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POROSITY OF THE FREE BOUNDARY IN A MINIMUM PROBLEM

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Abstract. Given a bounded domain $\Omega \subset \mathbb{R}^N (N \geq 2)$, a positive constant λ, and functions $q, h \in L^\infty(\Omega)$, we study geometric properties of non-negative minimizers of the minimum problem

$$\mathscr{J}(u) = \int_\Omega \left(A(|\nabla u|) + qF\left(u^+\right) + hu + \lambda \chi_{\{u>0\}}\right) \mathrm{d}x \to \min$$

over certain class $\mathscr{K}$ in the framework of Orlicz-Sobolev spaces, where u^+ denotes the positive part of u, $\chi_{\{\cdot\}}$ is the standard characteristic function, and the functions A and F satisfy the structural conditions of Lieberman-Tolksdorf's type. In particular, F is allowed to grow with a subcritical exponent. By using the technique of blow-up and the Harnack's inequality, we firstly prove the non-degeneracy of non-negative minimizers near the free boundary $\Gamma^+ := \partial\{u > 0\} \cap \Omega$, and then we show that the free boundary Γ^+ is locally porous. Furthermore, we also prove that $\{u > 0\}$ has a uniformly positive density.
Keywords. Density; Free boundary problem; Minimum problem; Orlicz space; Porosity.

1. INTRODUCTION

Let Ω be an open bounded domain in $\mathbb{R}^N (N \geq 2)$, q and h be functions that belong to $L^\infty(\Omega)$, and λ be a positive constant. Given functions a, $F \in C^1([0,+\infty);[0,+\infty))$ that satisfy $a(0) = F(0) = 0$ and the structural conditions of Lieberman-Tolksdorf's type (see [1, 2]), i.e.,

$$0 < \delta_0 \leq \frac{ta'(t)}{a(t)} \leq \delta_1, \quad \forall t > 0 \tag{1.1}$$

and

$$0 < 1 + \theta_0 \leq \frac{tF'(t)}{F(t)} \leq 1 + \theta_1, \quad \forall t > 0 \tag{1.2}$$

with some constants $\delta_0, \delta_1, \theta_0$, and θ_1 satisfying

$$1 + \delta_0 < N \quad \text{and} \quad 1 + \theta_0 \leq 1 + \theta_1 < \frac{N(1+\delta_0)}{N - (1+\delta_0)},$$

we consider the following minimum problem

$$\mathscr{J}(u) = \int_\Omega \left(A(|\nabla u|) + qF\left(u^+\right) + hu + \lambda \chi_{\{u>0\}}\right) \mathrm{d}x \to \min \tag{1.3}$$

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over the set

$$\mathscr{K} = \left\{ u \in W^{1,A}(\Omega);\, u-\varphi \in W_0^{1,A}(\Omega),\ u \geq 0 \text{ a.e. in } \Omega \right\},$$

where $A(t) := \int_0^t a(s)\mathrm{d}s$, φ is a non-negative function satisfying $\varphi \in W^{1,A}(\Omega) \cap L^\infty(\Omega)$, and $W^{1,A}(\Omega)$ and $W_0^{1,A}(\Omega)$ denote the Orlicz-Sobolev spaces corresponding to the function A (see Section 2 for definitions). Note that $\varphi \in \mathscr{K}$. Thus $\mathscr{K} \neq \emptyset$.

We observe that there is a wide range of functions a (or F) satisfying condition (1.1) (or (1.2)). A typical example of a corresponds to the p-Laplace case, namely,

$$a(t) = pt^{p-1}, \quad \forall p \in (1,+\infty). \tag{1.4}$$

Other examples of a include

$$a(t) = (1+t)\ln(1+t) - t,$$

for which (1.1) holds true with $\delta_0 = 1$ and $\delta_1 = 2$; and the case of a containing a variable exponent:

$$a(t) = \begin{cases} c_1 t^{p-1}, & 0 \leq t < t_0, \\ c_2 t^{g(t)-1}, & t \geq t_0, \end{cases}$$

where $t_0 > 1$, $c_1 > 0$, $c_2 > 0$, $p > 1$, and $g \in C^1[t_0,+\infty)$ satisfying

$$\begin{cases} c_3 \leq g'(t)t\ln t + g(t) - 1 \leq c_4, \quad \forall t \geq t_0 \\ p = g'(t_0)t_0 \ln t_0 + g(t_0) \\ c_1 = c_2 t_0^{g(t_0)-p} \end{cases}$$

with constants c_3 and c_4 satisfying $c_4 \geq c_3 > 0$. Note that $\frac{t(c_3 t^{g(t)-1})'}{c_3 t^{g(t)-1}} = tg'(t)\ln t + g(t) - 1$. By direct computations, one may verify that such a satisfies (1.1) with $\delta_0 = \min\{p-1, c_3\}$ and $\delta_1 = \max\{p-1, c_4\}$; see [3] for more examples of a or F.

For a governed by (1.4) and $F(t) = t^\gamma$ with $\gamma > 0$, minimum problem (1.3) is reduced to be the p-Laplace problem:

$$J(u) = \int_\Omega \left(|\nabla u|^p + q\left(u^+\right)^\gamma + hu + \lambda \chi_{\{u>0\}} \right) \mathrm{d}x \to \min, \tag{1.5}$$

which is often used to model the dynamics in the fields of, e.g., chemistry, physics, and aerodynamics. For example,

(i) the minimum problem (1.5) with $\lambda = 0$, $q \neq 0$, and $\gamma = 1$ refers to the obstacle problem, describing the problems of equilibrium of elastic membranes, fluid filtration in porous media, and control of temperature; see, e.g., [4];
(ii) the minimum problem (1.5) with $\lambda = 0$, $q \neq 0$, and general $\gamma \in (0,p)$ is known as the chemical reaction problem, which can be used to model the density of certain chemical specie in reaction with a porous catalyst pellet; see, e.g., [5];
(iii) the minimum problem (1.5) with $\lambda \neq 0$ and $q \equiv 0$ relates to the jets and cavities problem, which can be applied to the combustion theory [6] and the problems of dams [7] and heat flow [8], etc.

In the past few decades, great efforts have been devoted to investigating the existence and regularities of minimizers in Sobolev spaces or Orlicz-Sobolev spaces [9, 10, 11, 12, 13, 14, 15, 16, 17], as well as the geometric properties of the free boundary [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29], among which the latter brings more difficulties due to the fact that specific estimates of minimizers are needed, such as optimal growth and non-degeneracy of minimizers near the free boundary.

Regarding the minimum problem of the type of (1.5) with $\lambda = 0$, $q \neq 0$, and $\gamma = 1$, local porosity of the free boundary was obtained in [22] for $p \in (1,+\infty)$; and finite $(N-1)$-dimensional Hausdorff measure of the free boundary was studied in [18, 19] for $p = 2$, [23] for $p \in (2,+\infty)$, and [30] for $p \in (1,2)$, respectively. In a general case of problem (1.5), the minimum problem (1.3) with $F(u^+) = u^+$ and $\lambda = 0$ was considered in [31] and [32] under the framework of Orlicz-Sobolev spaces, and local porosity and finite $(N-1)$-dimensional Hausdorff measure of the free boundary were established, respectively. All the afore-mentioned problems were studied in the case of $h \equiv 0$.

As for the minimum problem of the type of (1.5) with $\lambda = 0$, $q \neq 0$, and $\gamma \in (0,1)$, finite $(N-1)$-dimensional Hausdorff measure of the free boundary was obtained in [27] and [28] for $p = 2$ and $h \equiv 0$, while geometric properties of the free boundary for the general $p \in (1,+\infty)$ and $\gamma \in (0, p)$ is less studied. It is worth mentioning that $C^{1,\alpha}$ regularity of minimizers in the two-phase case and $h \not\equiv 0$ was proved in [14], and [17, 29], under the framework of Sobolev spaces, and Sobolev-Orlicz spaces, respectively.

Concerning the minimum problem of the type of (1.5) with $\lambda > 0$, $q \equiv 0$, and $h \equiv 0$, it was shown in [11] and [33] that, in dimension 2, the free boundary is analytic when $p = 2$ and $p \in (2-\varepsilon_0,+\infty)$ with an absolute constant ε_0, respectively. In the general case of $N \geq 2$ and $p \in (1,+\infty)$, the authors of [20] overcame the non-uniform elliptic conditions and obtained the Lipschitz continuity and non-degeneracy of minimizers, which implied uniform density of positive sets and finite $(N-1)$-dimensional Hausdorff measure of the free boundary. It should be mentioned that the authors of [25] considered the problem containing (1.5) with $p \in [2,N)$, $q \not\equiv 0$, and $\gamma \in [1, p)$, and established the uniform density of positive sets and finite $(N-1)$-dimensional Hausdorff measure of the free boundary. In the setting of Orlicz-Sobolev spaces, for the minimum problem having a form like (1.3) with $\lambda > 0$, $q \equiv 0$, and $h \equiv 0$, the authors of [26] obtained several results on the regularity of minimizers and the free boundary, including Lipschitz continuity and non-degeneracy of minimizers, uniform density of positive sets, and finite $(N-1)$-dimensional Hausdorff measure of the free boundary, etc., while the authors of [34] proved that the free boundary is a real analytic hyperplane when $N \in (2, n_0)$ with $n_0 \in [5,7]$.

Although there has been a considerable effort devoted to the study of geometric properties of the free boundary, to the best of our knowledge, few results were reported on the porosity of the free boundary for minimum problem (1.3), which has a more general form and allows γ in (1.5) to be a subcritical exponent. It is worth mentioning that the study of porosity of sets appears naturally in some problems in real analysis, especially in the differentiation theory; see [35] for comprehensive surveys. As stated in [35], the notion of *porosity* of a set $E \subset \mathbb{R}^N$ at a point $x \in \mathbb{R}^N$ concerns the size of "hole" in the set E near to x. In particular, for a porous set E, it is not only nowhere dense but is "small" due to the fact that "holes" near to each point $x \in E$ are "big" in a certain sense.

In this paper, we aim at studying the porosity of the free boundary for the minimum problem (1.3) with $\lambda > 0$ and a subcritical exponent in Orlicz-Sobolev spaces. More specifically, by virtue of the results of existence and regularity of minimizers obtained in [17], we firstly establish non-degeneracy of minimizers near the free boundary by using the technique of blow-up and the Harnack's inequality. Then, we prove local porosity of the free boundary by means of optimal growth and non-degeneracy of minimizers. In addition, based on the porosity of the free boundary, we also show that the positive set is uniformly dense. It is worth noting that the *a priori* estimates obtained in [17] provide the foundation for us to apply the Harnack's inequality to our problem with subcritical exponents.

In the rest of the paper, we introduce first some notations. In Section 2, we introduce basic concepts of Orlicz-Sobolev spaces and some technical lemmas needed in the proofs of the main results. In Section 3, we establish the non-degeneracy of minimizers near the free boundary. In Section 4, we state and prove the main results obtained in this paper, namely, local porosity of the free boundary and uniform density of the positive set.

Notation. Denote by $B_r(x)$ a ball in Ω with centre $x \in \Omega$ and radius $r > 0$. If not explicitly stated otherwise, denote balls in Ω by B_r and B_R with the same centre, and radius r and R, respectively.

For a minimizer u of the functional $\mathscr{J}(u)$, let $\{u > 0\} := \{x \in \Omega;\ u(x) > 0\}$, and $\Gamma^+ := (\partial\{x \in \Omega;\ u(x) > 0\}) \cap \Omega$, which is the so-called free boundary.

For a measurable set $E \subset \mathbb{R}^N$, $\mathscr{L}^N(E)$ denotes the N-dimensional Lebesgue measure of E.

2. Preliminaries

2.1. **Properties of a and F.** Since a is strictly increasing in $[0,+\infty)$, the inverse of a exists. In the sequel, we denote the inverse of a by a^{-1}. Furthermore, define $\widetilde{A}(t) := \int_0^t a^{-1}(s)\mathrm{d}s$ for any $t \in [0,+\infty)$.

The following lemma presents basic properties of a and F.

Lemma 2.1 ([17, 26])**.** *The functions A, F, and a^{-1} satisfy the following properties:*

(i) $\min\left\{s^{\frac{1}{\delta_0}}, s^{\frac{1}{\delta_1}}\right\}a^{-1}(t) \le a^{-1}(st) \le \max\left\{s^{\frac{1}{\delta_0}}, s^{\frac{1}{\delta_1}}\right\}a^{-1}(t), \quad \forall s,t \ge 0;$

(ii) $\min\left\{s^{1+\delta_0}, s^{1+\delta_1}\right\}\frac{A(t)}{1+\delta_1} \le A(st) \le \max\left\{s^{1+\delta_0}, s^{1+\delta_1}\right\}(1+\delta_1)A(t), \quad \forall s,t \ge 0;$

(iii) $\min\left\{s^{1+\theta_0}, s^{1+\theta_1}\right\}F(t) \le F(st) \le \max\left\{s^{1+\theta_0}, s^{1+\theta_1}\right\}F(t), \quad \forall s,t \ge 0.$

2.2. **The Orlicz-Sobolev spaces $W^{1,A}(\Omega)$ and $W_0^{1,A}(\Omega)$.** Recall that the functional

$$\|u\|_{L^A(\Omega)} := \inf\left\{k > 0;\ \int_\Omega A\left(\frac{|u(x)|}{k}\right)\mathrm{d}x \le 1\right\}$$

is a norm on the Orlicz space $L^A(\Omega)$, which is the linear hull of the Orlicz class $\mathscr{K}_A(\Omega)$, that is, the smallest vector space containing $\mathscr{K}_A(\Omega)$ defined by

$$\mathscr{K}_A(\Omega) := \left\{u \text{ is measurable};\ \int_\Omega A(|u(x)|)\mathrm{d}x < \infty\right\}.$$

The Orlicz-Sobolev space $W^{1,A}(\Omega)$ is defined by

$$W^{1,A}(\Omega) := \left\{u \in L^A(\Omega);\ \nabla u \text{ exists in the weak sense and } |\nabla u| \in L^A(\Omega)\right\}$$

endowed with the norm

$$\|u\|_{W^{1,A}(\Omega)} := \|u\|_{L^A(\Omega)} + \|\nabla u\|_{L^A(\Omega)}.$$

Note that $L^A(\Omega)$ and $W^{1,A}(\Omega)$ are reflexive Banach spaces; see [26]. As in the case of ordinary Sobolev spaces, $W_0^{1,A}(\Omega)$ is taken to be the closure of $C_0^\infty(\Omega)$ in $W^{1,A}(\Omega)$.

The following imbedding theorem is used in this paper.

Lemma 2.2 ([26]). *$L^A(\Omega) \hookrightarrow L^{1+\delta_0}(\Omega)$ continuously.*

The following lemma presents a property of solutions to the A-harmonic equation.

Lemma 2.3 ([36]). *Let $u \in W^{1,A}(B_R)$. Suppose that v is a bounded weak solution of*

$$\operatorname{div} \frac{a(|\nabla v|)}{|\nabla v|}\nabla v = 0 \text{ in } B_R, \quad v - u \in W_0^{1,A}(B_R).$$

Then, for any $\kappa \in (0,N)$, there exists a positive constant C depending only on $\kappa, N, \delta_0, \delta_1$, and $\|v\|_{L^\infty(B_R)}$ such that

$$\int_{B_R} A(|\nabla u - \nabla v|)\,dx \leq C\left(\int_{B_R} (A(|\nabla u|) - A(|\nabla v|))\,dx + R^{\frac{\kappa}{2}}\left(\int_{B_R} (A(|\nabla u|) - A(|\nabla v|))\,dx\right)^{\frac{1}{2}}\right).$$

The following lemma is concerned with the Harnack's inequality for the A-Laplace equation.

Lemma 2.4 (Harnack's inequality, [1]). *Let $\hat{h} \in L^\infty(B_R)$. Suppose that $v \in W^{1,A}(B_R)$ with $0 \leq v \leq M$ is a weak solution to*

$$\operatorname{div} \frac{a(|\nabla v|)}{|\nabla v|}\nabla v = \hat{h} \text{ in } B_R.$$

Then, there exists a positive constant C depending only on N, δ_0, δ_1, M, and $R-r$ such that

$$\sup_{B_r} v \leq C\left(\inf_{B_r} v + a^{-1}\left(\|\hat{h}\|_{L^\infty(B_R)} R\right) R\right), \quad \forall r \in (0,R).$$

2.3. **Properties of minimizers of $\mathscr{J}(u)$.** First, we state a result on the existence of a minimizer for the functional $\mathscr{J}(u)$.

Lemma 2.5 ([17]). *The functional $\mathscr{J}(u)$ admits at least one minimizer $u \in \mathscr{K}$. Moreover, u is a weak solution to the equation*

$$\operatorname{div} \frac{a(|\nabla u|)}{|\nabla u|}\nabla u = qf(u) + h \text{ in } \{u > 0\},$$

where the function f is the derivative of F, i.e., $f(t) = F'(t)$ for all $t \geq 0$.

The following lemma states that the minimizers of $\mathscr{J}(u)$ are uniformly bounded in $L^\infty(\Omega) \cap W^{1,A}(\Omega)$.

Lemma 2.6 ([17]). *Let u be a minimizer of $\mathscr{J}(u)$ over the set $\mathscr{K}$. Then, $u \in L^\infty(\Omega)$. Furthermore, there exists a positive constant C_0 depending only on N, θ_0, θ_1, δ_0, δ_1, λ, $A(1)$, $\widetilde{A}(1)$, $\|h\|_{L^\infty(\Omega)}$, $\|q\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,A}(\Omega)}$, and the diameter of Ω such that*

$$\|u\|_{L^\infty(\Omega)} + \|u\|_{W^{1,A}(\Omega)} \leq C_0$$

holds true for all minimizers $u \in \mathscr{K}$.

The next lemma provides the local Log-Lipschitz regularity of minimizers of $\mathscr{J}(u)$, which will be used in the proof of main results.

Lemma 2.7 ([17]). *Let u be a minimizer of $\mathscr{J}(u)$ over the set $\mathscr{K}$. Then, u is locally Log-Lipschitz continuous. More precisely, for any $\Omega' \Subset \Omega$, there exists a positive constant C depending only on N, θ_0, θ_1, δ_0, δ_1, λ, $F(1)$, $A(1)$, $\widetilde{A}(1)$, $\|h\|_{L^\infty(\Omega)}$, $\|q\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,A}(\Omega)}$, Ω', and the diameter of Ω such that*

$$|u(x)-u(y)| \leq C|x-y||\log|x-y||, \quad \forall x,y \in \Omega'.$$

Therefore, $u \in C^{0,\tau}_{loc}(\Omega)$ for any $\tau \in (0,1)$.

The following result indicates that the minimizers of $\mathscr{J}(u)$ can not grow too fast near the free boundary.

Lemma 2.8 ([17]). *Let u be a minimizer of $\mathscr{J}(u)$ over the set $\mathscr{K}$. Let $x_0 \in \partial\{u>0\}$ and $B_{r_0}(x_0) \Subset \Omega$ with some $r_0 > 0$. Then, there exists a positive constant C_1 depending only on r_0, N, θ_0, θ_1, δ_0, δ_1, λ, $F(1)$, $A(1)$, $\widetilde{A}(1)$, $\|h\|_{L^\infty(\Omega)}$, $\|q\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,A}(\Omega)}$, and the diameter of Ω such that $|u(x)| \leq C_1|x-x_0|$ holds true for all $x \in B_r(x_0)$ and for all $r \in (0,r_0)$.*

3. Non-Degeneracy of Minimizers of $\mathscr{J}(u)$

As mentioned in the introduction, to obtain the geometric properties of the free boundary, it is necessary to first establish optimal growth and non-degeneracy of minimizers in the minimum problem. Since the optimal growth has been guaranteed by Lemma 2.8, we shall prove the non-degeneracy of minimizers of the functional $\mathscr{J}(u)$ near the free boundary Γ^+.

Proposition 3.1. *Let u be a minimizer of $\mathscr{J}(u)$ over the set $\mathscr{K}$. There exists a positive constant $\underline{c}$ depending only on N, θ_0, θ_1, δ_0, δ_1, λ, $F(1)$, $A(1)$, $\widetilde{A}(1)$, $\|h\|_{L^\infty(\Omega)}$, $\|q\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,A}(\Omega)}$, and the diameter of Ω such that, for any $x_0 \in \{u>0\} \cap \Omega'$ with arbitrary $\Omega' \Subset \Omega$, it holds that*

$$u(x_0) \geq \underline{c}\operatorname{dist}\left(x_0, \Gamma^+\right). \tag{3.1}$$

Proof. Given $x_0 \in \{u>0\} \cap \Omega'$ with $\Omega' \Subset \Omega$, it suffices to show that (3.1) holds true for $x_0 \in \{u>0\} \cap \Omega'$ satisfying $0 < \operatorname{dist}(x_0, \Gamma^+) \ll \Theta$ with a certain positive constant Θ depending only on N, θ_0, θ_1, δ_0, δ_1, λ, $F(1)$, $A(1)$, $\widetilde{A}(1)$, $\|h\|_{L^\infty(\Omega)}$, $\|q\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,A}(\Omega)}$, and the diameter of Ω. Let $d := \operatorname{dist}(x_0, \Gamma^+)$ and

$$v(x) := \frac{1}{d}u(x_0 + dx).$$

It is clear that v is a minimizer to the following functional

$$J(\xi) := \int_{B_1(0)} \left(A(|\nabla\xi|) + q(x_0+dx)F(d\xi^+) + dh(x_0+dx)\xi + \lambda\chi_{\{\xi>0\}}\right) \mathrm{d}x.$$

Moreover, v satisfies the following equation

$$\operatorname{div} \frac{a(|\nabla v|)}{|\nabla v|}\nabla v = dq(x_0+dx)f(dv(x)) + dh(x_0+dx) \text{ in } \{v>0\} \cap B_1(0).$$

The thesis of Proposition 3.1 is equivalent to proving that $v(0)$ is bounded away from zero. Without loss of generality, we assume that $d<1$. Since $F \in C^1([0,+\infty);[0,+\infty))$ and $\|u\|_{L^\infty(\Omega)} <$

C_0 (see Lemma 2.6), then $f(\cdot)$ is bounded over $[0,C_0]$. Then, applying Lemma 2.4 and Lemma 2.1(i), we arrive at

$$\begin{aligned}
\sup_{B_{2/3}(0)} v(x) \leq & C_2\left(\inf_{B_{2/3}(0)} v(x)+a^{-1}\left(\|dq\left(x_0+dx\right)f(dv)+dh\left(x_0+dx\right)\|_{L^\infty(B_1(0))}\right)\right)\\
\leq & C_2\left(\inf_{B_{2/3}(0)} v(x)+a^{-1}\left(d\|q\|_{L^\infty(\Omega)}\|f(u)\|_{L^\infty(\Omega)}+d\|h\|_{L^\infty(\Omega)}\right)\right)\\
\leq & C_2\left(\inf_{B_{2/3}(0)} v(x)+a^{-1}\left(dM_1\|q\|_{L^\infty(\Omega)}+d\|h\|_{L^\infty(\Omega)}\right)\right)\\
\leq & C_2\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}\left(M_1\|q\|_{L^\infty(\Omega)}+\|h\|_{L^\infty(\Omega)}\right)\right)\\
\leq & C_2\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right),
\end{aligned} \tag{3.2}$$

where $M_2 := M_1\|q\|_{L^\infty(\Omega)}+\|h\|_{L^\infty(\Omega)}$, $M_1 := \max_{s\in[0,C_0]} f(s)$, and C_2 depends only on N, θ_0, θ_1, δ_0, δ_1, λ, $F(1)$, $A(1)$, $\widetilde{A}(1)$, $\|h\|_{L^\infty(\Omega)}$, $\|q\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,A}(\Omega)}$, and the diameter of Ω. Let ψ be a non-negative, smooth, and radially symmetric cut-off function satisfying $\psi\equiv 0$ in $B_{1/5}(0)$ and $\psi\equiv 1$ in $B_1(0)\setminus B_{2/5}(0)$. Define the test function Ψ over $B_1(0)$ by

$$\Psi(x) := \min\left\{v,\ C_2\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)\psi(x)\right\},\quad \forall x\in B_1(0).$$

We observe that $\Psi\in W^{1,A}(B_1(0))$ and $\Psi\equiv v$ in $B_{2/3}(0)\setminus B_{2/5}(0)$. By the definition of $\Psi(x)$, we have

$$B_{1/5}(0)\subset\Pi := \left\{y\in B_{2/5}(0);\ C_2\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)\psi(y)<v(y)\right\}\subset B_{2/5}(0).$$

By the minimality of v, we obtain

$$\begin{aligned}
&\int_\Pi \left(A\left(|\nabla v|\right)+q\left(x_0+dx\right)F(dv)+dh\left(x_0+dx\right)v+\lambda\chi_{\{v>0\}}\right)\mathrm{d}x\\
\leq &\int_\Pi \left(A\left(|\nabla \Psi|\right)+q\left(x_0+dx\right)F(d\Psi)+dh\left(x_0+dx\right)\Psi+\lambda\chi_{\{\Psi>0\}}\right)\mathrm{d}x.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\int_\Pi \left(\lambda\left(1-\chi_{\{\Psi>0\}}\right)+q\left(x_0+dx\right)\left(F(dv)-F(d\Psi)\right)+dh\left(x_0+dx\right)(v-\Psi)\right)\mathrm{d}x\\
\leq &\int_\Pi \left(A\left(|\nabla\Psi|\right)-A\left(|\nabla v|\right)\right)\mathrm{d}x.
\end{aligned} \tag{3.3}$$

Note that A is a non-negative function. By Lemma 2.1(ii), we estimate the right-hand side of (3.3) as below:

$$
\begin{aligned}
&\int_{\Pi}\left(A\left(|\nabla\Psi|\right)-A\left(|\nabla v|\right)\right)\mathrm{d}x\\
\leq&\int_{\Pi}A\left(|\nabla\Psi|\right)\mathrm{d}x\\
=&\int_{\Pi}A\left(C_2\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)|\nabla\psi(x)|\right)\mathrm{d}x\\
\leq& C_3\max\left\{\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)^{1+\delta_0},\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)^{1+\delta_1}\right\}\int_{\Pi}A\left(\|\nabla\psi\|_{L^\infty(B_{2/5}(0))}\right)\mathrm{d}x\\
:=& C_4\max\left\{\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)^{1+\delta_0},\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)^{1+\delta_1}\right\},
\end{aligned}
\tag{3.4}
$$

where C_3 depends only on C_2, δ_0, and δ_1. The left-hand side of (3.3) becomes

$$
\int_{\Pi}\lambda\left(1-\chi_{\{\Psi>0\}}\right)\mathrm{d}x=\int_{\Pi}\lambda\chi_{\{\Psi=0\}}\mathrm{d}x\geq\lambda|B_{1/5}(0)|. \tag{3.5}
$$

By (3.2) and the definition of Π, we have

$$
\begin{aligned}
\int_{\Pi}dh\left(x_0+dx\right)\left(v-\Psi\right)\mathrm{d}x\leq&2\int_{\Pi}d\|h\|_{L^\infty(\Omega)}|v|\mathrm{d}x\\
\leq& C_2\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)\int_{B_{2/5}(0)}2d\|h\|_{L^\infty(\Omega)}\mathrm{d}x\\
:=& C_5d\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right).
\end{aligned}
\tag{3.6}
$$

Similarly, by the definition of F, Lemma 2.1(iii) and (3.2), we obtain

$$
\begin{aligned}
&\int_{\Pi}q\left(x_0+dx\right)\left(F\left(dv\right)-F\left(d\Psi\right)\right)\mathrm{d}x\\
\leq&2\int_{\Pi}|q\left(x_0+dx\right)|F\left(dv\right)\mathrm{d}x\\
\leq&2\int_{\Pi}\|q\|_{L^\infty(\Omega)}F\left(dv\right)\mathrm{d}x\\
\leq& C_6\max\left\{\left(dv\right)^{1+\theta_0},\left(dv\right)^{1+\theta_1}\right\}\int_{\Pi}F(1)\mathrm{d}x\\
\leq& C_7\max\left\{\left(d\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)\right)^{1+\theta_0},\left(d\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)\right)^{1+\theta_1}\right\},
\end{aligned}
\tag{3.7}
$$

where C_6 depends only on $\|q\|_{L^\infty(\Omega)}$, and C_7 depends only on C_6, C_2, N, θ_0, θ_1, and $F(1)$. Thus, based on (3.4), (3.5), (3.6), and (3.7), we derive

$$\begin{aligned}\lambda|B_{1/5}(0)| \leq & C_5 d\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)\\ & + C_4\max\left\{\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)^{1+\delta_0}, \left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)^{1+\delta_1}\right\}\\ & + C_7\max\left\{\left(d\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)\right)^{1+\theta_0}, \left(d\left(v(0)+d^{\frac{1}{\delta_1}}a^{-1}(M_2)\right)\right)^{1+\theta_1}\right\}.\end{aligned}$$

Now choosing sufficiently small Θ, which implies that d is sufficiently small, we conclude that $v(0)\geq \underline{c}>0$, where $\underline{c}$ is a positive constant depending only on N, θ_0, θ_1, δ_0, δ_1, λ, $F(1)$, $A(1)$, $\widetilde{A}(1)$, $\|h\|_{L^\infty(\Omega)}$, $\|q\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,A}(\Omega)}$, and the diameter of Ω. □

By virtue of Proposition 3.1, we prove the following non-degeneracy of minimizers near the free boundary.

Proposition 3.2. *Let u be a minimizer of $\mathscr{J}(u)$ over the set $\mathscr{K}$. There exists a positive constant $\underline{C}$ depending only on N, θ_0, θ_1, δ_0, δ_1, λ, $F(1)$, $A(1)$, $\widetilde{A}(1)$, $\|h\|_{L^\infty(\Omega)}$, $\|q\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,A}(\Omega)}$, and the diameter of Ω, such that, for any $x_0\in\partial\{u>0\}\cap\Omega'$ with $\Omega'\Subset\Omega$, it holds that*

$$\sup_{B_r(x_0)} u\geq \underline{C}r, \quad \forall r\in\left(0,\operatorname{dist}(\partial\Omega',\partial\Omega)\right).$$

Proof. By continuity, it suffices to show u is non-degenerate in the set $\Omega'^+ := \{u>0\}\cap\Omega'$. We initially prove that there exists a constant $\omega_0>0$ depending only on N, θ_0, θ_1, δ_0, δ_1, λ, $F(1)$, $A(1)$, $\widetilde{A}(1)$, $\|h\|_{L^\infty(\Omega)}$, $\|q\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,A}(\Omega)}$, and the diameter of Ω such that, for $x\in\Omega'^+$,

$$\sup_{B_{d(x)}(x_0)} u\geq (1+\omega_0)u(x_0),$$

where $d(x):=\operatorname{dist}(x,\Gamma^+)$. We prove by contradiction. Suppose that such ω_0 does not exist. Then there exist sequences of $\omega_j=o(1)$ and $x_j\in\Omega'^+$ such that

$$\sup_{B_{d_j}(x_j)} u\leq (1+\omega_j)u(x_j),$$

where $d_j:=\operatorname{dist}(x_j,\Gamma^+)=o(1)$. Let $\widetilde{\rho}_j(z):=\frac{u(x_j+d_jz)}{d_j}$. It is clear that $0\leq\widetilde{\rho}_j(z)\leq\frac{(1+\omega_j)u(x_j)}{d_j}$ in $B_1(0)$. Define $\Phi_j(z):=\frac{(1+\omega_j)u(x_j)}{d_j}-\widetilde{\rho}_j(z)$ in $B_1(0)$. Then $0\leq\Phi_j(z)\leq\frac{(1+\omega_j)u(x_j)}{d_j}$ in $B_1(0)$.

Note that $\Phi_j(z)$ satisfies

$$
\begin{aligned}
|\operatorname{div}_z \frac{a\left(|\nabla\Phi_j(z)|\right)}{|\nabla\Phi_j(z)|}\nabla\Phi_j(z)| =& |\operatorname{div}_z \frac{a(|\nabla\widetilde{\rho}_j(z)|)}{|\nabla\widetilde{\rho}_j(z)|}\nabla\widetilde{\rho}_j(z)| \\
=& |\operatorname{div}_y \frac{a(|\nabla u(y)|)}{|\nabla u(y)|}\nabla u(y)|d_j \\
=& |q\left(x_j+d_jz\right)f(u)+h\left(x_j+d_jz\right)|d_j \\
:=& \hat{h}d_j.
\end{aligned}
$$

By Lemma 2.4, for any $r_1<1$, we arrive at

$$
\begin{aligned}
0\le& \sup_{B_{r_1}(0)}\Phi_j(z) \\
\le& C_8\left(\inf_{B_{r_1}(0)}\Phi_j(z)+a^{-1}\left(d_j\|\hat{h}\|_{L^\infty(B_1(0))}\right)\right) \\
\le& C_8\left(\Phi_j(0)+a^{-1}\left(d_j\left(M_1\|q\|_{L^\infty(\Omega)}+\|h\|_{L^\infty(\Omega)}\right)\right)\right) \\
\le& C_8\left(\Phi_j(0)+d_j^{\frac{1}{\delta_1}}a^{-1}\left(M_1\|q\|_{L^\infty(\Omega)}+\|h\|_{L^\infty(\Omega)}\right)\right),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
0\le& \sup_{B_{r_1}(0)}\left(\frac{(1+\omega_j)u(x_j)}{d_j}-\widetilde{\rho}_j(z)\right) \\
\le& C_8\left(\left(\frac{(1+\omega_j)u(x_j)}{d_j}-\widetilde{\rho}_j(0)\right)+d_j^{\frac{1}{\delta_1}}a^{-1}\left(M_1\|q\|_{L^\infty(\Omega)}+\|h\|_{L^\infty(\Omega)}\right)\right), \qquad (3.8)
\end{aligned}
$$

where C_8 depends only on N, θ_0, θ_1, δ_0, δ_1, λ, $F(1)$, $A(1)$, $\widetilde{A}(1)$, $\|h\|_{L^\infty(\Omega)}$, $\|q\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,A}(\Omega)}$, and the diameter of Ω. Define $\rho_j(z):=\frac{d_j}{u(x_j)}\widetilde{\rho}_j(z)$ in $B_1(0)$. By (3.8) and observing $\rho_j(0)=1$, we have

$$
\begin{aligned}
0\le& \left(\left(1+\omega_j\right)-\rho_j(z)\right) \\
\le& C_8\left(1+\omega_j-\rho_j(0)+d_j^{\frac{1}{\delta_1}}a^{-1}\left(M_1\|q\|_{L^\infty(\Omega)}+\|h\|_{L^\infty(\Omega)}\right)\frac{d_j}{u(x_j)}\right) \\
\le& C_8\left(\omega_j+\frac{1}{\underline{c}}d_j^{\frac{1}{\delta_1}}a^{-1}\left(M_1\|q\|_{L^\infty(\Omega)}+\|h\|_{L^\infty(\Omega)}\right)\right), \qquad (3.9)
\end{aligned}
$$

where in the last inequality we used Proposition 3.1. For any z_1 and $z_2\in B_{r_1}(0)$, by (3.9), it holds that

$$
\begin{aligned}
|\rho_j(z_1)-\rho_j(z_2)|\le& |(1+\omega_j)-\rho_j(z_1)|+|(1+\omega_j)-\rho_j(z_2)| \\
\le& 2C_8\left(\omega_j+\frac{1}{\underline{c}}d_j^{\frac{1}{\delta_1}}a^{-1}\left(M_1\|q\|_{L^\infty(\Omega)}+\|h\|_{L^\infty(\Omega)}\right)\right)\to 0 \text{ as } j\to\infty.
\end{aligned}
$$

Therefore, ρ_j is equi-continuous in $B_{r_1}(0)$. Then, we deduce by (3.9) that $\rho_j(z)\to\rho(z)\equiv 1$ in $B_{r_1}(0)$, and the arbitrariness of r_1 implies that $\rho(z)\equiv 1$ in $B_1(0)$. In order to obtain a contradiction, let $y_j\in\Gamma^+$ be such that $d_j=|x_j-y_j|$. Up to subsequence, there would hold

$\rho_j\left(\frac{y_j-x_j}{d_j}\right)=0$, which is a contradiction for $j\gg 1$. At last, we derive the non-degeneracy of minimizers by using an argument of the Caffarelli's polygonal curve. We construct a polygonal curve along which u grows linearly. Starting from $x_0=x$, we find a sequence of points $\{x_i\}_{i\geq 0}$ satisfying

(i) $u(x_i)\geq(1+\omega_0)^i u(x_0)$,
(ii) $|x_i-x_{i-1}|\leq \operatorname{dist}(x_{i-1},\Gamma^+)$,
(iii) $u(x_i)-u(x_{i-1})\geq c_i|x_i-x_{i-1}|$, in particular, $u(x_i)-u(x_0)\geq c|x_i-x_0|$,

where c_i and c depend only on N, θ_0, θ_1, δ_0, δ_1, λ, $F(1)$, $A(1)$, $\widetilde{A}(1)$, $\|h\|_{L^\infty(\Omega)}$, $\|q\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,A}(\Omega)}$, and the diameter of Ω. Since $u(x_i)\to\infty$ as $i\to\infty$, the above process must be finite, that is, there exists a last point x_{i_0} in $B_r(x_0)$. For such a last point, there is a positive constant c' such that $|x_{i_0}-x_0|\geq c'r$. Finally, we have

$$\sup_{B_r(x_0)} u\geq u(x_{i_0})\geq u(x_0)+c|x_{i_0}-x_0|\geq cc'r:=\underline{C}r.$$

□

4. Main Results

In this section, by virtue of the optimal growth and non-degeneracy of minimizers (see Lemma 2.8 and Proposition 3.2, respectively), we show that the free boundary is locally porous. Furthermore, we show that the positive set has a uniform density.

We firstly state the concept of porosity.

Definition 4.1. A set E is said to be porous with porosity constant $\sigma\in(0,1)$ if there exists a constant $r_2>0$ such that

$$\forall x\in E,\ \forall r\in(0,r_2)\Rightarrow \exists y\in\mathbb{R}^N \text{ s.t. } B_{\sigma r}(y)\subset B_r(x)\setminus E.$$

It is well known that the Hausdorff dimension of a porous set does not exceed $N-C\sigma^N$, where $C=C(N)>0$ is a constant that depends only on N. Thus, the N-dimensional Lebesgue measure of the porous set is zero; see, e.g., [22, 37].

The following theorem is the first main result obtained in this paper.

Theorem 4.1. *Let u be a minimizer of $\mathscr{J}(u)$ over the set $\mathscr{K}$. Then, for every compact set $K\subset\Omega$, the intersection $K\cap\partial\{u>0\}$ is porous with porosity constant σ depending only on N, θ_0, θ_1, δ_0, δ_1, λ, $F(1)$, $A(1)$, $\widetilde{A}(1)$, $\|h\|_{L^\infty(\Omega)}$, $\|q\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,A}(\Omega)}$, and the diameter of Ω.*

Proof. The proof can be proceeded in the same way as in [22]. For convenience of readers, we provide the details.

Without loss of generality, we assume that the compact K is the closed unit ball $\overline{B_1(0)}$, and that $\overline{B_2(0)}\subset\Omega$. Assume that $r_0=1$ in Lemma 2.8. For any $x\in\overline{B_1(0)}\cap\{u>0\}$, we define

$$d_x:=\operatorname{dist}\left(x,\overline{B_1(0)}\setminus\Omega^+\right).$$

Let $z\in B_1(0)\cap\partial\{u>0\}$. For any $r\in\left(0,\frac{1}{3}\right)$, by the non-degeneracy of minimizers, there exist a positive constant $\underline{C}$ determined by Proposition 3.2 and a point $Y\in B_r(z)\cap\Omega^+$, such that

$$u(Y)\geq\underline{C}r. \tag{4.1}$$

For such Y, we take $z_Y \in \overline{B_1(0)} \cap \partial\{u>0\}$ with $|Y - z_Y| = d_Y$. Note that $d_Y \le r$. By Lemma 2.8, we have

$$u(x) \le C_1 |x - z_Y|, \quad \forall x \in B_r(z_Y). \tag{4.2}$$

We infer from (4.1) and (4.2) that

$$\underline{C} r \le u(Y) \le C_1 d_Y.$$

Let $\sigma := \frac{\underline{C}}{C_1}$. Then $\sigma r \le d_Y$ and $\sigma \le 1$. Therefore, $B_{\sigma r}(Y) \cap B_r(z) \subset \{u>0\}$. Let y_0 be on the line with z and Y as endpoints and satisfy $|y_0 - Y| = \frac{\sigma r}{2}$. We claim that

$$B_{\sigma r/2}(y_0) \subset B_{\sigma r}(Y) \cap B_r(z) \subset B_r(z) \setminus (\partial\{u>0\}) \subset B_r(z) \setminus \left(\overline{B_1(0)} \cap \partial\{u>0\} \right).$$

This shows that $\overline{B_1(0)} \cap \partial\{u>0\}$ is porous with the porosity constant $\frac{\sigma}{2}$. Indeed, on the one hand, for every $y \in B_{\sigma r/2}(y_0)$, it follows that

$$|y - Y| \le |y - y_0| + |y_0 - Y| < \frac{\sigma r}{2} + \frac{\sigma r}{2} = \sigma r.$$

On the other hand, it holds that

$$|y - z| \le |y - y_0| + |y_0 - z| = |y - y_0| + (|z - Y| - |y_0 - Y|) < \frac{\sigma r}{2} + \left(r - \frac{\sigma r}{2} \right) = r.$$

□

Now we state the second main result obtained in this paper.

Theorem 4.2. *Let $x_0 \in \partial\{u>0\} \cap \Omega'$ with $\Omega' \Subset \Omega$. Then, for any $0 < r \ll 1$, it holds that*

$$c \le \frac{\mathscr{L}^N (B_r(x_0) \cap \{u>0\})}{\mathscr{L}^N (B_r(x_0))} \le 1 - c, \tag{4.3}$$

where $c \in (0,1)$ is a constant depending only on N, θ_0, θ_1, δ_0, δ_1, λ, $F(1)$, $A(1)$, $\widetilde{A}(1)$, $\|h\|_{L^\infty(\Omega)}$, $\|q\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,A}(\Omega)}$, and the diameter of Ω.

Proof. In view of the local porosity of the free boundary, there are a positive constant σ and a ball $B_{\sigma r}$ with radius σr such that

$$\frac{\mathscr{L}^N (B_r(x_0) \cap \{u>0\})}{\mathscr{L}^N (B_r(x_0))} \ge \frac{\mathscr{L}^N (B_{\sigma r})}{\mathscr{L}^N (B_r(x_0))} = \sigma^N,$$

which implies the first inequality presented in (4.3).

In order to prove the second inequality presented in (4.3), we argue by contradiction. Assume that there exists a sequence of r_j with $r_j \to 0$ as $j \to \infty$ such that

$$\mathscr{L}^N \left(B_{r_j}(x_0) \cap \{u=0\} \right) = o\left(r_j^N \right).$$

Define

$$v_j(z) := \frac{u(x_0 + r_j z)}{r_j}, \quad \forall z \in B_1(0).$$

It follows that

$$\mathscr{L}^N \left(B_1(0) \cap \{v_j = 0\} \right) = o(1).$$

In view of the optimal growth of u (Lemma 2.8), we observe that v_j is uniformly bounded with respect to j. Moreover, v_j is a minimizer of the following functional

$$J_j(\xi) := \int_{B_1(0)} \left(A\left(|\nabla\xi|\right) + q\left(x_0 + r_j x\right) F\left(r_j \xi^+\right) + r_j h\left(x_0 + r_j x\right)\xi + \lambda\chi_{\{\xi>0\}}\right) \mathrm{d}x.$$

Let $w_j \in W^{1,A}(B_1(0))$ be the solution to the equation

$$\operatorname{div} \frac{a(|\nabla w_j|)}{|\nabla w_j|}\nabla w_j = 0 \text{ in } B_1(0),\ w_j = v_j \text{ on } \partial B_1(0).$$

Note that w_j is uniformly bounded due to the fact that v_j is uniformly bounded. In addition, by comparison principle (see [26, Lemma 2.8]) and regularity of w_j (see [1, Thereom 1.7]), we have $w_j \geq 0$ in $B_1(0)$. By the minimality of v_j, we obtain

$$\begin{aligned}
&\int_{B_1(0)} \left(A\left(|\nabla v_j|\right) - A\left(|\nabla w_j|\right)\right) \mathrm{d}x \\
\leq &\int_{B_1(0)} \lambda\left(\chi_{\{w_j>0\}} - \chi_{\{v_j>0\}}\right) \mathrm{d}x + \int_{B_1(0)} q\left(x_0 + r_j x\right)\left(F\left(r_j w_j\right) - F\left(r_j v_j\right)\right) \mathrm{d}x \\
&+ \int_{B_1(0)} r_j h\left(x_0 + r_j x\right)(w_j - v_j)\mathrm{d}x \\
= &\int_{B_1(0)\cap\{v_j>0\}} \lambda\left(\chi_{\{w_j>0\}} - \chi_{\{v_j>0\}}\right) \mathrm{d}x + \int_{B_1(0)\cap\{v_j=0\}} \lambda\left(\chi_{\{w_j>0\}} - \chi_{\{v_j>0\}}\right) \mathrm{d}x \\
&+ \int_{B_1(0)} q\left(x_0 + r_j x\right)\left(F\left(r_j w_j\right) - F\left(r_j v_j\right)\right) \mathrm{d}x + \int_{B_1(0)} r_j h\left(x_0 + r_j x\right)(w_j - v_j)\mathrm{d}x \\
\leq &\lambda\mathscr{L}^N\left(B_1(0)\cap\{v_j = 0\}\right) + \int_{B_1(0)} \|q\|_{L^\infty(\Omega)}\left(F(r_j w_j) + F(r_j v_j)\right) \mathrm{d}x \\
&+ \int_{B_1(0)} r_j \|h\|_{L^\infty(\Omega)}(w_j + v_j)\mathrm{d}x \\
\leq &o(1) + r_j^{1+\theta_0} \int_{B_1(0)} \|q\|_{L^\infty(\Omega)}\left(F\left(\|w_j\|_{L^\infty(B_1(0))}\right) + F\left(\|v_j\|_{L^\infty(B_1(0))}\right)\right) \mathrm{d}x \\
&+ r_j \int_{B_1(0)} \|h\|_{L^\infty(\Omega)}\left(\|w_j\|_{L^\infty(B_1(0))} + \|v_j\|_{L^\infty(B_1(0))}\right) \mathrm{d}x \\
\leq &o(1) + r_j^{1+\theta_0} \int_{B_1(0)} 2\|q\|_{L^\infty(\Omega)} F\left(\|v_j\|_{L^\infty(B_1(0))}\right) \mathrm{d}x + r_j \int_{B_1(0)} 2\|h\|_{L^\infty(\Omega)}\|v_j\|_{L^\infty(B_1(0))}\mathrm{d}x.
\end{aligned}$$

Then, we deduce that

$$\int_{B_1(0)} \left(A\left(|\nabla v_j|\right) - A\left(|\nabla w_j|\right)\right) \mathrm{d}x \to 0 \text{ as } j \to \infty,$$

which along with Lemma 2.3 implies that

$$\int_{B_1(0)} A\left(|\nabla v_j - \nabla w_j|\right) \mathrm{d}x \to 0 \text{ as } j \to \infty.$$

Note that $\mu_j := v_j - w_j \in W_0^{1,A}(B_1(0))$. By Lemma 2.2, we have $\mu_j = v_j - w_j \in W_0^{1,1+\delta_0}(B_1(0))$. Hence $\mu_j \to 0$ in $W_0^{1,1+\delta_0}(B_1(0))$. For any ball B with centre 0 and satisfying $B \Subset B_1(0)$, we infer from Lemma 2.7 and [1, Thereom 1.7] that

$$\|v_j\|_{C^\alpha(B)} \leq C' \quad \text{and} \quad \|w_j\|_{C^{1,\alpha}(B)} \leq C'',$$

where C' is a positive constant determined by C of Lemma 2.7, and C'' is a positive constant depending only on $\alpha, N, \delta_0, \delta_1, A(1)$, and $\|v_j\|_{L^\infty(B_1(0))}$. Note that v_j is uniformly bounded with respect to j. Thus, the constant C'' is independent of j. Therefore, there exist subsequences, denoted also by v_j and w_j, and functions $v_0 \in C(B')$ and $w_0 \in C^1(B')$ for any ball $B' \Subset B$ with centre 0 such that

$$\begin{aligned}
&v_j \to v_0 \text{ uniformly in } B,\\
&w_j \to w_0 \text{ uniformly in } B',\\
&\nabla w_j \to \nabla w_0 \text{ uniformly in } B',\\
&\nabla v_j \rightharpoonup \nabla v_0 \text{ in } L^{1+\delta_0}(B_1(0)),\\
&\mu_j = v_j - w_j \to 0 \text{ uniformly in } B'.
\end{aligned}$$

Thus, $v_0 = w_0$ in B'. By the Harnack's inequality, we have $\sup_{B_s(0)} w_j \leq C \inf_{B_s(0)} w_j$ for any positive constant s satisfying $B_s(0) \Subset B' \Subset B_1(0)$, where C is a positive constant determined by Lemma 2.4. Passing to the limit, we have $\sup_{B_s(0)} v_0 \leq C \inf_{B_s(0)} v_0$. Since $v_0(0) = 0$, we have $v_0 \equiv 0$ in $B_s(0)$.

On the other hand, by the non-degeneracy property (Proposition 3.2), for sufficiently small s, it holds that $\sup_{B_s(0)} v_j \geq \underline{C}s > 0$ with a positive constant $\underline{C}$, and hence $\sup_{B_s(0)} v_0 \geq \underline{C}s > 0$. We obtain a contradiction.

□

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