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OUTER APPROXIMATION FOR PSEUDO-CONVEX MIXED-INTEGER NONLINEAR PROGRAM PROBLEMS

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Abstract. Outer approximation (OA) for solving convex mixed-integer nonlinear programming (MINLP) problems is heavily dependent on the convexity of functions and a natural issue is to relax the convexity assumption. This paper is devoted to OA for dealing with a pseudo-convex MINLP problem. By solving a sequence of nonlinear subproblems, we use Lagrange multiplier rules via Clarke subdifferentials of subproblems to introduce a parameter and then equivalently reformulate such MINLP as the mixed-integer linear program (MILP) master problem. Then, an OA algorithm is constructed to find the optimal solution to the MNILP by solving a sequence of MILP relaxations. The OA algorithm is proved to terminate after a finite number of steps. Numerical examples are illustrated to test the constructed OA algorithm.
Keywords. Clarke Subdifferential; Mixed-integer nonlinear programming; MILP master program; Outer Approximation; Pseudo-convexity.

1. Introduction

Many practical optimization problems involving integer and continuous decision variables are modelled as mixed-integer nonlinear programming problems (MINLPs) and solution algorithms for solving these MINLPs have become an active research area over the past few decades; see, e.g. [6, 16, 19, 20, 25, 30, 31] and the references therein. In general, such MINLP can be expressed mathematically as follows:

$$(\mathrm{P})\begin{cases}\min\limits_{x,y}\ f(x,y)\\ \text{s.t.}\ \ g_i(x,y)\le 0,\ i=1,\cdots,m,\\ \qquad x\in X, y\in Y \text{ integer},\end{cases}$$

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where $f, g_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ $(i = 1, \cdots, m)$ are nonlinear functions, $X \subseteq \mathbb{R}^n$ is a bounded, closed, and convex set in $\mathbb{R}^n$, and $Y \subseteq \mathbb{Z}^p$ is a set of integers in. If objective and constraint functions are convex, such problem is known as a convex MINLP; otherwise, it is a non-convex MINLP.

MINLP problems have been extensively used in many diverse and important applications, such as portfolio optimization [21], block layout design in the manufacturing and service sectors [5], network design with queuing delay constraints [4], integrated design and control of chemical process [15], drinking water distribution systems security [23], and minimizing the environmental impact of utility plants [13]. Since MINLP has wide applications in practical problems, the study on solution algorithms is an active research direction in optimization and mathematical programming. To the best of our knowledge, the solution algorithms mainly rely on the branch-and-bound method, extended cutting plane, generalized Benders decomposition, outer approximation (OA), and so on. We refer the reader to [3, 10, 12, 14, 17, 18, 22, 28, 33, 37, 38] for details on these solution algorithms.

In this paper, we restrict our analysis to OA for MINLP problems. The OA method was first introduced by Duran and Grossmann [10] to solve MINLP problems in which f and g are affine in integer variables and convex in continuous variables. Subsequently, Fletcher and Leyffer [12] extended this OA to convex MINLP problems, where f and g are continuously differentiable. Drewes and Ulbrich [9] studied OA to deal with mixed-integer second order cone programming prblems. Eronen et al. [11] and the authors [33] generalized OA to nonsmooth convex MINLP problems in which f and g are convex but nonsmooth, and proposed outer approximation algorithms to find the optimal solution by solving a sequence of relaxed MILP problems. Recently, Delfino and de Oliveira [8] combined OA and bundle method algorithms for dealing with nonsmooth convex MINLP problems.

It is known that solving convex MINLP by OA is strongly dependent on convexity of functions since the convexity plays a key role in the fundamental insight behind the algorithm that MINLP is equivalent to a MILP of finite size. However, it is frequently and generally that non-convex MINLP problems appear in practical optimization problems, and from the theoretical viewpoint and for applications, it is natural and interesting to study solution algorithms for the general non-convex MINLP problems. Motivated by this observation, we consider a class of non-convex MINLP problems with pseudo-convex functions and apply OA to deal with such MINLP. The aim of this paper is to establish the OA algorithm for finding the optimal solution of such MINLP and then extend OA to solve MINLP from the convex case to the non-convex one.

The paper is organized as follows. Section 2 contains preliminaries and some results on Clarke subdifferentials used in this paper. Section 3 is devoted to Lagrange multiplier rules via Clarke subdifferentials for generalized optimization problems. In Section 4, we study a pseudo-convex MINLP problem and its solution algorithm via the OA method. Using Lagrange multiplier rules for optimization problems, we introduce a parameter to reformulate the pseudo-convex MINLP problem as MILP master program and prove that the MILP master program is equivalent to MINLP. Then we construct an outer approximation algorithm to find optimal solutions of MINLP by solving (feasible/infeasible) subproblems and a sequence of relaxed MILP problems. The termination of this algorithm after a finite number of steps is also proved therein. Section 5 contains several numerical examples to test the constructed algorithm. The conclusion of this paper is given in Section 6, the last section.

2. PRELIMINARIES

In this section we briefly discuss some generalized differential constructions of our study and review some results widely used in what follows; see [2, 7, 26, 27] for more details.

For any subset A of $\mathbb{R}^n$, we denote by $\mathrm{cl}(A)$ and $\mathrm{conv}(A)$ the closure and convex hull of A, respectively. We denote by $d_A(\cdot)$ the distance function which is defined by

$$d_A(x) := \inf\{\|x-u\| : u \in A\} \ \ \forall x \in \mathbb{R}^n.$$

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a local Lipschitz function and $\bar{x} \in \mathbb{R}^n$. For any direction $h \in \mathbb{R}^n$, we denote by

$$\phi^\circ(\bar{x}; h) := \limsup_{x\to\bar{x}, t\downarrow 0} \frac{\phi(x+th)-\phi(x)}{t}$$

the Clarke directional derivative of ϕ at $\bar{x}$ along direction h. We denote by

$$\partial_c\phi(\bar{x}) := \{\alpha \in \mathbb{R}^n : \ \langle \alpha, h\rangle \le \phi^\circ(\bar{x};h) \ \ \forall h \in \mathbb{R}^n\}$$

the Clarke subdifferential of ϕ at $\bar{x}$. It is known from [7] that

$$\partial_c\phi(\bar{x}) = \mathrm{conv}\big\{\lim_k \nabla\phi(x_k) : x_k \to \bar{x} \text{ and } x_k \notin \Omega_\phi\big\},$$

where Ω_ϕ is the set of zero measure in which ϕ is not Fréchet differentiable.

For the case that ϕ is convex, the Clarke subdifferential reduces to that in the sense of convex analysis; that is

$$\partial_c\phi(\bar{x}) = \partial\phi(\bar{x}) = \{\alpha \in \mathbb{R}^n : \ \langle \alpha, x-\bar{x}\rangle \le \phi(x)-\phi(\bar{x}) \ \ \forall x \in \mathbb{R}^n\}.$$

Let Ω be a closed subset of $\mathbb{R}^n$ and $\bar{x} \in \Omega$. We denote by $T_c(\Omega,\bar{x})$ and $T(\Omega,\bar{x})$ the Clarke tangent cone and the contingent (Bouligand) cone of A at $\bar{x}$, respectively and they are defined by

$$T_c(\Omega,\bar{x}) := \operatorname*{Liminf}_{x\xrightarrow{\Omega}\bar{x}, t\to 0^+} \frac{\Omega-x}{t} \ \text{ and } \ T(\Omega,\bar{x}) := \operatorname*{Limsup}_{t\to 0^+} \frac{\Omega-\bar{x}}{t},$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. Thus, $v \in T_c(\Omega,\bar{x})$ if and only if for any $x_k \xrightarrow{\Omega} \bar{x}$ and any $t_k \to 0^+$, there exists $v_k \to v$ such that $x_k + t_k v_k \in \Omega$ for all $k \in \mathbb{N}$, and $v \in T(\Omega,\bar{x})$ if and only if there exist $v_k \to v$ and $t_k \to 0^+$ such that $\bar{x} + t_k v_k \in \Omega$ for all $k \in \mathbb{N}$.

We denote by $N_c(\Omega,\bar{x})$ the Clarke normal cone of Ω at $\bar{x}$ which is defined as

$$N_c(\Omega,\bar{x}) := \big\{\alpha \in \mathbb{R}^n : \ \langle \alpha, h\rangle \le 0 \ \forall h \in T_c(\Omega,\bar{x})\big\}$$

It is known from [7] that

$$N_c(\Omega,\bar{x}) = \mathrm{cl}(\mathbb{R}_+\partial_c d_\Omega(\bar{x})).$$

For the case when Ω is convex, it is known that Clarke tangent cone coincides with contingent cone; that is,

$$T_c(\Omega,\bar{x}) = T(\Omega,\bar{x}) = \mathrm{cl}(\mathbb{R}_+(\Omega-x))$$

and Clarke normal cone reduces to that in the sense of convex analysis; that is

$$N_c(\Omega,\bar{x}) = N(\Omega,\bar{x}) = \{\alpha \in \mathbb{R}^n : \langle \alpha, x-\bar{x}\rangle \le 0 \ \ \forall x \in \Omega\}.$$

The following lemmas on the Clarke subdifferential are cited from [7, Theorem 2.9.8] and [7, Proposition 2.3.12] which are useful in the analysis.

Lemma 2.1. *Let $\phi_1, \phi_2 : \mathbb{R}^n \to \mathbb{R}$ be local Lipschitz functions and $\bar{x} \in \mathbb{R}^n$. Then*

$$\partial_c(\phi_1 + \phi_2)(\bar{x}) \subseteq \partial_c\phi_1(\bar{x}) + \partial_c\phi_2(\bar{x}).$$

Lemma 2.2. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a local Lipschitz function and $\bar{x} \in \mathbb{R}^n$ be such that $\phi(\bar{x}) = 0$. Define $\phi_+(x) := \max\{\phi(x), 0\}$ for all $x \in \mathbb{R}^n$. Then ϕ_+ is local Lipschitz around $\bar{x}$ and*

$$\partial_c\phi_+(\bar{x}) = [0,1]\partial_c\phi(\bar{x})$$

where $[0,1]\partial_c\phi(\bar{x}) := \{t\alpha : t \in [0,1] \text{ and } \alpha \in \partial_c\phi(\bar{x})\}$.

3. Lagrange Multiplier Rules for Optimization Problems

In this section, we discuss one optimization problem posed by the general mathematical program and mainly focus on the familiar and useful mathematical technique, known as the Lagrange multiplier rule. We begin with such optimization problem.

Suppose that $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 0, 1, \cdots, m)$ are local Lipschitz (not convex necessarily) functions and A is a bounded closed subset of $\mathbb{R}^n$. We consider the following optimization problem:

$$\begin{cases} \min \ \varphi_0(x) \\ \text{s.t.} \ \ \varphi_i(x) \le 0, i = 1, \cdots, m, \\ \qquad x \in A, \end{cases} \tag{3.1}$$

The following proposition is the Lagrange multiplier rule for an optimal solution to (3.1). This result follows from [7, Proposition 2.4.2 and Theorem 6.1.1].

Proposition 3.1. *Suppose that $\bar{x}$ is an optimal solution to (3.1). Then $\bar{x}$ is feasible to (3.1) and there exist nonnegative multipliers $(\lambda_0, \lambda_1, \cdots, \lambda_m) \in \mathbb{R}^{m+1} \backslash \{0\}$ such that*

$$0 \in \lambda_0\partial_c\varphi_0(\bar{x}) + \sum_{i=1}^{m} \lambda_i\partial_c\varphi_i(\bar{x}) + N_c(A, \bar{x}) \ \ \textit{and} \ \ \lambda_i\varphi_i(\bar{x}) = 0, i = 1, \cdots, m. \tag{3.2}$$

The necessary conditions in Proposition 3.1 are regarded as being degenerated for the case when the multiplier corresponding to φ_0 (which we have labelled λ_0) vanishes, since then the objective function φ_0 being minimized is not involved. Various supplementary conditions have been proposed to ensure the existence of the "normal" multiplier (i.e. with $\lambda_0 = 1$). These conditions are named as constraint qualifications.

For the convex optimization problem in (3.1) where each φ_i $(i = 0, 1, \cdots, m)$ is convex and $A = \mathbb{R}^n$, optimality condition (3.2) with normal multiplier $\lambda_0 = 1$ is named as Karush-Kuhn-Tucker (KKT) conditions and further KKT conditions are also sufficient to the optimal solution in this case. To ensure KKT conditions, several type of constraint qualifications are well recognized and studied. We refer the reader to [1, 24, 29, 32, 36] for more details on these constraint qualifications.

When it comes to optimization problem (3.1), we discuss two types of constraint qualifications ensure the normal multiplier $\lambda_0 = 1$ in (3.2) for an optimal solution to (3.1). The first type is named as Mangasarian-Fromowitz constraint qualification (see (3.3) below) in the classic case where each φ_i $(i = 0, 1, \cdots, m)$ is continuously differentiable and $A = \mathbb{R}^n$.

Proposition 3.2. *Suppose that each φ_i $(i = 0, 1, \cdots, m)$ is continuously differentiable and $A = \mathbb{R}^n$ in (3.1). Suppose that $\bar{x}$ is an optimal solution to (3.1) and there is a vector $v \in \mathbb{R}^n$ such that*

$$\langle \nabla\varphi_i(\bar{x}), v\rangle < 0, \ \ \forall i \in I(\bar{x}) \tag{3.3}$$

where $I(\bar{x}) := \{i \in \{1, \cdots, m\} : \varphi_i(\bar{x}) = 0\}$ *is the active index set. Then there exist nonnegative multipliers* $(\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^m \backslash \{0\}$ *such that*

$$\nabla \varphi_0(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla \varphi_i(\bar{x}) = 0 \ \textit{and} \ \lambda_i \varphi_i(\bar{x}) = 0, i = 1, \cdots, m.$$

For convenience to present the other type of constraint qualifications for (3.1), we consider a parameterized family of optimization problems and the calm property of optimization problem (3.1).

Let $z = (z_1, \cdots, z_m) \in \mathbb{R}^m$ be given. We define optimization problem $(\mathrm{OP})_z$ as follows:

$$(\mathrm{OP})_z \begin{cases} \min \ \varphi_0(x) \\ \text{s.t.} \ \ \varphi_i(x) \le z_i, i = 1, \cdots, m, \\ \qquad x \in A, \end{cases}$$

and define $v(z)$ as the optimal value of $(\mathrm{OP})_z$; that is,

$$v(z) := \inf \left\{ \varphi_0(x) : \varphi_i(x) \le z_i, i = 1, \cdots, m \ \text{ and } \ x \in A \right\} \tag{3.4}$$

for all $z = (z_1, \cdots, z_m) \in \mathbb{R}^m$. It is known that $(\mathrm{OP})_z$ reduces to optimization problem (3.1) if $z = (0, \cdots, 0) \in \mathbb{R}^m$.

Let $\bar{x}$ be an optimal solution to (3.1). Recall from [7] that optimization problem (3.1) is said to be calm at $\bar{x}$ if there exist $\delta, M > 0$ such that $\varphi_0(x) - \varphi_0(\bar{x}) + M\|z\| \ge 0$ holds for any $z \in \mathbb{R}^n$ with $\|z\| < \delta$ and any $x \in B(\bar{x}, \delta)$ feasible for $(\mathrm{OP})_z$.

With the assumption of calm property for (3.1), the normal multiplier $\lambda_0 = 1$ in (3.2) exists if $\bar{x}$ is an optimal solution to (3.1). This result follows from [7, Proposition 6.4.4 and Theorem 6.5.2].

Proposition 3.3. *Suppose that* $\bar{x}$ *is an optimal solution to* (3.1) *and optimization problem* (3.1) *is calm at* $\bar{x}$. *Then there exist nonnegative multipliers* $(\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^m \backslash \{0\}$ *such that*

$$0 \in \partial_c \varphi_0(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial_c \varphi_i(\bar{x}) + N_c(A, \bar{x}) \ \textit{and} \ \lambda_i \varphi_i(\bar{x}) = 0, i = 1, \cdots, m.$$

The calm property of optimization problem (3.1) is necessary to the normal multiplier $\lambda_0 = 1$ in (3.2). The following proposition, cited from [7, Proposition 6.4.2], provides a sufficient condition for the calm property at any optimal solution to (3.1).

Proposition 3.4. *Suppose that* $v(0)$ *is finite and*

$$\liminf_{z \to 0} \frac{v(z) - v(0)}{\|z\|} > -\infty. \tag{3.5}$$

Then (3.1) *is calm at any optimal solution to* (3.1).

Proposition 3.5. *Suppose that* $\varphi_i (i = 0, 1, \cdots, m)$ *are convex,* A *is convex, and optimization problem* (3.1) *satisfies the Slater constraint qualification. Then* (3.1) *is calm at any optimal solution to* (3.1).

Proof. Based on Proposition 3.4, we next prove that (3.5) holds. By applying [35, Proposition 3.4], one can verify that $v(\cdot)$ in (3.4) is a lower semicontinuous convex function on $\mathbb{R}^m$ and the

Slater constraint qualification guarantees that $v(\cdot)$ is continuous at $0 = (0,\cdots,0) \in \mathbb{R}^m$. Thus we can take $\xi \in \partial v(0)$ and then

$$\liminf_{z\to 0} \frac{v(z)-v(0)}{\|z\|} \geq -\|\xi\| > -\infty,$$

which implies that (3.5) holds. The proof is complete. □

The following proposition is immediate from Propositions 3.3 and 3.5. This result is a main tool used in this paper.

Proposition 3.6. *Suppose that $v(0)$ is finite and (3.5) holds. Then, for any optimal solution x to (3.1), there exist nonnegative multipliers $(\lambda_1,\cdots,\lambda_m) \in \mathbb{R}^m \backslash \{0\}$ such that*

$$0 \in \partial_c \varphi_0(x) + \sum_{i=1}^{m} \lambda_i \partial_c \varphi_i(x) + N_c(A,x) \ \ \textit{and} \ \ \lambda_i \varphi_i(x) = 0, i = 1,\cdots,m.$$

4. Pseudo-Convex MINLP and the Outer Approximation Algorithm

In this section, we study a pseudo-convex MINLP problem and its solution algorithm by outer approximation(OA). Due to lack of convexity, it is necessary to make some modification to the OA method when solving such pseudo-convex MINLP problem. The main work of this section is to establish a modified OA algorithm for finding the optimal solution to the pseudo-convex MINLP problem. This pseudo-convex MINLP problem is defined as follows:

$$(\mathrm{P}) \begin{cases} \min\limits_{x,y} f(x,y) \\ \text{ s.t. } g_i(x,y) \leq 0, i = 1,\cdots,m, \\ \qquad x \in X, y \in Y \text{ integer}, \end{cases} \tag{4.1}$$

where

(a) *$f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ is continuously differentiable and convex;*
(b) *$g_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R} (i = 1,\cdots,m)$ are differentiable pseudo-convex functions; that is, for all $z_1 = (x_1,y_1), z_2 = (x_2,y_2) \in \mathbb{R}^n \times \mathbb{R}^p$, one has*

$$\begin{cases} g_i(z_2) > g_i(z_1), \ \textit{if} \ \nabla g_i(z_1)^T (z_2 - z_1) > 0, \\ g_i(z_2) \geq g_i(z_1), \ \textit{if} \ \nabla g_i(z_1)^T (z_2 - z_1) = 0; \end{cases}$$

(c) *$X \subseteq \mathbb{R}^n$ is a bounded and closed convex set in and $Y \subseteq \mathbb{Z}^p$ is a set of integers.*

Note that the main idea of OA for solving convex MINLP problems is to reformulate the MINLP as an equivalent MILP master program and construct an OA algorithm to solve a finite sequence of relaxed MILP master problems so as to find the optimal solution. Along this idea, we consider to reformulate the pseudo-convex MINLP problem in (4.1) as an equivalent MILP master program.

For any $y \in Y$, we study the following the nonlinear subproblem $P(y)$:

$$P(y) \begin{cases} \min\limits_{x} \ f(x,y) \\ \text{s.t. } \ g_i(x,y) \leq 0, \ i = 1,\cdots,m, \\ \qquad x \in X, \end{cases}$$

We divide Y as two disjoint subsets:

$$T := \{y \in Y : P(y) \text{ is feasible}\} \ \text{ and } \ S := \{y \in Y : P(y) \text{ is infeasible}\}.$$

Throughout of this section, we suppose the following two assumptions to be hold:

A1 *For any $y_j \in T$, nonlinear subproblem $P(y_j)$ is calm at any optimal solution;*

A2 *For any $y_l \in S$, there exists $I_l \subseteq \{1, \cdots, m\}$ such that the following nonlinear subproblem $F(y_l)$ is calm at any optimal solution:*

$$F(y_l) \begin{cases} \min\limits_{x} \sum\limits_{i \in I_l} \max\{g_i(x, y_l), 0\} \\ \text{s.t. } g_i(x, y_l) \le 0, \; i \in J_l := \{1, \cdots, m\} \backslash I_l, \\ \qquad x \in X. \end{cases} \tag{4.2}$$

4.1. **Infeasible Nonlinear Subproblems.** Let $y_l \in S$. By the assumption **A2**, we consider the nonlinear subproblem $F(y_l)$ as (4.2). Suppose that $x_l \in X$ is an optimal solution to $F(y_l)$ (the existence of x_l is due to that X is bounded and closed and all g_i are continuous). Then by assumption **A2** and Proposition 3.6, there exist $\lambda_{l,i} \ge 0 (\forall i \in J_l)$ such that

$$0 \in \partial_c \varphi(x_l) + \sum_{i \in J_l} \lambda_{l,i} \triangledown_x g_i(x_l, y_l) + N(X, x_l) \text{ and } \lambda_{l,i} g_i(x_l, y_l) = 0, \; \forall i \in J_l \tag{4.3}$$

where $\varphi(x) := \sum_{i \in I_l} \max\{g_i(x, y_l), 0\}$ for any $x \in X$. We denote

$$\begin{cases} J_l^1 := \{i \in I_l : g_i(x_l, y_l) > 0\}, \\ J_l^2 := \{i \in I_l : g_i(x_l, y_l) = 0\}, \\ J_l^3 := \{i \in I_l : g_i(x_l, y_l) < 0\}. \end{cases}$$

It follows from (4.3) that there exist $\lambda_{l,i} \in [0, 1] (\forall i \in I_l)$ such that

$$\begin{cases} 0 \in \sum\limits_{i \in J_l} \lambda_{l,i} \triangledown_x g_i(x_l, y_l) + \sum\limits_{i \in I_l} \lambda_{l,i} \triangledown_x g_i(x_l, y_l) + N(X, x_l), \\ \lambda_{l,i} = 1, i \in J_l^1, \\ \lambda_{l,i} = 0, i \in J_l^3. \end{cases} \tag{4.4}$$

The following proposition shows that constraints can be added to exclude integers which produce infeasible nonlinear subproblems.

Proposition 4.1. *Let x_l solve nonlinear subproblem $F(y_l)$. Then the following constraints*

$$\begin{cases} g_i(x_l, y_l) + \triangledown g_i(x_l, y_l)^T \begin{pmatrix} x - x_l \\ y - y_l \end{pmatrix} \le 0, i = 1, \cdots, m, \\ x \in X, y \in Y, \end{cases} \tag{4.5}$$

could exclude the infeasible integer y_l.

Proof. Note that $P(y_l)$ is infeasible and thus $\varphi(x_l) > 0$. Suppose on the contrary that there exists $\hat{x} \in X$ such that $(\hat{x}, y_l)$ is feasible to the constraints in (4.5). Then

$$g_i(x_l, y_l) + \triangledown_x g_i(x_l, y_l)^T (\hat{x} - x_l) \le 0,$$

and consequently

$$\sum_{i=1}^{m} (\lambda_{l,i} g_i(x_l, y_l) + \lambda_{l,i} \triangledown_x g_i(x_l, y_l)^T (\hat{x} - x_l)) \le 0. \tag{4.6}$$

Noting that $\hat{x} - x_l \in T(X, x_l)$ by the convexity of X, it follows from (4.4) that

$$\sum_{i=1}^{m} \lambda_{l,i} \triangledown_x g_i(x_l, y_l)^T (\hat{x} - x_l) \ge 0.$$

This and (4.6) imply that $\sum_{i=1}^m \lambda_{l,i} g_i(x_l, y_l) \le 0$. On the other hand, by (4.3) and (4.4), one has

$$\sum_{i=1}^m \lambda_{l,i} g_i(x_l, y_l) = \sum_{i \in I_l} \lambda_{l,i} g_i(x_l, y_l) = \varphi(x_l) \le 0,$$

which is a contradiction as $\varphi(x_l) > 0$. The proof is complete. □

4.2. **Feasible nonlinear subproblems.** Let $y_j \in T$. Then $P(y_j)$ is feasible and we suppose that x_j is an optimal solution to subproblem $P(y_j)$ (the existence of x_j is due to the compactness of X and the continuity of f, g_i). By assumption **A1** and Proposition 3.6, there exist $\lambda_{j,1}, \cdots, \lambda_{j,m} \ge 0$ such that

$$\begin{cases} 0 \in \nabla_x f(x_j, y_j) + \sum\limits_{i=1}^m \lambda_{j,i} \nabla_x g_i(x_j, y_j) + N(X, x_j), \\ \lambda_{j,i} g_i(x_j, y_j) = 0, i = 1, \cdots, m. \end{cases} \tag{4.7}$$

The following proposition is useful in the reformulation of the pseudo-convex MINLP problem in (4.1).

Proposition 4.2. *Let x_j solve nonlinear subproblem $P(y_j)$. Then, for any $r > 0$,*

$$\nabla_x f(x_j, y_j)^T (x - x_j) \ge 0 \tag{4.8}$$

holds for all $x \in X$ with $g_i(x_j, y_j) + r \nabla_x g_i(x_j, y_j)^T (x - x_j) \le 0, i = 1, \cdots, m$.

Proof. Let $r > 0$ and $x \in X$ be such that

$$g_i(x_j, y_j) + r \nabla_x g_i(x_j, y_j)^T (x - x_j) \le 0, \ \ \forall i = 1, \cdots, m.$$

Then $x - x_j \in T(X, x_j)$ and it follows from (4.7) that

$$\left(\nabla_x f(x_j, y_j) + \sum_{i=1}^m \lambda_{j,i} \nabla_x g_i(x_j, y_j) \right)^T (x - x_j) \ge 0$$

and

$$r \sum_{i=1}^m \lambda_{j,i} \nabla_x g_i(x_j, y_j)^T (x - x_j) \le 0.$$

Both inequalities imply that (4.8) holds as $r > 0$. The proof is complete. □

4.3. **Reformulation of the pseudo-convex MINLP problem.** Let $y_j \in T$ and suppose that $x_j \in X$ solves the nonlinear subproblem $P(y_j)$. We denote by

$$I(x_j) := \{i : g_i(x_j, y_j) = 0\}$$

the active index set and let $J(x_j) := \{1, \cdots, m\} \backslash I(x_j)$.

If $J(x_j) \ne \emptyset$, we consider the following continuous subproblem:

$$\textit{sub-}P(x_j, y_j) \begin{cases} \min\limits_{x,y,\theta} \ \theta \\ \text{ s.t. } \nabla g_i(x_j, y_j)^T \begin{pmatrix} x - x_j \\ y - y_j \end{pmatrix} \le \theta, i \in J(x_j) \\ \qquad x \in X, y \in \operatorname{conv}(Y). \end{cases}$$

We denote by β_j the optimal value of *sub-*$P(x_j, y_j)$. It is easy to verify that

$$\beta_j = \max_{i \in J(x_j)} \max_{(x,y) \in X \times \operatorname{conv}(Y)} \left\{ \nabla g_i(x_j, y_j)^T \begin{pmatrix} x - x_j \\ y - y_j \end{pmatrix} \right\}.$$

We select a parameter $\alpha_j > 0$ as follows:

$$\alpha_j := \begin{cases} \dfrac{-\max\limits_{i\in J(x_j)} g_i(x_j,y_j)}{\beta_j}, & \text{if } J(x_j)\neq\emptyset, \beta_j>0, \\ 1, & \text{otherwise.} \end{cases} \tag{4.9}$$

To equivalently reformulate the pseudo-convex MINLP in (4.1), we consider the following mixed-integer linear program (MILP) problem:

$$\begin{cases} \min\limits_{x,y,\theta}\ \theta \\ \text{s.t. } f(x_j,y_j)+\nabla f(x_j,y_j)^T\begin{pmatrix} x-x_j \\ y-y_j\end{pmatrix}\leq\theta\ \ \forall y_j\in T, \\ \quad g_i(x_j,y_j)+\alpha_j\nabla g_i(x_j,y_j)^T\begin{pmatrix} x-x_j \\ y-y_j\end{pmatrix}\leq 0, i=1,\cdots,m,\ \forall y_j\in T, \\ \quad x\in X, y\in T \text{ integer.} \end{cases} \tag{4.10}$$

The following theorem demonstrates that it is feasible to reformulate the pseudo-convex MINLP prolbem in (4.1) as an equivalent MILP problem.

Theorem 4.1. *The pseudo-convex MINLP problem (P) in* (4.1) *is equivalent to the MILP problem in* (4.10) *in the sense that both problems have the same optimal value and that the optimal solution* $(\bar{x},\bar{y})$ *to problem (P) corresponds to the optimal solution* $(\bar{x},\bar{y},\bar{\theta})$ *to MILP problem of* (4.10) *with* $\bar{\theta}=f(\bar{x},\bar{y})$.

Proof. Suppose that $(\bar{x},\bar{y},\bar{\theta})$ is an optimal solution MILP problem of (4.9) and (x_{j_0},y_{j_0}) solves the pseudo-convex MINLP problem (P) in (4.1). We need to prove that $\bar{\theta}=f(x_{j_0},y_{j_0})$.

We assume that $\bar{y}=y_{j_1}$ for some $y_{j_1}\in T$. Then

$$f(x_{j_1},y_{j_1})+(\nabla_x f(x_{j_1},y_{j_1}),\nabla_y f(x_{j_1},y_{j_1}))^T\begin{pmatrix}\bar{x}-x_{j_1}\\ 0\end{pmatrix}\leq\bar{\theta} \tag{4.11}$$

and

$$g_i(x_{j_1},y_{j_1})+\alpha_j(\nabla_x g_i(x_{j_1},y_{j_1}),\nabla_y g_i(x_{j_1},y_{j_1}))^T\begin{pmatrix}\bar{x}-x_{j_1}\\ 0\end{pmatrix}\leq 0, i=1,\cdots,m.$$

This and Proposition 4.2 imply that $\nabla_x f(x_{j_1}^T(\bar{x}-x_{j_1})\geq 0$. In view of (4.11), one has

$$\bar{\theta}\geq f(x_{j_1},y_{j_1})\geq f(x_{j_0},y_{j_0}).$$

We next prove that $\bar{\theta}\leq f(x_{j_0},y_{j_0})$. To this aim, it suffices to show that (x_{j_0},y_{j_0}) satisfies all constraints with respect to g_i in (4.10). Granting this, it follows that $((x_{j_0},y_{j_0},f(x_{j_0},y_{j_0}))$ is feasible to problem of (4.9) as f is convex and consequently $\bar{\theta}\leq f(x_{j_0},y_{j_0})$.

Let $y_j\in T$. We only need to consider the case $\beta_j>0$. We first claim that

$$g_i(x_j,y_j)+\alpha_j\nabla g_i(x_j,y_j)^T\begin{pmatrix}x_{j_0}-x_j\\ y_{j_0}-y_j\end{pmatrix}\leq 0,\ \forall i\in I(x_j). \tag{4.12}$$

(Otherwise there exists $i\in I(x_j)$ such that

$$g_i(x_j,y_j)+\alpha_j\nabla g_i(x_j,y_j)^T\begin{pmatrix}x_{j_0}-x_j\\ y_{j_0}-y_j\end{pmatrix}>0$$

and consequently

$$\nabla g_i(x_j,y_j)^T \begin{pmatrix} x_{j_0}-x_j \\ y_{j_0}-y_j \end{pmatrix} > 0.$$

Hence $g_i(x_{j_0},y_{j_0}) > g_i(x_j,y_j) = 0$ since g_i is pseudo-convex, which contradicts $g_i(x_{j_0},y_{j_0}) \le 0$).

Let $k \in J(x_j)$. By the choice of α_j in (4.9), one has

$$\begin{aligned} & g_k(x_j,y_j) + \alpha_j \nabla g_k(x_j,y_j)^T \begin{pmatrix} x_{j_0}-x_j \\ y_{j_0}-y_j \end{pmatrix} \\ = \;& g_k(x_j,y_j) + \frac{-\max\limits_{i\in J(x_j)} g_i(x_j,y_j)}{\beta_j} \nabla g_k(x_j,y_j)^T \begin{pmatrix} x_{j_0}-x_j \\ y_{j_0}-y_j \end{pmatrix} \\ \le \;& g_k(x_j,y_j) + \Big(-\max_{i\in J(x_j)} g_i(x_j,y_j)\Big) \\ \le \;& 0. \end{aligned}$$

This and (4.12) imply that (x_{j_0},y_{j_0}) satisfies all constraints with respect to g_i in (4.9). Hence $\bar{\theta} = f(x_{j_0},y_{j_0})$. The proof is complete. □

Based on Theorem 4.1, we have the following theorem in which the OA method enables the pseudo-convex MINLP problem to be reformulated as an equivalent MILP master program.

Theorem 4.2. *For any $y_j \in T$, let x_j solve subproblem $P(y_j)$ and select a parameter α_j as (4.9), and for any $y_l \in S$, let x_l solve subproblem $F(y_l)$. Consider the following MILP master problem (MP):*

$$(\text{MP}) \begin{cases} \min\limits_{x,y,\theta} \; \theta \\ \text{s.t. } \; f(x_j,y_j) + \nabla f(x_j,y_j)^T \begin{pmatrix} x-x_j \\ y-y_j \end{pmatrix} \le \theta \;\; \forall y_j \in T, \\ \quad g_i(x_j,y_j) + \alpha_j \nabla g_i(x_j,y_j)^T \begin{pmatrix} x-x_j \\ y-y_j \end{pmatrix} \le 0, i=1,\cdots,m, \;\; \forall y_j \in T, \\ \quad g_i(x_l,y_l) + \nabla g_i(x_j,y_j)^T \begin{pmatrix} x-x_l \\ y-y_l \end{pmatrix} \le 0, i=1,\cdots,m, \;\; \forall y_l \in S, \\ \quad x \in X, y \in Y \text{ integer}. \end{cases} \tag{4.13}$$

Then master program (MP) is equivalent to the pseudo-convex MINLP problem (P) of (4.1) in the sense that both problems have the same optimal value and that the optimal solution $(\bar{x},\bar{y})$ to problem (P) corresponds to the optimal solution $(\bar{x},\bar{y},\bar{\eta})$ to (MP) with $\bar{\theta} = f(\bar{x},\bar{y})$.

Remark 4.1. Theorem 4.2 is an extension of [3, Theorem 1] and [34, Theorem 3.5] in the sense of reformulating MINLP as one equivalent MILP master problem from the convex case to the non-convex one. Furthermore, this result demonstrates that any optimal solution to problem (P) of (4.1) gives rise to an optimal solution to (MP) of (4.13). However, the converse of this may not be valid necessarily even for the convex case as some optimal solution to the reformulated MILP may be infeasible to the original MINLP (see [3, Example 1] and [33, Remark 3.1] for more details).

4.4. **An outer approximation algorithm.** In this subsection, a modified OA algorithm is developed to find the optimal solution of the pseudo-convex MINLP problem (P) of (4.1) by solving a finite number of MILP relaxations of (MP) in (4.13).

Let $y_0, y_1, \cdots, y_k$ be given. At iteration k, we define

$$T^k := T \cap \{y_0, y_1, \cdots, y_k\} \text{ and } S^k := S \cap \{y_0, y_1, \cdots, y_k\}.$$

Check the nonlinear subproblem $P(y_k)$ and exactly one of the following cases occurs:

(a) $y_k \in T^k$. Solve subproblem $P(y_k)$ to obtain the optimal solution x_k and select the parameter $\alpha_k > 0$ as said in (4.9).

(b) $y_k \in S^k$. Solve subproblem $F(y_k)$ as said in (4.2) to obtain the optimal solution x_k.

Let $UBD^k := \min\{f(x_j, y_j) : j \in T^k\}$. We consider the following relaxed master program MP^k:

$$MP^k \begin{cases} \min\limits_{x,y,\theta} \ \theta \\ \text{s.b.} \ \ \theta < UBD^k \\ \quad f(x_j,y_j) + \nabla f(x_j,y_j)^T \begin{pmatrix} x - x_j \\ y - y_j \end{pmatrix} \leq \theta \ \ \forall y_j \in T^k, \\ \quad g_i(x_j,y_j) + \alpha_j \nabla g_i(x_j,y_j)^T \begin{pmatrix} x - x_j \\ y - y_j \end{pmatrix} \leq 0, i = 1, \cdots, m, \ \ \forall y_j \in T^k, \\ \quad g_i(x_l,y_l) + \nabla g_i(x_l,y_l)^T \begin{pmatrix} x - x_l \\ y - y_l \end{pmatrix} \leq 0, i = 1, \cdots, m, \ \ \forall y_l \in S^k, \\ \quad x \in X, y \in Y \text{ integer.} \end{cases} \tag{4.14}$$

Solve MP^k to obtain a new integer assignment y_{k+1} and then the procedure is repeated iteratively until the relaxed master program is infeasible.

Now, we are in a position to present the detail description of an OA algorithm for solving the pseudo-convex MINLP problem (P) of (4.1) as follows.

Algorithm 1 (Outer Approximation Algorithm)

1: Initialization. Given an initial $y_0 \in Y$, set $T^0 = S^0 := \emptyset$, $UBD^0 := \infty$ and let $k := 0$
2: **for** $k = 0, 1, 2, \cdots,$ **do**
3:     Check subproblem $P(y_k)$
4:     **if** $P(y_k)$ is feasible **then**
5:         Solve $P(y_k)$ and obtain a solution x_k
            Select the parameter α_k as (4.9)
            Set $T^k := T^{k-1} \cup \{y_k\}$, $S^k := S^{k-1}$ and $UBD^k := \min\{UBD^{k-1}, f(x_k, y_k)\}$
6:     **else**
7:         Solve subproblem $F(y_k)$ as (4.2) and obtain a solution x_k
            Set $S^k := S^{k-1} \cup \{y_k\}$, $T^k := T^{k-1}$ and $UBD^k := UBD^{k-1}$
8:     **end if**
9:     Solve the relaxation MP^k and obtain a new integer y_{k+1}
        Set $k := k + 1$ and go back to line 3
10: **end for**

Remark 4.2. The constraint $\theta < UBD^k$ appearing in MP^k of (4.14) can be used to exclude any y_j $(j \in T_k)$ from being the optimal solution to the relaxed master program MP^k. In practice, this constraint would be replaced by the constraint $\theta \leq UBD^k - \varepsilon$, where $\varepsilon > 0$ is selected as some convergence tolerance parameter. This algorithm can only construct an ε-optimal solution to MINLP problem (P).

Next, we pay main attention to the termination criterion of Algorithm 1. If some optimal solution $(\hat{x}, \hat{y})$ to MP^k is repeatedly generated by the procedure, the constructed algorithm may not terminate finitely even under the assumption that both constraint sets X and Y are with finite elements. Fortunately, as shown by the following theorem, this case will not occur in the procedure of Algorithm 1.

Theorem 4.3. *Let Algorithm 1 be defined as above. Then any integer variable in Y will not be generated twice by the Algorithm 1.*

Proof. Let $k \in \mathbb{N}$ and $(\hat{x}, \hat{y}, \hat{\theta})$ be an optimal solution to MP^k of (4.14). By Proposition 4.1, one can verify that $\hat{y} \notin S^k$. To complete the proof, it suffices to prove that $\hat{y} \notin T^k$.

Suppose to the contrary that $\hat{y} = y_{j_k}$ for some $y_{j_k} \in T^k$. Then $(\hat{x}, y_{j_k}, \hat{\theta})$ is feasible to the relaxed master program MP^k and

$$\begin{cases} \hat{\theta} < UBD^k \leq f(x_{j_k}, y_{j_k}), \\ f(x_{j_k}, y_{j_k}) + \nabla f(x_{j_k}, y_{j_k})^T \begin{pmatrix} \hat{x} - x_{j_k} \\ 0 \end{pmatrix} \leq \hat{\theta}, \\ g_i(x_{j_k}, y_{j_k}) + \alpha_{j_k} \nabla g_i(x_{j_k}, y_{j_k})^T \begin{pmatrix} \hat{x} - x_{j_k} \\ 0 \end{pmatrix} \leq 0, i = 1, \cdots, m. \end{cases} \tag{4.15}$$

Note that $\alpha_{j_k} > 0$ and it follows from Proposition 4.2 that

$$\nabla_x f(x_{j_k}, y_{j_k})^T (\hat{x} - x_{j_k}) \geq 0.$$

Substituting this into (4.15), we can conclude $f(x_{j_k}, y_{j_k}) \leq \hat{\theta}$, which contradicts $\hat{\theta} < f(x_{j_k}, y_{j_k})$ in (4.15). Hence $\hat{y} \notin T^k$. The proof is complete. □

The following theorem on finite convergence of Algorithm 1 can be obtained from Theorem 4.3.

Theorem 4.4. *Assume that the cardinality of Y is finite. Then or pseudo-convex MINLP problem (P) is infeasible or Algorithm 1 terminates in a finite number of steps at an optimal solution to problem (P) of* (4.1).

Proof. We consider the case that MINLP problem (P) is feasible. It is not hard to verify that the termination of Algorithm 1 after a finite number of steps follows from the finite cardinality of Y and Theorem 4.3.

Suppose that Algorithm 1 terminate at k_0-th step for some $k_0 \in \mathbb{N}$. Then the relaxed master program MP^{k_0} is infeasible. Let $(\bar{x}, \bar{y})$ be an optimal solution to the pseudo-convex MINLP problem (P) with the optimal value $f(\bar{x}, \bar{y})$. If there exists some $y_j \in T^{k_0 - 1}$ such that $f(x_j, y_j) = f(\bar{x}, \bar{y})$, then the conclusion holds. Next, we assume that $f(x_j, y_j) > f(\bar{x}, \bar{y})$ for all $y_j \in T^{k_0 - 1}$. Then $UBD^{k_0 - 1} > f(\bar{x}, \bar{y})$. We claim that

$$k_0 \in T^{k_0} \text{ and } f(x_{k_0}, y_{k_0}) = f(\bar{x}, \bar{y}). \tag{4.16}$$

where x_{k_0} is an optimal solution to $P(y_{k_0})$.

Indeed, if $k_0 \in S^{k_0}$, then $UBD^{k_0} = UBD^{k_0-1}$ by Algorithm 1. Noting that $g_i(\bar{x},\bar{y}) \leq 0$ and $f(\bar{x},\bar{y}) < UBD^{k_0-1}$, by using the proof of Theorem 4.1, one can verify that $(\bar{x},\bar{y},f(\bar{x},\bar{y}))$ is feasible to MP^{k_0}, which is a contradiction as MP^{k_0} is infeasible. This implies that $y_{k_0} \in T^{k_0}$ and thus $f(x_{k_0},y_{k_0}) \geq f(\bar{x},\bar{y})$. We claim that $f(x_{k_0},y_{k_0}) = f(\bar{x},\bar{y})$. Otherwise $f(x_{k_0},y_{k_0}) > f(\bar{x},\bar{y})$. If $f(x_{k_0},y_{k_0}) \leq UBD^{k_0-1}$, then $f(\bar{x},\bar{y}) < f(x_{k_0},y_{k_0}) = UBD^{k_0}$ and consequently $(\bar{x},\bar{y},f(\bar{x},\bar{y}))$ is feasible for MP^{k_0}, which contradicts that MP^{k_0} is infeasible. If $f(x_{k_0},y_{k_0}) > UBD^{k_0-1}$, then $f(\bar{x},\bar{y}) < UBD^{k_0-1} = UBD^{k_0}$. Thus $(\bar{x},\bar{y},f(\bar{x},\bar{y}))$ is also feasible to MP^{k_0}, a contradiction as MP^{k_0} is infeasible. This means $f(x_{k_0},y_{k_0}) = f(\bar{x},\bar{y})$ and consequently (4.16) holds. The proof is complete. □

Remark 4.3. (i) When restricted to the case of convex and smooth MINLPs, Algorithm 1 could recapture the corresponding algorithms established in [3, 12, 33, 34]. Further, Algorithm 1 and Theorem 4.4 extend theoretically the OA method in the sense of solving MINLP problems from the convex case to the non-convex one.

(ii) For the convex MINLP, one can easily take parameters $\alpha_j \equiv 1$ in (MP) of (4.13) and the convexity can guarantee that this (MP) is equivalent to the convex MINLP. Further, one can still choose parameters α_j as said in (4.9) to equivalently reformulate the convex MINLP. Then a nature question arisen herein is to compare numerical performance of solving the convex MINLP between two different methods by choices of parameters α_j. This will be one part of our work in future.

5. Numerical Examples

In this section, we apply Algorithm 1 to pseudo-convex MINLP examples. These examples are all solved by BARON version 16.10.6 (cf. [31]). The first example is given as follows.

Example 5.1. Consider the following pseudo-convex MINLP problem:

$$\begin{cases} \min\limits_{x,y} f(x,y) = x^2 - 4y \\ \text{s.t. } g(x,y) = x^3 + x - 5y \leq 0, \\ \qquad x \in [-3,10], y \in [0,10] \cap \mathbb{Z}. \end{cases}$$

Choose the tolerance $\varepsilon = 0.005$, and let $y_1 = 4$ be the initial point.

At the first iteration. Check subproblem $P(y_1)$ that is feasible. Solve subproblem $P(y_1)$ and denote the optimal solution by $x_1 = 0$. Solve subproblem *sub-*$P(x_1,y_1)$ and obtain the optimal value $\beta_1 = 20$. Then select parameter $\alpha_1 = 1$ as (4.9). Solve the following relaxed MILP problem:

$$MP^1 \begin{cases} \min\limits_{x,y,\theta} \theta \\ \text{s.t. } \ \theta \leq f(x_1,y_1) - \varepsilon, \\ \qquad f(x_1,y_1) + \nabla f(x_1,y_1)^T \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix} \leq \theta, \\ \qquad g(x_1,y_1) + \alpha_1 \nabla g(x_1,y_1)^T \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix} \leq 0, \\ \qquad x \in [-3,10], y \in [0,10] \cap \mathbb{Z}. \end{cases}$$

Denote the optimal solution by $(\hat{x}, \hat{y}, \hat{\theta}) = (-0.1429, 10, -48.1429)$.

In the second iteration, we set $y_2 = \hat{y} = 10$ and check subproblem $P(y_2)$ by Algorithm 1; $P(y_2)$ is feasible. Solve subproblem $P(y_2)$ and denote the optimal solution by $x_2 = 0$. Solve subproblem $sub-P(x_2, y_2)$ and obtain the optimal value $\beta_2 = 50$. Then select parameter $\alpha_2 = 1$ as (4.9). According to Algorithm 1, consider the following constraints:

$$\begin{cases} \theta \leq f(x_2, y_2) - \varepsilon, \\ f(x_2, y_2) + \nabla f(x_2, y_2))^T \begin{pmatrix} x - x_2 \\ y - y_2 \end{pmatrix} \leq \theta, \\ g(x_2, y_2) + \alpha_2 \nabla g(x_2, y_2)^T \begin{pmatrix} x - x_2 \\ y - y_2 \end{pmatrix} \leq 0. \end{cases}$$

Add these constraints into MP^1 and obtain relaxed MILP problem MP^2. As MP^2 is infeasible, the algorithm stops and an ε-optimal solution is obtained.

The following example is taken from [38].

Example 5.2. Consider the following pseudo-convex MINLP problem:

$$\begin{cases} \min\limits_{x,y} f(x,y) = \dfrac{(x-3)^2 - 10x}{3x + y + 1} \\ \text{s.t. } g_1(x,y) = (x-7)^3 - 5y \leq 0, \\ \qquad g_2(x,y) = x - 1.8y \leq 0, \\ \qquad x \geq 1, y \in [0,8] \cap \mathbb{Z}. \end{cases}$$

Choose the tolerance $\varepsilon = 0.005$, and let $y_1 = 4$ be the initial point.

At the first iteration. Check subproblem $P(y_1)$ that is feasible. Solve subproblem $P(y_1)$ and denote the optimal solution by $x_1 = 4.5337$. Solve subproblem $sub\text{-}P(x_1, y_1)$ and obtain the optimal value $\beta_1 = 13.3276$. Then select parameter $\alpha_1 = \frac{2.6663}{13.3276}$ as (4.9). Solve the following relaxed MILP problem:

$$MP^1 \begin{cases} \min\limits_{x,y,\theta} \theta \\ \text{s.t. } \theta \leq f(x_1, y_1) - \varepsilon, \\ \qquad f(x_1, y_1) + \nabla f(x_1, y_1)^T \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix} \leq \theta, \\ \qquad g_i(x_1, y_1) + \alpha_1 \nabla g_i(x_1, y_1)^T \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix} \leq 0, i = 1, 2, \\ \qquad x \geq 1, y \in [0,8] \cap \mathbb{Z}. \end{cases}$$

Denote the optimal solution by $(\hat{x}, \hat{y}, \hat{\theta}) = (1, 0, -2.4610)$.

In the second iteration, we set $y_2 = \hat{y} = 0$ and check subproblem $P(y_2)$ by Algorithm 1; $P(y_2)$ is infeasible. Consider the following subproblem $F(y_2)$:

$$F(y_2) \begin{cases} \min\limits_{x} g_2(x, y_2) \\ \text{s.t. } g_1(x, y_2) \leq 0, \\ \qquad x \geq 1. \end{cases}$$

Solve subproblem $F(y_2)$ and denote the optimal solution by $x_2 = 7$. According to Algorithm 1, consider the following constraints:

$$g_i(x_2,y_2) + \nabla g_i(x_2,y_2)^T \begin{pmatrix} x - x_2 \\ y - y_2 \end{pmatrix} \leq 0, i = 1,2.$$

Add these two constraints into MP^1 and obtain relaxed MILP problem MP^2. Solve MP^2 and denote the optimal solution by $(\hat{x},\hat{y},\hat{\theta}) = (1,1,-2.4235)$. The second iteration is concluded.

In the third iteration, we set $y_3 = \hat{y} = 1$ and check subproblem $P(y_3)$ by Algorithm 1; $P(y_3)$ is infeasible. Consider the following subproblem $F(y_3)$:

$$F(y_3) \begin{cases} \min\limits_x g_2(x,y_3) \\ \text{ s.t. } g_1(x,y_3) \leq 0, \\ \qquad x \geq 1. \end{cases}$$

Solve subproblem $F(y_3)$ and denote the optimal solution by $x_3 = 4.7639$. According to Algorithm 1, consider the following constraints:

$$g_i(x_3,y_3) + \nabla g_i(x_3,y_3)^T \begin{pmatrix} x - x_3 \\ y - y_3 \end{pmatrix} \leq 0, i = 1,2.$$

Add these two constraints into MP^2 and obtain relaxed MILP problem MP^3. Solve MP^3 and denote the optimal solution by $(\hat{x},\hat{y},\hat{\theta}) = (2.5279,3,-2.3484)$. The third iteration is concluded.

In the fourth iteration, we set $y_4 = \hat{y} = 3$ and check subproblem $P(y_4)$ that is feasible. Solve subproblem $P(y_4)$ and denote the optimal solution by $x_1 = 4.3333$. Solve subproblem *sub*-$P(x_4,y_4)$ and obtain the optimal value $\beta_4 = 7.5666$. Then select parameter $\alpha_4 = \frac{1.0667}{7.5666}$ as (4.9). According to Algorithm 1, consider the following constraints:

$$\begin{cases} \theta \leq f(x_4,y_4) - \varepsilon, \\ f(x_4,y_4) + \nabla f(x_4,y_4))^T \begin{pmatrix} x - x_4 \\ y - y_4 \end{pmatrix} \leq \theta, \\ g_i(x_4,y_4) + \alpha_4 \nabla g_i(x_4,y_4)^T \begin{pmatrix} x - x_4 \\ y - y_4 \end{pmatrix} \leq 0, i = 1,2. \end{cases}$$

Add these constraints into MP^3 and obtain relaxed MILP problem MP^4. As MP^4 is infeasible, the algorithm stops and an ε-optimal solution is obtained.

6. Conclusions

The main work of this paper is to solve a pseudo-convex MINLP problem by an outer approximation algorithm. When to solve convex MINLP by the outer approximation, the convexity was proved to play a key role in the reformulation of MINLP as an equivalent MILP as well as the finite termination and convergence analysis. By contrast, our work is to weaken the convexity assumption when to reformulate MINLP as an equivalent MILP master program. This extends the outer approximation for solving MINLP from the convex case to the pseudo-convex case. The next step of our future work would be to consider the outer approximation for dealing with a broader class of non-convex MINLP problems.

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