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A CONTINUITY RESULT FOR THE NASH EQUILIBRIUM OF A CLASS OF NETWORK GAMES

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Abstract. We investigate a class of Nash equilibrium problems on networks where both the utility function and the strategy space of each player are parametrized by means of continuous functions of a vector parameter. We provide a variational inequality reformulation of the Nash equilibrium problem which turns out to be much simpler than the original problem. Moreover, under additional assumptions, the continuity of the solution with respect to the parameter is proved.

Keywords. Network games; Nash equilibrium; Set convergence; Variational inequality.

1. Introduction

Network games are a kind of games characterized by the fact that players are represented by nodes of a graph, and the direct relationship between any two players is represented by an arc connecting them. Relationships can be asymmetric so the resulting graph is, in general, a directed weighted graph. To express the fact that neighbors of a given player can influence her action, her utility function is modeled so as to depend on her own action variable as well as on the action variables of her neighbors, although further terms can be added to take into account global effects in the network. Analytic properties and graph-theoretical aspects are thus intermingled in the investigation of the Nash equilibria of these games. This model has been pioneered in the papers [3, 4], where the authors, considering quadratic utility functions, were able to obtain exact results connecting the Nash equilibrium with spectral properties of the adjacency matrix of the graph. We refer the interested reader to the excellent survey [10], where many extensions of the results in [3] are presented, but the models considered are mostly quadratic and with interior solutions, so as to obtain closed-form solutions. While the above mentioned papers focus on the key-player identification, a different point of view has been considered in [1, 2] where a new player joins a group of players which is interacting in a non-cooperative way through a generalized Nash game and can face three different situations: playing together with the other players in a generalized Nash game, playing first and waiting for the response of the opponent group, or letting the group play first and act then as a follower.

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More recently, some authors investigated games played on networks by means of the variational inequality approach put forward long time ago in [9], based on finding Nash equilibria by solving a variational inequality, where the operator (called the *pseudogradient* of the game) is made up with the partial gradients of the utility functions of all the players. In this respect, a relevant paper is [17], where the properties of the operator which appears in the variational inequality formulation have been studied in detail. Along the same lines, in [18, 19, 21], the authors focused on the case of bounded strategy space and also of generalized Nash equilibrium problems, where no analytical solutions are available.

In this note, we use a variational inequality reformulation of network games which does not involve the pseudogradient of the game. This point of view has been considered in the interesting paper [22], where the authors consider a nonlinear complementary problem formulation of a class of games with unbounded strategy space, which was previously deeply investigated with the use of best-response functions (see, e.g., [5]). In our contribution, we consider a class of utility functions where the interaction matrix is parametrized by continuous functions of a vector parameter and the strategy space is bounded and can be perturbed by means of some continuous functions of the same parameter. Under a simple set of assumptions, we are then able to prove the continuity of the unique Nash equilibrium with respect to the parameter.

The paper is organized as follows. In the following Section 2, we recall the basic elements of a Nash equilibrium problem on networks and introduce the class of utility functions of our model. Section 3 is devoted to the equivalent variational inequality formulation of the problem. In Section 4 we first prove a general continuity result for variational inequalities which fits our framework and also investigate the monotonicity properties of the relevant operator. Then, we sum up all our previous results to state the continuity theorem of the unique Nash equilibrium of our game and illustrate our findings with the help of a numerical example. In the short concluding section we mention some further research perspectives.

2. Nash Equilibrium Problems on Networks

We consider a non-cooperative game, where the set of players is denoted by $\{1,\dots,n\}$ and corresponds to the set of nodes of a directed graph. We denote with $A_i \subset \mathbb{R}$ the action space of player i, while $A = A_1 \times \cdots \times A_n$. A vector $x = (x_1,\dots,x_n) \in A$ is called an *action profile*. We also use the common notations $x_{-i} = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ and $x = (x_i,x_{-i})$ when we wish to distinguish the action of player i from the action of all the other players. For any given player i, the set $N(i)$ denotes her neighbors in the graph, that is a player j is a neighbor of i if (i,j) is an arc of the graph. The intensity of the relationship between and two neighbors is described by the interaction matrix $\mathscr{F}$, whose diagonal elements are zero. The off-diagonal elements f_{ij} can be either positive or negative, but for tractability reasons it is often assumed that they are all of the same sign.

Each player i is endowed with a payoff function $u_i : A \to \mathbb{R}$ that she wishes to maximize. We now recall the definition of a Nash equilibrium, which is one of the most common solution concepts in Game Theory.

Definition 2.1. An action profile $x^* \in A$ is a Nash Equilibrium (NE) of the game iff for each $i \in \{1,\dots,n\}$:

$$u_i(x_i^*,x_{-i}^*) \geq u_i(x_i,x_{-i}^*), \qquad \forall\, x_i \in A_i. \tag{2.1}$$

As mentioned above, it is convenient, for tractability reasons, to consider games where the neighbors of a player influence the player's behavior in the same direction for all players. We make this concept precise with the help of the marginal utility function.

Definition 2.2. The network game has the property of strategic complements if:

$$\frac{\partial^2 u_i}{\partial x_j \partial x_i}(x) > 0, \qquad \forall\,(i,j): f_{ij} \neq 0,\ \forall\, x \in A. \tag{2.2}$$

Definition 2.3. The network game has the property of strategic substitutes if:

$$\frac{\partial^2 u_i}{\partial x_j \partial x_i}(x) < 0, \qquad \forall\,(i,j): f_{ij} \neq 0,\ \forall\, x \in A. \tag{2.3}$$

The standard variational inequality approach to Nash equilibrium problems is recalled in the following theorem. For an account of variational inequalities the interested reader can refer to [7, 14, 16].

Theorem 2.1. *For each $i \in \{1,\dots,n\}$, let u_i be a continuously differentiable function on A and $u_i(\cdot,x_{-i})$ be concave with respect to its own action x_i, for each $x_{-i} \in A_{-i}$. Moreover, let A be closed and convex. Then, x^* is a Nash equilibrium if and only if it solves the variational inequality $VI(T,A)$: find $x^* \in A$ such that*

$$T(x^*)^\top (x - x^*) \geq 0, \qquad \forall\, x \in A, \tag{2.4}$$

where the operator

$$[T(x)]^\top := -\left(\frac{\partial u_1}{\partial x_1}(x), \dots, \frac{\partial u_n}{\partial x_n}(x)\right) \tag{2.5}$$

is also called the pseudogradient of the game.

In our model, we will not use the variational inequality involving the pseudogradient but another, simpler, variational inequality. In both cases, it is important to recall some useful monotonicity properties of the relevant operator.

Definition 2.4. An operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone on A iff:

$$[T(x) - T(y)]^\top (x - y) \geq 0, \qquad \forall\, x, y \in A.$$

If the equality holds only when $x = y$, T is said to be strictly monotone on A.
T is said to be τ-strongly monotone on A iff there exists $\tau > 0$ such that

$$[T(x) - T(y)]^\top (x - y) \geq \tau \|x - y\|^2, \qquad \forall\, x, y \in A.$$

Conditions that ensure the unique solvability of a variational inequality problem are given by the following theorem (see, e.g., [16]).

Theorem 2.2. *If $K \subset \mathbb{R}^n$ is a compact convex set and $T : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on K, then the variational inequality problem $VI(T,K)$ admits at least one solution. In the case that K is unbounded, existence of a solution may be established under the following coercivity condition:*

$$\lim_{\|x\| \to +\infty} \frac{[T(x) - T(x_0)]^\top (x - x_0)}{\|x - x_0\|} = +\infty,$$

for $x \in K$ and some $x_0 \in K$.
Furthermore, the solution is unique if T is strictly monotone on K.

We now specify the class of parametric utility functions investigated in this paper. Consider a vector of coefficients $c \in \mathbb{R}^n_{++}$, and for a fixed $\alpha \in \mathbb{R}^n$, positive numbers $L_i(\alpha)$ and let $A_i(\alpha) = [0, L_i(\alpha)]$, for any $i \in \{1, \dots, n\}$, and $A(\alpha) = [0, L_1(\alpha)] \times \ldots \times [0, L_n(\alpha)]$. Moreover, in order to model the fact that the network of relationships can change, we consider a parameter dependent interaction matrix $\mathscr{F}(\alpha)$, with elements $f_{ij}(\alpha)$. The payoff function of player i is defined as follows:

$$u_i(\alpha, x) = v_i\left(x_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} f_{ij}(\alpha) x_j\right) - c_i x_i. \tag{2.6}$$

The form of the function above reflects the following model: the term $c_i x_i$ is the cost that player i faces when playing x_i. The first term, if positive, represents her revenue and depends on her action x_i and on a weighted sum of the actions of her neighbors. The influence of neighbors on the utility of player i can thus be both positive or negative according to the definitions (2.2) and (2.3). This kind of utility functions (with fixed interaction matrix) has been investigated in [5], with the use of best-response functions, to model the choice of public goods on networks (see also [8]). To progress in the analysis of the game further hypotheses are required, as specified in the theorems of the following section.

3. Variational Inequality formulation

Theorem 3.1. *Let $\alpha \in \mathbb{R}^n$ be given. Consider the network game where, for any $i \in \{1, \dots, n\}$, $[0, L_i(\alpha)]$ is the action space of player i and the utility function u_i is defined in (2.6), where the function $v_i : \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:*

(i) $v_i \in C^2(\mathbb{R}_+)$,
(ii) v_i *is strictly concave on* $\mathbb{R}_+$,
(iii) $v_i'(0) > c_i$,
(iv) $\lim_{t\to\infty} v_i'(t) < c_i$.

Then, $x^*(\alpha)$ is a NE if and only if, for any $i \in \{1, \dots, n\}$, it satisfies the following system:

$$\begin{cases} x_i^*(\alpha) + \sum_{j\neq i} f_{ij}(\alpha) x_j^*(\alpha) = b_i & \text{if } x_i^*(\alpha) \in]0, L_i(\alpha)[, \\ x_i^*(\alpha) + \sum_{j\neq i} f_{ij}(\alpha) x_j^*(\alpha) \geq b_i & \text{if } x_i^*(\alpha) = 0, \\ x_i^*(\alpha) + \sum_{j\neq i} f_{ij}(\alpha) x_j^*(\alpha) \leq b_i & \text{if } x_i^*(\alpha) = L_i(\alpha), \end{cases} \tag{3.1}$$

where b_i is the unique solution of the equation

$$v_i'(t) = c_i.$$

Proof. By the definition of Nash equilibrium, we have that, for any $i \in \{1, \dots, n\}$,

$$x_i^*(\alpha) = \arg\max_{x_i \in [0, L_i(\alpha)]} u_i(\alpha, x_i, x_{-i}^*(\alpha)),$$

which is equivalent to:

$$\frac{\partial u_i}{\partial x_i}(\alpha, x^*(\alpha)) \begin{cases} = 0 & \text{if } x_i^*(\alpha) \in]0, L_i(\alpha)[, \\ \leq 0 & \text{if } x_i^*(\alpha) = 0, \\ \geq 0 & \text{if } x_i^*(\alpha) = L_i(\alpha), \end{cases}$$

which, in turn, is equivalent to:

$$v_i'(x_i^*(\alpha)+\sum_{j\neq i} f_{ij}(\alpha)x_j^*(\alpha)) \begin{cases} = c_i = v_i'(b_i) & \text{if } x_i^*(\alpha)\in]0,L_i(\alpha)[, \\ \leq c_i = v_i'(b_i) & \text{if } x_i^*(\alpha)=0, \\ \geq c_i = v_i'(b_i) & \text{if } x_i^*(\alpha)=L_i(\alpha). \end{cases}$$

Since v_i is a strictly concave function by assumptions, the function v_i' is strictly decreasing, hence the above conditions are equivalent to:

$$x_i^*(\alpha)+\sum_{j\neq i} f_{ij}(\alpha)x_j^*(\alpha) \begin{cases} = b_i & \text{if } x_i^*(\alpha)\in]0,L_i(\alpha)[, \\ \geq b_i & \text{if } x_i^*(\alpha)=0, \\ \leq b_i & \text{if } x_i^*(\alpha)=L_i(\alpha). \end{cases}$$

□

The following theorem establishes an equivalent variational inequality formulation of (3.1).

Theorem 3.2. *Let $g:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ be defined as follows:*

$$g_i(\alpha,x)=x_i+\sum_{j\neq i} f_{ij}(\alpha)x_j-b_i, \qquad \forall\, i\in\{1,\dots,n\}, \tag{3.2}$$

and let $A(\alpha)=[0,L_1(\alpha)]\times\ldots\times[0,L_n(\alpha)]$, where each $L_i:\mathbb{R}^n\to\mathbb{R}_{++}$ is an arbitrary positive function. For each $\alpha\in\mathbb{R}^n$, consider the variational inequality problem of finding $x^*(\alpha)\in A(\alpha)$ such that:

$$\sum_{i=1}^n g_i(\alpha,x^*(\alpha))(x_i-x_i^*(\alpha))\geq 0, \qquad \forall\, x\in A(\alpha). \tag{3.3}$$

We then have that $x^*(\alpha)$ is a solution of (3.3) if and only if it is a solution of the system (3.1) for any $i\in\{1,\dots,n\}$.

Proof. We first prove that (3.3) $\implies$ (3.1). Let $x^*(\alpha)$ be a solution of (3.3).

(1) If $x_i^*(\alpha)\in]0,L_i(\alpha)[$ we choose x in (3.3) such that $x_j=x_j^*(\alpha)$, for any $j\neq i$ and $x_i=x_i^*(\alpha)\pm\varepsilon$, which is feasible, for ε small enough, and yields $\pm\varepsilon g_i(\alpha,x^*(\alpha))\geq 0$ which, in turn, entails $x_i^*(\alpha)+\sum_{j\neq i} f_{ij}(\alpha)x_j^*(\alpha)-b_i=0$.
(2) If $x_i^*(\alpha)=0$, by choosing $x_j=x_j^*(\alpha)$, for any $j\neq i$, and $x_i=L_i(\alpha)$ we deduce that $x_i^*(\alpha)+\sum_{j\neq i} f_{ij}(\alpha)x_j^*(\alpha)-b_i\geq 0$.
(3) If $x_i^*(\alpha)=L_i(\alpha)$, by choosing $x_j=x_j^*(\alpha)$, for any $j\neq i$, and $x_i=0$ we deduce that $x_i^*(\alpha)+\sum_{j\neq i} f_{ij}(\alpha)x_j^*(\alpha)-b_i\leq 0$.

To prove that (3.1) $\implies$ (3.3), assume that $x^*(\alpha)$ satisfies (3.1) and consider the three set of indices defined as follows:

$$\begin{aligned} I_0 &= \{i:\ x_i^*(\alpha)\in]0,L_i(\alpha)[\}, \\ I_+ &= \{i:\ x_i^*(\alpha)=0\}, \\ I_- &= \{i:\ x_i^*(\alpha)=L_i(\alpha)\}. \end{aligned}$$

If we split the sum in (3.3) accordingly, then

$$\sum_{i=1}^{n} g_i(\alpha, x^*(\alpha))(x_i - x_i^*(\alpha))$$
$$= 0 + \sum_{i\in I_+}\left(x_i^*(\alpha) + \sum_{j\neq i} f_{ij}(\alpha)x_j^*(\alpha) - b_i\right)x_i$$
$$+ \sum_{i\in I_-}\left(x_i^*(\alpha) + \sum_{j\neq i} f_{ij}(\alpha)x_j^*(\alpha) - b_i\right)(x_i - L_i(\alpha)) \geq 0$$

holds for any $x \in A(\alpha)$. □

We note that Theorem 3.2 allows to reformulate the Nash equilibrium problem in the game with non-quadratic utility functions (2.6) as an equivalent affine variational inequality $VI(g(\alpha,\cdot), A(\alpha))$, where the map

$$g(x,\alpha) = [I + \mathscr{F}(\alpha)]x - b.$$

4. Continuity Result

We now prove a continuity result for parametric variational inequalities which will allow us to obtain the continuity of the Nash equilibrium. The literature on stability of solutions of variational inequalities is vast, (see, e.g., [13] and the references therein) and the results obtained are not always of straightforward application. The result which we present here is an adaptation from [6] and fits very well our problem, while using a simple set of hypotheses.

We first recall the definition of set convergence in the sense of Kuratowski-Painlevé, often called just Kuratowski-convergence. Let S be a metric space, and $\{M_n\}_{n\in\mathbb{N}}$ a sequence of subsets of S. Consider the sets:

$$\limsup_n M_n = \Big\{ y \in S :\ \exists\, n_1 < n_2 < \ldots < n_i < \ldots, \text{ with } y_{n_i} \in M_{n_i} \text{ and } y = \lim_{i\to\infty} y_{n_i} \Big\},$$

$$\liminf_n M_n = \Big\{ y \in S :\ \exists\, n_0 \in \mathbb{N} :\ \forall\, n > n_0, \exists y_n \in M_n \text{ and } y = \lim_{n\to\infty} y_n \Big\}.$$

Definition 4.1. Let M be a subset of S. The sequence of sets $\{M_n\}$ is said to be Kuratowski-convergent to M iff:

$$\limsup_n M_n = \liminf_n M_n = M, \tag{4.1}$$

and we write $M_n \xrightarrow{K} M$.

Since $\liminf_n M_n \subseteq \limsup_n M_n$ always holds, to verify (4.1), it is enough to check whether

$$\limsup_n M_n \subseteq M \subseteq \liminf_n M_n.$$

Thus, to check whether $M_n \xrightarrow{K} M$ we have to check the two following conditions:

(a) for each subsequence $\{k_n\} \subset \mathbb{N}$ such that $v_{k_n} \in M_{k_n}$ and $\lim_n v_{k_n} = v$, then $v \in M$.
(b) For each $v \in M$ it exists a sequence $\{v_n\}$ such that $v_n \in M_n$, for n large enough, and $\lim_n v_n = v$.

Theorem 4.1. *Let S be a metric space and $K \subseteq S$. Consider a map $G : S \times \mathbb{R}^n \to \mathbb{R}^n$. Assume that G is continuous in $K \times \mathbb{R}^n$ and, for each $t \in K$, let $C(t)$ be a nonempty closed and convex subset of $\mathbb{R}^n$. For each $t \in K$ consider the variational inequality problem of finding $x(t) \in C(t)$ such that:*

$$\sum_{i=1}^{n} G_i(t,x(t))(y_i - x_i(t)) \geq 0, \qquad \forall\, y \in C(t). \tag{4.2}$$

Let us also assume that $G(t,\cdot)$ be τ-strongly monotone on $\mathbb{R}^n$, uniformly with respect to $t \in K$, that is:

$$\sum_{i=1}^{n} (G_i(t,x) - G_i(t,y))(x_i - y_i) \geq \tau\, \|x-y\|^2, \qquad \forall\, t \in K,\ \forall\, x,y \in \mathbb{R}^n. \tag{4.3}$$

Moreover, assume that for each $\{t_n\} \subset K$ such that $\lim_n t_n = t$, it follows $C(t_n) \xrightarrow{K} C(t)$. Then, the solution map $t \mapsto x(t)$ is continuous in t.

Proof. We will prove that, for an arbitrarily fixed $t \in K$ and for any sequence $\{t_n\} \subset K$ such that $\lim_n t_n = t$, we get $\lim_n x(t_n) = x(t)$.

We first prove that the sequence $\{x(t_n)\}$ is bounded. Because of part (b) of the Kuratowski convergence of $C(t_n)$ to $C(t)$, it exists a sequence of elements $\{v_n\}$ such that $v_n \in C(t_n)$, for n large enough, and $\lim_n v_n = x(t)$. The continuity of G then entails $\lim_n G(t_n, v_n) = G(t,x(t))$.

Consider now (4.2) for $t = t_n$ and, for n large enough, we can choose $y = v_n$ and get:

$$\sum_{i=1}^{n} G_i(t_n,x(t_n))((v_n)_i - x_i(t_n)) \geq 0. \tag{4.4}$$

From the uniform strong monotonicity of G, we get:

$$\begin{aligned}
\|x(t_n) - v_n\|^2 &\leq \frac{1}{\tau} \sum_{i=1}^{n} (G_i(t_n,x(t_n)) - G_i(t_n,v_n))(x_i(t_n) - (v_n)_i) \\
&= \frac{1}{\tau} \sum_{i=1}^{n} (G_i(t_n,x(t_n)))(x_i(t_n) - (v_n)_i) + \frac{1}{\tau} \sum_{i=1}^{n} (G_i(t_n,v_n))((v_n)_i - x_i(t_n)) \\
&\leq \frac{1}{\tau} \sum_{i=1}^{n} (G_i(t_n,v_n))((v_n)_i - x_i(t_n)) \leq \frac{1}{\tau}\, \|G(t_n,v_n)\|\, \|x(t_n) - v_n\|.
\end{aligned}$$

Thus, we have that:

$$\|x(t_n)\| \leq \|x(t_n) - v_n\| + \|v_n\| \leq \frac{1}{\tau} \|G(t_n,v_n)\| + \|v_n\|,$$

hence the boundedness of $\{x(t_n)\}$ follows from the convergence of $G(t_n,v_n)$ and v_n.

Let us now recall that under the monotonicity hypothesis, the Minty's Lemma holds true, that is, the variational inequality (4.2) is equivalent to the following variational inequality problem of finding $x(t) \in C(t)$ such that:

$$\sum_{i=1}^{n} G_i(t,y)(y_i - x_i(t)) \geq 0, \qquad \forall\, y \in C(t). \tag{4.5}$$

Let us now denote with $\{w(t_n)\}$ an arbitrary subsequence of $\{x(t_n)\}$. From $w(t_n)$ we can extract a subsequence which converges to some $v \in \mathbb{R}^n$. We thus have:

$$w(t_{k_n}) \in C(t_{k_n}), \quad \lim_n w(t_{k_n}) = v.$$

The part (a) of the Kuratowski convergence of $C(t_n)$ to $C(t)$ entails $v \in C(t)$. We now prove that $v = x(t)$. To this end, we apply Minty's lemma by writing (4.5) for $t = t_{k_n}$:

$$\sum_{i=1}^{n} G_i(t_{k_n}, y)(y_i - w_i(t_{k_n})) \geq 0, \qquad \forall\, y \in C(t_{k_n}). \tag{4.6}$$

For the part (b) of the Kuratowski convergence of $C(t_n)$ to $C(t)$, we get that for any $z \in C(t)$, it exists a sequence $\{z_n\}$ such that, for n large enough, $z_n \in C(t_n)$ and $\lim_n z_n = z$. We can the test (4.6) with $y = z_{k_n}$ and obtain:

$$\sum_{i=1}^{n} G_i(t_{k_n}, z_{k_n})((z_{k_n})_i - w_i(t_{k_n})) \geq 0,$$

and passing to limit for $n \to \infty$ we get $\sum_{i=1}^{n} G_i(t,z)(z_i - v_i(t)) \geq 0$, which, using Minty's lemma again, and having arbitrarily chosen $z \in C(t)$, yields to

$$\sum_{i=1}^{n} G_i(t, v)(z_i - v_i) \geq 0, \qquad \forall z \in C(t).$$

The uniqueness of the solution of (4.2) entails $v = x(t)$ and, finally, from the general principle of convergence, $\lim_n x(t_n) = x(t)$. □

Remark 4.1. The uniform strong monotonicity hypothesis has been used only to prove the boundedness of the sequence $\{x(t_n)\}$. It follows that, in the case where the sets $C(t)$ are all subsets of some bounded set, we can simply require the strict monotonicity of $G(t,\cdot)$, for each t, in order to have both the uniqueness of the solution and the applicability of Minty's lemma.

The following theorem provides a Kuratowski-convergence result for a sequence of polyhedra of $\mathbb{R}^n$ (see [6] for the proof).

Theorem 4.2. *Let m be a positive integer, and $\{a_{ij}(t)\}$ and $\{b_i(t)\}$, $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$, be continuous functions defined on a subset K of a metric space S. For each $t \in K$ and for each $i \in \{1,\dots,m\}$, define the vector: $a_i(t) = (a_{i1}(t),\dots,a_{in}(t))$, and denote with $P(t)$ the polyhedron in $\mathbb{R}^n$ defined by:*

$$a_i(t)^\top x \leq b_i(t), \qquad i \in \{1,\dots,m\}.$$

Furthermore, let $\{t_n\} \subseteq K$ be a sequence such that $\lim_n t_n = t$, for some $t \in K$. Assume now that for every set of different indices $\{i_1,\dots,i_k\} \in \{1,\dots,m\}$ the matrix functions whose rows are given by $\{a_i(s)\}_{i=i_1,\dots,i_k}$ be of constant rank in a neighborhood of t. It then follows that $P(t_n) \xrightarrow{K} P(t)$.

In order to apply Theorem 4.1 to our class of utility functions, it is then necessary to check the monotonicity properties of the operator in the variational inequality formulation (3.3).

To this end, we first recall a general result, which can be applied to our formulation, and then provide a condition which ensures the uniform strong monotonicity for a class of parametric interaction matrices.

The following lemma characterizes the monotonicity of the mapping g in (3.3).

Lemma 4.1.

(1) Let $\alpha \in \mathbb{R}^n$ be given and $\lambda_{min}\left(\frac{\mathscr{F}(\alpha)+\mathscr{F}(\alpha)^\top}{2}\right)$ denote the minimum eigenvalue of the symmetric part of the matrix $\mathscr{F}(\alpha)$. Then, the affine map $g(\alpha,\cdot)$ is monotone on $\mathbb{R}^n$ if and only if

$$\lambda_{min}\left(\frac{\mathscr{F}(\alpha)+\mathscr{F}(\alpha)^\top}{2}\right) \geq -1$$

and $g(\alpha,\cdot)$ is strongly monotone on $\mathbb{R}^n$ if and only if

$$\lambda_{min}\left(\frac{\mathscr{F}(\alpha)+\mathscr{F}(\alpha)^\top}{2}\right) > -1.$$

(2) If there exists $\gamma > 0$ such that

$$\lambda_{min}\left(\frac{\mathscr{F}(\alpha)+\mathscr{F}(\alpha)^\top}{2}\right) \geq -1+\gamma, \qquad \forall\, \alpha \in \mathbb{R}^n, \tag{4.7}$$

then $g(\alpha,\cdot)$ is strongly monotone on $\mathbb{R}^n$ uniformly with respect to α.

Proof. See, e.g., [7]. □

Proposition 4.1. *Consider the game defined in (2.6) with interaction terms given by $f_{ij}(\alpha) = h_i(\alpha)f(\alpha_i-\alpha_j)$ or $f_{ij}(\alpha) = h_i(\alpha)f(|\alpha_i-\alpha_j|)$, where $f:\mathbb{R}\to\mathbb{R}$ is non-expansive and there is $\delta_0 \in \mathbb{R}_+$ such that $f(\delta_0)=0$. Then, the map $g(\alpha,\cdot)$ is strongly monotone on $\mathbb{R}^n$, uniformly with respect to α, if there exists $\gamma>0$ such that*

$$n\,h(\alpha)\,(\sqrt{2}\,\sigma_\alpha+\delta_0) \leq 1-\gamma, \qquad \forall\, \alpha \in \mathbb{R}^n,$$

where $h(\alpha) = \max_{i=1,\dots,n} |h_i(\alpha)|$ and σ_α is the standard deviation of $(\alpha_1,\dots,\alpha_n)$.

Proof. Since

$$\lambda_{min}\left(\frac{\mathscr{F}(\alpha)+\mathscr{F}(\alpha)^T}{2}\right) = -\lambda_{max}\left(\frac{-\mathscr{F}(\alpha)-\mathscr{F}(\alpha)^T}{2}\right),$$

condition (4.7) is equivalent to

$$\lambda_{max}\left(\frac{-\mathscr{F}(\alpha)-\mathscr{F}(\alpha)^T}{2}\right) \leq 1-\gamma, \qquad \forall\, \alpha \in \mathbb{R}^n. \tag{4.8}$$

We now assume that $f_{ij}(\alpha) = h_i(\alpha)f(\alpha_i-\alpha_j)$. Since f is non-expansive and $f(\delta_0)=0$ we have:

$$|f(\alpha_i-\alpha_j)| = |f(\alpha_i-\alpha_j)-f(\delta_0)| \;\leq |\alpha_i-\alpha_j-\delta_0|.$$

If we denote with $\rho(M)$ and $\|M\|_F$ the spectral radius and the Frobenius norm of a matrix M, then the following chain of equalities and inequalities holds:

$$\begin{aligned}
&\lambda_{max}\left(\frac{-\mathscr{F}(\alpha)-\mathscr{F}(\alpha)^T}{2}\right) \\
&\leq \rho\left(\frac{-\mathscr{F}(\alpha)-\mathscr{F}(\alpha)^T}{2}\right) = \rho\left(\frac{\mathscr{F}(\alpha)+\mathscr{F}(\alpha)^T}{2}\right) \\
&\leq \left\|\frac{\mathscr{F}(\alpha)+\mathscr{F}(\alpha)^T}{2}\right\|_F \leq \|\mathscr{F}(\alpha)\|_F \\
&= \sqrt{\sum_{i=1}^n\sum_{j=1}^n [f_{ij}(\alpha)]^2} = \sqrt{\sum_{i=1}^n\sum_{j=1}^n h_i(\alpha)^2[f(\alpha_i-\alpha_j)]^2} \\
&\leq h(\alpha)\sqrt{\sum_{i=1}^n\sum_{j=1}^n [f(\alpha_i-\alpha_j)]^2} \\
&\leq h(\alpha)\sqrt{\sum_{i=1}^n\sum_{j=1}^n (\alpha_i-\alpha_j-\delta_0)^2} \\
&= h(\alpha)\sqrt{\sum_{i=1}^n\sum_{j=1}^n (\alpha_i-\alpha_j)^2+n^2\delta_0^2-2\delta_0\sum_{i=1}^n\sum_{j=1}^n(\alpha_i-\alpha_j)} \\
&= h(\alpha)\sqrt{\sum_{i=1}^n\sum_{j=1}^n (\alpha_i-\alpha_j)^2+n^2\delta_0^2} \\
&\leq h(\alpha)\left[\sqrt{\sum_{i=1}^n\sum_{j=1}^n (\alpha_i-\alpha_j)^2}+n\delta_0\right] \\
&= h(\alpha)\left[\sqrt{2n^2\sigma_\alpha^2}+n\delta_0\right] = n\,h(\alpha)\,(\sqrt{2}\,\sigma_\alpha+\delta_0).
\end{aligned}$$

Therefore, the assumptions imply that condition (4.8) holds, hence the uniform strong monotonicity of $g(\alpha,\cdot)$ follows from Lemma 4.1.

When $f_{ij}(\alpha)=h_i(\alpha)f(|\alpha_i-\alpha_j|)$, the proof is essentially the same as the previous case, but in this case we need to use the following inequality:

$$\begin{aligned}
h(\alpha)&\sqrt{\sum_{i=1}^n\sum_{j=1}^n (\alpha_i-\alpha_j)^2+n^2\delta_0^2-2\delta_0\sum_{i=1}^n\sum_{j=1}^n|\alpha_i-\alpha_j|} \\
&\leq h(\alpha)\sqrt{\sum_{i=1}^n\sum_{j=1}^n (\alpha_i-\alpha_j)^2+n^2\delta_0^2}.
\end{aligned}$$

□

Taking into account the previous results we can now state the continuity theorem of the Nash equilibrium, whose proof is thus straightforward.

Theorem 4.3. *Consider the game with utility functions defined in (2.6), where v_i satisfy the set of hypotheses in Theorem 3.1 and let $f_{ij}(\alpha)$ and $L_i(\alpha)$ be continuous functions of α. Let the*

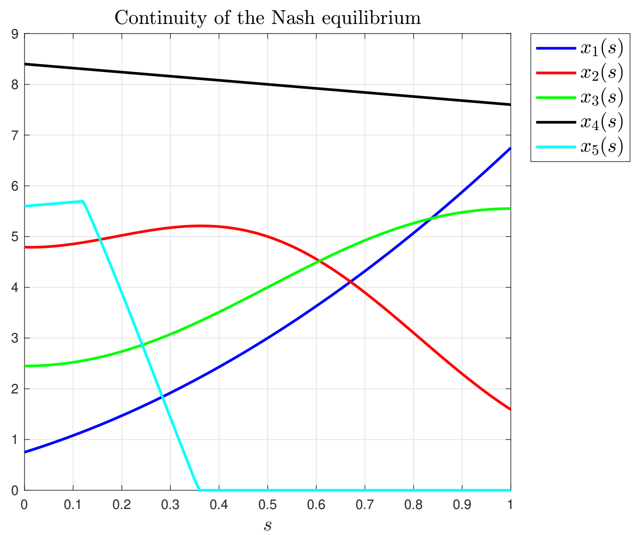

FIGURE 1. Continuity of the Nash equilibrium with respect to parameter s.

map $g(\alpha,\cdot)$ defined in (3.2) be strongly monotone on $\mathbb{R}^n$, uniformly with respect to α. It then follows that the unique Nash equilibrium of the game, $x^* : \alpha \mapsto x^*(\alpha)$, is a continuous function of α.

We conclude this section with an illustrative example of Theorem 4.3.

Example 4.1. Consider a game with $n = 5$ players, where the utility functions are defined as in (2.6), with $c_i = 0.1$, $v_i(t) = \sqrt{\varepsilon + t}$ and $\varepsilon = 0.01$ for any $i \in \{1,\dots,5\}$. We assume that the interaction terms are given by $f_{ij}(\alpha) = \alpha_i - \alpha_j$, for any $i, j \in \{1,\dots,5\}$, so that the matrix $\mathscr{F}(\alpha)$ is skew-symmetric. Lemma 4.1 guarantees that the map $g(\alpha,\cdot)$ is strongly monotone on $\mathbb{R}^5$ uniformly with respect to α. Moreover, we assume that the continuous functions $L_i(\alpha)$ are defined as follows:

$$\begin{cases} L_1(\alpha_1) = 3\,\alpha_1^2, \\ L_2(\alpha_2) = 4\,\alpha_2 + \cos(2\pi\alpha_2), \\ L_3(\alpha_3) = 2\,\alpha_3 + \sin(2\pi\alpha_3), \\ L_4(\alpha_4) = 4\,\alpha_4, \\ L_5(\alpha_5) = 2\,\alpha_5. \end{cases}$$

In turn, the parameters $\alpha_1,\dots,\alpha_5$ are supposed to be affine functions of a scalar parameter $s \in [0,1]$ as follows:

$$\begin{cases} \alpha_1(s) = 0.5 + s, \\ \alpha_2(s) = 1.4 - 0.8\,s, \\ \alpha_3(s) = 1.7 + 0.6\,s, \\ \alpha_4(s) = 2.1 - 0.2\,s, \\ \alpha_5(s) = 2.8 + 0.4\,s. \end{cases}$$

Figure 1 shows the continuity of the five components of the Nash equilibrium of the game as function of parameter s.

5. CONCLUSIONS

In this paper, we considered a class of games played on networks, where both the interaction matrix and the strategy space could be perturbed by means of continuous functions of a vector parameter. We provided a convenient variational inequality formulation of the game which allowed us to prove the continuity of the unique Nash equilibrium. In future investigations, we plan to consider models where these parameters are random variables, along the same lines as in [11, 12, 20]. Moreover, we plan to extend our results to the case of generalized Nash equilibrium problems, by using the duality theory developed in [15].

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REFERENCES

[1] D. Aussel, T.C. Thanh Cong, R. Riccardi, Strategic decision in a two-period game using a multi-leader-follower approach. Part 1 – General setting and weighted Nash equilibrium, J. Appl. Numer. Optim. 4 (2022), 67-85.
[2] Aussel, T.C. Thanh Cong, R. Riccardi, Strategic decision in a two-period game using a multi-leader-follower approach. Part 2 – Decision making for the new player, J. Appl. Numer. Optim. 4 (2022), 87-101.
[3] C. Ballester, A. Calvo-Armengol, Y. Zenou, Who's Who in Networks. Wanted: The Key Player, Econometrica 74 (2006), 1403-1417.
[4] C. Ballester, A. Calvo-Armengol, Interactions with hidden complementarities, Reg. Sci. Urban Econ. 40 (2010), 397-406.
[5] Y. Bramoullé, R. Kranton, Public goods in networks, J. Econ. Theory 135 (2007), 478-494.
[6] A. Caruso, A. A. Khan, F. Raciti, Continuity Results for a Class of Variational Inequalities with Applications to Time-Dependent Network Problems, Numer. Funct. Analysis Optim. 30 (2009), 1272-1288.
[7] F. Facchinei, J.-S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems, Springer, Berlin, 2003.
[8] A. Galeotti, S. Goyal, M. O. Jackson, F. Vega-Redondo, L. Yariv, Network games Rev. Econ. Stud. 77 (2010), 218-244.
[9] D. Gabay, H. Moulin, On the uniqueness and stability of Nash equilibria in noncooperative games in: Applied stochastic control in econometrics and management science, pp. 271-294, North-Holland, Amsterdam, 1980.
[10] M. O. Jackson, Y. Zenou, Games on Networks in: Handbook of Game Theory with Economic Applications, pp. 95-163, Elsevier, 2015.
[11] B. Jadamba, M. Pappalardo, F.Raciti, Efficiency and Vulnerability Analysis for Congested Networks with Random Data, J. Optim. Theory Appl. 177 (2018), 563-583.
[12] B. Jadamba, F. Raciti, On the modelling of some environmental games with uncertain data, J. Optim. Theory Appl. 167 (2015), 959-968.
[13] J. Kyparisis, Sensitivity Analysis framework for variational inequalities, Math. Program. 38 (1987), 203-213.
[14] I. V. Konnov, Equilibrium Models and Variational Inequalities, Elsevier, 2007.
[15] G. Mastroeni, M. Pappalardo, F. Raciti, Generalized Nash equilibrium problems and variational inequalities in Lebesgue spaces, Minimax Theory Appl. 5 (2020), 47-64.
[16] A. Nagurney, Network Economics. A Variational Inequality Approach, Springer, 1999.

[17] F. Parise and A. Ozdaglar, A variational inequality framework for network games: Existence, uniqueness, convergence and sensitivity analysis, Games Econ. Behav. 114 (2019), 47-82.

[18] M. Passacantando, F. Raciti, A Note on Network Games with Strategic Complements and the Katz-Bonacich Centrality Measure, in: Optimization and Decision Science, Cerulli et al. (eds.), pp. 51-61, 2021.

[19] M. Passacantando, F. Raciti, A note on generalized Nash games played on networks, in: Nonlinear Analysis, Differential Equations, and Applications, T.M. Rassias (ed.), Springer Optimization and Its Applications, vol. 173, pp. 365-380, Springer, Cham 2021.

[20] M. Passacantando, F. Raciti, Optimal road maintenance investment in traffic networks with random demands, Optim. Lett. 15 (2021), 1799-1819.

[21] M. Passacantando, F. Raciti, A Variational Inequality Approach to a Class of Network Games with Local Complementarities and Global Congestion, in: Optimization in Artificial Intelligence and Data Sciences, Amorosi, Dell'Olmo, Lari (eds.), pp. 1-11, 2022.

[22] Z. Zhou, B. Yolken, R. A. Miura-Ko, N. Banbos, A game-theoretical formulation of influence networks, in: American Control Conference (ACC), 3802-3807, 2016. doi: 10.1109/ACC.2016.7525505