{R}^3}[2sV(x)+(\nabla V(x),x)]u^2dx\\ &\quad+\frac{1}{2(4s+2t-3)}\int_{\mathbb{R}^3}(I_\mu*F(u))[2(s+t)f(u)u-(4s+2t+\mu)F(u)]dx+\frac{s}{3}\int_{\mathbb{R}^3}|u|^{2_s^*}dx\end{aligned}$$

which yields $m\ge 0$. If $m=0$, then there exists a critical point sequence $\{u_n\}$ of Φ with $\Phi(u_n)\to 0$, which implies

$$\lim_{n\to\infty}\|u_n\|_{L^{2_s^*}(\mathbb{R}^3)}^{2_s^*}=0. \tag{4.24}$$

Similar as (3.14), we infer that

$$\int_{\mathbb{R}^3}(I_\mu*F(u_n)f(u_n)u_n\le\varepsilon\|u_n\|_{L^2(\mathbb{R}^3)}^{2+\frac{2\mu}{3}}+C_\varepsilon\|u_n\|_{L^{2_s^*}(\mathbb{R}^3)}^{2q},$$

which implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}(I_\mu*F(u_n)f(u_n)u_n=0,$$

as $\varepsilon\to 0$. Combining with (4.24) and $\langle\Phi'(u_n),u_n\rangle=0$, we obtain $\lim_{n\to\infty}\|u_n\|=0$, which contradicts $\|u_n\|\ge\rho$. Therefore, $0<m<+\infty$. Then let $\{u_n\}$ be a sequence satisfying $\Phi'(u_n)=0,\Phi(u_n)\to m$. Applying the same argument as above, we can obtain that $\{u_n\}$ is bounded. Similar to the proof of Lemma 4.4, we have that there exists $u\in H^s(\mathbb{R}^3)$ such that $\Phi'(u)=0,\ \Phi(u)=m$. □

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