J. Nonlinear Var. Anal. 8 (2024), No. 1, pp. 95-108
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.8.2024.1.05

ON TOPOLOGICAL PROPERTIES OF SOLUTION SETS OF SEMILINEAR FRACTIONAL DIFFERENTIAL INCLUSIONS WITH NON-CONVEX RIGHT-HAND SIDE

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Abstract. In this paper, we study the Cauchy problem for a semilinear fractional order differential inclusion with a nonconvex-valued almost lower semicontinuous nonlinearity and a linear closed operator generating a C_0-semigroup in a separable Banach space. By using the theory of measure of noncompactness and condensing operators, we study topological properties of the solution set of this problem. We prove that the solution set of the Cauchy problem possesses the classical Kneser connectedness property.
Keywords. Almost lower semicontinuous multioperator; Cauchy problem; Condensing operator, Gerasimov-Caputo fractional derivative, Measure of noncompactness; Semilinear differential inclusion.

1. INTRODUCTION

The theory of differential equations and inclusions of fractional order attracts the attention of a large number of researchers due to its numerous applications in mathematical physics, biology, economics, engineering, ecology, and other branches of natural sciences (see, e.g., [1, 2, 3, 4, 5] and the references therein). Up to the present moment, various approaches to the solvability of differential equations and inclusions of a fractional order $q \in (0,1)$ have been developed. The Cauchy type problems for differential equations of fractional order $q \in (0,1)$ were solved in [6]-[11]. In [12, 13], the trajectories of differential inclusions of fractional order $q \in (0,1)$ obeying generalized boundary conditions expressed in the form of operator inclusions were studied. In [14]-[18], the solvability of periodic boundary value problems for fractional differential inclusions was studied, and the corresponding results for antiperiodic problems were presented in [19]-[22]. Approximation methods for fractional differential equations and inclusions were described in [23]-[28].

In the present paper, we consider the Cauchy problem for a semilinear fractional differential inclusion in a separable Banach space E of the following form:

$$ {}^{C}D_0^q x(t) \in Ax(t) + F(t, x(t)),\ t \in [0,T], \tag{1.1}$$

$$ x(0) = x_0, \tag{1.2}$$

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Received 23 May 2023; Accepted 12 September 2023; Published online 13 December 2023

where ${}^{C}D_0^q$, $0 < q < 1$, is the Gerasimov–Caputo fractional derivative, $F : [0,T] \times E \multimap E$ is an almost lower semicontinuous multivalued map with compact values, $A : D(A) \subset E \to E$ is a closed linear (not necessarily bounded) operator in E, and $x_0 \in E$. By using the theory of measures of noncompactness and condensing operators, we study the topological properties of the solution set of problem (1.1)-(1.2).

2. Preliminaries

2.1. **Fractional integral and derivative.**

Definition 2.1. (See, e.g., [1, 2]). The fractional integral of order $q \in (0,1)$ of a function $g \in L^1([0,T];E)$ is the function $I_0^q g$ of the following form

$$I_0^q g(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s)\, ds,$$

where Γ is Euler's gamma-function

$$\Gamma(q) = \int_0^\infty x^{q-1} e^{-x} dx.$$

Definition 2.2. The Gerasimov–Caputo fractional derivative of the order $q \in (N-1,N]$ of a function $g \in C^N([0,T];E)$ is the function ${}^{C}D_0^q g$ of the following form

$${}^{C}D_0^q g(t) = \frac{1}{\Gamma(N-q)} \int_0^t (t-s)^{N-q-1} g^{(N)}(s)\, ds.$$

2.2. **Multivalued maps.** Let us recall some concepts (see, for example, [29, 30]).

We denote by $\mathscr{E}$ is a Banach space and introduce the following notation:

- $P(\mathscr{E}) = \{A \subseteq \mathscr{E} : A \neq \varnothing\}$ is the collection of all non-empty subsets of $\mathscr{E}$;
- $Pb(\mathscr{E}) = \{A \in P(\mathscr{E}) : A \text{ is bounded}\}$;
- $Pv(\mathscr{E}) = \{A \in P(\mathscr{E}) : A \text{ is convex}\}$;
- $C(\mathscr{E}) = \{A \in P(\mathscr{E}) : A \text{ is closed}\}$;
- $K(\mathscr{E}) = \{A \in P(\mathscr{E}) : A \text{ is compact}\}$;
- $Kv(\mathscr{E}) = \{Pv(\mathscr{E}) \cap K(\mathscr{E})\}$ is the collection of all convex and non-empty compact subsets of $\mathscr{E}$;
- $Cv(\mathscr{E}) = \{Pv(\mathscr{E}) \cap C(\mathscr{E})\}$ is the collection of all convex and non-empty closed subsets of $\mathscr{E}$.

Definition 2.3. [31] Let $(\mathscr{A}, \geq)$ be a partially ordered set. A function $\beta : Pb(\mathscr{E}) \to \mathscr{A}$ is called the measure of noncompactness (MNC) in $\mathscr{E}$ if, for each set $\Omega \in Pb(\mathscr{E})$, $\beta(\overline{\mathrm{co}}\,\Omega) = \beta(\Omega)$, where $\overline{\mathrm{co}}\,\Omega$ is the closure of the convex hull of Ω.

A measure of noncompactness β is called:

1) *monotone* if, for all $\Omega_0, \Omega_1 \in Pb(\mathscr{E})$, $\beta(\Omega_0) \leq \beta(\Omega_1)$ for $\Omega_0 \subseteq \Omega_1$;
2) *nonsingular* if, for all $a \in E$ and each $\Omega \in Pb(\mathscr{E})$, $\beta(\{a\} \cup \Omega) = \beta(\Omega)$.

If $\mathscr{A}$ is a cone in a Banach space, then the MNC β is called:

3) *regular* if $\beta(\Omega) = 0$ is equivalent to the relative compactness of $\Omega \in Pb(\mathscr{E})$;
4) *real* if $\mathscr{A}$ is the set of all real numbers $\mathbb{R}$ with the natural ordering.

The example of a real MNC obeying all above properties is the Hausdorff MNC $\chi(\Omega)$:

$$\chi(\Omega) = \inf\{\varepsilon > 0, \text{for which } \Omega \text{ has a finite } \varepsilon\text{-net in } \mathscr{E}\ \}.$$

Notice that the Hausdorff MNC satisfies the semi-homogeneity condition, i.e., $\chi(\lambda\Omega) = |\lambda|\chi(\Omega)$, for each $\lambda \in \mathbb{R}$ and each $\Omega \in Pb(\mathscr{E})$. For a set $M \subset \mathscr{E}$, define $\|M\| = \sup_{x\in M} \|x\|_{\mathscr{E}}$.

Let X be a metric space and Y be a normed space.

Definition 2.4. A multivalued map (multimap) $\mathscr{F} : X \to P(Y)$ is said to be lower semicontinuous (l.s.c.) at a point $x \in X$ if, for each open set $V \subset Y$ such that $\mathscr{F}(x) \cap V \neq \emptyset$, there exists an open neighborhood $U(x)$ of the point x such that $\mathscr{F}(x') \cap V \neq \emptyset$ for all $x' \in U(x)$.

A multimap is lower semicontinuous if it is lower semicontinuous at every point $x \in X$.

Definition 2.5. A multimap $\mathscr{F} : [0,T] \times X \to K(Y)$ is said to be almost lower semicontinuous (a.l.s.c.) if there exists a sequence of disjoint compact sets $I_n \subseteq [0,T]$ such that

(i) $meas([0,T] \setminus I) = 0$, where $I = \cup_n I_n$;
(ii) the restriction of $\mathscr{F}$ on each set $J_n = I_n \times Y$ is l.s.c.

Definition 2.6. A multivalued map $\mathscr{F} : X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) at a point $x \in X$ if, for every open set $V \subset Y$ such that $\mathscr{F}(x) \subset V$, there exists a neighborhood $U(x)$ of x such that $\mathscr{F}(U(x)) \subset V$.

A multimap is upper semicontinuous if it is upper semicontinuous at every point $x \in X$.

Proposition 2.1. (see, e.g., [30, Theorem 1.2.37]) *Let $F : X \to P(Y)$ be a u.s.c. multimap. If $A \subset X$ is a connected set and $F(x)$ is connected for every $x \in A$, then the image $F(A)$ is a connected subset of Y.*

Definition 2.7. A multimap $\mathscr{F} : X \to P(Y)$ is called closed if its graph $G_{\mathscr{F}} = \{(x,y) : x \in X, y \in \mathscr{F}(x)\}$ is a closed subset of $X \times Y$.

Definition 2.8. A continuous map $f : X \subseteq \mathscr{E} \to \mathscr{E}$ is called condensing with respect to a MNC β (or β–condensing) if, for each bounded set $\Omega \subseteq X$ which is not relatively compact, $\beta(f(\Omega)) \ngeq \beta(\Omega)$.

2.3. **Measurable multifunctions.** Let E be a separable Banach space. Recall some notions (see, e.g., [29, 30]).

Definition 2.9. Let $p \geq 1$. A multifunction $G : [0,T] \to K(E)$ is called:

- L^p–integrable if G admits an L^p–Bochner integrable selection, i.e., there exists a function $g \in L^p([0,T];E)$ such that $g(t) \in G(t)$ for a.e. $t \in [0,T]$;
- L^p–integrably bounded if there exists a function $\xi \in L^p([0,T])$ such that $\|G(t)\| \leq \xi(t)$ for a.e. $t \in [0,T]$.

The set of all L^p–integrable selections of a multifunction $G : [0,T] \to K(E)$ is denoted by $\mathscr{S}^p_G$. A multifunction $G : [0,T] \to K(E)$ is called measurable if, for every open subset $V \subset E$, $G^{-1}(V)$ is Lebesgue measurable. Every multimap $\mathscr{F} : [0,T] \times E \to K(E)$ generates a correspondence assigning to every function $q : [0,T] \to E$. The multifunction $\Phi : [0,T] \to P(E)$ is defined by the formula $\Phi(t) = \mathscr{F}(t, q(t))$. If, for every measurable function q, Φ is measurable, then $\mathscr{F}$ is said to be superpositionally measurable.

Lemma 2.1. (see [29, Proposition 1.3.1], and [30, Theorem 1.5.19]) *If a multimap $\mathscr{F} : [0,T] \times E \to P(E)$ is a.l.s.c., then it is superpositionally measurable.*

Lemma 2.2. (see [29, Theorem 4.2.1]) *Let a sequence of functions $\{\xi_n\} \subset L^1([0,T];E)$ be L^1–integrably bounded and $\chi(\{\xi_n\}(t)) \leq \alpha(t)$ a.e. $t \in [0,T]$, for all $n = 1,2,...$, where $\alpha \in L^1_+([0,T])$. Then, for each $\delta > 0$, there exist a compact set $K_\delta \subset E$, a set $m_\delta \subset [0,T]$ of a Lebesgue measure $m_\delta < \delta$, and a set of functions $G_\delta \subset L^1([0,T];E)$ with values in K_δ such that, for each $n \geq 1$, there exists a function $b_n \in G_\delta$ for which*

$$\|\xi_n(t) - b_n(t)\|_E \leq 2\alpha(t) + \delta, \qquad t \in [0,T] \setminus m_\delta.$$

Moreover, $\{b_n\}$ may be chosen so that $b_n \equiv 0$ on m_δ, and this sequence is weakly compact.

In the sequel, we need the following important property on the χ-estimation of the integral of a multifunction.

Lemma 2.3. (see [29, Theorem 4.2.3]) *Let E be a separable Banach space, and let $G : [0,T] \to K(E)$ be an integrable, integrably bounded multifunction such that $\chi(G(t)) \leq v(t)$ for a.e. $t \in [0,T]$, where χ is the Hausdorff MNC in E and $v(\cdot) \in L^1_+(0,T)$. Then*

$$\chi\Big(\int_0^t G(s)\,ds\Big) \leq \int_0^t v(s)\,ds.$$

Definition 2.10. A sequence of functions $\{\xi_n\} \subset L^p([0,T];E)$, $p \geq 1$, is called L^p–semicompact if it is L^p–integrably bounded and the set $\{\xi_n(t)\}$ is relatively compact in E for a.e. $t \in [0,T]$.

3. Auxiliary Results

Consider Cauchy problem (1.1) - (1.2) in a separable Banach space E.

We assume that the multimap $F : [0,T] \times E \to K(E)$ obeys the following conditions:

$(F1)$ $F : [0,T] \times E \to K(E)$ is a.l.s.c.;

$(F2)$ for each $r > 0$, there exists a function $\omega_r \in L^\infty([0,T])$ such that, for each $x \in E$, $\|x\| \leq r$, $\|F(t,x)\| \leq \omega_r(t)$ for a.e. $t \in [0,T]$;

$(F3)$ there exists a function $\mu \in L^\infty([0,T])$ such that, for each nonempty bounded set $Q \subset E$, $\chi(F(t,Q)) \leq \mu(t)\chi(Q)$ for a.e. $t \in [0,T]$, where χ is the Hausdorff MNC in$E$.

(A) $A : D(A) \subset E \to E$ is an infinitesimal generator of a bounded C_0–semigroup $\{U(t)\}_{t\geq 0}$ of linear operators in E. Denote $M = \sup\{\|U(t)\|; t \in [0;T]\}$.

For $x \in C([0,\tau];E), 0 < \tau \leq T$, consider the multifunction

$$\Phi_F : [0,\tau] \to K(E), \qquad \Phi_F(t) = F(t,x(t)).$$

From conditions $(F1)-(F2)$ and Lemma 2.1, it follows that Φ_F is measurable and L^p–integrable for $p \geq 1$. Then, the superposition multimap $\mathscr{P}^\infty_F : C([0,\tau];E) \multimap L^\infty([0,\tau];E)$ defined as follows form $\mathscr{P}^\infty_F(x) = \mathscr{S}^\infty_{\Phi_F}$ is well-defined.

Definition 3.1. A mild solution to Cauchy problem (1.1) - (1.2) on an interval $[0,\tau]$, $\tau \in (0,T]$ is a function $x \in C([0,\tau];E)$ such that

$$x(t) = \mathscr{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\phi(s)ds, \qquad t \in [0,\tau],$$

where $\phi \in \mathscr{P}_F^\infty(x)$ and

$$\mathscr{G}(t) = \int_0^\infty \xi_q(\theta)U(t^q\theta)d\theta, \qquad \mathscr{T}(t) = q\int_0^\infty \theta\xi_q(\theta)U(t^q\theta)d\theta,$$

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\Psi_q(\theta^{-1/q}),$$

and

$$\Psi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \theta \in \mathbb{R}^+.$$

Remark 3.1. (See, e.g., [11]) $\xi_q(\theta) \geq 0$, $\int_0^\infty \xi_q(\theta)\,d\theta = 1$, and $\int_0^\infty \theta\xi_q(\theta)\,d\theta = \frac{1}{\Gamma(q+1)}$.

Lemma 3.1. (See [11, Lemma 3.4]) *The operator functions $\mathscr{G}$ and $\mathscr{T}$ possess the following properties:*

1) *for all $t \in [0,T]$, $\mathscr{G}(t)$ and $\mathscr{T}(t)$ are bounded linear operators. More precisely, for every $x \in E$,*

$$\|\mathscr{G}(t)x\|_E \leq M\,\|x\|_E\,;$$

and

$$\|\mathscr{T}(t)x\|_E \leq \frac{qM}{\Gamma(1+q)}\,\|x\|_E\,; \tag{3.1}$$

2) *the operator functions $\mathscr{G}(\cdot)$ and $\mathscr{T}(\cdot)$ are strongly continuous, i.e., functions $t \in [0,T] \to \mathscr{G}(t)x$ and $t \in [0,T] \to \mathscr{T}(t)x$ are continuous for all $x \in E$.*

Consider the operator $S : L^\infty([0,T];E) \to C([0,T];E)$ defined as

$$S(\phi)(t) = \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\phi(s)ds.$$

Lemma 3.2. *(see [25], Lemma 7) Let $\{\xi_n\}$ be an L^∞-semicompact sequence in $L^\infty([0,\tau];E)$. Then $\{S\xi_n\}$ is compact in $C([0,\tau];E)$.*

Now, we consider the multioperator $\Upsilon_F : C([0,\tau];E) \multimap C([0,\tau];E)$, given as

$$\Upsilon_F = \mathscr{G}(t)x_0 + S\circ\mathscr{P}_F^\infty(x), \qquad t \in [0,\tau].$$

In [32], the following existence theorems for problem (1.1) - (1.2) were proved.

Theorem 3.1. *Under conditions (A), $(F1)-(F3)$,there exists $\tau \in (0,T]$ such that the set $\Sigma_{x_0}^F[0,\tau]$ of all mild solutions to Cauchy problem (1.1) - (1.2) on the interval $[0,\tau]$ is a non-empty subset of the space $C([0,\tau];E)$.*

Theorem 3.2. *Under conditions (A), $(F1)$, $(F3)$, we suppose that condition $(F2)$ has the following form:*

$(F'2)$ *there exists $\alpha \in L_+^\infty([0,T])$ such that $\|F(t,x)\|_E \leq \alpha(t)(1+\|x\|_E)$ for a.e. $t \in [0,T]$.*

If $\frac{2MT^q}{\Gamma(1+q)}k < 1$, where $k = \max\{\|\alpha\|_\infty, \|\mu\|_\infty\}$ and functions α and μ are from conditions $(F'2)$ and $(F3)$, respectively, then problem (1.1) - (1.2) has a mild solution.

4. Main Results

Consider the question on the topological structure of the solutions set of problem (1.1) - (1.2). We prove that this set possesses the classical Kneser connectedness property.

Suppose that the linear part of inclusion (1.1) satisfies condition (A), and the multivalued nonlinearity F obeys condition $(F1)$. Moreover, assume for F the following slightly more strict conditions of boundedness and χ-regularity:

$(F2_L)$ there exists a constant $K > 0$ such that $\|F(t,x)\|_E \leq K(1+\|x\|_E)$ for a.e. $t \in [0,T]$;

$(F3_L)$ there exists a function $\mu \in L^\infty([0,T])$ such that, for each bounded set $Q \subset E$,

$$\lim_{\tau \to +0} \chi(F(J_{t,\tau} \times Q)) \leq \mu(t)\chi(Q), \text{ for a.e. } t \in [0,T],$$

where χ is the Hausdorff MNC in E and $J_{t,\tau} = [t-\tau, t+\tau] \cap [0,T]$.

Consider a multivalued map $\tilde{F} : [0,T] \times E \to K(E)$, which is defined by $\tilde{F}(t,x) = \cap_{\varepsilon > 0} F^\varepsilon(t,x)$, where $F^\varepsilon(t,x) = \overline{co}\{F(s,y) : |s-t| < \varepsilon, \|y-x\| < \varepsilon\}$. Let us show that $\tilde{F}$ satisfies the χ–regularity condition:

$$\chi(\tilde{F}(t,Q)) \leq \mu(t)\chi(Q), \tag{4.1}$$

for a.e. $t \in [0,T]$ and for each bounded set $Q \subset E$.

In fact for any $t \in [0,T]$, for which estimation $(F3_L)$ holds, take arbitrary $\delta > 0$ and choose τ, $0 < \tau < \delta$ such that

$$\chi(F(J_{t,\tau} \times W_\delta(Q))) \leq \mu(t)\chi(W_\delta(Q)) + \delta \leq \mu(t)(\chi(Q)+\delta) + \delta,$$

where $W_\delta(Q)$ is a δ-neighbourhood of set Q.

Now

$$\chi(\tilde{F}(t,Q)) \leq \chi(F^\tau(t,Q)) \leq \chi(F(J_{t,\tau} \times W_\delta(Q))) \leq \mu(t)(\chi(Q)+\delta) + \delta.$$

Estimate (4.1) follows from the arbitrariness of δ.

Lemma 4.1. *Under the conditions above, there exists a non-empty compact convex set $X \subset C([0,T];E)$ such that*

$$x(0) = x_0, \quad \textit{for all} \quad x \in X; \tag{4.2}$$

$$\mathscr{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\overline{co}\tilde{F}(s,X(s))ds \subseteq X(t) \quad \textit{for all} \quad t \in [0,T];$$

and

$$\Sigma_{x_0}^F \subset X. \tag{4.3}$$

Proof. Assume without loss of generality that $\tilde{F}$ satisfies the following condition of global integral boundedness:

$(F''2)$

$$\left\|\tilde{F}(t,x)\right\|_E \leq \gamma(t) \text{ for a.e. } t \in [0,T], x \in E$$

for a given function $\gamma(\cdot) \in L_+^\infty([0,T])$.

Notice that in this case a solution x to problem (1.1) - (1.2) satisfies the estimate $\|x\|_{C([0,T];E)} \leq N$, where

$$N = M\Big(\|x_0\|_E + \frac{\|\gamma\|_\infty T^q}{\Gamma(q+1)}\Big).$$

Construct a decreasing sequence of closed and convex sets $\{X^n\}_{n=1}^{\infty} \subset C([0,T];E)$ by the following inductive process. Set

$$X^0 = \left\{ x \in C([0,T];E) : x(0) = x_0, \|x\|_{C([0,T];E)} \le N \right\}. \tag{4.4}$$

Further, let $X^n = \overline{Y^n}, n \ge 1$, where

$$Y^n = \left\{ y \in C([0,T];E) : y(t) = \mathscr{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)f(s)ds, f \in \mathscr{P}^{\infty}_{\overline{\mathrm{co}}\tilde{F}(\cdot,X^{n-1}(\cdot))} \right\}.$$

Notice that $X^n, n \ge 1$ are non-empty sets since $\Sigma^F_{x_0}[0,T] \subset X^n$ for every $n \ge 0$. In $C([0,T];E)$, one introduces the following MNC:

$$\psi(\Omega) = \sup_{t\in[0,T]} e^{-\vartheta t}\chi(\Omega(t)),$$

where the constant $\vartheta > 0$ is chosen in the following way. Take an arbitrary, but fixed constant $d > 0$ satisfying the inequality

$$\frac{M\|\mu\|_\infty}{\Gamma(q+1)} \cdot d^q < \frac{1}{4}$$

and then take ϑ so large that

$$\frac{M\|\mu\|_\infty}{\Gamma(q+1)} \cdot \frac{1}{\vartheta d^{1-q}} < \frac{1}{4}.$$

Notice that the MNC ψ is monotone and non-singular. From condition $(F3)$, we have, for $0 \le s \le t \le T$,

$$\begin{aligned}
\chi\left(\mathscr{T}(t-s)\overline{\mathrm{co}}F\left(s,X^{n-1}(s)\right)\right) &\le \|\mathscr{T}(t-s)\|\,\chi\left(\overline{\mathrm{co}}F\left(s,X^{n-1}(s)\right)\right)\\
&\le \frac{qM}{\Gamma(q+1)}\chi\left(F\left(s,X^{n-1}(s)\right)\right)\\
&\le \frac{qM}{\Gamma(q+1)}\mu(s)\chi\left(X^{n-1}(s)\right)\\
&\le \frac{qM}{\Gamma(q+1)}\mu(s)\psi\left(X^{n-1}\right)\cdot e^{\vartheta s}.
\end{aligned}$$

From (3.1) and $(F''2)$, it follows that, for each $t \in [0,T]$, the multifunction

$$s \multimap \mathscr{T}(t-s)\overline{\mathrm{co}}F\left(s,X^{n-1}(s)\right),\ s \in [0,t],$$

is integrable and a.e. bounded, so we find from Lemma 2.3 that, for an arbitrary $t \in [0,T]$,

$$\begin{aligned}
e^{-\vartheta t}\chi\left(X^{n-1}(t)\right) &= e^{-\vartheta t}\chi\left(\mathscr{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\overline{\mathrm{co}}F\left(s,X^{n-1}(s)\right)ds\right)\\
&= e^{-\vartheta t}\chi\left(\int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\overline{\mathrm{co}}F\left(s,X^{n-1}(s)\right)ds\right)\\
&\le e^{-\vartheta t}\frac{qM\|\mu\|_\infty}{\Gamma(q+1)}\int_0^t (t-s)^{q-1}e^{\vartheta s}\psi\left(X^{n-1}\right)ds.
\end{aligned}$$

Now, for the case that $t \leq d$, the last expression may be estimated as

$$\frac{qM\|\mu\|_\infty}{\Gamma(q+1)}\psi\left(X^{n-1}\right)\int_0^t (t-s)^{q-1}e^{-\vartheta(t-s)}ds \leq \frac{qM\|\mu\|_\infty}{\Gamma(q+1)}\psi\left(X^{n-1}\right)\frac{t^q}{q}$$
$$\leq \frac{M\|\mu\|_\infty}{\Gamma(q+1)}\cdot d^q\cdot\psi\left(X^{n-1}\right).$$

In case that $t > d$, we estimate the same expression as

$$\frac{qM\|\mu\|_\infty}{\Gamma(q+1)}\psi\left(X^{n-1}\right)\left(\int_0^{t-d}(t-s)^{q-1}e^{\vartheta s}ds + \int_{t-d}^t (t-s)^{q-1}e^{\vartheta s}ds\right)$$
$$\leq \frac{qM\|\mu\|_\infty}{\Gamma(q+1)}\psi\left(X^{n-1}\right)\left(e^{-\vartheta t}\frac{1}{d^{1-q}}\frac{e^{\vartheta(t-d)}-1}{\vartheta}+\frac{d^q}{q}\right)$$
$$\leq \frac{qM\|\mu\|_\infty}{\Gamma(q+1)}\psi\left(X^{n-1}\right)\left(\frac{1}{\vartheta d^{1-q}}+\frac{d^q}{q}\right).$$

Therefore, in both cases, we obtain the estimate $\psi(Y^n) \leq \frac{1}{2}\psi\left(X^{n-1}\right)$, from which it follows that $\psi(X^n) \leq \frac{1}{2}\psi\left(X^{n-1}\right)$. This implies $\psi(X^n) \to 0$ as $n \to \infty$. Consider the set $\widetilde{X} = \cap_{n\geq 1}X^n$. From the monotonicity of the MNC ψ, it follows that $\psi(\widetilde{X}) = 0$ and hence $\chi(\widetilde{X}(t)) = 0$ for all $t \in [0,T]$. Moreover, inequality (4.1) implies

$$\chi\left(\tilde{F}\left(t,\widetilde{X}(t)\right)\right) = 0 \text{ for each } t \in [0,T]$$

and, by applying $(F''2)$, we see that every sequence $\{f_n\} \subset \mathscr{P}^\infty_{\overline{\mathrm{co}}\tilde{F}(\cdot,\widetilde{X}(\cdot))}$ is semicompact in $L^\infty([0,T];E)$.

Now, we define the set $X \subseteq \widetilde{X}$ as

$$X = \left\{y \in C([0,T];E) : y(t) = \mathscr{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)f(s)ds, f \in \mathscr{P}^\infty_{\overline{\mathrm{co}}\tilde{F}(\cdot,\widetilde{X}(\cdot))}\right\}.$$

Applying Lemma 3.2, we obtain that X is a compact set and hence it is a desirable one. To verify (4.3), it is sufficient to remember that by construction

$$F(t,x) \subseteq \tilde{F}(t,x) \quad \text{for all} \quad t \in [0,T]\times E. \tag{4.5}$$

□

Consider now a compact set $D \subset [0,T]\times E$ by $D = \{(s,y) : s \in [0,T], y = x(s), x \in X\}$. Let $r > 0$ be the radius of the ball X^0, defined by (4.4). For $(t,x) \in D, \varepsilon > 0$ and $W > \sigma = MK(1+r)$, consider the set

$$V(t,x,\varepsilon) = \{(s,y) \in D : t \leq s < t+\varepsilon, \|y-x\| \leq \frac{W(s-t)^q}{\Gamma(1+q)}\}.$$

We need the following assertions (see [33, 34]).

Lemma 4.2. *The family of sets $\{V(t,x,\varepsilon) : (t,x) \in D, \varepsilon > 0\}$ form a basis of closed-open nieghborhoods for a topology $\mathfrak{T}^+$ on D, stronger than the usual metric topology.*

Lemma 4.3. *The multifunction $F : D \to K(E)$ admits an almost $\mathfrak{T}^+$–continuous selection $\gamma : D \to E$ in the sense that γ is $\mathfrak{T}^+$–continuous on every set $D_n = J_n \cap D$.*

Now, for $(t,x) \in D_n$, set

$$G_n(t,x) = \cap_{\varepsilon>0} G_n^{\varepsilon}(t,x) = \cap_{\varepsilon>0} \overline{co}\{\gamma(s,y) : (s,y) \in D_j, |s-t| < \varepsilon, \|y-x\| < \varepsilon\},$$

and, for $(t,x) \in D$, define

$$G(t,x) = \begin{cases} G_n(t,x), & (t,x) \in D_n \\ \{\gamma(t,x)\}, & (t,x) \notin \cap_n D_n \end{cases}$$

From the construction, it clearly follows that

$$\gamma(t,x) \in G(t,x) \tag{4.6}$$

and $G(t,x) \subseteq \tilde{F}(t,x)$ for all $(t,x) \in D$. From condition $(F3_L)$ and the compactness of sets D_n, it follows that, for every sequence $\varepsilon_k \to +0$, the corresponding sequence $\chi(F_n^{\varepsilon_k}(t,x))$ tends to zero and hence every $G(t,x)$ is a nonempty compact convex set.

The following assertion holds (see, e.g., [29, Lemma 5.5.6]).

Lemma 4.4. *The multimap G is u.s.c. on $\cup_n D_n$.*

Conider now the following problem on the compact set D

$$^C D_0^q x(t) \in Ax(t) + G(t,x(t)), \ (t,x) \in D, \tag{4.7}$$

$$x(0) = x_0, \tag{4.8}$$

It is clear that the multimap G saisfies the following conditions

- $(G1)$ for each $x \in E$, the multifunction $G : [0,T] \to K(E)$ admits a strongly continious selection;
- $(G2)$ for a.e. $t \in [0,T]$, the multimap $G : E \to K(E)$ is u.s.c.;
- $(G3)$ $\|G(t,x)\|_E \leq \gamma(t)$ for a.e. $t \in [0,T]$, $x \in E$ and a given $\gamma(\cdot) \in L_+^{\infty}([0,T])$;
- $(G4)$ there exists a function $\mu \in L^{\infty}([0,T])$ such that, for every bounded subset $Q \subset E$, $\chi(G(t,Q)) \leq \mu(t)\chi(Q)$, for a.e. $t \in [0,T]$, where χ denote the Hausdorff MNC in E.

Since G is also obviously integrably bounded, the integral multioperator Υ_G is defined and u.s.c. on X. Choose an arbitrary function $x \in X$. From the construction of X, it is easy to see that, for each $\eta > 0$, the function x can be uniformly η-aproximated by the function

$$\tilde{x}(t) = \mathscr{G}(t)x_0 + \int_0^t (t-s)^{q-1} \mathscr{T}(t-s)\vartheta(s)ds,$$

where $\vartheta(\cdot) \in L^{\infty}([0,T];E)$ and $\|\vartheta(t)\| \leq K(1+r)$ for a.e. $t \in [0,T]$.

Lemma 4.5. *The function x satisfies the following condition:*

$$\{(s,y) \in D : t \leq s < t+\varepsilon; y \in \overline{B}_{\frac{(s-t)^q(W-\sigma)}{\Gamma(1+q)}}(x(s))\} \subset V(t,x(t),\varepsilon).$$

In other words, for points s, close enough to t, $V(t,x(t),\varepsilon)$ is a metric neighborhood of $(s,x(s))$ (in the relative topology of the space D).

Proof. Let $(s,y) \in D$ and $y \in \overline{B}_{\frac{(s-t)^q(W-\sigma)}{\Gamma(1+q)}}(x(s))$. Then

$$\|y - x(t)\| \leq \|y - \tilde{x}(t)\| + \|\tilde{x}(t) - x(t)\| \leq \frac{(s-t)^q(W-\sigma)}{\Gamma(1+q)} + 2\eta.$$

Since η is arbitrary, it follows that

$$\|y - x(t)\| \leq \frac{(s-t)^q (W-\sigma)}{\Gamma(1+q)}.$$

This completes the proof. □

Consider now an arbitrary function $z \in \Upsilon_G(x)$ with the form

$$z(t) = \mathscr{G}(t)x_0 + \int_0^t (t-s)^{q-1} \mathscr{T}(t-s) g(s) ds,$$

were $g \in \mathscr{P}_G(x)$.

Lemma 4.6. *The following equality holds:* $g(t) = \gamma(t, x(t))$ *for a.e.* $t \in [0,t]$.

Proof. For each index value n, consider the set $I_n^* \subseteq I_n$ consisting of all $t \in I_n$ with the properties that:

(i) $g(t) \in G_n(t, x(t))$,

(ii) there exists a sequence $\{t_k \subset I_n\}$ strictly decreasing to t such that $g(t_k) \in G_n(t, x(t_k))$ for all k and $g(t_k) \to g(t)$.

It is known (see [34, Lemma 2.3]) that $meas\,(I_n \backslash I_n^*) = 0$. Following [33], we prove that $g(t) = \gamma(t, x(t))$ for all $t \in I_n^*$. Using a contradiction argument, let us suppose that there exists a point $t \in I_n^*$ and $\varepsilon > 0$ such that

$$\|g(t) - \gamma(t, x(t))\| = \varepsilon. \tag{4.9}$$

Since γ is $\mathfrak{T}^*$–continuos on D_n, one sees that there exist $\delta > 0$ such that

$$\|\gamma(s,y) - \gamma(t, x(t))\| < \frac{\varepsilon}{2}$$

for all $(s,y) \in V(t, x(t), \delta)$, $s \in I_n$. Now let $\{t_k \subset I_n\}$ be a sequence with the properties described in (ii). Choose k_0 such that, for $k \geq k_0$, $0 < t_k - t < \delta$ and

$$\|g(t_k) - g(t)\| < \frac{\varepsilon}{2}. \tag{4.10}$$

From Lemma 4.5, we know that $V(t, x(t), \delta)$ is a neighborhood of $(t_k, x(t_k))$ in the usual relative metric topology of D for all $k \geq k_0$. So, let the ω–neighborhood W_ω of $(t_k, x(t_k))$ in D be contained in $V(t, x(t), \delta)$. Then, for $k \geq k_0$,

$$g(t_k) \in G_n(t_k, x(t_k)) \subseteq \overline{co}\{\gamma(s,y) : (s,y) \in W_\omega, s \in I_n\}$$

and

$$\subset \overline{co}\{\gamma(s,y) : (s,y) \in V(t, x(t), \delta), s \in I_n\} \subset B_{\frac{\varepsilon}{2}}(\gamma(t, x(t))),$$

i.e.,

$$\|\gamma(t, x(t)) - g(t_k)\| < \frac{\varepsilon}{2}. \tag{4.11}$$

It is clear that (4.10) and (4.11) give the contradiction to (4.9) that proves the lemma. □

From the above lemma, it follows that the integral multioperator Υ_G is single-valued on X. Moreover, since it has the form

$$\Upsilon_G = \mathscr{G}(t)x_0 + \int_0^t (t-s)^{q-1} \mathscr{T}(t-s) \gamma(s, x(s)) ds,$$

it is a continuous selection of the integral multioperator Υ_F. Consequently, Υ_G transforms the set X into itself since Υ_F has the same property (see (4.2) and (4.5)). Moreover, every mild solution of problem (4.7) - (4.8) is a mild solution to (1.1) - (1.2).

Now we pass from problem (4.7) - (4.8) to the Cachy problem for a differential inclusion whose the right-hand side nonlinearity is defined on the whole $[0,T]\times E$.

To this aim, we consider a metric projection $P:[0,T]\times E\to Kv(E)$ by

$$P(t,x)=\{y\in X(t),\|x-y\|=dist(x,X(t))\}$$

and the multimap $\tilde{G}:[0,T]\times E\to Kv(E)$ defined by $\tilde{G}(t,x)=\overline{co}G(t,P(t,x))$. Following the lines of [29, Lemma 5.5.9], we have the following statement.

Lemma 4.7. *The multimap $\tilde{G}$ satisfies the following conditions:*

- (i) *the multifunction $\tilde{G}(\cdot,x):[0,T]\to Kv(E)$ admits a measurable selection for every $x\in E$;*
- (ii) *the multimap $\tilde{G}(t,\cdot):E\to Kv(E)$ is u.s.c. for a.e. $t\in[0,T]$;*
- (iii) *there exists a constant $R>0$ such that $\|\tilde{G}(t,x)\|\leqslant R$ for a.e $t\in[0,T]$ and $x\in E$;*
- (iv) *the multimap $\tilde{G}(t,\cdot):E\to Kv(E)$ is compact for a.e $t\in[0,T]$.*

We can consider now the Cauchy problem on $[0,T]\times E$:

$$^{C}D_0^q x(t)\in Ax(t)+\tilde{G}(t,x(t)),\ t\in[0,T], \tag{4.12}$$

$$x(0)=x_0. \tag{4.13}$$

From Lemma 4.7 and corresponding existence result ([7, Theorem 3]), we conclude that the set $\Sigma_{x_0}^{\tilde{G}}$ of all mild solutions to (4.12) - (4.13) is a nonempty and compact subset of $C([0,d];E)$. Moreover, from the result on the topological structure of the solution set (see [25, Theorem 3]), it follows that the set $\Sigma_{x_0}^{\tilde{G}}$ is connected.

Now, we can prove that the solution sets of problem (4.7) - (4.8) and (4.12) - (4.13) coincide.

Lemma 4.8. $\Sigma_{x_0}^{\tilde{G}}=\Sigma_{x_0}^{G}$.

Proof. In fact, let $x\in\Sigma_{x_0}^{\tilde{G}}$. Then

$$\begin{aligned}x(t)&\in\mathscr{G}(t)x_0+\int_0^t(t-s)^{q-1}\mathscr{T}(t-s)\tilde{G}(s,x(s))ds\\&=\mathscr{G}(t)x_0+\int_0^t(t-s)^{q-1}\mathscr{T}(t-s)\overline{co}G(s,P(s,x(s)))ds\\&\subseteq\mathscr{G}(t)x_0+\int_0^t(t-s)^{q-1}\mathscr{T}(t-s)\overline{co}G(s,X(s))ds\subset X(t).\end{aligned}$$

Hence $P(t,x(t))=\{x(t)\}$, so we have

$$x(t)=\mathscr{G}(t)x_0+\int_0^t(t-s)^{q-1}\mathscr{T}(t-s)f(s)ds,$$

where $f\in\mathscr{S}_{\tilde{G}(\cdot,x(\cdot))}=\mathscr{S}_{G(\cdot,x(\cdot))}$. Thus $x\in\Sigma_{x_0}^{G}$. The inclusion $\Sigma_{x_0}^{G}\subseteq\Sigma_{x_0}^{\tilde{G}}$ follows easily from the observation that $\Sigma_{x_0}^{G}\subset X$. $\square$

Problem (4.12) - (4.13) with u.s.c. nonlinear part is said to be associated with achy problem (1.1) - (1.2).

Theorem 4.1. *Under conditions* (A), $(F1)$, $(F2_L)$, $(F3_L)$, *the set* $\Sigma_{x_0}^F$ *of all mild solutions to problem* (1.1) - (1.2) *is connected. In particular, each set* $P(t) = \{x(t) : x \in \Sigma_{x_0}^F, t \in [0,T]\}$ *is also connected.*

Proof. In fact, let $x^1, x^2 \in \Sigma_{x_0}^F$ be mild solutions. They have the form

$$x^i(t) = \mathscr{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\phi^i(s)ds,$$

where $\phi^i \in \mathscr{P}_F(x^i)$, $i = 1,2$. Consider the multimaps $F^i(t,x) \subset F(t,x)$, $i = 1,2$ defined by

$$F^i(t,x) = \begin{cases} \{\phi^i(t)\}, & x \in x^i(t), \\ F(t,x), & x \notin x^i(t). \end{cases}$$

Since each function ϕ^i is measurable, one sees that there exists a sequence of disjoint compact sets $\{I_k\}, I_k \subset [0,T]$ such that $meas([0,T] \setminus \cup_k I_k) = 0$ and the restriction of ϕ^i on each I_k is continuous. Hence, each multimap F^i, $i = 1,2$ is a.l.s.c. and satisfies properties $(F2_L)$, $(F3_L)$. In accordance to Lemma 4.1, we construct a nonempty compact convex subset $X \subset C([0,T];E)$ containing mild solutions x^1, x^2 and invariant with respect to the action of Υ_F. Consequently, Υ_{F^i}, $i = 1,2$. For each of semilinear inclusions with nonlinearities F^1, F^2, let us pass to the associated differential inclusions with u.s.c. nonlinear parts $\tilde{G}^i$, $i = 1,2$. From construction (remind (4.6)), it follows that

$$\phi^i(t) = F^i(t, x^i(t)) = \gamma^i(t, x^i(t)) \in \tilde{G}^i(t, x^i(t))$$

for a.e. $t \in [0,T]$ and $i = 1,2$. Hence, each of x^i is the solution to the associated problem with $\tilde{G}^i$, $i = 1,2$.

Consider now the parametrized family of semilinear differential inclusions

$$^C D_0^q x(t) \in Ax(t) + \tilde{G}_\lambda(t, x(t)), \quad t \in [0,T], \lambda \in [0,1] \tag{4.14}$$

$$x(0) = x_0, \tag{4.15}$$

where the one-parameter family $\tilde{G}_\lambda$ is defined as

$$\tilde{G}_\lambda(t,x) = \begin{cases} \tilde{G}^1(t,x), & t \in [0, \lambda T]; \\ \overline{co}(\tilde{G}^1(t,x) \cup \tilde{G}^2(t,x)), & t = \lambda T; \\ \tilde{G}^2(t,x), & t \in [\lambda T, T]. \end{cases}$$

From [7, Theorem 3] and [25, Theorem 3], it follows that, for every $\lambda \in [0,1]$, $\Sigma_{x_0}^{\tilde{G}_\lambda}$ of mild solutions to (4.14) - (4.15) is a nonempty, compact, and connected subset of $C([0,T];E)$. It is easy to see that $\Sigma_{x_0}^{\tilde{G}_\lambda} \subset \Sigma_{x_0}^F$ for all $\lambda \in [0,1]$.

Moreover, family (4.14) - (4.15) satisfies conditions of the theorem on continuous dependence of the solutions set on a parameter (see [7, Theorem 4]), so the multimap $\lambda \multimap \Sigma_{x_0}^{\tilde{G}_\lambda}$ for all $\lambda \in [0,1]$, is u.s.c. and the set $\cup_{\lambda \in [0,1]} \Sigma_{x_0}^{\tilde{G}_\lambda}$ is connected (Proposition 2.1). It remains to observe only that $x^1 \in \Sigma_{x_0}^{\tilde{G}_1}$ and $x^2 \in \Sigma_{x_0}^{\tilde{G}_2}$. $\square$

Acknowledgments

The work was supported by the Russian Science Foundation (Project Number 22-71-10008). The work of J. C. Yao was partially supported by the grant MOST 111-2115-M-039-001-MY2.

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