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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR CRITICAL KIRCHHOFF-CHOQUARD EQUATIONS INVOLVING THE FRACTIONAL p-LAPLACIAN ON THE HEISENBERG GROUP

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Abstract. In this paper, we study the existence and multiplicity of solutions for the following Kirchhoff-Choquard type equation involving the fractional p-Laplacian on the Heisenberg group:

$$M(\|u\|_\mu^p)(\mu(-\Delta)_p^s u + V(\xi)|u|^{p-2}u) = f(\xi,u) + \int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta |u|^{Q_\lambda^*-2}u \quad \text{in } \mathbb{H}^N,$$

where $(-\Delta)_p^s$ is the fractional p-Laplacian on the Heisenberg group $\mathbb{H}^N$, M is the Kirchhoff function, $V(\xi)$ is the potential function, $0<s<1$, $1<p<\frac{N}{s}$, $\mu>0$, $f(\xi,u)$ is the nonlinear function, $0<\lambda<Q$, $Q=2N+2$, and $Q_\lambda^* = \frac{2Q-\lambda}{Q-2}$ is the Sobolev critical exponent. Using the Krasnoselskii genus theorem, the existence of infinitely many solutions is obtained if μ is sufficiently large. In addition, using the fractional version of the concentrated compactness principle, we prove that problem has m pairs of solutions if μ is sufficiently small. As far as we know, the results of our study are new even in the Euclidean case.
Keywords. Fractional concentration-compactness principle; Krasnoselskii genus; Kirchhoff-Choquard type equations; Heisenberg group.

1. INTRODUCTION

In this paper, we study the existence and multiplicity of solutions for the following Kirchhoff-Choquard type equation involving the fractional p-Laplacian on the Heisenberg group of the form:

$$M(\|u\|_\mu^p)(\mu(-\Delta)_p^s u + V(\xi)|u|^{p-2}u) = f(\xi,u) + \int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta |u|^{Q_\lambda^*-2}u \quad \text{in } \mathbb{H}^N, \tag{1.1}$$

where $(-\Delta)_p^s$ is the fractional p-Laplacian on the Heisenberg group $\mathbb{H}^N$, M is the Kirchhoff function, $V(\xi)$ is the potential function, $0<s<1$, $1<p<\frac{N}{s}$, $f(\xi,u)$ is the nonlinear function, $\mu>0$, $0<\lambda<Q$, $Q=2N+2$, and $Q_\lambda^* = \frac{2Q-\lambda}{Q-2}$ is the Sobolev critical exponent.

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Suppose that the Kirchhoff function M and potential function V satisfy the following assumptions:
(M) $M \in C(\mathbb{R},\mathbb{R})$, and there exist $\tau \in (1, \frac{Q_\lambda^*}{p}]$ and $0 < m_0 \leq m_1$ satisfying

$$m_0 t^{\tau-1} \leq M(t) \leq m_1 t^{\tau-1} \quad \text{for every } t \in \mathbb{R}_0^+,$$

that is, M is non-degenerate.

(V_1) $V(\xi) \in C(\mathbb{H}^N,\mathbb{R})$ with $V(\xi) \geq \min V(\xi) = 0$;

(V_2) there exists $R > 0$ satisfying $\lim_{|\eta|\to\infty} \mathrm{meas}(\{\xi \in B_R(\eta) : V(\xi) \leq c\}) = 0$ for every $c > 0$, where $\mathrm{meas}(\cdot)$ denotes the Lebesgue measure on $\mathbb{H}^N$.

The nonlinearity $f(\cdot,\cdot) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which requires different assumptions for critical growth and subcritical growth, respectively. For the case of critical exponent $\tau = \frac{Q_\lambda^*}{p}$, f satisfies the following assumptions:

(f_1) there exists $q \in (p, Q_\lambda^*)$ such that, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ satisfying

$$|f(\xi,t)| \leq p\varepsilon|t|^{p-1} + qC_\varepsilon|t|^{q-1} \quad \text{a.e. } \xi \in \mathbb{H}^N \quad \text{and} \quad \text{for every } t \in \mathbb{R};$$

(f_2) there exist $a_1 > 0$, $q_1 \in (p, Q_\lambda^*)$ satisfying

$$F(\xi,t) = \int_0^t f(\xi,s)ds \geq a_1|t|^{q_1} \quad \text{a.e. } \xi \in \mathbb{H}^N \quad \text{and} \quad \text{for every } t \in \mathbb{R}.$$

For the case of subcritical exponent $\tau \in (1, \frac{Q_\lambda^*}{p})$, the following conditions should be satisfied for f:

$(f_1)'$ there exists $q \in (\tau p, Q_\lambda^*)$ such that, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ satisfying

$$|f(\xi,t)| \leq \tau p\varepsilon|t|^{\tau p-1} + qC_\varepsilon|t|^{q-1} \quad \text{a.e. } \xi \in \mathbb{H}^N \quad \text{and} \quad \text{for every } t \in \mathbb{R};$$

$(f_2)'$ there exist $a_2 > 0$ and $q_2 \in (\tau p, Q_\lambda^*)$ satisfying

$$F(\xi,t) \geq a_2|t|^{q_2} \quad \text{a.e. } \xi \in \mathbb{H}^N \quad \text{and} \quad \text{for every } t \in \mathbb{R};$$

$(f_3)'$ there exists $q_0 \in (\frac{m_1\tau p}{m_0}, Q_\lambda^*)$ satisfying $q_0F(\xi,t) \leq f(\xi,t)t$ for every $(\xi,t) \in \mathbb{H}^N \times \mathbb{R}$, where m_0 and m_1 are the numbers from the condition (M).

Pohožaev [15] was the first to study Kirchhoff equation problems, and he proved the unique solvability of the mixed problems of quasi-linear hyperbolic Kirchhoff equations with Dirichlet boundary conditions. Since then, Kirchhoff type problems have received increasing attention, especially in various models of biological and physical systems. More recently, Fiscella and Valdinoci [6] discussed in detail the physical significance of the fractional Kirchhoff problem and its applications, and proposed a stable Kirchhoff variational problem as a very realistic model. If the nonlinear term has the convolution form, many interesting results were obtained for this kind of problem. For example, Fan [5] considered the following fractional Choquard-Kirchhoff equation with subcritical or critical nonlinearity of the form:

$$\begin{cases} M([u]_s^2)(-\Delta)^s u = \lambda \int_\Omega \frac{|u|^p}{|x-y|^\mu}dy|u|^{p-2}u + |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.2}$$

where $M(t) = a + b^{\theta-1}$, $\theta \in (1, \frac{2_s^*}{2})$, $0 < s < 1$, $\lambda > 0$ is a positive parameter, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $0 < \mu < N$, $N > 2s$, $\theta < p < 2_{\mu,s}^* = \frac{2N-\mu}{N-2s}$, and $2\theta <$

$q \le 2_s^* = \frac{2N}{N-2s}$. The existence of solutions for problem (1.2) was obtained by using variational methods and Nehari manifolds.

By using the concentration-compactness lemma and variational methods, Goel and Sreenadh [8] proved the existence and multiplicity of positive solutions of the Choquard-Kirchhoff equation:

$$\begin{cases} -M(\|u\|_2^2)\Delta u = \lambda f(x)|u|^{q-2}u + \int_\Omega \frac{|u|^{2_\mu^*}}{|x-y|^\mu}dy|u|^{2_\mu^*-2}u, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases}$$

where $M(t) = a + \varepsilon^p t^{\theta-1}$, $2_\mu^* = \frac{2N-\mu}{N-2}$, f is a continuous real valued sign changing function, and $1 < q \le 2$.

Liang et al. [12] considered the following Choquard-Kirchhoff equations with Hardy-Littlewood-Sobolev critical exponent:

$$-\left(a + b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u = \alpha k(x)|u|^{q-2}u + \beta\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy\right)|u|^{2_\mu^*-2}u, \quad x \in \mathbb{R}^N,$$

where $a > 0$, $b \ge 0$, $0 < \mu < 4$, $N \ge 3$, α, β are real parameters, $2_\mu^* = \frac{2N-\mu}{N-2}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, and $k(x) \in L^r(\mathbb{R}^N)$ with $r = \frac{2^*}{2^*-q}$. For the cases $1 < q < 2$, $q = 2$, $2 < q < 2^*$, and $4 < q < 2 \cdot 2_\mu^*$, they obtained the existence and multiplicity results by using the Symmetric Mountain Pass Theorem and genus theory under suitable conditions.

On the other hand, the study of nonlinear partial differential equations on the Heisenberg group has brought about widespread attention of many researchers. At the same time, some authors tried to establish the existence and multiplicity of solutions for partial differential equation solutions on the Heisenberg group. For example, Liang and Pucci [11] applied the Symmetric Mountain Pass Theorem to consider a class of the critical Kirchhoff-Poisson systems on the Heisenberg group. Pucci and Temperini [19] proved the existence of entire nontrivial solutions for the (p,q) critical systems on the Heisenberg group by an application of variational methods. Pucci [17] applied the Mountain Pass Theorem and the Ekeland variational principle to prove the existence of nontrivial nonnegative solutions of the Schrödinger-Hardy system on the Heisenberg group. However, once we turn our attention to the critical Choquard equation on the Heisenberg group, we immediately notice that the literature is relatively sparse. We note that Goel and Sreenadh [7] proved the regularity of solutions and the nonexistence of solutions for the critical Choquard equation on the Heisenberg group by using the Linking Theorem and the Mountain Pass Theorem.

Sun et al. [22] studied the following critical Choquard-Kirchhoff problem on the Heisenberg group:

$$M(\|u\|^2)(-\Delta_{\mathbb{H}}u + u) = \int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta|u|^{Q_\lambda^*-2}u + \mu f(\xi, u),$$

where f is a Carathéodory function, M is the Kirchhoff function, $\Delta_{\mathbb{H}}$ is the Kohn Laplacian on the Heisenberg group $\mathbb{H}^N$, $\mu > 0$ is a parameter, and $Q_\lambda^* = \frac{2Q-\lambda}{Q-2}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality. They were the first to establish a new version of the concentration-compactness principle on the Heisenberg group. Moreover, the existence of nontrivial solutions were obtained under non-degenerate and degenerate conditions. For more fascinating results, we refer to An and Liu [1], Bordoni and Pucci [3], Liu et al. [13], Liu and Zhang [14], Pucci [16, 17], and Pucci and Temperini [18, 19].

Inspired by the above results, we prove that problem (1.1) has infinitely many solutions for μ large enough. We also prove that this equation has m pairs of solutions for μ small enough and odd nonlinear function $f(x,\cdot)$. In particular, it should be pointed out that our results are new even in the Euclidean case.

Before stating the main results of this paper, we present some notions about the Heisenberg group $\mathbb{H}^N$. If $\xi = (x,y,t) \in \mathbb{H}^N$, then the definition of this group operation is

$$\tau_\xi(\xi') = \xi \circ \xi' = (x+x', y+y', t+t'+2(x'y-y'x)) \text{ for every } \xi, \xi' \in \mathbb{H}^N.$$

Next, $\xi^{-1} = -\xi$ is the inverse and therefore $(\xi')^{-1} \circ \xi^{-1} = (\xi \circ \xi')^{-1}$.

The definition of a natural group of dilations on $\mathbb{H}^N$ is $\delta_s(\xi) = (sx, sy, s^2 t)$ for every $s > 0$. Hence, $\delta_s(\xi_0 \circ \xi) = \delta_s(\xi_0) \circ \delta_s(\xi)$. It can be easily proved that the Jacobian determinant of dilations $\delta_s : \mathbb{H}^N \to \mathbb{H}^N$ is constant and equal to s^Q for every $\xi = (x,y,t) \in \mathbb{H}^N$. The natural number $Q = 2N+2$ is called the homogeneous dimension of $\mathbb{H}^N$ and the critical exponents is $Q^* := \frac{2Q}{Q-2}$. We define the Korányi norm as follows

$$|\xi|_H = \left[(x^2+y^2)^2 + t^2\right]^{\frac{1}{4}} \text{ for every } \xi \in \mathbb{H}^N,$$

and we derive this norm from the Heisenberg group's anisotropic dilation. Hence, the homogeneous degree of the Korányi norm is equal to 1, in terms of dilations

$$\delta_s : (x,y,t) \mapsto (sx, sy, s^2 t) \text{ for every } s > 0.$$

The set

$$B_H(\xi_0, r) = \{\xi \in \mathbb{H}^N : d_H(\xi_0, \xi) < r\},$$

denotes the Korányi open ball of radius r centered at ξ_0. For the sake of simplicity, we denote $B_r = B_r(O)$, where $O = (0,0)$ is the natural origin of $\mathbb{H}^N$.

The following vector fields

$$T = \frac{\partial}{\partial t},\ X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t},\ Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},$$

generate the real Lie algebra of left invariant vector fields for $j = 1, \cdots, n$, which forms a basis satisfying the Heisenberg regular commutation relation on $\mathbb{H}^N$. This means that

$$[X_j, Y_j] = -4\delta_{jk} T,\ [Y_j, Y_k] = [X_j, X_k] = [Y_j, T] = [X_j, T] = 0.$$

The so-called horizontal vector field is just a vector field with the span of $[X_j, Y_j]_{j=1}^n$.

The Heisenberg gradient on $\mathbb{H}^N$ is

$$\nabla_H = (X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n),$$

and the Kohn Laplacian on $\mathbb{H}^N$ is given by

$$\Delta_H = \sum_{j=1}^N X_j^2 + Y_j^2 = \sum_{j=1}^N \left[\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial x_j \partial t} + 4(x_j^2 + y_j^2) \frac{\partial^2}{\partial t^2}\right].$$

The Haar measure is invariant under the left translations of the Heisenberg group and is Q-homogeneous in terms of dilations. More precisely, it is consistent with the $(2n+1)$-dimensional Lebesgue measure. Hence, as shown by Leonardi and Masnou [10], the topological dimension

$2N+1$ of $\mathbb{H}^N$ is strictly less than its Hausdorff dimension $Q = 2N+2$. Next, $|\Omega|$ denotes the $(2N+1)$-dimensional Lebesgue measure of any measurable set $\Omega \subseteq \mathbb{H}^N$. Therefore,

$$|\delta_s(\Omega)| = s^Q|\Omega|, \ d(\delta_s\xi) = s^Q d\xi \text{ and } |B_H(\xi_0, r)| = \alpha_Q r^Q, \text{ where } \alpha_Q = |B_H(0,1)|.$$

For the case of critical exponent $\tau = \frac{Q_\lambda^*}{p}$, we have the following theorem.

Theorem 1.1. *Let $\tau = \frac{Q_\lambda^*}{p}$, $2 < p < \frac{N}{s}$, and suppose that condition (M) is satisfied. Assume that the nonlinearity $f(\xi,t)$ is odd in t for fixed ξ and satisfies conditions (f_1) and (f_2), and the potential function V satisfies conditions (V_1) and (V_2). Then problem (1.1) has infinitely many solutions for μ large enough.*

For the case of subcritical exponent $\tau \in (1, \frac{Q_\lambda^*}{p})$, we also have the following result.

Theorem 1.2. *Let $\tau \in (1, \frac{Q_\lambda^*}{p})$ and suppose that condition (M) is satisfied. Assume that $f(\cdot,\cdot)$ satisfies conditions $(f_1)'$, $(f_2)'$, and $(f_3)'$, and the potential function V satisfies conditions (V_1) and (V_2). Then*

(i) for every $\mu > 0$, there exists $\mu^* > 0$ such that problem (1.1) has at least one nontrivial solution u_λ with the following estimate, for every $\mu \in (0, \mu^*]$,

$$\|u_\lambda\|_\mu^p \le \left(\frac{\tau p q_0}{m_0 q_0 - m_1 \tau p}\right)^{\frac{1}{\tau}} \rho^{\frac{1}{\tau}} \mu^{\frac{Q_\lambda^*}{Q_\lambda^* - \tau p}} \tag{1.3}$$

and

$$\|u_\lambda\|_{H_{Q_\lambda^*}}^{Q_\lambda^*} \le \rho \frac{2q_0 Q_\lambda^*}{2Q_\lambda^* - q_0} \mu^{\frac{\tau Q_\lambda^*}{Q_\lambda^* - \tau p}}, \tag{1.4}$$

where $\rho = \frac{1}{q_0}(1 - \frac{m_1}{m_0}) + \frac{1}{\tau p} - \frac{1}{2Q_\lambda^*}$.

(ii) if $f(\xi,t)$ is odd with respect to t, then, for every $m \in \mathbb{N}$, there exists $\mu_m > 0$ such that problem (1.1) has at least m pairs of solutions $u_{\lambda,j}$ and $u_{\lambda,-j}$ $(j = 1, 2, \cdots, m)$ for $0 < \mu < \mu_m$, which satisfy (1.3) and (1.4).

The paper is organized as follows. In Section 2, we review some necessary definitions and useful lemmas related to our main proof. In Section 3, we mainly discuss the critical case $\tau = \frac{Q_\lambda^*}{p}$, and give the proof of Theorem 1.1. Finally, in Section 4, we discuss the subcritical case and prove Theorem 1.2.

2. Preliminaries

In this section, we review some necessary definitions and useful lemmas related to our main proof. First, let $u : \mathbb{H}^N \to \mathbb{R}$ be a measurable function. We set

$$[u]_{s,p} = \left(\int\int_{\mathbb{H}^{2N}} \frac{|u(\xi) - u(\eta)|^p}{|\xi - \eta|^{N+ps}} d\xi d\eta\right)^{\frac{1}{p}}$$

and define the fractional Sobolev space $S^{s,p}(\mathbb{H}^N)$ on the Heisenberg group as follows:

$$S^{s,p}(\mathbb{H}^N) = \{u \in L^p(\mathbb{H}^N) : \ u \text{ is a measurable function with } [u]_{s,p} < \infty\}$$

and the norm

$$\|u\|_{S^{s,p}(\mathbb{H}^N)} = ([u]_{s,p}^p + |u|_p^p)^{\frac{1}{p}} \text{ with } |u|_p = \left(\int_{\mathbb{H}^N} |u|^p d\xi \right)^{\frac{1}{p}}.$$

Moreover, for $\mu > 0$, let S_μ be the closure of $C_0^\infty(\mathbb{H}^N)$ with respect to the following norm

$$\|u\|_\mu = \left(\mu [u]_{s,p}^p + \|u\|_{p,V}^p \right)^{\frac{1}{p}} \text{ with } \|u\|_{p,V} = \left(\int_{\mathbb{H}^N} V(\xi)|u|^p d\xi \right)^{\frac{1}{p}}$$

in the presence of potential $V(\xi)$.

It follows that $(S_\mu, \|\cdot\|_\mu)$ is a uniformly convex Banach space, which was proved in Pucci et al. [20]. Now, we can define the weak solution of problem (1.1).

Definition 2.1. We call $u \in S_\mu$ a weak solution to problem (1.1) if

$$M(\|u\|_\mu^p)(\mu \int\int_{\mathbb{H}^{2N}} \frac{|u(\xi)-u(\eta)|^{p-2}(u(\xi)-u(\eta))}{|\xi-\eta|^{N+ps}} (\varphi(\xi)-\varphi(\eta)) d\xi d\eta + \int_{\mathbb{H}^N} V(\xi)|u|^{p-2} u \varphi d\xi)$$
$$= \int_{\mathbb{H}^N} f(\xi,u)\varphi d\xi + \int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta |u(\xi)|^{Q_\lambda^*-2} u(\xi)\varphi(\xi) d\xi \quad \text{for every } \varphi \in S_\mu. \tag{2.1}$$

The corresponding energy functional $I_\mu(u) : S_\mu \to \mathbb{R}$ of problem (1.1) is

$$I_\mu(u) = \frac{1}{p}\widetilde{M}(\|u\|_\mu^p) - \frac{1}{2Q_\lambda^*} \int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi - \int_{\mathbb{H}^N} F(\xi,u) d\xi,$$

where $\widetilde{M}(t) = \int_0^t M(s)ds$. It is easy to prove that $I_\mu \in C^1(S_\mu, R)$ and its critical points are solutions to problem (1.1).

Next, we define

$$H_{Q_\lambda^*} = \inf_{u \in S_\mu \setminus \{0\}} \frac{[u]_{s,p}^p}{\left(\int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi \right)^{\frac{p}{Q_\lambda^*}}} \tag{2.2}$$

and

$$\|u\|_{H_{Q_\lambda^*}}^{Q_\lambda^*} = \int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi.$$

By (2.2), we know that $H_{Q_\lambda^*}$ is positive. Let S denote the completion of $C_0^\infty(\mathbb{H}^N)$ with respect to the norm

$$\|u\|_S = \left([u]_{s,p}^p + \|u\|_{p,V}^p \right)^{\frac{1}{p}} \text{ with } \|u\|_{p,V} = \left(\int_{\mathbb{H}^N} V(\xi)|u|^p d\xi \right)^{\frac{1}{p}}.$$

Note that, for every fixed $\mu > 0$, the norm $\|u\|_W$ is equivalent to $\|u\|_\mu$. Invoking Bordoni and Pucci [3] and Pucci [17], we have the following embedding result.

Lemma 2.1. *Let $V(\xi)$ satisfy condition (V_1). Then, for every $\gamma \in [p, Q_\lambda^*]$, the embedding*

$$S_\mu \hookrightarrow S^{s,p}(\mathbb{H}^N) \hookrightarrow L^\gamma(\mathbb{H}^N)$$

is continuous. Moreover, for every $\gamma \in [p, Q_\lambda^*)$, the embedding $S_\mu \hookrightarrow L^\gamma(\mathbb{H}^N)$ is compact. In addition, there exists a constant $C_\gamma > 0$ satisfying $|u|_\gamma \le C_\gamma \|u\|_\mu$ for every $u \in S_\mu$.

Lemma 2.2. *Let V satisfy conditions (V_1) and (V_2), and let $\gamma \in [p, Q^*_\lambda)$ be a fixed exponent. Then, for every bounded sequence $\{u_n\}_n$ in S_μ, which up to a subsequence satisfies $u_n \to u$ in $L^\gamma(\mathbb{H}^N)$ as $n \to \infty$.*

Next, let $D^{s,p}(\mathbb{H}^N)$ be the completion of $C_0^\infty(\mathbb{H}^N)$ with respect to the Gagliardo semi-norm $[\cdot]_{s,p}$. Similarly to the proof of Sun et al. [22, Theorem 3.1], we obtain the following lemma.

Lemma 2.3. *For every $0 \le sp$, let $\{u_n\}_n \subset D^{s,p}(\mathbb{H}^N)$ be a bounded sequence satisfying*

$$\begin{cases} u_n \rightharpoonup u, \\ \int_{\mathbb{H}^N} \frac{|u_n(\xi)-u_n(\eta)|^p}{|\xi-\eta|^{N+ps}} d\eta \rightharpoonup \kappa \ge \int_{\mathbb{H}^N} \frac{|u(\xi)-u(\eta)|^p}{|\xi-\eta|^{N+ps}} d\eta + \sum_{j\in J} \kappa_j \delta_{x_j}, \\ \int_{\mathbb{H}^N} \frac{|u_n(\eta)|^{Q^*_\lambda}}{|\eta^{-1}\xi|^\lambda} d\eta |u_n(\xi)|^{Q^*_\lambda} \rightharpoonup \nu = \int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q^*_\lambda}}{|\eta^{-1}\xi|^\lambda} d\eta |u(\xi)|^{Q^*_\lambda} + \sum_{j\in J} \nu_j \delta_{x_j}, \end{cases}$$

where J is an at most countable index set, $x_j \in \mathbb{H}^N$, and δ_{x_j} is the Dirac mass at x_j. Furthermore, let

$$\kappa_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{\xi\in\mathbb{H}^N : |\xi|>R\}} \int_{\mathbb{H}^N} \frac{|u_n(\xi)-u_n(\eta)|^p}{|\xi-\eta|^{N+ps}} d\eta d\xi$$

and

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{\xi\in\mathbb{H}^N : |\xi|>R\}} \int_{\mathbb{H}^N} \frac{|u_n(\eta)|^{Q^*_\lambda}}{|\eta^{-1}\xi|^\lambda} d\eta |u_n(\xi)|^{Q^*_\lambda} d\xi.$$

Then, for the energy at infinity,

$$\limsup_{n\to\infty} \int_{\mathbb{H}^N} \int_{\mathbb{H}^N} \frac{|u_n(\xi)-u_n(\eta)|^p}{|\xi-\eta|^{N+ps}} d\eta d\xi = \int_{\mathbb{H}^N} d\kappa + \kappa_\infty \tag{2.3}$$

and

$$\limsup_{n\to\infty} \int_{\mathbb{H}^N} \int_{\mathbb{H}^N} \frac{|u_n(\eta)|^{Q^*_\lambda} |u_n(\xi)|^{Q^*_\lambda}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi = \int_{\mathbb{H}^N} d\nu + \nu_\infty. \tag{2.4}$$

In addition,

$$\kappa_j \ge H_{Q^*_\lambda} \nu_j^{\frac{p}{Q^*_\lambda}}$$

and

$$\kappa_\infty \ge H_{Q^*_\lambda} \nu_\infty^{\frac{p}{Q^*_\lambda}}.$$

3. Proof of Theorem 1.1

For the case of the critical exponent $\tau = \frac{Q^*_\lambda}{p}$ and the Kirchhoff function $M(\cdot)$ satisfying condition (M), we use this section to prove the existence of an infinite number of solutions to problem (1.1).

Lemma 3.1. *Let $\tau = \frac{Q^*_\lambda}{p}$, $2 < p < \frac{N}{s}$, and let condition (M) be satisfied. Suppose that the nonlinearity $f(\xi, t)$ is odd in t for fixed ξ, $f(\cdot,\cdot)$ satisfies conditions (f_1) and (f_2), and the potential function V satisfies conditions (V_1) and (V_2). Then I_μ is bounded from below for $\mu > \frac{2^p H_{Q^*_\lambda}^{-Q^*_\lambda/p}}{m_0}$ and I_μ is even.*

Proof. By (f_1), there exists $C_0 > 0$ satisfying

$$|f(\xi,t)| \le p|t|^{p-1} + C_0 q|t|^{q-1}$$

and

$$F(\xi,t) \le |t|^p + C_0|t|^q \quad \text{for a.e. } \xi \in \mathbb{H}^N \text{ and all } t \in \mathbb{R}.$$

For every $u \in S_\mu$, by condition (M), we have

$$I_\mu(u) \ge \frac{m_0}{p\tau}\|u\|_\mu^{p\tau} - \frac{1}{2Q_\lambda^*}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q_\lambda^*}|u(\xi)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi - \int_{\mathbb{H}^N}|u|^p dx - C_0\int_{\mathbb{H}^N}|u|^q dx.$$

By the definition of $H_{Q_\lambda^*}$ and Lemma 2.1, for $\tau = \frac{Q_\lambda^*}{p}$, we have

$$\begin{aligned} I_\mu(u) &\ge \frac{m_0}{p\tau}\|u\|_\mu^{p\tau} - \frac{1}{2Q_\lambda^*}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q_\lambda^*}|u(\xi)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi - C_1\|u\|_\mu^p - C_1\|u\|_\mu^q \\ &\ge \left(\frac{m_0}{Q_\lambda^*} - \frac{\mu^{-1}}{2Q_\lambda^*}H_{Q_\lambda^*}^{-Q_\lambda^*/p}\right)\|u\|_\mu^{Q_\lambda^*} - C_1\|u\|_\mu^p - C_1\|u\|_\mu^q. \end{aligned}$$

Since $\mu > \frac{2^p H_{Q_\lambda^*}^{-Q_\lambda^*/p}}{m_0} > \frac{H_{Q_\lambda^*}^{-Q_\lambda^*/p}}{2m_0}$ and $2 \le p < q < Q_\lambda^*$, we can deduce that $\frac{m_0}{Q_\lambda^*} - \frac{\mu^{-1}}{2Q_\lambda^*}H_{Q_\lambda^*}^{-Q_\lambda^*/p} = M_1 > 0$. There exists a small constant ε_1 such that $\varepsilon_1 C_2 < M_1$. By Young's inequality, we can deduce that

$$I_\mu(u) \ge (M_1 - \varepsilon_1 C_2)\|u\|_\mu^{Q_\lambda^*} - C_3.$$

Thus $I_\mu(u) \ge -C_3$. Moreover, since $f(\xi,t)$ is odd in t for fixed ξ, we obtain that I_μ is even. The proof of Lemma 3.1 is complete. □

Lemma 3.2. *Under the assumptions of Lemma 3.1, I_μ satisfies $(PS)_c$ condition for every $\mu > \frac{2^p H_{Q_\lambda^*}^{-Q_\lambda^*/p}}{m_0}$.*

Proof. Take $\{u_n\}_n \subset S_\mu$ to be a $(PS)_c$ sequence of the functional I_μ, that is,

$$I_\mu(u_n) \to c, \quad I'_\mu(u_n) \to 0 \quad \text{as } n \to \infty.$$

Then we claim that $\{u_n\}_n$ is bounded in S_μ. In fact, from the proof of Lemma 3.1, we can conclude that

$$c + o(1) \ge I_\mu(u_n) \ge (M_1 - \varepsilon_1 C_2)\|u_n\|_\mu^{Q_\lambda^*} - C_3.$$

Note that $M_1 - \varepsilon_1 C_2 > 0$ when ε_1 small enough, so $\{u_n\}_n$ is uniformly bounded in S_μ. This means that there is a subsequence of $\{u_n\}_n$ and $u \in S_\mu$ satisfying

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } S_\mu \text{ and in } L^{Q_\lambda^*}(\mathbb{H}^N), \\ &u_n \to u \quad \text{a.e. in } \mathbb{H}^N, \\ &|u_n|^{Q_\lambda^*-2}u_n \rightharpoonup |u|^{Q_\lambda^*-2}u \quad \text{in } L^{\frac{Q_\lambda^*}{Q_\lambda^*-1}}(\mathbb{H}^N) \end{aligned} \tag{3.1}$$

as $n \to \infty$. By the Brézis-Lieb type inequality, we can obtain

$$\|u_n - u\|_{H_{Q_\lambda^*}}^{Q_\lambda^*} = \|u_n\|_{H_{Q_\lambda^*}}^{Q_\lambda^*} - \|u\|_{H_{Q_\lambda^*}}^{Q_\lambda^*} + o(1).$$

Therefore, we have

$$\lim_{n\to\infty}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{(|u_n(\xi)|^{Q_\lambda^*}|u_n(\eta)|^{Q_\lambda^*-2}u_n(\eta)-|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*-2}u(\eta))(u_n(\eta)-u(\eta))}{|\eta^{-1}\xi|^\lambda}d\eta d\xi$$
$$=\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u_n(\xi)-u(\xi)|^{Q_\lambda^*}|u_n(\eta)-u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi+o(1). \tag{3.2}$$

Now, we shall demonstrate that

$$\lim_{n\to\infty}\int_{\mathbb{H}^N}(f(\xi,u_n)-f(\xi,u))(u_n-u)d\xi=0. \tag{3.3}$$

Up to a subsequence, it follows by Lemma 2.2 that $u_n\to u$ in $L^\gamma(\mathbb{H}^N)$ for $\gamma=p,q\in[p,Q_\lambda^*)$. By (f_1), we have

$$|f(\xi,t)|\le|t|^{p-1}+C_f|t|^{q-1}\quad\text{a.e. }\xi\in\mathbb{H}^N\quad\text{and}\quad\text{for every }t\in\mathbb{R}.$$

By the Hölder inequality, we see that

$$\begin{aligned}&\left|\int_{\mathbb{H}^N}(f(\xi,u_n)-f(\xi,u))(u_n-u)d\xi\right|\\&\le\int_{\mathbb{H}^N}\left((|u_n|^{p-1}+|u|^{p-1})|u_n-u|+C_f(|u_n|^{q-1}+|u|^{q-1})|u_n-u|\right)d\xi\\&\le(|u_n|_p^{p-1}+|u|_p^{p-1})|u_n-u|_p+C_f(|u_n|_q^{q-1}+|u|_q^{q-1})|u_n-u|_q,\end{aligned}$$

which implies that (3.3) holds.

Next, for every fixed $u\in S_\mu$, we define the following linear function $L(u)$ on S_μ:

$$\langle L(u),\varphi\rangle$$
$$=\mu\int\int_{\mathbb{H}^{2N}}\frac{|u(\xi)-u(\eta)|^{p-2}(u(\xi)-u(\eta))}{|\eta^{-1}\xi|^{N+ps}}(\varphi(\xi)-\varphi(\eta))d\xi d\eta+\int_{\mathbb{H}^N}V(\xi)|u|^{p-2}u\varphi d\xi$$

for every $\varphi\in S_\mu$.

Next, we show that $L(u)$ is bounded. In fact, it follows by the Hölder inequality that

$$\begin{aligned}|\langle L(u),\varphi\rangle|&\le\mu[u]_{s,p}^{p-1}[v]_{s,p}+\left(\int_{\mathbb{H}^N}V(\xi)|u|^pd\xi\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{H}^N}V(\xi)|v|^pd\xi\right)^{\frac{1}{p}}\\&\le\left([u]_{s,p}^{p-1}+\left(\int_{\mathbb{H}^N}V(\xi)|u|^pd\xi\right)^{\frac{p-1}{p}}\right)\|\varphi\|_\mu\end{aligned}$$

and the following equality (3.4) holds, due to $u_n\rightharpoonup u$ in S_μ,

$$\lim_{n\to\infty}\langle L(u),u_n-u\rangle=0. \tag{3.4}$$

We begin to prove that $\|u_n-u\|_\mu \to 0$ as $n\to\infty$. Let us assume that, in general, $\lim_{n\to\infty}\|u_n-u\|_\mu = d\neq 0$. Since $\{u_n\}_n$ is a $(PS)_c$ sequence, by (3.2), (3.3), and (3.4), we obtain that

$$\begin{aligned}
&o(1)\\
&=\langle I'_\mu(u_n),u_n-u\rangle-\langle I'_\mu(u),u_n-u\rangle\\
&=M(\|u_n\|_\mu^p)\langle L(u_n),u_n-u\rangle-M(\|u\|_\mu^p)\langle L(u),u_n-u\rangle\\
&-\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{(|u_n(\xi)|^{Q_\lambda^*}|u_n(\eta)|^{Q_\lambda^*-2}u_n(\eta)-|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*-2}u(\eta))(u_n(\eta)-u(\eta))}{|\eta^{-1}\xi|^\lambda}d\eta d\xi\\
&=M(\|u_n\|_\mu^p)\langle L(u_n)-L(u),u_n-u\rangle-\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u_n(\xi)-u(\xi)|^{Q_\lambda^*}|u_n(\eta)-u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi.
\end{aligned}\tag{3.5}$$

Let us consider each term on the right of the above formula separately. By the Brézis-Lieb type inequality, we obtain

$$[u_n-u]_{s,p}^p=[u_n]_{s,p}^p-[u]_{s,p}^p+o(1)$$

and

$$|V(\xi)^{\frac{1}{p}}(u_n-u)|_p^p=|V(\xi)^{\frac{1}{p}}u_n|_p^p-|V(\xi)^{\frac{1}{p}}u|_p^p+o(1).$$

Thus, for $M(\|u_n\|_\mu^p)$, we have

$$M(\|u_n\|_\mu^p)=M(\|u_n-u\|_\mu^p+\|u\|_\mu^p)+o(1).\tag{3.6}$$

For $\langle L(u_n)-L(u),u_n-u\rangle$, we apply the following inequality (see Kichenassamy and Veron [9])

$$|\xi-\eta|^p\le 2^p(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta)\quad\text{for } p\ge 2,\tag{3.7}$$

for every $\xi,\eta\in\mathbb{H}^{\mathbb{N}}$.

Next, we put (3.6) and (3.7) together into (3.5) to obtain the following estimate:

$$\begin{aligned}
&o(1)+\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u_n(\xi)-u(\xi)|^{Q_\lambda^*}|u_n(\eta)-u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi\\
&=M(\|u_n-u\|_\mu^p+\|u\|_\mu^p)\langle L(u_n)-L(u),u_n-u\rangle\\
&\ge m_0(\|u_n-u\|_\mu^p+\|u\|_\mu^p)^{\tau-1}\frac{\|u_n-u\|_\mu^p}{2^p}\\
&\ge\frac{m_0}{2^p}\|u_n-u\|_\mu^{\tau p}=\frac{m_0}{2^p}\|u_n-u\|_\mu^{Q_\lambda^*}.
\end{aligned}\tag{3.8}$$

For convolution term $\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u_n(\xi)-u(\xi)|^{Q_\lambda^*}|u_n(\eta)-u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi$, by (2.2), one has

$$\begin{aligned}
\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u_n(\xi)-u(\xi)|^{Q_\lambda^*}|u_n(\eta)-u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi &\le H_{Q_\lambda^*}^{-Q_\lambda^*/p}[u_n-u]_{s,p}^{Q_\lambda^*}\\
&\le\mu^{-1}H_{Q_\lambda^*}^{-Q_\lambda^*/p}\|u_n-u\|_\mu^{Q_\lambda^*}.
\end{aligned}\tag{3.9}$$

Finally, we put (3.9) into (3.8), and arrive at

$$o(1)+\mu^{-1}H_{Q_\lambda^*}^{-Q_\lambda^*/p}\|u_n-u\|_\mu^{Q_\lambda^*}\ge\frac{m_0}{2^p}\|u_n-u\|_\mu^{Q_\lambda^*}.$$

Letting $n\to\infty$, one has

$$\mu^{-1}2^pH_{Q_\lambda^*}^{-Q_\lambda^*/p}d^{Q_\lambda^*}\ge m_0d^{Q_\lambda^*},$$

which implies that $d = 0$ or

$$\mu \leq \frac{2^p}{m_0} H_{Q_\lambda^*}^{-Q_\lambda^*/p}.$$

This contradicts $\mu > \frac{2^p}{m_0} H_{Q_\lambda^*}^{-Q_\lambda^*/p}$, which implies that $d = 0$. Thus, $u_n \to u$ in S_μ when $\mu > \frac{2^p}{m_0} H_{Q_\lambda^*}^{-Q_\lambda^*/p}$. This completes the proof of Lemma 3.2. □

In order to prove Theorem 1.1 for problem (1.1) under critical conditions, we first review some basic results on the Krasnoselskii genus (see Clark [4] and Rabinowitz [21]). Let Y be a Banach space and $Z_2 = \{id, -id\}$ the symmetric group. We set

$$Z = \{X \subset Y \setminus \{0\} : X \text{ is closed and } X = -X\}.$$

Definition 3.1. For any $X \in Z$, the Krasnoselskii genus of X is defined as follows:

$$\gamma(X) = \inf\{m : \text{there exists } h \in (C, \mathbb{H}^m \setminus \{0\}) \text{ and } h \text{ is odd}\}.$$

We define $\gamma(X) = \infty$ if such k does not exist, and we set $\gamma(\emptyset) = 0$.

Lemma 3.3. *Let $Y = \mathbb{H}^N$ and denote the boundary of $\Omega \in \mathbb{H}^N$ by $\partial\Omega$, which is a symmetric bounded open subset. Then $\gamma(\partial\Omega) = N$.*

We denote the unit sphere in $\mathbb{H}^N$ by $\mathbb{S}^{N-1}$. We deduce by Lemma 3.3 that $\gamma(\mathbb{S}^{N-1}) = N$. The following result helps to prove the existence of an infinite number of solutions to problem (1.1).

Lemma 3.4. *(see Clark [4]) Let $I \in C^1(Y, \mathbb{R})$ satisfy the Palais-Smale condition. In addition, we assume that:*
(i) I is bounded from below and even;
(ii) there exists a compact set $K \in Z$ satisfying $\gamma(K) = k$ and $\sup_{x \in K} I(u) < I(0)$.
Then the critical value of I is less than $I(0)$, and I has at least k pairs of distinct critical points.

Proof of Theorem 1.1. Let $e_1, e_2, \cdots,$ be a basis for S_μ. For each $k \in \mathbb{N}$, k vectors $e_1, e_2, \cdots, e_k$ generate $Y_k = \text{span}\{e_1, e_2, \cdots, e_k\}$, which is a subspace of S_μ. Since, $p < q_1 < Q_\lambda^*$, we deduce that $Y_k \hookrightarrow L^{q_1}(\mathbb{H}^N)$. Considering that all norms of finite-dimensional Banach spaces are equivalent, we see that there is a positive $C(k)$ that depends only on k and satisfies

$$\|u\|_\mu^{q_1} \leq C(k) \int_{\mathbb{H}^N} |u|^{q_1} d\xi \quad \text{for every } u \in Y_k.$$

Under $\tau = \frac{Q_\lambda^*}{p}$, for every $u \in Y_k$, by conditions (M) and (f_2), we can deduce that

$$\begin{aligned} I_\mu(u) &\leq \frac{m_1}{p\tau} \|u\|_\mu^{p\tau} - \frac{1}{2Q_\lambda^*} \int_{\mathbb{H}^N} \int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q_\lambda^*} |u(\xi)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi - a_1 C(k) \|u\|_\mu^{q_1} \\ &\leq \left(\frac{m_1}{Q_\lambda^*} \|u\|_\mu^{Q_\lambda^* - q_1} - a_1 C(k) \right) \|u\|_\mu^{q_1}. \end{aligned}$$

Taking a sufficiently small constant $R > 0$ satisfying

$$\frac{m_1}{Q_\lambda^*} R^{Q_\lambda^* - q_1} < a_1 C(k),$$

we obtain, for every $r \in (0,R)$ and $u \in \Lambda = \{u \in Y_k : \|u\|_\mu = r\}$, the following equality

$$I_\mu(u) \leq r^{q_1}\left(\frac{m_1}{Q_\lambda^*}r^{Q_\lambda^*-q_1} - a_1C(k)\right) \leq R^{q_1}\left(\frac{m_1}{Q_\lambda^*}R^{Q_\lambda^*-q_1} - a_1C(k)\right) < 0 = I_\mu(0),$$

which implies that $\sup_{u\in\Lambda} I_\mu(u) < 0$. Note that Λ is a homeomorphism of $\mathbb{S}^{k-1}$, so $\gamma(\Lambda) = k$ by Lemma 3.3. It can be deduced from Lemma 3.4 that I_μ has at least k pairs of distinct critical points. Since k is arbitrary, we have an infinite number of pairs of distinct critical points for I_μ in S_μ. The proof of Theorem 1.1 is complete. □

4. Proof of Theorem 1.2

For the case of critical exponent $\tau \in (1, \frac{Q_\lambda^*}{p})$ and the Kirchhoff function $M(\cdot)$ satisfying condition (M), we prove in this section the existence and multiplicity of solutions to problem (1.1).

Lemma 4.1. *Let $\tau \in (1, \frac{Q_\lambda^*}{p})$ and suppose that the Kirchhoff function $M(\cdot)$ satisfies condition (M). Assume that the nonlinearity $f(\cdot,\cdot)$ satisfies condition $(f_3)'$. If $\{u_n\}_n$ is a $(PS)_c$ sequence of the functional I_μ, then $\{u_n\}_n$ is bounded in S_μ.*

Proof. Let $\{u_n\}_n$ be a $(PS)_c$ sequence. By conditions (M) and $(f_3)'$, we have

$$\begin{aligned} c + o(1)(1+\|u_n\|_\mu) &\geq I_\mu(u_n) - \frac{1}{q_0}\langle I'_\mu(u_n), u_n\rangle \\ &\geq \left(\frac{m_0}{p\tau} - \frac{m_1}{q_0}\right)\|u\|_\mu^{p\tau} + \left(\frac{1}{q_0} - \frac{1}{2Q_\lambda^*}\right)\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q_\lambda^*}|u(\xi)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi \\ &\quad + \int_{\mathbb{H}^N}\left(\frac{f(\xi,u_n)u_n}{q_0} - F(\xi,u_n)\right)d\xi \\ &\geq \left(\frac{m_0}{p\tau} - \frac{m_1}{q_0}\right)\|u_n\|_\mu^{p\tau}. \end{aligned}$$

Since $\left(\frac{m_0}{p\tau} - \frac{m_1}{q_0}\right) > 0$, we obtain that $\{u_n\}_n$ is bounded in S_μ. This completes the proof of Lemma 4.1. □

Lemma 4.2. *Let $\tau \in (1, \frac{Q_\lambda^*}{p})$ and suppose that the Kirchhoff function $M(\cdot)$ satisfies condition (M). Assume that $f(\cdot,\cdot)$ satisfies conditions $(f_1)'$-$(f_3)'$, and the potential function V satisfies conditions (V_1) and (V_2). Then, for every $\mu > 0$ and all $c \in (0, \rho(m_0\mu^\tau H_{Q_\lambda^*}^\tau)^{\frac{Q_\lambda^*}{Q_\lambda^*-\tau p}})$, I_μ satisfies the $(PS)_c$ condition, where $\rho = \frac{1}{\tau p} - \frac{1}{2Q_\lambda^*} + \frac{1}{q_0}(1 - \frac{m_1}{m_0})$.*

Proof. We need to divide the proof into two cases, due to the degenerate nature of problem (1.1): either $\inf_{n\in N}\|u_n\|_\mu = l > 0$ or $\inf_{n\in N}\|u_n\|_\mu = l = 0$.

Case I: $\inf_{n\in N}\|u_n\|_\mu = l > 0$. Since $\{u_n\}_n$ is a $(PS)_c$ sequence, we can deduce from Lemma 4.1 that $\{u_n\}_n$ is bounded in S_μ. Next, by Lemma 2.3, up to a subsequence, there is a nonnegative function $u \in S_\mu$ satisfying $u_n \rightharpoonup u$ in S_μ

$$\int_{\mathbb{H}^N}\frac{|u_n(\xi)-u_n(\eta)|^p}{|\xi-\eta|^{N+ps}}d\eta \rightharpoonup \kappa \geq \int_{\mathbb{H}^N}\frac{|u(\xi)-u(\eta)|^p}{|\xi-\eta|^{N+ps}}d\eta + \sum_{j\in J}\kappa_j\delta_{x_j} \tag{4.1}$$

and

$$\int_{\mathbb{H}^N}\frac{|u_n(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta|u_n(\xi)|^{Q_\lambda^*}\rightharpoonup \nu=\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta|u(\xi)|^{Q_\lambda^*}+\sum_{j\in J}\nu_j\delta_{x_j} \tag{4.2}$$

in the sense of measure, where $x_j\in\mathbb{H}^N$ and δ_{x_j} is the Dirac mass at x_j. In addition, we have

$$\kappa_j\geq H_{Q_\lambda^*}\nu_j^{\frac{p}{Q_\lambda^*}},\ \text{for every } j\in J. \tag{4.3}$$

Next, we prove that $\nu_j=0$ for every $j\in J$. For this purpose, let ξ_j be a singular point of the measures κ and ν, and we define $\psi_{\varepsilon,j}=\psi(\frac{\xi-\xi_j}{\varepsilon})$ as a cut-off function. Moreover, the hypotheses $0\leq\psi(\xi)\leq 1$,

$$\begin{cases}\psi(\xi)=1 & \text{in } B_1(0),\\ \psi(\xi)=0 & \text{in } \mathbb{H}^N\backslash B_2(0),\\ |\nabla_H\psi(\xi)|\leq 2 & \text{in } \mathbb{H}^N\end{cases}$$

hold, where $\psi\in C_0^\infty(\mathbb{H}^N)$. Now by the boundedness of $\{\psi_{\varepsilon,j}u_n\}$ in W_μ, we have $\langle I_\mu'(u_n),\psi_{\varepsilon,j}u_n\rangle\to 0$ as $n\to\infty$. Furthermore, we have

$$\begin{aligned}&M(\|u_n\|_\mu^p)\langle L(u_n),\psi_{\varepsilon,j}u_n\rangle\\&=\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u_n(\xi)|^{Q_\lambda^*}|u_n(\eta)|^{Q_\lambda^*}\psi_{\varepsilon,j}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi+\int_{\mathbb{H}^N}f(\xi,u_n)u_n\psi_{\varepsilon,j}d\xi+o(1),\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}&\langle L(u_n),\psi_{\varepsilon,j}u_n\rangle\\&=\mu\int\int_{\mathbb{H}^{2N}}\frac{|u_n(\xi)-u_n(\eta)|^p\psi_{\varepsilon,j}(\xi)}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta+\int_{\mathbb{H}^N}V(\xi)|u_n|^p\psi_{\varepsilon,j}d\xi\\&\quad+\mu\int\int_{\mathbb{H}^{2N}}\frac{|u_n(\xi)-u_n(\eta)|^{p-2}(u_n(\xi)-u_n(\eta))u_n(\eta)(\psi_{\varepsilon,j}(\xi)-\psi_{\varepsilon,j}(\eta))}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta.\end{aligned}$$

Similarly to the proof of Xiang et al. [23, Lemma 2.3], we have

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty}\left(\int\int_{\mathbb{H}^{2N}}\frac{|\left(\psi_{\varepsilon,j}(\xi)-\psi_{\varepsilon,j}(\eta)\right)u_n(\xi)|^p}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta\right)^{\frac{1}{p}}=0.$$

It follows from the Hölder inequality that

$$\begin{aligned}&\lim_{\varepsilon\to0}\limsup_{n\to\infty}M(\|u_n\|_\mu^p)\left(\mu\int\int_{\mathbb{H}^{2N}}\frac{|u_n(\xi)-u_n(\eta)|^{p-2}(u_n(\xi)-u_n(\eta))u_n(\eta)(\psi_{\varepsilon,j}(\xi)-\psi_{\varepsilon,j}(\eta))}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta\right)\\&\leq C\lim_{\varepsilon\to0}\limsup_{n\to\infty}\left(\int\int_{\mathbb{H}^{2N}}\frac{|u_n(\xi)-u_n(\eta)|^p}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta\right)^{\frac{p-1}{p}}\left(\int\int_{\mathbb{H}^{2N}}\frac{|u_n(\eta)(\psi_{\varepsilon,j}(\xi)-\psi_{\varepsilon,j}(\eta))|^p}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta\right)^{\frac{1}{p}}\\&\leq C\lim_{\varepsilon\to0}\limsup_{n\to\infty}\left(\int\int_{\mathbb{H}^{2N}}\frac{|u_n(\eta)(\psi_{\varepsilon,j}(\xi)-\psi_{\varepsilon,j}(\eta))|^p}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta\right)^{\frac{1}{p}}=0.\end{aligned} \tag{4.5}$$

Thus, by (4.1), (4.3), and condition (M), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{n\to\infty} M(\|u_n\|_\mu^p)\left(\mu\int\int_{\mathbb{H}^{2N}}\frac{|u_n(\xi)-u_n(\eta)|^p\psi_{\varepsilon,j}(\xi)}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta+\int_{\mathbb{H}^N}V(\xi)|u_n|^p\psi_{\varepsilon,j}d\xi\right)\\
&\geq \lim_{\varepsilon\to 0}\limsup_{n\to\infty} m_0\left(\mu\int\int_{\mathbb{H}^{2N}}\frac{|u_n(\xi)-u_n(\eta)|^p\psi_{\varepsilon,j}(\xi)}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta\right)^\tau\\
&\geq \lim_{\varepsilon\to 0} m_0\left(\mu\int\int_{\mathbb{H}^{2N}}\frac{|u(\xi)-u(\eta)|^p\psi_{\varepsilon,j}(\xi)}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta+\mu\kappa_j\right)^\tau\\
&= m_0(\mu\kappa_j)^\tau\geq m_0(\mu H_{Q_\lambda^*}v_j^{\frac{p}{Q_\lambda^*}})^\tau.
\end{aligned}\tag{4.6}$$

Moreover, it follows from (4.2) that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u_n(\eta)|^{Q_\lambda^*}|u_n(\xi)|^{Q_\lambda^*}\psi_{\varepsilon,j}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi\\
&=\lim_{\varepsilon\to 0}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q_\lambda^*}|u(\xi)|^{Q_\lambda^*}\psi_{\varepsilon,j}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi+v_j=v_j
\end{aligned}\tag{4.7}$$

and by Lemma 2.1, since $W_\mu\hookrightarrow L^\gamma(\mathbb{H}^N)$ is a compact embedding for every $\gamma\in[1,Q_\lambda^*)$, we have

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{H}^N}f(\xi,u_n)u_n\psi_{\varepsilon,j}d\xi=\lim_{\varepsilon\to 0}\int_{\mathbb{H}^N}f(\xi,u)u\psi_{\varepsilon,j}d\xi=0.\tag{4.8}$$

Now we put (4.5), (4.6), (4.7), and (4.8) into (4.4) and obtain

$$v_j\geq m_0(\mu H_{Q_\lambda^*}v_j^{\frac{p}{Q_\lambda^*}})^\tau=m_0\mu^\tau H_{Q_\lambda^*}^\tau v_j^{\frac{p\tau}{Q_\lambda^*}},$$

which implies that either $v_j=0$ or $v_j\geq(m_0\mu^\tau H_{Q_\lambda^*}^\tau)^{\frac{Q_\lambda^*}{Q_\lambda^*-p\tau}}$.

Now we can prove the possible concentration of mass at infinity. Similarly to the above argument, we define $\phi_R\in C_0^\infty(\mathbb{H}^N)$ as a cut-off function. Moreover, the hypotheses $0\leq\phi_R\leq 1$,

$$\begin{cases}\phi_R(\xi)=0 & \text{in } B_R(0),\\ \phi_R(\xi)=1 & \text{in } \mathbb{H}^N\backslash B_{2R}(0),\\ |\nabla_H\phi_R(\xi)|\leq\frac{2}{R} & \text{in } \mathbb{H}^N\end{cases}$$

hold. Next, by Lemma 2.3, one has

$$\kappa_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int\int_{\mathbb{H}^{2N}}\frac{|u_n(\xi)-u_n(\eta)|^p\phi_R(\xi)^p}{|\xi-\eta|^{N+ps}}d\eta d\xi\tag{4.9}$$

and

$$\begin{aligned}
v_\infty&=\lim_{R\to\infty}\limsup_{n\to\infty}\int\int_{\mathbb{H}^{2N}}\frac{|u_n(\eta)|^{Q_\lambda^*}|u_n(\xi)|^{Q_\lambda^*}\phi_R(\eta)^{2Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi\\
&=\lim_{R\to\infty}\limsup_{n\to\infty}\int\int_{\mathbb{H}^{2N}}\frac{|u_n(\eta)|^{Q_\lambda^*}|u_n(\xi)|^{Q_\lambda^*}\phi_R(\eta)}{|\eta^{-1}\xi|^\lambda}d\eta d\xi.
\end{aligned}\tag{4.10}$$

In addition, one has $\kappa_\infty \geq H_{Q_\lambda^*} \nu_\infty^{\frac{p}{Q_\lambda^*}}$. Since $\langle I'_\mu(u_n), \phi_R u_n\rangle \to 0$ as $n\to\infty$, we have

$$\begin{aligned} &M(\|u_n\|_\mu^p)\langle L(u_n), \phi_R u_n\rangle \\ &= \int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u_n(\xi)|^{Q_\lambda^*}|u_n(\eta)|^{Q_\lambda^*}\phi_R u_n}{|\eta^{-1}\xi|^\lambda} d\eta d\xi + \int_{\mathbb{H}^N} f(\xi,u_n)u_n\phi_R d\xi + o(1), \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} \langle L(u_n), \phi_R u_n\rangle &= \mu\int\int_{\mathbb{H}^{2N}} \frac{|u_n(\xi)-u_n(\eta)|^p\phi_R(\xi)}{|\eta^{-1}\xi|^{N+ps}} d\xi d\eta + \int_{\mathbb{H}^N} V(\xi)|u_n|^p\phi_R d\xi \\ &+\mu\int\int_{\mathbb{H}^{2N}} \frac{|u_n(\xi)-u_n(\eta)|^{p-2}(u_n(\xi)-u_n(\eta))u_n(\eta)(\phi_R(\xi)-\phi_R(\eta))}{|\eta^{-1}\xi|^{N+ps}} d\xi d\eta. \end{aligned}$$

Similarly to the proof of Xiang et al. [23], one has

$$\lim_{R\to\infty}\limsup_{n\to\infty}\left(\int\int_{\mathbb{H}^{2N}} \frac{|(\phi_R(\xi)-\phi_R(\eta))\,u_n(\xi)|^p}{|\xi-\eta|^{N+ps}} d\xi d\eta\right)^{\frac{1}{p}} = 0.$$

It follows from the Hölder inequality that

$$\begin{aligned} &\lim_{R\to\infty}\limsup_{n\to\infty} M(\|u_n\|_\mu^p)\left(\mu\int\int_{\mathbb{H}^{2N}} \frac{|u_n(\xi)-u_n(\eta)|^{p-2}(u_n(\xi)-u_n(\eta))u_n(\eta)(\phi_R(\xi)-\phi_R(\eta))}{|\eta^{-1}\xi|^{N+ps}} d\xi d\eta\right) \\ &\leq C\lim_{R\to\infty}\limsup_{n\to\infty}\left(\int\int_{\mathbb{H}^{2N}} \frac{|u_n(\xi)-u_n(\eta)|^p}{|\eta^{-1}\xi|^{N+ps}} d\xi d\eta\right)^{\frac{p-1}{p}}\left(\int\int_{\mathbb{H}^{2N}} \frac{|u_n(\eta)(\phi_R(\xi)-\phi_R(\eta))|^p}{|\eta^{-1}\xi|^{N+ps}} d\xi d\eta\right)^{\frac{1}{p}} \\ &\leq C\lim_{R\to\infty}\limsup_{n\to\infty}\left(\int\int_{\mathbb{H}^{2N}} \frac{|u_n(\eta)(\phi_R(\xi)-\phi_R(\eta))|^p}{|\eta^{-1}\xi|^{N+ps}} d\xi d\eta\right)^{\frac{1}{p}} = 0. \end{aligned} \tag{4.12}$$

For the first term on the right hand side of (4.11), we have by (2.3), (4.9), and condition (M) that

$$\begin{aligned} &\lim_{R\to\infty}\limsup_{n\to\infty} M(\|u_n\|_\mu^p)\left(\mu\int\int_{\mathbb{H}^{2N}} \frac{|u_n(\xi)-u_n(\eta)|^p\phi_R(\xi)}{|\eta^{-1}\xi|^{N+ps}} d\xi d\eta + \int_{\mathbb{H}^N} V(\xi)|u_n|^p\phi_R d\xi\right) \\ &\geq \lim_{R\to\infty}\limsup_{n\to\infty} m_0\left(\mu\int\int_{\mathbb{H}^{2N}} \frac{|u_n(\xi)-u_n(\eta)|^p\phi_R(\xi)^p}{|\eta^{-1}\xi|^{N+ps}} d\xi d\eta\right)^{\tau-1} \\ &\quad\left(\mu\int_{\{\xi\in\mathbb{H}^N:|\xi|>2R\}}\int_{\mathbb{H}^N} \frac{|u_n(\xi)-u_n(\eta)|^p}{|\eta^{-1}\xi|^{N+ps}} d\xi d\eta\right) \geq m_0\mu^\tau\kappa_\infty^{\tau-1}(\int_{\mathbb{H}^N} d\kappa + \kappa_\infty) \geq m_0\mu^\tau\kappa_\infty^\tau. \end{aligned} \tag{4.13}$$

For the second term on the right hand side of (4.11), since $(f_1)'$ and $S_\mu \hookrightarrow L^\gamma(\mathbb{H}^N)$ is a compact embedding for every $\gamma\in[1,Q_\lambda^*)$, it is easy to obtain that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{H}^N} f(\xi,u_n)u_n\phi_R dx = \lim_{R\to\infty}\int_{\mathbb{H}^N} f(\xi,u)u\phi_R d\xi = 0. \tag{4.14}$$

Now we put (4.10), (4.12), (4.13), and (4.14) into (4.11) to obtain that

$$\nu_\infty \geq m_0(\mu H_{Q_\lambda^*}\nu_\infty^{\frac{p}{Q_\lambda^*}})^\tau = m_0\mu^\tau H_{Q_\lambda^*}^\tau \nu_\infty^{\frac{p\tau}{Q_\lambda^*}},$$

which implies that either $v_\infty = 0$ or $v_\infty \geq (m_0\mu^\tau H^\tau_{Q^*_\lambda})^{\frac{Q^*_\lambda}{Q^*_\lambda - p\tau}}$. In the sequel, we prove that $v_j \geq (m_0\mu^\tau H^\tau_{Q^*_\lambda})^{\frac{Q^*_\lambda}{Q^*_\lambda - p\tau}}$ and $v_\infty \geq (m_0\mu^\tau H^\tau_{Q^*_\lambda})^{\frac{Q^*_\lambda}{Q^*_\lambda - p\tau}}$ is impossible. Indeed, if $v_j \geq (m_0\mu^\tau H^\tau_{Q^*_\lambda})^{\frac{Q^*_\lambda}{Q^*_\lambda - p\tau}}$, by Lemma 2.3 and (4.3), we would have

$$\begin{aligned}
c &= \lim_{n\to\infty} I_\mu(u_n) - \frac{1}{q_0}\langle I'_\mu(u_n), u_n\rangle \\
&\geq \left(\frac{m_0}{p\tau} - \frac{m_1}{q_0}\right)\left(\mu \int\int_{\mathbb{H}^{2N}} \frac{|u_n(\xi)-u_n(\eta)|^p}{|\eta^{-1}\xi|^{N+ps}} d\xi d\eta\right)^\tau \\
&\quad + \left(\frac{1}{q_0} - \frac{1}{2Q^*_\lambda}\right)\int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u_n(\eta)|^{Q^*_\lambda}|u_n(\xi)|^{Q^*_\lambda}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi \\
&\geq \left(\frac{m_0}{p\tau} - \frac{m_1}{q_0}\right)(\mu\kappa_j)^\tau + \left(\frac{1}{q_0} - \frac{1}{2Q^*_\lambda}\right) v_j \\
&\geq \left(\frac{1}{\tau p} - \frac{1}{2Q^*_\lambda} + \frac{1}{q_0}(1 - \frac{m_1}{m_0})\right)(m_0\mu^\tau H^\tau_{Q^*_\lambda})^{\frac{Q^*_\lambda}{Q^*_\lambda - p\tau}} = \rho(m_0\mu^\tau H^\tau_{Q^*_\lambda})^{\frac{Q^*_\lambda}{Q^*_\lambda - p\tau}},
\end{aligned}$$

where $\rho = \frac{1}{\tau p} - \frac{1}{2Q^*_\lambda} + \frac{1}{q_0}(1 - \frac{m_1}{m_0})$, which contradicts $c \in (0, \rho(m_0\mu^\tau H^\tau_{Q^*_\lambda})^{\frac{Q^*_\lambda}{Q^*_\lambda - \tau p}})$. This means that $v_j = 0$ for every $j \in J$. Similarly, $v_\infty = 0$. Thus, it follows (2.4) that

$$\lim_{n\to\infty}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u_n(\eta)|^{Q^*_\lambda}|u_n(\xi)|^{Q^*_\lambda}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi = \int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q^*_\lambda}|u(\xi)|^{Q^*_\lambda}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi.$$

Invoking the Brézis-Lieb Lemma, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u_n(\eta)-u(\eta)|^{Q^*_\lambda}|u_n(\xi)-u(\xi)|^{Q^*_\lambda}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi = 0. \tag{4.15}$$

Finally, we prove that $u_n \to u$ in S_μ. Take $\{u_n\}_n \subset S_\mu$ to be a $(PS)_c$ sequence of the functional I_μ, and define

$$\begin{aligned}
&\langle L(u), \varphi\rangle \\
&= \mu \int\int_{\mathbb{H}^{2N}} \frac{|u(\xi)-u(\eta)|^{p-2}(u(\xi)-u(\eta))(\varphi(\xi)-\varphi(\eta))}{|\eta^{-1}\xi|^{N+ps}} d\xi d\eta + \int_{\mathbb{H}^N} V(\xi)|u|^{p-2}u\varphi d\xi
\end{aligned}$$

for every $\varphi \in S_\mu$. Then, we have

$$\begin{aligned}
&o(1) = \langle I'_\mu(u_n) - I'_\mu(u), u_n - u\rangle \\
&= M(\|u_n\|^p_\mu)\langle L(u_n), u_n - u\rangle - M(\|u\|^p_\mu)\langle L(u), u_n - u\rangle \\
&- \int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{(|u_n(\xi)|^{Q^*_\lambda}|u_n(\eta)|^{Q^*_\lambda - 2}u_n(\eta) - |u(\xi)|^{Q^*_\lambda}|u(\eta)|^{Q^*_\lambda - 2}u(\eta))(u_n(\eta)-u(\eta))}{|\eta^{-1}\xi|^\lambda} d\eta d\xi \\
&- \int_{\mathbb{H}^N}(f(\xi,u_n) - f(\xi,u))(u_n - u)d\xi.
\end{aligned} \tag{4.16}$$

For the fourth term on the right hand side of (4.16), similarly to (3.3), we have

$$\lim_{n\to\infty}\int_{\mathbb{H}^N}(f(\xi,u_n)-f(\xi,u))(u_n-u)d\xi=0. \tag{4.17}$$

For the third term on the right hand side of (4.16), it follows from (3.1) and (4.15) that

$$\lim_{n\to\infty}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{(|u_n(\xi)|^{Q_\lambda^*}|u_n(\eta)|^{Q_\lambda^*-2}u_n(\eta)-|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*-2}u(\eta))(u_n(\eta)-u(\eta))}{|\eta^{-1}\xi|^\lambda}d\eta d\xi$$
$$=0. \tag{4.18}$$

Now we put (4.17) and (4.18) into (4.16) to obtain

$$\begin{aligned}o(1)=&M(\|u_n\|_\mu^p)(\langle L(u_n),u_n-u\rangle-\langle L(u),u_n-u\rangle)+M(\|u_n\|_\mu^p)\langle L(u),u_n-u\rangle\\&-M(\|u_n\|_\mu^p)\langle L(u),u_n-u\rangle.\end{aligned}$$

Since $u_n \rightharpoonup u$ and $\{u_n\}_n$ is bounded in S_μ, we can deduce that $\lim_{n\to\infty}M(\|u_n\|_\mu^p)\langle L(u),u_n-u\rangle=0$. Hence

$$\lim_{n\to\infty}M(\|u_n\|_\mu^p)(\langle L(u_n),u_n-u\rangle-\langle L(u),u_n-u\rangle)=0.$$

Since $\inf_{n\in N}\|u_n\|_\mu=l>0$, we have

$$\begin{aligned}o(1)&=(\langle L(u_n),u_n-u\rangle-\langle L(u),u_n-u\rangle)\\&=\mu\langle u_n-u,u_n-u\rangle_{s,p}^p+\int_{\mathbb{H}^N}V(\xi)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)d\xi\\&=B_1+B_2,\end{aligned} \tag{4.19}$$

where

$$B_1=\mu\langle u_n-u,u_n-u\rangle_{s,p}^p,\quad B_2=\int_{\mathbb{H}^N}V(\xi)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)d\xi,$$

and

$$\langle u,\varphi\rangle_{s,p}^p=\int\int_{\mathbb{H}^{2N}}\frac{|u(\xi)-u(\eta)|^{p-2}(u(\xi)-u(\eta))(\varphi(\xi)-\varphi(\eta))}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta.$$

Thus this gives us $B_1\geq 0$. We invoke some elementary inequalities (see, e.g., Kichenassamy and Veron [9]): For every $p>1$, there exist positive constants $C_1=C(p,n)>0$ and $C_2=C(p,n)>0$ such that

$$|\xi-\eta|^p\leq\begin{cases}C_1(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta) & \text{if } p\geq 2,\\ C_2[(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta)]^{\frac{p}{2}}(|\xi|^p+|\eta|^p)^{\frac{2-p}{2}} & \text{if } 1<p<2\end{cases} \tag{4.20}$$

for every $\xi,\eta\in\mathbb{R}$. Thus, $B_2\geq 0$. It follows by (4.19) that $B_1=B_2=o(1)$. For $p\geq 2$, one has

$$\begin{aligned}\mu[u_n-u]_{s,p}^p&=\mu\int\int_{\mathbb{H}^{2N}}\frac{|(u_n(\xi)-u_n(\eta))-(u(\xi)-u(\eta))|^p}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta\\&\leq C_1\mu\left(\langle u_n,u_n-u\rangle_{s,p}^p-\langle u,u_n-u\rangle_{s,p}^p\right)=o(1)\end{aligned}$$

and

$$\|u_n-u\|_V^p\leq C_1\int_{\mathbb{H}^N}V(\xi)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)d\xi=o(1),$$

which implies that $\|u_n-u\|_\mu=o(1)$. For $1<p<2$, we utilize the following inequality:

$$(a+b)^s\leq a^s+b^s,\quad \text{for every } a,b>0,\ s\in(0,1).$$

From (4.20), $B_1 = o(1)$, and the Hölder inequality, we obtain

$$\mu[u_n - u]_{s,p}^p = \mu \int\int_{\mathbb{H}^{2N}} \frac{|(u_n(\xi) - u_n(\eta)) - (u(\xi) - u(\eta))|^p}{|\eta^{-1}\xi|^{N+ps}} d\xi d\eta$$
$$\leq C_2\mu \left(\langle u_n, u_n - u\rangle_{s,p}^p - \langle u, u_n - u\rangle_{s,p}^p\right)^{\frac{p}{2}} \left([u_n]_{s,p}^{\frac{p(2-p)}{2}} + [u]_{s,p}^{\frac{p(2-p)}{2}}\right)$$
$$\leq C_2\mu \left(\langle u_n, u_n - u\rangle_{s,p}^p - \langle u, u_n - u\rangle_{s,p}^p\right)^{\frac{p}{2}} = o(1).$$

Similarly to (4.20) and $B_2 = o(1)$, one has

$$\|u_n - u\|_V^p \leq C \int_{\mathbb{H}^N} V(\xi)[(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)]^{\frac{p}{2}} (|u_n|^p + |u|^p)^{\frac{2-p}{2}} d\xi$$
$$\leq C_2 \left(\int_{\mathbb{H}^N} V(\xi)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\right)^{\frac{p}{2}} \left(\int_{\mathbb{H}^N} V(\xi)(|u_n|^p + |u|^p) d\xi\right)^{\frac{2-p}{2}}$$
$$\leq C_2 \left(\int_{\mathbb{H}^N} V(\xi)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\right)^{\frac{p}{2}} = o(1),$$

which implies $\|u_n - u\|_\mu = o(1)$.

Case II: $\inf_{n\in N} \|u_n\|_\mu = l = 0$. If 0 is an accumulation point of the sequence $\{u_n\}_n$, then there is a subsequence of $\{u_n\}_n$ that converges strongly to $u = 0$, so we reach the desired result. If 0 is an isolated point of the sequence $\{u_n\}_n$, then there is a subsequence, still denoted by $\{u_n\}_n$ satisfying $\inf_{n\in N} \|u_n\|_\mu = l > 0$, which was considered in Case I. This completes the proof of Lemma 4.2. □

Under assumptions $M(\cdot)$, $V(\cdot)$, and $f(\cdot,\cdot)$, we can now prove that the function has the mountain pass geometry.

Lemma 4.3. *Let $\tau \in (1, \frac{Q_\lambda^*}{p})$ and suppose that the Kirchhoff function $M(\cdot)$ satisfies condition (M). Assume that $f(\cdot,\cdot)$ satisfies condition $(f_1)'$, and the potential function $V(\xi)$ satisfies conditions (V_1) and (V_2). Then, for every $\mu > 0$, there exist $\alpha, \sigma > 0$ satisfying $I_\mu(u) > 0$ for $u \in B_\sigma \setminus \{0\}$, and $I_\mu(u) \geq \alpha$ for every $u \in S_\mu$ with $\|u\|_\mu = \sigma$, where $B_\sigma = \{u \in S_\mu : \|u\|_\mu < \sigma\}$.*

Proof. By condition $(f_1)'$, there exists $C_\varepsilon > 0$ (for every $\varepsilon > 0$) satisfying

$$F(\xi,t) \leq \varepsilon|t|^{\tau p} + C_\varepsilon|t|^q \quad \text{for a.e. } \xi \in \mathbb{H}^N \text{and all } t \in \mathbb{R}.$$

For any $u \in W_\mu$, by condition (M), we have

$$I_\mu(u) \geq \frac{m_0}{p\tau}\|u\|_\mu^{p\tau} - \frac{1}{2Q_\lambda^*}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q_\lambda^*}|u(\xi)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi - \varepsilon\int_{\mathbb{H}^N} |u|^{\tau p} d\xi - C_\varepsilon \int_{\mathbb{H}^N} |u|^q d\xi.$$

By taking $\varepsilon \in (0, \frac{m_0}{2\tau pc})$ and applying the definition of $H_{Q_\lambda^*}$, we have

$$I_\mu(u) \geq \frac{m_0}{p\tau}\|u\|_\mu^{p\tau} - \frac{\mu^{-1}}{2Q_\lambda^*} H_{Q_\lambda^*}^{-Q_\lambda^*/p} \|u\|_\mu^{Q_\lambda^*} - C\varepsilon\|u\|_\mu^{p\tau} - CC_\varepsilon\|u\|_\mu^q$$
$$\geq \left(\frac{m_0}{2\tau p} - \frac{\mu^{-1}}{2Q_\lambda^*} H_{Q_\lambda^*}^{-Q_\lambda^*/p} \|u\|_\mu^{Q_\lambda^* - \tau p} - CC_\varepsilon\|u\|_\mu^{q-\tau p}\right)\|u\|_\mu^{\tau p}.$$

Here, we used the fractional Sobolev embedding $|u|_{\tau p} \leq C\|u\|_\mu$ and $|u|_q \leq C\|u\|_\mu$ in the last inequality.

Next, we define

$$g(t) = \frac{m_0}{2\tau p} - \frac{\mu^{-1}}{2Q_\lambda^*} H_{Q_\lambda^*}^{-Q_\lambda^*/p} t^{Q_\lambda^* - \tau p} - CC_\varepsilon t^{q-\tau p} \quad \text{for every } t \geq 0.$$

Since $Q_\lambda^* > \tau p$ and $q > \tau p$, it is clear that $\lim_{t\to 0^+} g(t) = \frac{m_0}{2\tau p}$. Pick $\sigma = \|u\|_\mu$ small enough such that

$$\frac{\mu^{-1}}{2Q_\lambda^*} H_{Q_\lambda^*}^{-Q_\lambda^*/p} \sigma^{Q_\lambda^* - \tau p} + CC_\varepsilon \sigma^{q-\tau p} < \frac{m_0}{2\tau p},$$

and hence $I_\mu(u) \geq g(\sigma)\sigma^{\tau p} = \alpha$. The proof of Lemma 4.3 is thus complete. □

Lemma 4.4. *Let $\tau \in (1, \frac{Q_\lambda^*}{p})$ and suppose that Kirchhoff function $M(\cdot)$ satisfies condition (M). Assume that $f(\cdot,\cdot)$ satisfies condition $(f_1)'$, and the potential function $V(\xi)$ satisfies conditions (V_1) and (V_2). Then, for every $\mu > 0$, there exists $e \in S_\mu$ with $\|e\|_\mu > \sigma$ satisfying $I_\mu(e) < 0$, where σ is given by Lemma 4.3.*

Proof. By condition $(f_2)'$, we know that $F(\xi,t) \geq 0$ for a.e. $\xi \in \mathbb{H}^N$. Choose a function $u_0 \in S_\mu$ satisfying

$$\|u_0\|_\mu = 1 \quad \text{and} \quad \frac{1}{2Q_\lambda^*} \int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u_0(\eta)|^{Q_\lambda^*}|u_0(\xi)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi > 0.$$

By condition (M) and $F(\xi,t) \geq 0$ for a.e. $\xi \in \mathbb{H}^N$, one has

$$I_\mu(tu_0) \leq \frac{m_1 t^{\tau p}}{p\tau}\|u_0\|_\mu^{p\tau} - \frac{t^{2Q_\lambda^*}}{2Q_\lambda^*} \int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u_0(\eta)|^{Q_\lambda^*}|u_0(\xi)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi.$$

Since $2Q_\lambda^* > \tau p$, there exists $t \geq 1$ large enough satisfying $\|tu_0\|_\mu > \sigma$ and $I_\mu(tu_0) < 0$. Taking $e = tu_0$, the proof of Lemma 4.4 is complete. □

Note that function I_μ does not satisfy the $(PS)_c$ condition for every $c > 0$. Therefore, we can find a special finite-dimensional subspace to construct sufficiently small minimax levels.

Next, we obtain by assumption (V_1) that there is $\xi_0 \in \mathbb{H}^N$ satisfying $V(\xi_0) = \min_{\xi\in\mathbb{H}^N} V(\xi) = 0$. In general, we set $\xi_0 = 0$. By conditions (M) and $(f_2)'$, for $u \in S_\mu$, one has

$$\begin{aligned} I_\mu(u) &\leq \frac{m_1}{p\tau}\|u\|_\mu^{p\tau} - \frac{1}{2Q_\lambda^*}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q_\lambda^*}|u(\xi)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi - a_0 \int_{\mathbb{H}^N} |u|^{q_2} d\xi \\ &\leq \frac{m_1}{p\tau}\|u\|_\mu^{p\tau} - a_0 \int_{\mathbb{H}^N} |u|^{q_2} d\xi. \end{aligned}$$

We define the function $J_\mu : S_\mu \to \mathbb{R}$ as follows:

$$J_\mu(u) = \frac{m_1}{p\tau}\|u\|_\mu^{p\tau} - a_0 \int_{\mathbb{H}^N} |u|^{q_2} d\xi.$$

Thus, $I_\mu(u) \leq J_\mu(u)$, and we only need to construct small minimax levels of $J_\mu(u)$. For any $0 < \chi < 1$, we choose $\delta_\chi \in C_0^\infty(\mathbb{H}^N)$ with $|\delta_\chi|_{q_2} = 1$ and $\operatorname{supp}\delta_\chi \subset B_{r_\chi}(0)$ satisfying $[\delta_\chi]_{s,p}^p < \chi$. In the sequel, we make a scaling argument. Letting

$$e_\mu = \delta_\chi(\mu^{-\frac{\tau Q_\lambda^*}{Q(Q_\lambda^* - \tau p)}}\xi),$$

we have $\operatorname{supp} e_\mu \subset B_{\mu^{\frac{\tau Q_\lambda^*}{Q(Q_\lambda^*-\tau p)}} r_\chi}(0)$. Thus, for $\mu \in (0,1)$, $\tau > 1$ and $t \geq 0$,

$$
\begin{aligned}
J_\mu(te_\mu) &= \frac{t^{\tau p}}{p\tau}\|e_\mu\|_\mu^{p\tau} - a_0 t^{q_2}\int_{\mathbb{H}^N}|e_\mu|^{q_2}d\xi \\
&\leq \mu^{\frac{\tau Q_\lambda^*}{Q_\lambda^*-\tau p}}\left[\frac{t^{\tau p}}{p\tau}\left(\int\int_{\mathbb{H}^{2N}}\frac{|\delta_\chi(\xi)-\delta_\chi(\eta)|^p}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta + \int_{\mathbb{H}^N}V(\mu^{\frac{\tau Q_\lambda^*}{Q(Q_\lambda^*-\tau p)}}\xi)|\delta_\chi|^p d\xi\right)^\tau\right. \\
&\quad \left. -a_0 t^{q_2}\int_{\mathbb{H}^N}|\delta_\chi|^{q_2}d\xi\right] = \mu^{\frac{\tau Q_\lambda^*}{Q_\lambda^*-\tau p}}\Psi_\mu(t\delta_\chi).
\end{aligned}
$$

We define $\Psi_\mu \in C_1(S_\mu, \mathbb{H}^N)$ as follows:

$$
\Psi_\mu(u) = \frac{1}{p\tau}\left(\int\int_{\mathbb{H}^{2N}}\frac{|u(\xi)-u(\eta)|^p}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta + \int_{\mathbb{H}^N}V(\mu^{\frac{\tau Q_\lambda^*}{Q(Q_\lambda^*-\tau p)}}\xi)|u|^p d\xi\right)^\tau - a_0\int_{\mathbb{H}^N}|u|^{q_2}d\xi
$$

for every $u \in S_\mu$. It is clear that

$$
\begin{aligned}
&\max_{t\geq 0}\Psi_\mu(t\delta_\chi) \\
&= \frac{q_2-\tau p}{q_2\tau p(q_2 a_0)^{\frac{\tau p}{q_2-\tau p}}}\left(\int\int_{\mathbb{H}^{2N}}\frac{|\delta_\chi(\xi)-\delta_\chi(\eta)|^p}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta + \int_{\mathbb{H}^N}V(\mu^{\frac{\tau Q_\lambda^*}{Q(Q_\lambda^*-\tau p)}}\xi)|\delta_\chi|^p d\xi\right)^{\frac{\tau q_2}{q_2-\tau p}}.
\end{aligned}
$$

Due to condition (M), we know that $V(0)=0$ and $V \in (\mathbb{H}^N, \mathbb{R})$, so there is a constant $\Lambda_\chi > 0$ satisfying

$$
0 \leq V(\mu^{\frac{\tau Q_\lambda^*}{Q(Q_\lambda^*-\tau p)}}\xi) \leq \frac{\chi}{\delta_\chi},
$$

for every $|\xi| \leq r_\chi$ and $0 < \mu \leq \Lambda_\chi$. Since $[\delta_\chi]_{s,p}^p < \chi$, we have

$$
\max_{t\geq 0}\Psi_\mu(t\delta_\chi) \leq \frac{q_2-\tau p}{q_2\tau p(q_2 a_0)^{\frac{\tau p}{q_2-\tau p}}}(2\chi)^{\frac{\tau q_2}{q_2-\tau p}},
$$

therefore

$$
\max_{t\geq 0} I_\mu(t\delta_\chi) \leq \frac{q_2-\tau p}{q_2\tau p(q_2 a_0)^{\frac{\tau p}{q_2-\tau p}}}(2\chi)^{\frac{\tau q_2}{q_2-\tau p}}\mu^{\frac{\tau Q_\lambda^*}{Q_\lambda^*-\tau p}}, \tag{4.21}
$$

for every $\mu \in (0, \Lambda_\chi]$. To be more precise, we state the following lemma.

Lemma 4.5. *Under conditions of Lemma 4.3, there exists a constant $\Lambda > 0$ satisfying the following hypotheses: for every fix $\mu \in (0,\Lambda)$, there exists $\widetilde{e_\mu} \in S_\mu$ with $\|\widetilde{e_\mu}\|_\mu > \sigma$ satisfying $I_\mu(\widetilde{e_\mu}) < 0$ and*

$$
\max_{t\in[0,1]} I_\mu(t\widetilde{e_\mu}) \leq \rho\mu^{\frac{\tau Q_\lambda^*}{Q_\lambda^*-\tau p}},
$$

where $\rho = \frac{1}{\tau p} - \frac{1}{2Q_\lambda^*} + \frac{1}{q_0}(1-\frac{m_1}{m_0})$.

Proof. Let χ be small enough such that

$$\frac{q_2-\tau p}{q_2\tau p(q_2a_0)^{\frac{\tau p}{q_2-\tau p}}}(2\chi)^{\frac{\tau q_2}{q_2-\tau p}}\leq\rho.$$

For all $t\geq t_1$, take $\Lambda=\Lambda_\chi$ and choose $t_1>0$ satisfying $t_1\|e_\mu\|_\mu>\sigma$ and $I_\mu(te_\mu)<0$. Letting $\widetilde{e_\mu}=t_1e_\mu$, we obtain the desired result immediately. This completes the proof of Lemma 4.5. □

For any $m\in\mathbb{N}$, $1\leq i\neq j\leq m$, select functions $\delta_\chi^i\in C_0^\infty(\mathbb{H}^N)$ satisfying $\mathrm{supp}\delta_\chi^i\bigcap\mathrm{supp}\delta_\chi^j=\emptyset$, $|\delta_\chi|_{q_2}=1$, and $[\delta_\chi]_{s,p}^p<\chi$. There is $r_\chi^m>0$ satisfying $\mathrm{supp}\delta_\chi^i\subset B_{r_\chi^m}(0)$ for $i=1,2,\cdots,m$:

$$e_\mu=\delta_\chi(\mu^{-\frac{\tau Q_\lambda^*}{Q(Q_\lambda^*-\tau p)}}\xi)$$

and

$$E_{\mu,\chi}^m=\mathrm{span}\{e_\mu^1,e_\mu^2,\cdots,e_\mu^m\}.$$

Note that $u=\sum_{i=1}^m c^ie_\mu^i\in E_{\mu,\chi}^m$, so we obtain

$$\int\int_{\mathbb{H}^{2N}}\frac{|u_n(\xi)-u_n(\eta)|^p}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta=\sum_{i=1}^m|c^i|^p\int\int_{\mathbb{H}^{2N}}\frac{|e_\mu^i(\xi)-e_\mu^i(\eta)|^p}{|\eta^{-1}\xi|^{N+ps}}d\xi d\eta,$$

$$\int_{\mathbb{H}^N}V(\xi)|u|^pd\xi=\sum_{i=1}^m|c^i|^p\int_{\mathbb{H}^N}V(\xi)|e_\mu^i|^pd\xi,$$

$$\frac{1}{2Q_\lambda^*}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q_\lambda^*}|u(\xi)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi=\frac{1}{2Q_\lambda^*}\sum_{i=1}^m|c^i|^{2Q_\lambda^*}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|e_\mu^i(\eta)|^{Q_\lambda^*}|e_\mu^i(\xi)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi,$$

and

$$\int_{\mathbb{H}^N}F(\xi,u)d\xi=\sum_{i=1}^m\int_{\mathbb{H}^N}F(\xi,c^ie_\mu^i)d\xi.$$

Thus $I_\mu(u)=\sum_{i=1}^m I_\mu(c^ie_\mu^i)$ and

$$I_\mu(c^ie_\mu^i)\leq\mu^{\frac{\tau Q_\lambda^*}{Q_\lambda^*-\tau p}}\Psi_\mu(c^ie_\mu^i).$$

Take $\beta=\max|\delta_\chi^i|_p^p$, $i=1,2,\cdots,m$ and for every $|x|\leq r_\chi^m$ and $\mu\leq\Lambda_{m,\chi}$, there exists $\Lambda_{m,\chi}>0$ satisfying $V(\mu^{\frac{\tau Q_\lambda^*}{N(Q_\lambda^*-\tau p)}}\xi)\leq\frac{\chi}{\beta}$. From (4.21), we derive that

$$\max_{u\in E_{\mu,\chi}^m}I_\mu(u)\leq\frac{q_2-\tau p}{q_2\tau p(q_2a_0)^{\frac{\tau p}{q_2-\tau p}}}(2\chi)^{\frac{\tau q_2}{q_2-\tau p}}\mu^{\frac{\tau Q_\lambda^*}{Q_\lambda^*-\tau p}}$$

for every $\mu\in(0,\Lambda_{m,\chi}]$. Thus, we have the following lemma.

Lemma 4.6. *Under the conditions of Lemma 4.3, for every $m\in\mathbb{N}$, there exists $\Lambda_m>0$ satisfying the following hypothesis: for every $\mu\in(0,\Lambda_m]$, there exists an m-dimensional subspace E_μ^m satisfying*

$$\max_{u\in E_\mu^m}I_\mu(u)\leq\rho\mu^{\frac{\tau Q_\lambda^*}{Q_\lambda^*-\tau p}}.\tag{4.22}$$

Proof. Let χ small enough to satisfy

$$\frac{q_2-\tau p}{q_2\tau p(q_2a_0)^{\frac{\tau p}{q_2-\tau p}}}(2\chi)^{\frac{\tau q_2}{q_2-\tau p}} \leq \rho$$

and choose $E^m_\mu = E^m_{\mu,\chi}$. Then, we can obtain the desired result immediately. This completes the proof of Lemma 4.6. □

Proof of Theorem 1.2. Apply Lemma 4.5 for every $\mu > 0$, consider the function I_μ. Let $\mu^* = \Lambda_\chi$ and define for every $\mu \leq \mu^*$ the min-max value $c^\mu = \inf_{h\in\Gamma_\mu}\max_{t\in[0,1]} I_\mu(h(t))$, where

$$\Gamma_\mu = \{h \in C([0,1],W_\mu) : h(0) = 0 \text{ and } h(1) = \widetilde{e_\mu}\}.$$

By Lemma 4.3 and Lemma 4.5, one has $\alpha \leq c_\mu < \rho\mu^{\frac{\tau Q^*_\lambda}{Q^*_\lambda-\tau p}}$. By Lemma 4.2, we know that I_μ satisfies the $(PS)_{c_\mu}$ condition, and we can deduce that there is $u_1 \in S_\mu$ satisfying $I_\mu(u_1) \to c_\mu$, $I'_\mu(u_1) \to 0$. Therefore, u_1 is a solution of (2.1). Since u_1 is a critical point of I_μ, we have

$$\begin{aligned}\rho\mu^{\frac{\tau Q^*_\lambda}{Q^*_\lambda-\tau p}} &\geq I_\mu(u_1) = I_\mu(u_1) - \frac{1}{q_0}\langle I'_\mu(u_1),u_1\rangle\\ &\geq \left(\frac{m_0}{\tau p}-\frac{m_1}{q_0}\right)\|u_1\|^{\tau p}_\mu + \left(\frac{m_1}{q_0}-\frac{1}{2Q^*_\lambda}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u_1(\eta)|^{Q^*_\lambda}|u_1(\xi)|^{Q^*_\lambda}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi\right),\end{aligned}$$

which yields inequalities (1.3) and (1.4). This completes the proof of Theorem 1.2 (i).

Next, we are going to prove Theorem 1.2 (ii). We define

$$\Gamma = \{y \in C(S_\mu,S_\mu) : \ y \text{ is an odd homeomorphism}\},$$

and for every $B \in \Upsilon$, we define

$$i(B) = \min_{y\in\Gamma}\gamma(y(B)\bigcap\partial B_\sigma),$$

where $\sigma > 0$ is a constant defined in Lemma 4.3. Therefore, $i(B)$ is a version of the Benci pseudo-index Benci [2]. Let

$$c_j = \inf_{i(B)\leq j}\sup_{u\in B} I_\mu(u),\ j = 1,2,\cdots,m.$$

It is clear that $c_1 \leq c_2 \leq \cdots \leq c_m$. In the sequel, we are to going prove that $c_1 \geq \alpha$ and $c_m \leq \sup_{u\in E^m_\mu} I_\mu(u)$, where $\alpha > 0$ is the constant defined in Lemma 4.3. For all $B \in \Upsilon$, it follows from Benci [2, Theorem 1.4] that $i(B) \geq 1$, so we can deduce that $\gamma(B\bigcap\partial B_\sigma) \geq 1$. This implies that $B\bigcap\partial B_\sigma \neq \emptyset$. By Lemma 4.3, one has $I_\mu(u) > \alpha$ for every $\|u\|_\mu = \sigma$. Thus $\sup_{u\in B} I_\mu(u) > \alpha$ and $c_1 \geq \alpha$. Considering that the Krasnoselskii genus satisfies the dimension property Benci [2], we obtain

$$\gamma(y(E^m_\mu)\bigcap\partial B_\sigma) = \dim(E^m_\mu) = m, \text{ for every } y \in \Gamma,$$

which implies that $i(E^m_\mu) = m$. Hence, $c_m \leq \sup_{u\in E^m_\mu} I_\mu(u)$. By (4.22), one has

$$\alpha \leq c_1 \leq c_2 \leq \cdots \leq c_m \leq \sup_{u\in E^m_\mu} I_\mu(u) \leq \rho\mu^{\frac{\tau Q^*_\lambda}{Q^*_\lambda-\tau p}},$$

where $\rho > 0$ is a constant defined in Lemma 4.2. As can be seen from Lemma 4.2 that $I_\mu(u)$ satisfies the $(PS)_c$ condition at all levels $c \in (0, \rho(m_0\mu^\tau H^\tau_{Q^*_\lambda})^{\frac{Q^*_\lambda}{Q^*_\lambda - \tau p}})$. Finally, by using the general critical point theory, we obtain that all c_j of $1 \le j \le m$ are critical values of $I_\mu(u)$. Since $I_\mu(u)$ is even, one sees that $I_\mu(u)$ has at least m pairs of critical points. Therefore, $I_\mu(u)$ has at least m pairs of critical points as the solutions of problem (1.1). The proof of Theorem 1.2 is thus complete. □

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