J. Nonlinear Var. Anal. 8 (2024), No. 2, pp. 199-231
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.8.2024.2.02

SUB-ELLIPTIC SYSTEMS INVOLVING CRITICAL HARDY-SOBOLEV EXPONENTS AND SIGN-CHANGING WEIGHT FUNCTIONS ON CARNOT GROUPS

JINGUO ZHANG

Jiangxi Provincial Center for Applied Mathematics & School of Mathematics and Statistics, Jiangxi Normal University, Nanchang 330022, China

Abstract. This paper concerns the existence and multiplicity of positive solutions for the following subelliptic singular system on Carnot group:

$$\begin{cases} -\Delta_{\mathbb{G}} u = \dfrac{p_1}{p_1+p_2} h(\xi) \dfrac{\psi^{\alpha}|u|^{p_1-2}u|v|^{p_2}}{d(\xi)^{\alpha}} + \lambda f(\xi) \dfrac{\psi^{\beta}|u|^{q-2}u}{d(\xi)^{\beta}} & \text{in } \Omega, \\ -\Delta_{\mathbb{G}} v = \dfrac{p_2}{p_1+p_2} h(\xi) \dfrac{\psi^{\alpha}|u|^{p_1}|v|^{p_2-2}v}{d(\xi)^{\alpha}} + \mu g(\xi) \dfrac{\psi^{\beta}|v|^{q-2}v}{d(\xi)^{\beta}} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $-\Delta_{\mathbb{G}}$ is a sub-Laplacian on an arbitrary Carnot group $\mathbb{G}$, $0 \in \Omega$, d is the $\Delta_{\mathbb{G}}$-gauge, $\psi = |\nabla_{\mathbb{G}} d|$, Ω is a bounded domain in $\mathbb{G}$ with smooth boundary $\partial\Omega$, λ, $\mu > 0$, $1 < q < 2$, $0 \le \alpha < 2$, $0 \le \beta < 2$, p_1, $p_2 > 1$ satisfying $2 < p_1 + p_2 \le 2^*(\alpha)$ with $2^*(\alpha) = \frac{2(Q-\alpha)}{Q-2}$ as a critical Hardy-Sobolev exponent in the Stratified Lie context. For suitable assumptions on weight functions $f(\xi)$, $g(\xi)$, and $h(\xi)$, by using the variational methods and Nehari manifold, we prove that the subelliptic system admits at least two positive solutions when parameters pair (λ, μ) belongs to a certain subset of $\mathbb{R}^2_+$.

Keywords. Carnot groups; Critical Hardy-Sobolev exponent; Nehari manifold; Subelliptic system; Singular Hardy-type potentials.

1. INTRODUCTION

In this paper, we deal with the existence and multiplicity of positive solutions for a family of semilinear subelliptic systems defined on bounded domains of Carnot groups. Differential problems involving a sub-Laplacian operator on a bounded domain Ω of stratified groups have been intensively studied in recent years by many authors; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] for more details and applications.

*Corresponding author.
E-mail address: jgzhang@jxnu.edu.cn
Received 17 March 2023; Accepted 12 October 2023; Published online 16 February 2024

We start focusing our attention on the sub-Laplacian system with Hardy-Sobolev potentials of the form

$$\begin{cases} -\Delta_{\mathbb{G}} u = \dfrac{p_1}{p_1+p_2} h(\xi) \dfrac{\psi^{\alpha}|u|^{p_1-2}u|v|^{p_2}}{d(\xi)^{\alpha}} + \lambda f(\xi) \dfrac{\psi^{\beta}|u|^{q-2}u}{d(\xi)^{\beta}} & \text{in } \Omega, \\ -\Delta_{\mathbb{G}} v = \dfrac{p_2}{p_1+p_2} h(\xi) \dfrac{\psi^{\alpha}|u|^{p_1}|v|^{p_2-2}v}{d(\xi)^{\alpha}} + \mu g(\xi) \dfrac{\psi^{\beta}|v|^{q-2}v}{d(\xi)^{\beta}} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_{\mathbb{G}}$ is a sub-Laplacian operator on a Carnot group $\mathbb{G}$, Ω is a bounded domain in $\mathbb{G}$ with smooth boundary $\partial\Omega$, $0 \in \Omega$, $1 < q < 2$, $\lambda,\ \mu > 0$, $0 \leq \alpha < 2$, $0 \leq \beta < 2$, $p_1,\ p_2 > 1$ with $2 < p_1 + p_2 \leq 2^*(\alpha)$ and $2^*(\alpha) = \frac{2(Q-\alpha)}{Q-2}$ is the critical Hardy-Sobolev exponent in this context, Q being the homogeneous dimension of the space $\mathbb{G}$ with $Q > 3$, d is the natural gauge associated with the fundament solution of $-\Delta_{\mathbb{G}}$ on $\mathbb{G}$, and ψ is the weight function defined as $\psi(\xi) = |\nabla_{\mathbb{G}} d|$. The weight functions f, g, and $h : \Omega \to \mathbb{R}$ satisfy some additional conditions, which will be given later.

When $u = v$, $f = g$ in Ω, $p_1 = p_2 = \frac{2^*(\alpha)}{2}$, and $\lambda = \mu$, system (1.1) reduces to the scalar sub-Laplacian equation with critical nonlinearities

$$\begin{cases} -\Delta_{\mathbb{G}} u = \lambda f(\xi) \dfrac{\psi^{\beta}|u|^{q-2}u}{d(\xi)^{\beta}} + h(\xi) \dfrac{\psi^{\alpha}|u|^{2^*(\alpha)-2}u}{d(\xi)^{\alpha}} \quad \text{in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Observe that a great deal of interest has been paid in the literature to subelliptic equations on Stratifed Lie groups; see, e.g. [4, 5, 15, 16, 17, 18] and the references therein. Recently, numerous authors directed their attention to the study of semilinear sub-Laplacian problems with critical nonlinearities and Hardy-type potentials arising in the context of stratified groups; see, e.g., [9, 11, 19, 20, 21, 22] and the references therein. For example, the author in [20], by finding the minimizer of the corresponding energy functional on positive Nehari and sign-changing Nehari manifold, studied the existence and multiplicity of solutions of the following nonhomogeneous subelliptic problem

$$-\Delta_{\mathbb{G}} u = |u|^{2^*-2}u + f(\xi) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain of $\mathbb{G}$ with smooth boundary, $2^* = \frac{2Q}{Q-2}$ is the critical Sobolev exponent, and the inhomogeneous term f satisfies suitable summability assumptions. Meanwhile, Loiudice [8] studied the existence, nonexistence, and regularity of the problem (1.2) when $f(\xi) = h(\xi) = 1$, $\beta = 0$, and $q = 2$. In [11], Zhang considered the following doubly parameters problem involving the Hardy-type singularity and critical Hardy-Sobolev exponents

$$-\Delta_{\mathbb{G}} u = \lambda \frac{\psi^{\alpha}|u|^{2^*(\alpha)-2}u}{d(\xi)^{\alpha}} + \beta f(\xi)|u|^{p-2}u \text{ in } \mathbb{G}. \tag{1.3}$$

Applying the concentration compactness principle and the theory of genus, the author proved that problem (1.3) admit infinitely many nontrivial solutions. On the other hand, Zhang [13] investigated the following multiple critical subelliptic problem

$$\begin{cases} -\Delta_{\mathbb{G}}u-\mu\dfrac{\psi^2 u}{d(\xi)^2}=\dfrac{\psi^{\alpha}|u|^{2^*(\alpha)-2}u}{d(\xi)^{\alpha}}+\dfrac{\lambda\eta}{\eta+\theta}\dfrac{\psi^{\alpha}|u|^{\eta-2}u|v|^{\theta}}{d(\xi)^{\alpha}} & \text{in } \mathbb{G},\\ -\Delta_{\mathbb{G}}v-\mu\dfrac{\psi^2 u}{d(\xi)^2}=\dfrac{\psi^{\alpha}|v|^{2^*(\alpha)-2}v}{d(\xi)^{\alpha}}+\dfrac{\lambda\theta}{\eta+\theta}\dfrac{\psi^{\alpha}|u|^{\eta}|v|^{\theta-2}v}{d(\xi)^{\alpha}} & \text{in } \mathbb{G}.\end{cases} \tag{1.4}$$

Under the suitable assumptions on μ, α, λ, η ,and θ, the author adapted the mountain pass theorem and the refined version of the concentration-compactness principle to obtain a nontrivial solution of (1.4).

Loiudice [9] considered the sub-Laplacian Brezis-Nirenberg problem with Hardy potential

$$-\Delta_{\mathbb{G}}u-\mu\frac{\psi^2}{d(\xi)^2}u=u^{2^*-1}+\lambda u \text{ in } \Omega, \quad u=0 \text{ on } \partial\Omega, \tag{1.5}$$

where $Q\geq 3$, $\Omega\subset\mathbb{G}$ is a smooth bounded domain and $0\in\Omega$. The author proved that if $0<\mu\leq\mu_{\mathbb{G}}-1$, then problem (1.5) has a positive solution for all $\lambda\in(0,\lambda_1)$. If $\mu_{\mathbb{G}}-1<\mu<\mu_{\mathbb{G}}$, then there exists $\lambda_*\in(0,\lambda_1)$ such that problem (1.5) admits at least one positive solution for $\lambda\in(\lambda_*,\lambda_1)$. If $\lambda\leq 0$ and Ω is δ_γ-starshaped about the origin, then (1.5) has the trivial solution. Later, the author studied the following limit problem on $\mathbb{G}$:

$$-\Delta_{\mathbb{G}}u-\mu\frac{\psi^2}{d(\xi)^2}u=|u|^{2^*-2}u \text{ in } \mathbb{G}. \tag{1.6}$$

By means of regularity tools and Moser-type estimates on annuli, Loiudice showed that the ground state solution u of (1.8) satisfies

$$u(\xi)\sim\frac{1}{d(\xi)^{\sqrt{\mu_{\mathbb{G}}}+\sqrt{\mu_{\mathbb{G}}-\mu}}} \text{ as } d(\xi)\to\infty.$$

A question naturally arise—what effect does sign-changing weight functions and doubly parameters between Hardy-type singular term and convex-concave nonlinearities in form of (1.1) on Stratifed Lie group. To our knowledge, it seems that there are few results in the literature on this topic.

The variational formulation of (1.1) stands on the validity of the following sub-Laplacian Hardy-Sobolev inequality, which holds in any Carnot group of homogeneous dimension $Q\geq 3$. Assuming that $0\leq\alpha<2$, one sees that there exists a positive constant $C=C(\alpha,Q)$ such that

$$C\left(\int_{\mathbb{G}}\psi^{\alpha}\frac{|u|^{2^*(\alpha)}}{d(\xi)^{\alpha}}d\xi\right)^{\frac{2}{2^*(\alpha)}}\leq\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u|^2d\xi, \quad \forall u\in C_0^{\infty}(\mathbb{G}), \tag{1.7}$$

where $2^*(\alpha)=\frac{2(Q-\alpha)}{Q-2}$ and $\psi=|\nabla_{\mathbb{G}}d|$. The main difficulty in (1.1) is the lack of compactness in the related Hardy-Sobolev embedding, due to the invariance of the norms in (1.7) with respect to the following noncompact group of rescalings

$$u_\gamma(\xi)=\gamma^{\frac{Q-2}{2}}u(\delta_\gamma(\xi)), \quad \forall\gamma>0, \tag{1.8}$$

where δ_γ denotes the natural dilations of the group. Moreover, the function ψ appearing in the left hand side of (1.7) is δ_γ-homogeneous of degree 0 and ψ is a smooth function out of the origin, and then it is bounded on Ω.

This paper aims to deal with coupled sub-Laplacian systems with critical Hardy-Sobolev exponents and consider the combined effect of doubly parameters and convex-concave nonlinearities on the number of solutions. The main difficulty that one encounters when dealing with the sub-Laplacian singular problem (1.1) is that the explicit form of the Hardy-Sobolev extremals in the Carnot setting is not known, even for the Heisenberg group $\mathbb{H}^N$. This lack of information seems to make the known techniques, namely the Brezis-Nirenberg methods not directly applicable to problem (1.1). Nevertheless, as already recognized by the authors in [8], this difficulty can be overcome since the real ingredient which is needed to perform asymptotic expansions of Brezis-Nirenberg type is the knowledge of the asymptotic behavior of Hardy-Sobolev minimizers at ∞. We explicitly remark that such extremals behave at ∞ exactly as the fundamental solution of sub-Laplacian operator $\Delta_{\mathbb{G}}$.

In this paper, we denote by $S_0^1(\Omega)$ the Folland-Stein space defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{S_0^1(\Omega)} = \left(\int_\Omega |\nabla_{\mathbb{G}} u|^2 dz\right)^{\frac{1}{2}}.$$

By using the Nehari manifold and fibering map analysis, we establish the existence of at least two positive solutions for a subelliptic system involving critical nonlinearities with sign-changing weight functions with respect to the pair of parameters λ and μ belonging to a suitable subset of $\mathbb{R}_+^2$. Since the embedding $S_0^1(\Omega) \hookrightarrow L^{2^*(\alpha)}(\Omega, \frac{\psi^\alpha}{d(\xi)^\alpha} d\xi)$ $(\alpha \in [0,2))$ is not compact, then the corresponding energy functional does not satisfy the Palais-Smale condition in general. Therefore, it is difficult to obtain the critical points of energy functional by simple arguments, which are based on the compactness of the Sobolev embedding. To overcome this difficulty, we extract a Palais-Smale sequence in the Nehari manifold and show that the weak limit of this sequence is the required solution of problem (1.1).

To state our main results, we introduce

$$\Lambda_1 := \left(\frac{(2-q)S(\alpha,Q)^{\frac{p_1+p_2}{2}}}{(p_1+p_2-q)|h|_\infty}\right)^{\frac{2}{p_1+p_2-2}} \left(\frac{p_1+p_2-q}{p_1+p_2-2}\right)^{-\frac{2}{2-q}} S(\beta,Q)^{\frac{q}{2-q}}, \tag{1.9}$$

where $S(t,Q)$ $(t=\alpha,\beta)$ is the best constant that will be introduced in next section. Moreover, To construct our problem more precise, we give the following assumptions on the possibly sign-changing weight functions f, g, and h:

$(fg)_1$ $f,\, g \in L^{q_*}(\Omega, \frac{\psi^\beta}{d(\xi)^\beta} d\xi)$, $f^\pm = \max\{\pm f, 0\} \not\equiv 0$, and $g^\pm = \max\{\pm g, 0\} \not\equiv 0$ in $\overline{\Omega}$, where $q_* := \frac{2^*(\beta)}{2^*(\beta)-q}$.

$(h)_1$ $h \in L^\infty(\Omega)$ and $h^+ = \max\{h,0\} \not\equiv 0$ in $\overline{\Omega}$.

Define the set

$$\mathscr{C}_\Gamma = \left\{ (\lambda,\mu) \in \mathbb{R}_+^2 : \begin{array}{c} 0 < \left(\lambda |f|_{L^{q_*}(\Omega, \frac{\psi^\beta}{d(\xi)^\beta} d\xi)}\right)^{\frac{2}{2-q}} \\ + \left(\mu |g|_{L^{q_*}(\Omega, \frac{\psi^\beta}{d(\xi)^\beta} d\xi)}\right)^{\frac{2}{2-q}} < \Gamma \end{array} \right\}. \tag{1.10}$$

In the case of subcritical or critical nonlinearity, we prove the following existence results.

Theorem 1.1. *Assume that* $(fg)_1$, $(h)_1$ *hold. If* $1 \le q < 2$, $0 \le \alpha < 2$, $0 \le \beta < 2$, $2 < p_1 + p_2 \le 2^*(\alpha)$, *and* λ, $\mu > 0$ *satisfy* $(\lambda, \mu) \in \mathscr{C}_{\Lambda_1}$, *then system* (1.1) *has at least one positive solution in* $S_0^1(\Omega) \times S_0^1(\Omega)$.

Theorem 1.2. *Assume that* $(fg)_1$, $(h)_1$ *hold. If* $1 \le q < 2$, $0 \le \alpha < 2$, $0 \le \beta < 2$, $2 < p_1 + p_2 < 2^*(\alpha)$, *and* λ, $\mu > 0$ *satisfy* $(\lambda, \mu) \in \mathscr{C}_{\Lambda_2}$, *where* $\Lambda_2 = (\frac{q}{2})^{\frac{2}{2-q}} \Lambda_1$, *then system* (1.1) *has at least two positive solutions in* $S_0^1(\Omega) \times S_0^1(\Omega)$.

The critical case is more challenging and requires information about the asymptotic behavior of solutions of the following limiting problem at infinity:

$$-\Delta_{\mathbb{G}} u = \frac{\psi^\alpha |u|^{2^*(\alpha)-2} u}{d(\xi)^\alpha} \text{ and } u \ge 0 \text{ in } \mathbb{G}, \tag{1.11}$$

where $0 < \alpha < 2$, $2^*(\alpha) = \frac{2(Q-\alpha)}{Q-2}$. We get around the difficulty by working with certain asymptotic estimates for solution to (1.11) recently obtained by Loiudice [8]; see Lemma 6.1. In order to use the results of [8], we may assume $h(\xi) \equiv 1$ and the following extra assumptions on f and g:

$(fg)_2$ There exist a_0, b_0 and $r_0 > 0$ such that $B_d(\xi_0, 2r_0) \subset \Omega$ and $f(\xi) \ge a_0$, $g(\xi) \ge b_0$ for all $\xi \in B_d(0, 2r_0)$, where $B_d(\xi, r)$ is the d-ball with center at ξ and radius r.

In this case, problem (1.1) can be written as follows:

$$\begin{cases} -\Delta_{\mathbb{G}} u = \dfrac{p_1}{p_1+p_2} \dfrac{\psi^\alpha |u|^{p_1-2} u |v|^{p_2}}{d(\xi)^\alpha} + \lambda f(\xi) \dfrac{\psi^\beta |u|^{q-2} u}{d(\xi)^\beta} & \text{in } \Omega, \\ -\Delta_{\mathbb{G}} v = \dfrac{p_2}{p_1+p_2} \dfrac{\psi^\alpha |u|^{p_1} |v|^{p_2-2} v}{d(\xi)^\alpha} + \mu g(\xi) \dfrac{\psi^\beta |v|^{q-2} v}{d(\xi)^\beta} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

We then establish the following:

Theorem 1.3. *Assume that* $(fg)_1$, $(h)_1$, *and* $(fg)_2$ *hold. If* $1 < q < 2$, $0 \le \alpha < 2$, $0 \le \beta < 2$, $p_1 + p_2 = 2^*(\alpha)$, *and* λ, $\mu > 0$ *satisfy* $(\lambda, \mu) \in \mathscr{C}_{\Lambda_*}$, *where* Λ_* *is given in Section 6, then system* (1.12) *has at least two positive solutions in* $S_0^1(\Omega) \times S_0^1(\Omega)$.

The article is organized as follows. In Section 2, variational setting of problem (1.1) and some preliminary results are introduced. In Section 3, we give some results about the Nehari manifold and fibering map. In Section 4, we show that the Palais-Smale condition holds for the energy functional associated with (1.1) at energy level in a suitable range related to the best Sobolev constant. In Section 5, we prove the existence of Palais-Smale sequences and proof of Theorems 1.1 and 1.2. We give the detail of proof of Theorem 1.3 in Section 6, the last section.

2. Preliminaries

We briefly recall the relevant definitions and notations related the Carnot group functional setting. For a complete treatment, we refer the reader to the monograph [10, 17] and papers [5, 7, 14, 16].

A finite dimensional Lie algebra $\mathfrak{g}$ is said to be stratified of step $k \in \mathbb{N}$ if there exists subspace $V_1, \cdots, V_k$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k \text{ and } [V_1, V_i] = V_{i+1},\ i = 1, \cdots k-1; \quad [V_1, V_k] = \{0\}.$$

A connected and simply connected Lie group $\mathbb{G}$ is a Carnot group if its Lie algebra $\mathfrak{g}$ is finite dimensional and stratified. In any Carnot group, the exponential mapping $\exp : \mathfrak{g} \to \mathbb{G}$ is an analytic diffeomorphism. We use it to define analytic maps $\xi_i : \mathbb{G} \to V_i (i = 1, 2, \cdots, k)$, through the equation $g = \exp\xi(g)$, where $\xi(g) = \xi_1(g) + \cdots + \xi_k(g)$. Let $X = \{X_1, \cdots, X_m\}$ be a basis of V_1, with $m = \dim(V_1)$. The coordinates of ξ's projection in the basis $X_1, \cdots, X_m$ are denoted by $x_1 = x_1(g), \cdots, x_m = x_m(g)$, that is, $x_j(g) = \langle \xi(g), X_j \rangle,\ j = 1, \cdots, m$. We set $x = x(g) = (x_1, \cdots, x_m) \in \mathbb{R}^m$.Later we will need to exploit the properties of the exponential coordinates in the second layer of the stratification of $\mathfrak{g}$. We thus fix an orthonormal basis $Y_1, \cdots, Y_r$ of V_2 and define the exponential coordinates in the second layer V_2 of a point $g \in \mathbb{G}$ by setting $y_i(g) = \langle \xi(g), Y_i \rangle$, $i = 1, \cdots, r$, and $y = (y_1, \cdots, y_r) \in \mathbb{R}^r$.

Let $N = \sum_{i=1}^k \dim(V_i)$ be the topological dimension of $\mathbb{G}$, the decomposition $\mathbb{R}^N = \mathbb{R}^{\dim(V_1)} \times \mathbb{R}^{\dim(V_2)} \times \cdots \times \mathbb{R}^{\dim(V_k)}$ is valid, and for every $\gamma > 0$, the dilation $\delta_\gamma : \mathbb{G} \to \mathbb{G}$ given by

$$\delta_\gamma(x) = \delta_\gamma(x^{(1)}, x^{(2)}, \cdots, x^{(k)}) = (\gamma^1 x^{(1)}, \cdots, \gamma^k x^{(k)})$$

is an automorphism of the group $\mathbb{G}$, where $x^{(i)} \in \mathbb{R}^{\dim(V_i)}$ for $i = 1, 2, \cdots, k$. Here, by the automorphisms $\{\delta_\gamma\}_{\gamma>0}$, the homogeneous dimension of $\mathbb{G}$ is given by $Q = \sum_{i=1}^k i \cdot \dim(V_i)$.

Let $X = \{X_1, X_2, \cdots, X_m\}$ be a basis of V_1 with $m = \dim(V_1)$. From the Proposition 1.2.29 of [10], the left invariant vector field $X_i\,(k = 1, \cdots, m)$ has an explicit form as following

$$X_i = \frac{\partial}{\partial x_i^{(1)}} + \sum_{l=2}^{k} \sum_{r=1}^{\dim(V_l)} a_{i,r}^{(l)}(x^{(1)}, \cdots, x^{(l-1)}) \frac{\partial}{\partial x_r^{(l)}},$$

where $a_{i,r}^{(l)}$ is a homogeneous (with respect to δ_γ) polynomial function of degree $l-1$. Then, once a basis $X_1, X_2, \cdots, X_m$ of the horizontal layer is fixed, we define, for any function $u : \mathbb{G} \to \mathbb{R}$ for which the partial derivatives $X_j u$ exist, the horizontal gradient of u, denoted by $\nabla_{\mathbb{G}} u$, as the horizontal section $\nabla_{\mathbb{G}} u := \sum_{i=1}^m (X_i u) X_i$, whose coordinates are $(X_1 u, X_2 u, \cdots, X_m u)$. Moreover, if $\phi = (\phi_1, \phi_2, \cdots, \phi_m)$ is an horizontal section such that $X_j \phi_j \in L^1_{loc}(\mathbb{G})$ for $j = 1, \cdots, m$, we define $\mathrm{div}_{\mathbb{G}}\,\phi$ as the real valued function $\mathrm{div}_{\mathbb{G}}(\phi) := -\sum_{j=1}^m X_j^* \phi_j = \sum_{j=1}^m X_j \phi_j$. From the above results, the second-order differential operator $\Delta_{\mathbb{G}} := \sum_{j=1}^m X_j^2$ is called the (canonical) sub-Laplacian on $\mathbb{G}$. The sub-Laplacian $\Delta_{\mathbb{G}}$ is a left invariant homogeneous hypoelliptic differential operator, thanks to Hörmander's theorem, and $\Delta_{\mathbb{G}} u = \mathrm{div}_{\mathbb{G}}(\nabla_{\mathbb{G}} u)$. In addition, we can check that $\nabla_{\mathbb{G}}$ and $\Delta_{\mathbb{G}}$ are left-translation invariant with respect to the group action τ_z and δ_γ-homogeneous, respectively, of degree one and two, that is, $\nabla_{\mathbb{G}}(u \circ \tau_z) = \nabla_{\mathbb{G}} u \circ \tau_z$, $\nabla_{\mathbb{G}}(u \circ \delta_\gamma) = \gamma \nabla_{\mathbb{G}} u \circ \delta_\gamma$, and $\Delta_{\mathbb{G}}(u \circ \tau_z) = \Delta_{\mathbb{G}} u \circ \tau_z$, $\Delta_{\mathbb{G}}(u \circ \delta_\gamma) = \gamma^2 \Delta_{\mathbb{G}} u \circ \delta_\gamma$, where the left translation $\tau_z : \mathbb{G} \to \mathbb{G}$ is defined as $x \mapsto \tau_z x := z \circ x, \quad \forall x, z \in \mathbb{G}$.

From [16], when $Q \geq 3$, Carnot groups possess the following property: there exists a suitable homogeneous norm d on $\mathbb{G}$ such that

$$\Gamma(z) = \frac{C}{d(z)^{Q-2}}, \quad \forall z \in \mathbb{G}$$

is a fundamental solution of $-\Delta_{\mathbb{G}}$ with pole at 0, for a suitable constant $C > 0$. By definition, the homogeneous norm on $\mathbb{G}$ is a continuous function $d : \mathbb{G} \to [0, +\infty)$, smooth away from the origin, such that $d(\delta_\gamma(z)) = \gamma d(z)$ for every $\gamma > 0$ and $z \in \mathbb{G}$, $d(z^{-1}) = d(z)$ and $d(z) = 0$ iff $z = 0$. Moreover, if we define $d(z_1, z_2) := d(z_2^{-1} \circ z_1)$, then d is a pseudo-distence on $\mathbb{G}$. In particular, d satisfies the pseudo-triangular inequality:

$$d(z_1\,,z_2) \le c(d(z_1\,,z_3) + d(z_3\,,z_2)), \quad \forall z_1, z_2, z_3 \in \mathbb{G}$$

for a suitable positive constant c. Throughout this paper, we shall almost exclusively work with this homogeneous norm d, which is related to the fundamental solution of the sub-Laplace operator $-\Delta_{\mathbb{G}}$, and then we can introduce the balls associated with such norm d, calling them d-balls, defined as $B_d(z,R) = \{y \in \mathbb{G} : d(z,y) < R\}$.

In fact, the norm on $\mathbb{G}$ can be induced by the Euclidean distance $|\cdot|$ on $\mathfrak{g}$ through the exponential mapping, which also induces the homogeneous pseudo-norm $|\cdot|_{\mathfrak{g}}$ on $\mathfrak{g}$, namely, for $\xi \in \mathfrak{g}$ with $\xi = \xi_1 + \cdots + \xi_k$, where $\xi_i \in V_i$, define a pseudo-norm on $\mathfrak{g}$ as follows

$$|\xi|_{\mathfrak{g}} = |(\xi_1, \cdots, \xi_k)|_{\mathfrak{g}} := \Big(\sum_{i=1}^{k} |\xi_i|^{\frac{2k!}{i}}\Big)^{\frac{1}{2k!}}.$$

The induced norm on $\mathbb{G}$ has the form $|g|_{\mathbb{G}} = |\exp_{\mathbb{G}}^{-1}(g)|_{\mathfrak{g}}$ for all $g \in \mathbb{G}$. The function $|\cdot|_{\mathbb{G}}$ is usually known as the non-isotropic gauge. It defines a pseudo-distence on $\mathbb{G}$ given by

$$d(g,h) := |h^{-1} \circ g|_{\mathbb{G}}, \qquad \forall g, h \in \mathbb{G}.$$

Now, we define the function space corresponding to problem (1.1), posed in framework of Sobolev space $\mathscr{H} := S_0^1(\Omega) \times S_0^1(\Omega)$ with standard norm

$$\|(u,v)\| = \Big(\int_\Omega (|\nabla_{\mathbb{G}} u|^2 + |\nabla_{\mathbb{G}} v|^2) d\xi\Big)^{\frac{1}{2}}.$$

Then $\mathscr{H}$ is a Hilbert space. The energy functional $I_{\lambda,\mu} : \mathscr{H} \to \mathbb{R}$ associated to (1.1) is given by

$$\begin{aligned} I_{\lambda,\mu}(u,v) = \frac{1}{2}\|(u,v)\|^2 - \frac{1}{p_1+p_2}\int_\Omega h(\xi)\frac{\psi^\alpha |u|^{p_1}|v|^{p_2}}{d(\xi)^\alpha} d\xi \\ - \frac{1}{q}\int_\Omega \Big(\lambda f(\xi)\frac{\psi^\beta |u|^q}{d(\xi)^\beta} + \mu g(\xi)\frac{\psi^\beta |v|^q}{d(\xi)^\beta}\Big) d\xi. \end{aligned}$$

Under $(fg)_1$ and $(h)_1$, it is easy to see that $I_{\lambda,\mu}$ is well defined and continuously differentiable on $\mathscr{H}$. Moreover, the critical points of the functional $I_{\lambda,\mu}$ are the weak solutions to (1.1).

Definition 2.1. A function $(u,v) \in \mathscr{H}$ is called a weak solution to (1.1) if, for all $(\phi_1, \phi_2) \in \mathscr{H}$,

$$\begin{aligned} \int_\Omega \nabla_{\mathbb{G}} u \nabla_{\mathbb{G}} \phi_1 d\xi + \int_\Omega \nabla_{\mathbb{G}} v \nabla_{\mathbb{G}} \phi_2 d\xi = \frac{p_1}{p_1+p_2}\int_\Omega h(\xi)\frac{\psi^\alpha |u|^{p_1-2}u|v|^{p_2}\phi_1}{d(\xi)^\alpha} d\xi \\ + \frac{p_2}{p_1+p_2}\int_\Omega h(\xi)\frac{\psi^\alpha |u|^{p_1}|v|^{p_2-2}v\phi_2}{d(\xi)^\alpha} d\xi + \lambda\int_\Omega f(\xi)\frac{\psi^\beta |u|^{q-2}u\phi_1}{d(\xi)^\beta} d\xi \\ + \mu\int_\Omega g(\xi)\frac{\psi^\beta |v|^{q-2}v\phi_2}{d(\xi)^\beta} d\xi. \end{aligned}$$

Now we state the following inequality, which is used in the subsequent lemmas; see [18, Theorem 1.2] or [2, Theorem 1.4].

Lemma 2.1. *Let $p \in [2, 2^*(\alpha)]$ $(\alpha \in [0,2])$. Then there exists a constant $C_p > 0$ such that*

$$C_p \left(\int_\Omega \frac{\psi^\alpha |u|^p}{d(\xi)^\alpha} d\xi \right)^{\frac{2}{p}} \le \int_\Omega |\nabla_{\mathbb{G}} u|^2 d\xi, \quad \forall u \in S_0^1(\Omega). \tag{2.1}$$

Moreover, for $p = 2^*(\alpha)$, the best constant in (2.1) is denoted by $S(\alpha, Q)$, that is,

$$S(\alpha, Q) = \inf_{u \in S_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla_{\mathbb{G}} u|^2 d\xi}{\left(\int_\Omega \frac{\psi^\alpha |u|^{2^*(\alpha)}}{d(\xi)^\alpha} d\xi \right)^{\frac{2}{2^*(\alpha)}}}. \tag{2.2}$$

From [8, 18], it is well known that $S(\alpha, Q)$ is achieved if and only if $\Omega = \mathbb{G}$, and the extremal function $u \in S_0^1(\mathbb{G})$ for problem (2.2), up to a change of sign, is positive, and

$$u \in L^{\frac{2^*}{2}, \infty}(\mathbb{G}) \cap L^\infty(\mathbb{G}) \text{ and } u(\xi) \sim \frac{1}{d(\xi)^{Q-2}} \text{ as } d(\xi) \to \infty.$$

Let $u(\xi) > 0$ be a fixed extremal function for problem (2.2) and consider, for $\varepsilon > 0$, the family of rescaled functions $U_\varepsilon(\xi) = \varepsilon^{-\frac{Q-2}{2}} u(\delta_{\frac{1}{\varepsilon}}(\xi))$. Then, the functions $U_\varepsilon(\xi)$ are solutions, up to multiplicative constants, of the following equation

$$-\Delta_{\mathbb{G}} u = \frac{\psi^\alpha |u|^{2^*(\alpha)-2} u}{d(\xi)^\alpha} \quad \text{in } \mathbb{G},$$

and satisfies

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} U_\varepsilon|^2 d\xi = \int_{\mathbb{G}} \frac{\psi^\alpha |U_\varepsilon|^{2^*(\alpha)}}{d(\xi)^\alpha} d\xi = S(\alpha, Q)^{\frac{Q-\alpha}{2-\alpha}}, \quad \forall \varepsilon > 0.$$

For this, taking $\rho > 0$ small enough such that $B_d(0, \rho) \subset \Omega$. Choose the cut-off function $\eta \in C_0^\infty(B_d(0, \rho))$ such that $0 \le \eta \le 1$ and $\eta \equiv 1$ in $B_d(0, \frac{\rho}{2})$. Define the function

$$u_\varepsilon(\xi) = \eta(\xi) U_\varepsilon(\xi). \tag{2.3}$$

The following asymptotic expansions hold; see [8, Lemma 6.1].

Lemma 2.2. *Let the homogeneous dimension $Q \ge 4$, $0 \le \alpha < 2$. Then the following estimates hold when $\varepsilon \to 0$:*

$$\int_\Omega |\nabla_{\mathbb{G}} u_\varepsilon|^2 d\xi = S(\alpha, Q)^{\frac{Q-\alpha}{2-\alpha}} + O(\varepsilon^{Q-2}), \tag{2.4}$$

and

$$\int_\Omega \frac{\psi^\alpha |u_\varepsilon|^{2^*(\alpha)}}{d(\xi)^\alpha} d\xi = S(\alpha, Q)^{\frac{Q-\alpha}{2-\alpha}} + O(\varepsilon^{Q-\alpha}). \tag{2.5}$$

Taking into account the exact asymptotic behavior of Hardy-Sobolev extremals, we have the following results.

Lemma 2.3. *Assume that $0 \le s < 2$, $Q \ge 4$, and $1 \le q < 2^*(s)$. Then, as $\varepsilon \to 0$, the following estimates hold:*

$$\int_\Omega \frac{\psi^s |u_\varepsilon|^q}{d(\xi)^s} d\xi = \begin{cases} C\varepsilon^{Q-s-\frac{q(Q-2)}{2}}, & \text{if} \quad q > \dfrac{Q-s}{Q-2}, \\ C\varepsilon^{Q-s-\frac{q(Q-2)}{2}} |\ln \varepsilon|, & \text{if} \quad q = \dfrac{Q-s}{Q-2}, \\ C\varepsilon^{\frac{q(Q-2)}{2}}, & \text{if} \quad q < \dfrac{Q-s}{Q-2}. \end{cases} \tag{2.6}$$

Proof. For all $1 \le q < 2^*(s)$, as $\varepsilon \to 0$, it is easily seen that

$$\begin{aligned} \int_\Omega \frac{\psi^s |u_\varepsilon(\xi)|^q}{d(\xi)^s} d\xi &= \int_\Omega \frac{\psi^s |\eta(\xi)\varepsilon^{-\frac{Q-2}{2}} u(\delta_{\frac{1}{\varepsilon}}(\xi))|^q}{d(\xi)^s} d\xi \\ &\ge \varepsilon^{-\frac{q(Q-2)}{2}} \int_{B_d(0,\frac{\rho}{2})} \psi^s \frac{|u(\delta_{\frac{1}{\varepsilon}}(\xi))|^q}{d(\xi)^s} d\xi \\ &= \varepsilon^{-\frac{q(Q-2)}{2}} \int_{B_d(0,\frac{\rho}{2\varepsilon})} \psi^s \frac{|u(\delta_1(\zeta))|^q}{\varepsilon^s d(\zeta)^s} \varepsilon^Q d\zeta \\ &\ge C\varepsilon^{-\frac{q(Q-2)}{2}+Q-s} \int_{B_d(0,\frac{\rho}{2\varepsilon})\setminus B_d(0,\rho_0)} \frac{d(\zeta)^{-(Q-2)q}}{d(\zeta)^s} d\zeta \\ &\ge C\varepsilon^{-\frac{q(Q-2)}{2}+Q-s} \int_{\rho_0}^{\frac{\rho}{2\varepsilon}} \frac{1}{r^{(Q-2)\cdot q+s-Q+1}} dr, \end{aligned} \tag{2.7}$$

where $\rho_0 > 0$ is large enough such that $u(\xi) \ge C d(\xi)^{2-Q}$ for $d(\xi) \ge \rho_0$ and ε is small enough so that $\frac{\rho}{2\varepsilon} > \rho_0$.

(i) If $(Q-2)q+s-Q=0$, straightforward computations yield

$$\int_{\rho_0}^{\frac{\rho}{2\varepsilon}} \frac{1}{r^{(Q-2)q+s-Q+1}} dr = \int_{\rho_0}^{\frac{\rho}{2\varepsilon}} \frac{1}{r} dr = C|\ln \varepsilon|. \tag{2.8}$$

So, (2.7) and (2.8) yield that

$$\int_\Omega \frac{\psi^s |u_\varepsilon(\xi)|^q}{d(\xi)^s} d\xi \ge C\varepsilon^{Q-s-\frac{q(Q-2)}{2}} |\ln \varepsilon|. \tag{2.9}$$

(ii) If $(Q-2)q+s-Q<0$, then it follows that $(Q-2)q+s-Q+1<1$ and

$$\int_{\rho_0}^{\frac{\rho}{2\varepsilon}} \frac{1}{r^{(Q-2)q+s-Q+1}} dr = \int_{\rho_0}^{\frac{\rho}{2\varepsilon}} r^{Q-s-(Q-2)q-1} dr = C\varepsilon^{-(Q-s-(Q-2)q)}. \tag{2.10}$$

Then, inserting (2.10) into (2.7), we obtain

$$\int_\Omega \frac{\psi^s |u_\varepsilon(\xi)|^q}{d(\xi)^s} d\xi \ge C\varepsilon^{Q-s-\frac{q(Q-2)}{2}-Q+s+(Q-2)q} = C\varepsilon^{\frac{q(Q-2)}{2}}. \tag{2.11}$$

(iii) If $(Q-2)q+s-Q>0$, then $(Q-2)q+s-Q+1>1$ and there exists $C>0$ such that

$$\Big| \int_{\rho_0}^{\frac{\rho}{2\varepsilon}} \frac{1}{r^{(Q-2)q+s-Q+1}} dr \Big| \le C. \tag{2.12}$$

Therefore, by (2.7) and (2.12), one has

$$\int_{\Omega} \frac{\psi^{s}|u_{\varepsilon}(\xi)|^{q}}{d(\xi)^{s}} d\xi \geq C\varepsilon^{Q-s-\frac{q(Q-2)}{2}}. \tag{2.13}$$

Thus, (2.9), (2.11), and (2.13) imply that (2.6) holds. □

We conclude this section by introducing the following minimizing problem

$$S_{\alpha,p_1,p_2} = \inf_{(u,v)\in\mathscr{H}\setminus\{(0,0)\}} \frac{\int_{\Omega}(|\nabla_{\mathbb{G}}u|^2+|\nabla_{\mathbb{G}}v|^2)d\xi}{\left(\int_{\Omega}\frac{\psi^{\alpha}|u|^{p_1}|v|^{p_2}}{d(\xi)^{\alpha}}d\xi\right)^{\frac{2}{2^*(\alpha)}}} \text{ for all } \varepsilon>0, \tag{2.14}$$

where $p_1+p_2=2^*(\alpha)$. In light of Young's inequality $|u|^{p_1}|v|^{p_2} \leq \frac{p_1}{p_1+p_2}|u|^{p_1+p_2}+\frac{p_2}{p_1+p_2}|v|^{p_1+p_2}$ and Lemma 2.1, the best constant in (2.14) is well defined. Using the ideas from [23], we establish the following relationship between $S(\alpha,Q)$ and S_{α,p_1,p_2}.

Lemma 2.4. *For the constants $S(\alpha,Q)$ and S_{α,p_1,p_2} given in (2.2) and (2.14), respectively, it holds*

$$S_{\alpha,p_1,p_2} = \left[\left(\frac{p_1}{p_2}\right)^{\frac{p_2}{p_1+p_2}}+\left(\frac{p_2}{p_1}\right)^{\frac{p_1}{p_1+p_2}}\right] S(\alpha,Q). \tag{2.15}$$

In particular, S_{α,p_1,p_2} is achieved for $\Omega=\mathbb{G}$.

Proof. Define a function $\theta:\mathbb{R}^+\to\mathbb{R}^+$ as $\theta(x)=x^{\frac{2p_2}{p_1+p_2}}+(\frac{1}{x})^{\frac{2p_1}{p_1+p_2}}$. Then, θ attains its minimum at point $x_0=\sqrt{\frac{p_2}{p_1}}$, and

$$\min_{x\in\mathbb{R}^+}\theta(x)=\theta(x_0)=\left(\frac{p_1}{p_2}\right)^{\frac{p_2}{p_1+p_2}}+\left(\frac{p_2}{p_1}\right)^{\frac{p_1}{p_1+p_2}}.$$

Now, let $\{w_n\}\subset S_0^1(\Omega)$ be a minimizing sequence for $S(\alpha,Q)$, and let $u_n=sw_n$ and $v_n=tw_n$ for $s,t>0$. By definition of S_{α,p_1,p_2}, we have

$$\begin{aligned}
S_{\alpha,p_1,p_2} &\leq \frac{\|(u_n,v_n)\|^2}{\left(\int_{\Omega}\frac{\psi^{\alpha}|u_n|^{p_1}|v_n|^{p_2}}{d(\xi)^{\alpha}}d\xi\right)^{\frac{2}{p_1+p_2}}} \\
&= \frac{(s^2+t^2)}{s^{\frac{2p_1}{p_1+p_2}}t^{\frac{2p_2}{p_1+p_2}}} \frac{\|w_n\|^2}{\left(\int_{\Omega}\frac{\psi^{\alpha}|w_n|^{2^*(\alpha)}}{d(\xi)^{\alpha}}d\xi\right)^{\frac{2}{p_1+p_2}}} \\
&= \frac{(s^2+t^2)}{s^{\frac{2p_1}{p_1+p_2}}t^{\frac{2p_2}{p_1+p_2}}} S(\alpha,Q) = \left[\left(\frac{s}{t}\right)^{\frac{2p_2}{p_1+p_2}}+\left(\frac{t}{s}\right)^{\frac{2p_1}{p_1+p_2}}\right] S(\alpha,Q).
\end{aligned}$$

Choosing s and t such that $\frac{s}{t}=\sqrt{\frac{p_1}{p_2}}$ and letting $n\to\infty$ yield

$$S_{\alpha,p_1,p_2} \leq \left[\left(\frac{p_1}{p_2}\right)^{\frac{p_2}{p_1+p_2}}+\left(\frac{p_2}{p_1}\right)^{\frac{p_1}{p_1+p_2}}\right] S(\alpha,Q). \tag{2.16}$$

On the other hand, let $\{(u_n,v_n)\}$ be a minimizing sequence for S_{α,p_1,p_2}. Define $\omega_n=s_nv_n$ for some $s_n>0$ such that

$$\int_{\Omega}\frac{\psi^{\alpha}|u_n|^{p_1+p_2}}{d(\xi)^{\alpha}}d\xi = \int_{\Omega}\frac{\psi^{\alpha}|\omega_n|^{p_1+p_2}}{d(\xi)^{\alpha}}dx.$$

Then Young's inequality implies that

$$\int_\Omega \frac{\psi^\alpha |u_n|^{p_1}|\omega_n|^{p_2}}{d(\xi)^\alpha} d\xi \le \frac{p_1}{p_1+p_2}\int_\Omega \frac{\psi^\alpha |u_n|^{p_1+p_2}}{d(\xi)^\alpha} d\xi + \frac{p_2}{p_1+p_2}\int_\Omega \frac{\psi^\alpha |\omega_n|^{p_1+p_2}}{d(\xi)^\alpha} d\xi$$
$$= \int_\Omega \frac{\psi^\alpha |\omega_n|^{p_1+p_2}}{d(\xi)^\alpha} d\xi = \int_\Omega \frac{\psi^\alpha |u_n|^{p_1+p_2}}{d(\xi)^\alpha} d\xi.$$

Using this, we obtain

$$\begin{aligned}
&\frac{\|(u_n,v_n)\|^2}{\left(\int_\Omega \frac{\psi^\alpha |u_n|^{p_1}|v_n|^{p_2}}{d(\xi)^\alpha} d\xi\right)^{\frac{2}{p_1+p_2}}}\\
&= s_n^{\frac{2p_2}{p_1+p_2}}\left[\frac{\|u_n\|^2}{\left(\int_\Omega \frac{\psi^\alpha |u_n|^{p_1}|\omega_n|^{p_2}}{d(\xi)^\alpha} d\xi\right)^{\frac{2}{p_1+p_2}}} + \frac{\|v_n\|^2}{\left(\int_\Omega \frac{\psi^\alpha |u_n|^{p_1}|\omega_n|^{p_2}}{d(\xi)^\alpha} d\xi\right)^{\frac{2}{p_1+p_2}}}\right]\\
&\ge s_n^{\frac{2p_2}{p_1+p_2}}\frac{\|u_n\|^2}{\left(\int_\Omega \frac{\psi^\alpha |u_n|^{p_1+p_2}}{d(\xi)^\alpha} d\xi\right)^{\frac{2}{p_1+p_2}}} + s_n^{\frac{2p_2}{p_1+p_2}-2}\frac{\|\omega_n\|^2}{\left(\int_\Omega \frac{\psi^\alpha |\omega_n|^{p_1+p_2}}{d(\xi)^\alpha} d\xi\right)^{\frac{2}{p_1+p_2}}}\\
&\ge \left(s_n^{\frac{2p_2}{p_1+p_2}} + \left(\frac{1}{s_n}\right)^{\frac{2p_1}{p_1+p_2}}\right)S(\alpha,Q)\\
&= \theta(s_n)S(\alpha,Q) \ge \theta(x_0)S(\alpha,Q).
\end{aligned}$$

On passing to the limit as $n\to\infty$, we obtain

$$S_{\alpha,p_1,p_2} \ge \left[\left(\frac{p_1}{p_2}\right)^{\frac{p_2}{p_1+p_2}} + \left(\frac{p_2}{p_1}\right)^{\frac{p_1}{p_1+p_2}}\right] S(\alpha,Q). \tag{2.17}$$

Hence, from (2.16) and (2.17), we obtain the required result. □

3. Nehari Manifold for (1.1)

In this section, we study the nature of Nehari manifold associated with (1.1). In the case $p_1+p_2\ge 2$, the functional $I_{\lambda,\mu}$ is not bounded below on $\mathscr{H}$. We will show that it is bounded on some suitable subset of $\mathscr{H}$ and on minimizing $I_{\lambda,\mu}$ on these subsets, we obtain the nontrivial solutions of (1.1). We define the Nehari set $\mathscr{N}_{\lambda,\mu}$ as

$$\mathscr{N}_{\lambda,\mu} = \{(u,v)\in\mathscr{H}\setminus\{(0,0)\} : \langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\}.$$

Thus, $(u,v)\in\mathscr{N}_{\lambda,\mu}$ if and only if

$$\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = \|(u,v)\|^2 - H(u,v) - Q_{\lambda,\mu}(u,v) = 0, \tag{3.1}$$

where

$$H(u,v) := \int_\Omega h(\xi)\frac{\psi^\alpha |u|^{p_1}|v|^{p_2}}{d(\xi)^\alpha} d\xi,$$

and

$$Q_{\lambda,\mu}(u,v) := \lambda\int_\Omega f(\xi)\frac{\psi^\beta |u|^q}{d(\xi)^\beta} d\xi + \mu\int_\Omega g(\xi)\frac{\psi^\beta |v|^q}{d(\xi)^\beta} d\xi.$$

It is easy to see that $\mathscr{N}_{\lambda,\mu}$ contains every nonzero solution of (1.1). In fact, we show later that local minimizers of $\mathscr{N}_{\lambda,\mu}$ are the critical points of $I_{\lambda,\mu}$.

In order to study the properties of Nehari manifolds, we first give the following estimation results. By Young's inequality and Sobolev embedding theorem, we have

$$\begin{aligned} H(u,v) &\le |h|_\infty\Big(\frac{p_1}{p_1+p_2}\int_\Omega \frac{\psi^\alpha |u|^{p_1+p_2}}{d(\xi)^\alpha}d\xi + \frac{p_2}{p_1+p_2}\int_\Omega \frac{\psi^\alpha |v|^{p_1+p_2}}{d(\xi)^\alpha}d\xi\Big) \\ &\le |h|_\infty S(\alpha,Q)^{-\frac{p_1+p_2}{2}}\|(u,v)\|^{p_1+p_2}. \end{aligned} \tag{3.2}$$

Similarly, by Hölder and Hardy-Sobolev inequalities, for all $u\in S_0^1(\Omega)$, we have

$$\begin{aligned} \int_\Omega f(\xi)\frac{\psi^\beta|u|^q}{d(\xi)^\beta}d\xi &\le \left(\int_\Omega \frac{\psi^\beta |f|^{\frac{2^*(\beta)}{2^*(\beta)-q}}}{d(\xi)^\beta}d\xi\right)^{\frac{2^*(\beta)-q}{2^*(\beta)}} \left(\int_\Omega \frac{\psi^\beta|u|^{2^*(\beta)}}{d(\xi)^\beta}d\xi\right)^{\frac{q}{2^*(\beta)}} \\ &\le |f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)} S(\beta,Q)^{-\frac{q}{2}}\|u\|^q_{S_0^1(\Omega)}, \end{aligned} \tag{3.3}$$

where $q_* = \frac{2^*(\beta)}{2^*(\beta)-q}$. Then, for all $(u,v)\in\mathscr{H}$,

$$Q_{\lambda,\mu}(u,v) \le S(\beta,Q)^{-\frac{q}{2}}\left[(\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}} + (\mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}\|(u,v)\|^q. \tag{3.4}$$

Further, by (3.3) and Young's inequality, we have

$$\begin{aligned} Q_{\lambda,\mu} &\le S(\beta,Q)^{-\frac{q}{2}}\Big(\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\|u\|^q_{S_0^1(\Omega)} + \mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\|v\|^q_{S_0^1(\Omega)}\Big) \\ &= \Big(\Big[\frac{2}{q}\Big(\frac12-\frac{1}{p_1+p_2}\Big)\Big(\frac1q-\frac{1}{p_1+p_2}\Big)^{-1}\Big]^{\frac{q}{2}}\|u\|^q_{S_0^1(\Omega)}\Big) \\ &\times\Big(\Big[\frac{2}{q}\Big(\frac12-\frac{1}{p_1+p_2}\Big)\Big(\frac1q-\frac{1}{p_1+p_2}\Big)^{-1}\Big]^{-\frac{q}{2}} S(\beta,Q)^{-\frac{q}{2}}\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\Big) \\ &+\Big(\Big[\frac{2}{q}\Big(\frac12-\frac{1}{p_1+p_2}\Big)\Big(\frac1q-\frac{1}{p_1+p_2}\Big)^{-1}\Big]^{\frac{q}{2}}\|v\|^q_{S_0^1(\Omega)}\Big) \\ &\times\Big(\Big[\frac{2}{q}\Big(\frac12-\frac{1}{p_1+p_2}\Big)\Big(\frac1q-\frac{1}{p_1+p_2}\Big)^{-1}\Big]^{-\frac{q}{2}} S(\beta,Q)^{-\frac{q}{2}}\mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\Big) \\ &\le \Big(\frac12-\frac{1}{p_1+p_2}\Big)\Big(\frac1q-\frac{1}{p_1+p_2}\Big)^{-1}\|(u,v)\|^2 \\ &+C_*\Big[\Big(\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\Big)^{\frac{2}{2-q}} + \Big(\mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\Big)^{\frac{2}{2-q}}\Big], \end{aligned} \tag{3.5}$$

where

$$C_* = \frac{2-q}{2}\left(\frac{p_1+p_2-q}{p_1+p_2-2}\right)^{\frac{q}{2-q}} S(\beta,Q)^{-\frac{q}{2-q}} > 0. \tag{3.6}$$

Lemma 3.1. *The functional $I_{\lambda,\mu}$ is coercive and bounded below on $\mathscr{N}_{\lambda,\mu}$.*

Proof. Let $(u,v) \in \mathscr{N}_{\lambda,\mu}$. From (3.4), we have

$$I_{\lambda,\mu}(u,v) = \frac{p_1+p_2-2}{2(p_1+p_2)}\|(u,v)\|^2 - \frac{p_1+p_2-q}{q(p_1+p_2)}Q_{\lambda,\mu}(u,v) \geq \frac{p_1+p_2-2}{2(p_1+p_2)}\|(u,v)\|^2 - \frac{p_1+p_2-q}{q(p_1+p_2)}C_{\lambda,\mu}\|(u,v)\|^q, \tag{3.7}$$

where

$$C_{\lambda,\mu} = \left[\left(\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\right)^{\frac{2}{2-q}} + \left(\mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\right)^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}} S(\beta,Q)^{-\frac{q}{2}} > 0. \tag{3.8}$$

As $1 < q < 2 < p_1+p_2$, one sees that $I_{\lambda,\mu}$ is coercive on $\mathscr{N}_{\lambda,\mu}$. Now, take $a = \frac{p_1+p_2-2}{2(p_1+p_2)}$ and $b = \frac{p_1+p_2-q}{q(p_1+p_2)}C_1$, and consider the function $\rho : \mathbb{R}^+ \to \mathbb{R}$ as $\rho(t) = at^2 - bt^q$, $t > 0$. Then, one can easily see that $\rho'(t) = 0$ if and only if $t = (\frac{qb}{2a})^{\frac{1}{2-q}}$ and $\rho''((\frac{qb}{2a})^{\frac{1}{2-q}}) = 2a(2-q) > 0$. So, ρ attains its minimum at $(\frac{qb}{2a})^{\frac{1}{2-q}}$ and

$$\rho(t) \geq \rho((\frac{qb}{2a})^{\frac{1}{2-q}}) = -(2-q)\left(\frac{b}{2}\right)^{\frac{2}{2-q}}\left(\frac{q}{a}\right)^{\frac{q}{2-q}}. \tag{3.9}$$

Hence, (3.7) and (3.9) imply that

$$I_{\lambda,\mu}(u,v) \geq \rho(\|(u,v)\|) \geq \rho((\frac{qb}{2a})^{\frac{1}{2-q}}) = -(2-q)\left(\frac{b}{2}\right)^{\frac{2}{2-q}}\left(\frac{q}{a}\right)^{\frac{q}{2-q}},$$

which means that $I_{\lambda,\mu}$ is bounded below on $\mathscr{N}_{\lambda,\mu}$. □

Now, fixing $(u,v) \in \mathscr{H}$, we define $\Psi_{(u,v)} : t \mapsto I_{\lambda,\mu}(tu,tv)$ $(t > 0)$, much known as fiber maps, as

$$\Psi_{(u,v)}(t) = \frac{t^2}{2}\|(u,v)\|^2 - \frac{t^q}{q}Q_{\lambda,\mu}(u,v) - \frac{t^{p_1+p_2}}{p_1+p_2}H(u,v).$$

Thus,

$$\Psi'_{(u,v)}(t) = t\|(u,v)\|^2 - t^{q-1}Q_{\lambda,\mu}(u,v) - t^{p_1+p_2-1}H(u,v) \tag{3.10}$$

and

$$\Psi''_{(u,v)}(t) = \|(u,v)\|^2 - (q-1)t^{q-2}Q_{\lambda,\mu}(u,v) - (p_1+p_2-1)t^{p_1+p_2-2}H(u,v).$$

From (3.1) and (3.10), we know that the Nehari manifold $\mathscr{N}_{\lambda,\mu}$ is closely to the behavior of the function $\Psi_{(u,v)}$, namely, $(tu,tv) \in \mathscr{N}_{\lambda,\mu}$ if and only if $\Psi'_{(u,v)}(t) = 0$. Therefore it is natural to split $\mathscr{N}_{\lambda,\mu}$ into three parts corresponding to local minima, local maxima, and points of inflection, respectively. For this, we set

$$\mathscr{N}^-_{\lambda,\mu} := \{(u,v) \in \mathscr{N}_{\lambda,\mu} : \Psi''_{(u,v)}(1) < 0\},$$
$$\mathscr{N}^+_{\lambda,\mu} := \{(u,v) \in \mathscr{N}_{\lambda,\mu} : \Psi''_{(u,v)}(1) > 0\},$$
$$\mathscr{N}^0_{\lambda,\mu} := \{(u,v) \in \mathscr{N}_{\lambda,\mu} : \Psi''_{(u,v)}(1) = 0\}.$$

Furthermore, for each $(u,v) \in \mathcal{N}_{\lambda,\mu}$, we have the following equalities

$$\begin{aligned}\Psi''_{(u,v)}(1) &= 2\|(u,v)\|^2 - qQ_{\lambda,\mu}(u,v) - (p_1+p_2)H(u,v)\\ &= (2-q)\|(u,v)\|^2 - (p_1+p_2-q)H(u,v) \qquad (3.11)\\ &= (p_1+p_2-q)Q_{\lambda,\mu}(u,v) - (p_1+p_2-2)\|(u,v)\|^2. \qquad (3.12)\end{aligned}$$

In what follow, we study some basic properties of $\mathcal{N}^+_{\lambda,\mu}$, $\mathcal{N}^0_{\lambda,\mu}$, and $\mathcal{N}^-_{\lambda,\mu}$.

Lemma 3.2. *If (u_0^+,v_0^+), and (u_0^-,v_0^-) are local minimizer for $I_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$, respectively, then (u_0^+,v_0^+), (u_0^-,v_0^-) are nontrivial solutions to* (1.1).

Proof. Let (u_0^+,v_0^+) such that $I_{\lambda,\mu}(u_0^+,v_0^+) = \inf_{\mathcal{N}^+_{\lambda,\mu}} I_{\lambda,\mu}$, and define the set

$$\mathcal{V} = \left\{(u,v)\in\mathcal{H} :\ \langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle > 0\right\},$$

where $\Phi_{\lambda,\mu}(u,v) := \langle I'_{\lambda,\mu}(u,v),(u,v)\rangle$. Note that, for any $(u,v)\in\mathcal{H}$ satisfying $\Phi_{\lambda,\mu}(u,v)=0$, $\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle > 0$ if and only if $\Psi''_{(u,v)}(1) > 0$. Then we have

$$\mathcal{N}^+_{\lambda,\mu} = \left\{(u,v)\in\mathcal{V} :\ \Phi_{\lambda,\mu}(u,v)=0\right\}.$$

So, by the Lagrange multipliers, we have that there exists $\theta\in\mathbb{R}$ such that $I'_{\lambda,\mu}(u_0^+,v_0^+) = \theta\Phi'_{\lambda,\mu}(u_0^+,v_0^+)$. Since $(u_0^+,v_0^+)\in\mathcal{N}^+_{\lambda,\mu}$, $\langle I'_{\lambda,\mu}(u_0^+,v_0^+),(u_0^+,v_0^+)\rangle = 0$, and $\langle \Phi'_{\lambda,\mu}(u_0^+,v_0^+),(u_0^+,v_0^+)\rangle > 0$, then $\theta = 0$, i.e., (u_0^+,v_0^+) is a nontrivial weak solution to (1.1). Similarly, for $(u_0^-,v_0^-)\in\mathcal{N}^-_{\lambda,\mu}$ such that $I_{\lambda,\mu}(u_0^-,v_0^-) = \inf_{\mathcal{N}^-_{\lambda,\mu}} I_{\lambda,\mu}$ is a nontrivial weak solution to (1.1). This completes the proof. □

Moreover, from the definitions of $\mathcal{N}^-_{\lambda,\mu}$, $\mathcal{N}^+_{\lambda,\mu}$, (3.11), and (3.12) respectively, we have the following result.

Lemma 3.3. (i) *For any $(u,v)\in\mathcal{N}^-_{\lambda,\mu}$, $H(u,v)>0$;*
(ii) *For any $(u,v)\in\mathcal{N}^+_{\lambda,\mu}$, $Q_{\lambda,\mu}(u,v)>0$.*

Lemma 3.4. *If $(\lambda,\mu)\in\mathcal{C}_{\Lambda_1}$, then $\mathcal{N}^0_{\lambda,\mu}=\emptyset$, where set $\mathcal{C}_{\Lambda_1}$ is defined in* (1.10).

Proof. On contrary, assume that there exists $(\lambda,\mu)\in\mathbb{R}^2_+$ with

$$0 < (\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}} + (\mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}} < \Lambda_1,$$

such that $\mathcal{N}^0_{\lambda,\mu}\neq\emptyset$. Then, for $(u,v)\in\mathcal{N}^0_{\lambda,\mu}$,

$$\|(u,v)\|^2 = \frac{p_1+p_2-q}{2-q}H(u,v), \tag{3.13}$$

$$\|(u,v)\|^2 = \frac{p_1+p_2-q}{p_1+p_2-2}Q_{\lambda,\mu}(u,v). \tag{3.14}$$

By (3.2) and (3.13), we obtain

$$\|(u,v)\| \geq \left(\frac{2-q}{p_1+p_2-q}\frac{S(\alpha,Q)^{\frac{p_1+p_2}{2}}}{|h|_\infty}\right)^{\frac{1}{p_1+p_2-2}}. \tag{3.15}$$

From (3.4) and (3.14), we have

$$\|(u,v)\| \le \left(\frac{p_1+p_2-q}{p_1+p_2-2}\right)^{\frac{1}{2-q}} S(\beta,Q)^{-\frac{q}{2(2-q)}} \times \left[(\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}} + (\mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}} \right]^{\frac{1}{2}}. \tag{3.16}$$

On combining (3.15) and (3.16), we have

$$(\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}} + (\mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}$$
$$\ge \Lambda_1 := \left(\frac{(2-q)S(\alpha,Q)^{\frac{p_1+p_2}{2}}}{(p_1+p_2-q)|h|_\infty}\right)^{\frac{2}{p_1+p_2-2}} \left(\frac{p_1+p_2-q}{p_1+p_2-2}\right)^{-\frac{2}{2-q}} S(\beta,Q)^{\frac{q}{2-q}},$$

which is a contradiction. The proof is completed. □

The above result shows that $\mathcal{N}_{\lambda,\mu}$ is a manifold for suitable choice of (λ,μ). We now show that $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$ are nonempty. For this, we define some notations. For each $(u,v) \in \mathscr{H}\setminus\{(0,0)\}$, define $m_{(u,v)}(t) = t^{2-q}\|(u,v)\|^2 - t^{p_1+p_2-q}H(u,v)$ for all $t>0$. Clearly, $(tu,tv) \in \mathcal{N}_{\lambda,\mu}$ if and only if $m_{(u,v)}(t) = Q_{\lambda,\mu}(u,v)$. Since

$$m'_{(u,v)}(t) = (2-q)t^{1-q}\|(u,v)\|^2 - (p_1+p_2-q)t^{p_1+p_2-q-1}H(u,v),$$

for any $(tu,tv) \in \mathcal{N}_{\lambda,\mu}$, we obtain

$$t^{q-1}m'_{(u,v)}(t) = \Psi''_{(u,v)}(t) = t^{-2}\Psi''_{(tu,tv)}(1), \tag{3.17}$$

which implies that $(tu,tv) \in \mathcal{N}^+_{\lambda,\mu}$ (or $\mathcal{N}^-_{\lambda,\mu}$) if and only if $m'_{(u,v)}(t) > 0$ (or $m'_{(u,v)}(t) < 0$). Furthermore, if $H(u,v) > 0$, then $m_{(u,v)}(0) = 0$, $m_{(u,v)}(t) \to -\infty$ as $t \to \infty$ and $m_{(u,v)}$ attains its maximum at

$$t_{\max} = \left[\frac{(2-q)\|(u,v)\|^2}{(p_1+p_2-q)H(u,v)}\right]^{\frac{1}{p_1+p_2-2}} > 0.$$

So, $m_{(u,v)}$ is strictly increasing in $[0,t_{\max})$, strictly decreasing in $(t_{\max},\infty)$, and

$$m_{(u,v)}(t_{\max}) = \left[\frac{(2-q)\|(u,v)\|^2}{(p_1+p_2-q)H(u,v)}\right]^{\frac{2-q}{p_1+p_2-2}} \|(u,v)\|^2 - \left(\frac{(2-q)\|(u,v)\|^2}{(p_1+p_2-q)H(u,v)}\right)^{\frac{p_1+p_2-q}{p_1+p_2-2}} H(u,v)$$
$$= \|(u,v)\|^q \left(\frac{2-q}{p_1+p_2-q}\right)^{\frac{2-q}{p_1+p_2-2}} \left(\frac{p_1+p_2-2}{p_1+p_2-q}\right) \left(\frac{\|(u,v)\|^{p_1+p_2}}{H(u,v)}\right)^{\frac{2-q}{p_1+p_2-2}}$$
$$\ge \|(u,v)\|^q \left(\frac{2-q}{p_1+p_2-q}\right)^{\frac{2-q}{p_1+p_2-2}} \left(\frac{p_1+p_2-2}{p_1+p_2-q}\right) \left(\frac{S(\alpha,Q)^{\frac{p_1+p_2}{2}}}{|h|_\infty}\right)^{\frac{2-q}{p_1+p_2-2}}.$$

Using the same argument used above, if $Q_{\lambda,\mu}(u,v)>0$, we define

$$\mathfrak{m}_{(u,v)}(t)=t^{2-(p_1+p_2)}\|(u,v)\|^2-t^{q-(p_1+p_2)}Q_{\lambda,\mu}(u,v),\quad \forall t>0.$$

Clearly, $(tu,tv)\in\mathcal{N}_{\lambda,\mu}$ if and only if $\mathfrak{m}_{(u,v)}(t)=H(u,v)$. Since $1<q<2<p_1+p_2$, one has $\mathfrak{m}_{(u,v)}(t)\to-\infty$ as $t\to 0^+$, $\mathfrak{m}_{(u,v)}(t)\to 0$ as $t\to\infty$, $\mathfrak{m}$ attains its maximum at $\bar{t}_{\max}$, and $\mathfrak{m}_{(u,v)}$ is strictly increasing in $(0,\bar{t}_{\max})$ and strictly decreasing in $(\bar{t}_{\max},\infty)$, where

$$\bar{t}_{\max}=\left[\frac{(p_1+p_2-q)Q_{\lambda,\mu}(u,v)}{(p_1+p_2-2)\|(u,v)\|^2}\right]^{\frac{1}{2-q}}>0.$$

Moreover, if $(tu,tv)\in\mathcal{N}_{\lambda,\mu}$, we have $t^{p_1+p_2-1}\mathfrak{m}'_{(u,v)}(t)=\Psi''_{(u,v)}(t)=t^{-2}\Psi''_{(tu,tv)}(1)$, which implies that $(tu,tv)\in\mathcal{N}^+_{\lambda,\mu}$ (or $\mathcal{N}^-_{\lambda,\mu}$) if and only if $\mathfrak{m}'_{(u,v)}(t)>0$ (or $\mathfrak{m}'_{(u,v)}(t)<0$).

Now, based on the results above, we prove the following lemma.

Lemma 3.5. *Suppose that $(\lambda,\mu)\in\mathscr{C}_{\Lambda_1}$ and $(u,v)\in\mathscr{H}\backslash\{(0,0)\}$. Then the following assertions hold.*

(i) *If $H(u,v)>0$ and $Q_{\lambda,\mu}(u,v)\le 0$, then there exists a unique $t^->t_{\max}$ such that $(t^-u,t^-v)\in\mathcal{N}^-_{\lambda,\mu}$ and $I_{\lambda,\mu}(t^-u,t^-v)=\sup_{t\ge t_{\max}}I_{\lambda,\mu}(tu,tv)$.*

(ii) *If $H(u,v)>0$ and $Q_{\lambda,\mu}(u,v)>0$, then there exists a unique $0<t^+<t_{\max}<t^-$ such that $(t^+u,t^+v)\in\mathcal{N}^+_{\lambda,\mu}$, $(t^-u,t^-v)\in\mathcal{N}^-_{\lambda,\mu}$. Moreover, $I_{\lambda,\mu}(t^+u,t^+v)=\inf_{0\le t\le t_{max}}I_{\lambda,\mu}(tu,tv)$ and $I_{\lambda,\mu}(t^-u,t^-v)=\sup_{t\ge t_{\max}}I_{\lambda,\mu}(tu,tv)$.*

(iii) *If $Q_{\lambda,\mu}(u,v)>0$ and $H(u,v)\le 0$, then there exists a unique $0<t^+<\bar{t}_{\max}$ such that $(t^+u,t^+v)\in\mathcal{N}^+_{\lambda,\mu}$ and $I_{\lambda,\mu}(t^+u,t^+v)=\inf_{t\ge 0}I_{\lambda,\mu}(tu,tv)$.*

(iv) *If $Q_{\lambda,\mu}(u,v)<0$ and $H(u,v)<0$, then there does not exist any critical point.*

Proof. (i) Since $H(u,v)>0$ and $Q_{\lambda,\mu}(u,v)\le 0$, then there exists a unique $t^->t_{\max}>0$ such that

$$m_{(u,v)}(t^-)=Q_{\lambda,\mu}(u,v)\le 0,\quad m'_{(u,v)}(t^-)<0. \tag{3.18}$$

Then,

$$\begin{aligned}\langle I'_{\lambda,\mu}(t^-u,t^-v),(t^-u,t^-v)\rangle&=(t^-)^2\|(u,v)\|^2-(t^-)^qQ_{\lambda,\mu}(u,v)-(t^-)^{p_1+p_2}H(u,v)\\&=(t^-)^q[m_{(u,v)}(t^-)-Q_{\lambda,\mu}(u,v)]=0\end{aligned}$$

yields that $(t^-u,t^-v)\in\mathcal{N}_{\lambda,\mu}$, coupling with (3.17), and $(t^-u,t^-v)\in\mathcal{N}^-_{\lambda,\mu}$. Moreover, from (3.18) we have

$$\frac{d}{dt}I_{\lambda,\mu}(tu,tv)\Big|_{t=t^-}=(t^-)^{q-1}\left[m_{(u,v)}(t^-)-Q_{\lambda,\mu}(u,v)\right]=0,$$

and

$$\frac{d^2}{dt^2}I_{\lambda,\mu}(tu,tv)\Big|_{t=t^-}=(q-1)(t^-)^{q-2}[m_{(u,v)}(t^-)-Q_{\lambda,\mu}(u,v)]+(t^-)^{q-1}m'_{(u,v)}(t^-)<0.$$

Hence, we derive $I_{\lambda,\mu}(t^-u,t^-v)=\sup_{t\ge t_{\max}}I_{\lambda,\mu}(tu,tv)$.

(ii) If $H(u,v)>0$ and $Q_{\lambda,\mu}(u,v)>0$, for all $(\lambda,\mu)\in\mathscr{C}_{\Lambda_1}$, we have by (3.4) that

$$\begin{aligned}
m_{(u,v)}(0) &= 0 < Q_{\lambda,\mu}(u,v) \\
&\le S(\beta,Q)^{-\frac{q}{2}}\left((\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}+(\mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\|(u,v)\|^q \\
&< S(\beta,Q)^{-\frac{q}{2}}\|(u,v)\|^q \\
&\quad\times\Big[\Big(\frac{(2-q)S(\alpha,Q)^{\frac{p_1+p_2}{2}}}{(p_1+p_2-q)|h|_\infty}\Big)^{\frac{2}{p_1+p_2-2}}\Big(\frac{p_1+p_2-q}{p_1+p_2-2}\Big)^{-\frac{2}{2-q}}S(\beta,Q)^{\frac{q}{2-q}}\Big]^{\frac{2-q}{2}} \\
&= \Big(\frac{(2-q)S(\alpha,Q)^{\frac{p_1+p_2}{2}}}{(p_1+p_2-q)|h|_\infty}\Big)^{\frac{2-q}{p_1+p_2-2}}\Big(\frac{p_1+p_2-2}{p_1+p_2-q}\Big)\|(u,v)\|^q \\
&\le m_{(u,v)}(t_{\max}).
\end{aligned}$$

Then, there are unique t^+, t^- such that $0<t^+<t_{\max}<t^-$ and

$$\Phi_{(u,v)}(t^+)=Q_{\lambda,\mu}(u,v)=\Phi_{(u,v)}(t^-),\quad \Phi'_{(u,v)}(t^+)>0>\Phi'_{(u,v)}(t^-),$$

which implies $(t^+u,t^+v)\in\mathscr{N}^+_{\lambda,\mu}$, $(t^-u,t^-v)\in\mathscr{N}^-_{\lambda,\mu}$ and $\frac{d}{dt}I_{\lambda,\mu}(tu,tv)=0$ for $t\in\{t^+,t^-\}$, $\frac{d^2}{dt^2}I_{\lambda,\mu}(tu,tv)>0$ for all $t\in(0,t_{\max})$, and $\frac{d^2}{dt^2}I_{\lambda,\mu}(tu,tv)<0$ for all $t\in(t_{\max},\infty)$. This yields that $I_{\lambda,\mu}(t^+u,t^+v)=\inf_{0\le t\le t_{\max}}I_{\lambda,\mu}(tu,tv)$ and $I_{\lambda,\mu}(t^-u,t^-v)=\sup_{t\ge t_{\max}}I_{\lambda,\mu}(tu,tv)$.

(iii) If $Q_{\lambda,\mu}(u,v)>0$ and $H(u,v)\le 0$, based on the same argument used in parts (i), we have that there exists a unique $0<t^+<\bar{t}_{\max}$ such that

$$\mathfrak{m}_{(u,v)}(t^+)=H(u,v)\le 0,\quad \mathfrak{m}'_{(u,v)}(t^+)>0.$$

Further, from $\langle I'_{\lambda,\mu}(t^+u,t^+v),(t^+u,t^+v)\rangle=0$ and $\Psi''_{(u,v)}(t^+)>0$, we have $(t^+u,t^+v)\in\mathscr{N}_{\lambda,\mu}$ and $(t^+u,t^+v)\in\mathscr{N}^+_{\lambda,\mu}$. Moreover, $\frac{d^2}{dt^2}I_{\lambda,\mu}(t^+u,t^+v)>0$ and $\frac{d}{dt}I_{\lambda,\mu}(t^+u,t^+v)=0$ imply that

$$I_{\lambda,\mu}(t^+u,t^+v)=\inf_{t\ge 0}I_{\lambda,\mu}(tu,tv).$$

(iv) If $Q_{\lambda,\mu}(u,v)<0$ and $H(u,v)<0$, then $\Psi_{(u,v)}(0)=0$, $\Psi'_{(u,v)}(t)>0$ for all $t>0$. This implies $\Psi_{(u,v)}$ is strictly increasing function and does not have critical point. This completes the proof. □

From Lemma 3.4, if $(\lambda,\mu)\in\mathscr{C}_{\Lambda_1}$, then $\mathscr{N}_{\lambda,\mu}=\mathscr{N}^+_{\lambda,\mu}\cup\mathscr{N}^-_{\lambda,\mu}$ and $\mathscr{N}^+_{\lambda,\mu}\cap\mathscr{N}^-_{\lambda,\mu}=\emptyset$. Now we define

$$c_{\lambda,\mu}=\inf_{(u,v)\in\mathscr{N}_{\lambda,\mu}}I_{\lambda,\mu}(u,v),$$

$$c^+_{\lambda,\mu}=\inf_{(u,v)\in\mathscr{N}^+_{\lambda,\mu}}I_{\lambda,\mu}(u,v),$$

and

$$c^-_{\lambda,\mu}=\inf_{(u,v)\in\mathscr{N}^-_{\lambda,\mu}}I_{\lambda,\mu}(u,v).$$

Theorem 3.1. *The following facts hold:*

(i) *If* $(\lambda,\mu)\in\mathscr{C}_{\Lambda_1}$, *then* $c_{\lambda,\mu}\le c^+_{\lambda,\mu}<0$.

(ii) *Let* $\Lambda_2 := (\frac{q}{2})^{\frac{2}{2-q}}\Lambda_1$. *Then, for any* $(\lambda,\mu) \in \mathscr{C}_{\Lambda_2}$, $c^-_{\lambda,\mu} > c_0 > 0$, *where* c_0 *is a constant depending on* λ, μ, β, q, Q, $S(\alpha,Q)$, $S(\beta,Q)$, $|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}$, $|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}$, *and* $|h|_\infty$.

Proof. (i) Let $(u,v) \in \mathscr{N}^+_{\lambda,\mu}$. By (3.11), we have

$$H(u,v) < \frac{2-q}{p_1+p_2-q}\|(u,v)\|^2. \tag{3.19}$$

Using (3.1) and (3.19), one has

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= \Big(\frac{1}{2}-\frac{1}{q}\Big)\|(u,v)\|^2 + \Big(\frac{1}{q}-\frac{1}{p_1+p_2}\Big)H(u,v) \\ &< \Big[\Big(\frac{1}{2}-\frac{1}{q}\Big) + \Big(\frac{1}{q}-\frac{1}{p_1+p_2}\Big)\frac{2-q}{p_1+p_2-q}\Big]\|(u,v)\|^2 \\ &= -\frac{(2-q)(p_1+p_2-2)}{2q(p_1+p_2)}\|(u,v)\|^2 < 0. \end{aligned}$$

So, from the definitions of $c_{\lambda,\mu}$ and $c^+_{\lambda,\mu}$, we can deduce that $c_{\lambda,\mu} \le c^+_{\lambda,\mu} < 0$.

(ii) Let $(u,v) \in \mathscr{N}^-_{\lambda,\mu}$. From (3.11), we have

$$\frac{2-q}{p_1+p_2-q}\|(u,v)\|^2 < H(u,v), \tag{3.20}$$

which together with Hölder's inequality and Sobolev embedding theorem implies that

$$\|(u,v)\| > \Big(\frac{(2-q)S(\alpha,Q)^{\frac{p_1+p_2}{2}}}{(p_1+p_2-q)|h|_\infty}\Big)^{\frac{1}{p_1+p_2-2}}. \tag{3.21}$$

Then, it follows from (3.7) and (3.21) that

$$\begin{aligned} I_{\lambda,\mu}(u,v) &\ge \|(u,v)\|^q\Big[\frac{p_1+p_2-2}{2(p_1+p_2)}\|(u,v)\|^{2-q} - \frac{p_1+p_2-q}{q(p_1+p_2)}S(\nu,Q)^{-\frac{q}{2}} \\ &\qquad \times \Big(\big(\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\big)^{\frac{2}{2-q}} + \big(\mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\big)^{\frac{2}{2-q}}\Big)^{\frac{2-q}{2}}\Big] \\ &> \Big(\frac{(2-q)S(\alpha,Q)^{\frac{p_1+p_2}{2}}}{p_1+p_2-q}\Big)^{\frac{q}{p_1+p_2-2}}\ \Big[\frac{p_1+p_2-2}{2(p_1+p_2)}\Big(\frac{(2-q)S(\alpha,Q)^{\frac{p_1+p_2}{2}}}{(p_1+p_2-q)|h|_\infty}\Big)^{\frac{2-q}{p_1+p_2-2}} \\ &\qquad - \frac{p_1+p_2-q}{q(p_1+p_2)}S(\beta,Q)^{-\frac{q}{2}}\Big(\big(\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\big)^{\frac{2}{2-q}} \\ &\qquad\qquad + \big(\mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\big)^{\frac{2}{2-q}}\Big)^{\frac{2-q}{2}}\Big]. \end{aligned}$$

Thus, if

$$\begin{aligned} &\frac{p_1+p_2-2}{2(p_1+p_2)}\Big(\frac{(2-q)S(\alpha,Q)^{\frac{p_1+p_2}{2}}}{(p_1+p_2-q)|h|_\infty}\Big)^{\frac{2-q}{p_1+p_2-2}} - \frac{p_1+p_2-q}{q(p_1+p_2)}S(\beta,Q)^{-\frac{q}{2}} \\ &\times \Big(\big(\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\big)^{\frac{2}{2-q}} + \big(\mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\big)^{\frac{2}{2-q}}\Big)^{\frac{2-q}{2}} > 0, \end{aligned}$$

i.e.,

$$
\begin{aligned}
&(\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}+(\mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}\\
&<\Big[\frac{q(p_1+p_2)}{p_1+p_2-q}S(\beta,Q)^{\frac{q}{2}}\frac{p_1+p_2-2}{2(p_1+p_2)}\Big(\frac{(2-q)S(\alpha,Q)^{\frac{p_1+p_2}{2}}}{(p_1+p_2-q)|h|_\infty}\Big)^{\frac{2-q}{p_1+p_2-2}}\Big]^{\frac{2}{2-q}}\\
&=\Big(\frac{q(p_1+p_2-2)}{2(p_1+p_2-q)}\Big)^{\frac{2}{2-q}}S(\beta,Q)^{\frac{q}{2-q}}\Big(\frac{(2-q)S(\alpha,Q)^{\frac{p_1+p_2}{2}}}{(p_1+p_2-q)|h|_\infty}\Big)^{\frac{2}{p_1+p_2-2}}\\
&=\Big(\frac{q}{2}\Big)^{\frac{2}{2-q}}\Big(\frac{(2-q)S(\alpha,Q)^{\frac{p_1+p_2}{2}}}{(p_1+p_2-q)|h|_\infty}\Big)^{\frac{2}{p_1+p_2-2}}\Big(\frac{p_1+p_2-2}{p_1+p_2-q}\Big)^{\frac{2}{2-q}}S(\beta,Q)^{\frac{q}{2-q}}\\
&=\Big(\frac{q}{2}\Big)^{\frac{2}{2-q}}\Lambda_1=:\Lambda_2,
\end{aligned}
$$

then $I_{\lambda,\mu}(u,v)\geq c_0>0$ for all $(u,v)\in\mathscr{N}^-_{\lambda,\mu}$. This completes the proof. □

In the end of this section, we state the following lemmas, which provide a local parametrization around any point of $\mathscr{N}_{\lambda,\mu}$.

Lemma 3.6. *Assume $(\lambda,\mu)\in\mathscr{C}_{\Lambda_1}$. Then, for every $(u,v)\in\mathscr{N}_{\lambda,\mu}$, there exist $\delta>0$ and a differentiable mapping $\zeta: B_d((0,0),\delta)\subset\mathscr{H}\to\mathbb{R}^+$ such that $\zeta(0,0)=1$, $\zeta(w_1,w_2)((u,v)-(w_1,w_2))\in\mathscr{N}_{\lambda,\mu}$, and*

$$
\langle\zeta'(0,0),(w_1,w_2)\rangle=\frac{2\mathscr{B}_{(u,v)}(w_1,w_2)-q\mathscr{Q}_{(u,v)}(w_1,w_2)-2\mathscr{P}_{(u,v)}(w_1,w_2)}{(2-q)\|(u,v)\|^2-(p_1+p_2-q)H(u,v)},
$$

where

$$
\mathscr{B}_{(u,v)}(w_1,w_2)=\int_\Omega(\nabla_{\mathbb{G}}u\nabla_{\mathbb{G}}w_1+\nabla_{\mathbb{G}}v\nabla_{\mathbb{G}}w_2)d\xi,
$$

$$
\mathscr{Q}_{(u,v)}(w_1,w_2)=\lambda\int_\Omega f(\xi)\frac{\psi^\beta|u|^{q-2}uw_1}{d(\xi)^\beta}d\xi+\mu\int_\Omega g(\xi)\frac{\psi^\beta|v|^{q-2}vw_2}{d(\xi)^\beta}d\xi,
$$

and

$$
\mathscr{P}_{(u,v)}(w_1,w_2)=p_1\int_\Omega\frac{\psi^\alpha|u|^{p_1-2}|v|^{p_2}uw_1}{d(\xi)^\alpha}d\xi+p_2\int_\Omega\frac{\psi^\alpha|u|^{p_1}|v|^{p_2-2}vw_2}{d(\xi)^\alpha}d\xi.
$$

Proof. For $(u,v)\in\mathscr{N}_{\lambda,\mu}$, define $\mathscr{F}_{(u,v)}:\mathbb{R}\times\mathscr{H}\to\mathbb{R}$ by

$$
\begin{aligned}
\mathscr{F}_{(u,v)}(\zeta,(w_1,w_2))&=\langle I'_{\lambda,\mu}(\zeta((u,v)-(w_1,w_2))),\zeta((u,v)-(w_1,w_2))\rangle\\
&=\zeta^2\|(u-w_1,v-w_2)\|^2\\
&\quad-\zeta^q\int_\Omega\Big(\lambda f(\xi)\frac{\psi^\beta|u-w_1|^q}{d(\xi)^\beta}+\mu g(\xi)\frac{\psi^\beta|v-w_2|^q}{d(\xi)^\beta}\Big)d\xi\\
&\quad-\zeta^{p_1+p_2}\int_\Omega h(\xi)\frac{\psi^\alpha|u-w_1|^{p_1}|v-w_2|^{p_2}}{d(\xi)^\alpha}d\xi.
\end{aligned}
$$

So, $\mathscr{F}_{(u,v)}(1,(0,0)) = \langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = 0$ and

$$\begin{aligned}\frac{d}{d\zeta}\mathscr{F}_{(u,v)}(1,(0,0)) &= 2\|(u,v)\|^2 - qQ(u,v) - (p_1+p_2)H(u,v)\\ &= (2-q)\|(u,v)\|^2 - (p_1+p_2-q)H(u,v)\\ &= \Psi''_{(u,v)}(1) \neq 0,\end{aligned}$$

where $\Psi''_{(u,v)}(1)$ is given in (3.11). According to the implicit function theorem, there exist $\delta > 0$ and a differentiable function $\zeta : B_d((0,0),\varepsilon) \to \mathbb{R}^+$ such that $\zeta(0,0) = 1$ and

$$\langle \zeta'(0,0),(w_1,w_2)\rangle = \frac{2\mathscr{B}_{(u,v)}(w_1,w_2) - q\mathscr{Q}_{(u,v)}(w_1,w_2) - 2\mathscr{P}_{(u,v)}(w_1,w_2)}{(2-q)\|(u,v)\|^2 - (p_1+p_2-q)H(u,v)}$$

for all $(w_1,w_2) \in \mathscr{H}$. Moreover, we have $\mathscr{F}_{(u,v)}(\zeta(w_1,w_2),(w_1,w_2)) = 0$ for all $(w_1,w_2) \in B_d((0,0),\delta)$, which implies that

$$\langle I'_{\lambda,\mu}(\zeta(w_1,w_2)((u,v)-(w_1,w_2))), \zeta(w_1,w_2)((u,v)-(w_1,w_2))\rangle = 0,$$

that is, $\zeta(w_1,w_2)((u,v)-(w_1,w_2)) \in \mathscr{N}_{\lambda,\mu}$. □

Lemma 3.7. *Under the condition of Lemma 3.6, for every $(u,v) \in \mathscr{N}^-_{\lambda,\mu}$, there exist $\delta > 0$ and a differentiable map $\zeta^- : B_d((0,0),\delta) \subset \mathscr{H} \to \mathbb{R}^+$ such that $\zeta^-(0,0) = 1$, $\zeta^-(w_1,w_2)((u,v)-(w_1,w_2)) \in \mathscr{N}^-_{\lambda,\mu}$, and*

$$\langle (\zeta^-)'(0,0),(w_1,w_2)\rangle = \frac{2\mathscr{B}_{(u,v)}(w_1,w_2) - q\mathscr{Q}_{(u,v)}(w_1,w_2) - 2\mathscr{P}_{(u,v)}(w_1,w_2)}{(2-q)\|(u,v)\|^2 - (p_1+p_2-q)H(u,v)}$$

for all $(w_1,w_2) \in \mathscr{H}$, where $\mathscr{B}_{(u,v)}$, $\mathscr{Q}_{(u,v)}$, and $\mathscr{P}_{(u,v)}$ are defined as in Lemma 3.6.

Proof. Following the proof of Lemma 3.6, there exist $\delta > 0$ and a differentiable function $\zeta^- : B_d((0,0),\delta) \subset \mathscr{H} \to \mathbb{R}^+$ such that $\zeta^-(0,0) = 1$ and $\zeta^-(w_1,w_2)((u,v)-(w_1,w_2)) \in \mathscr{N}^-_{\lambda,\mu}$ for every $(w_1,w_2) \in B_d((0,0),\delta)$. Since $(u,v) \in \mathscr{N}^-_{\lambda,\mu}$, we have

$$\Psi''_{(u,v)}(1) = (2-q)\|(u,v)\|^2 - (p_1+p_2-q)H(u,v) < 0.$$

It then follow from the continuity of $\Psi''_{(u,v)}$ and ζ^- that

$$\lim_{(w_1,w_2)\to(0,0)} \Psi''_{\zeta^-(w_1,w_2)((u,v)-(w_1,w_2))}(1) = \Psi''_{(u,v)}(1) < 0,$$

so $\zeta^-(w_1,w_2)((u,v)-(w_1,w_2)) \in \mathscr{N}^-_{\lambda,\mu}$ for δ small enough. □

4. Compactness of the Palais-Smale Sequences

In this section, we show that the functional $I_{\lambda,\mu}$ satisfies the $(PS)_c$-conditions.

Definition 4.1. Let $c \in \mathbb{R}$, W be a Banach space, and let $I_{\lambda,\mu} \in C^1(W,\mathbb{R})$ be a C^1 function. Then $\{(u_n,v_n)\} \subset W$ is a Palais-Smale sequence at level c ($(PS)_c$-sequence) in W for $I_{\lambda,\mu}$ if $I_{\lambda,\mu}(u_n,v_n) = c + o_n(1)$ and $I'_{\lambda,\mu}(u_n,v_n) = o_n(1)$ strongly in W^{-1} as $n \to \infty$. $I_{\lambda,\mu}$ is said to satisfy the $(PS)_c$-condition if, for any Palais-Smale sequence $\{(u_n,v_n)\}$ in W, $I_{\lambda,\mu}$ has a convergent subsequence.

Lemma 4.1. *Suppose that $\{(u_n,v_n)\} \subset \mathscr{H}$ is a $(PS)_c$-sequence for $I_{\lambda,\mu}$ such that $(u_n,v_n) \rightharpoonup (u,v)$ weakly in $\mathscr{H}$. Then $I'_{\lambda,\mu}(u,v)=0$ and*

$$I_{\lambda,\mu}(u,v) \geq -C_*\Big((\lambda|f|_{L^{q^*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}} + (\mu|g|_{L^{q^*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}} \Big),$$

where $C_*>0$ is given in (3.6).

Proof. Let $\{(u_n,v_n)\}$ be a $(PS)_c$-sequence in $\mathscr{H}$. By using the standard argument, one can easily obtain $I'_{\lambda,\mu}(u,v)=0$, i.e. $\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = 0$. Using this and (3.5), we obtain

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= \Big(\frac{1}{2}-\frac{1}{p_1+p_2}\Big)\|(u,v)\|^2 - \Big(\frac{1}{q}-\frac{1}{p_1+p_2}\Big)Q_{\lambda,\mu}(u,v) \\ &\geq \frac{p_1+p_2-2}{2(p_1+p_2)}\|(u,v)\|^2 - \frac{(p_1+p_2-q)}{q(p_1+p_2)}\cdot\frac{p_1+p_2-2}{2(p_1+p_2)}\cdot\frac{q(p_1+p_2)}{p_1+p_2-q}\|(u,v)\|^2 \\ &\quad - C_*\Big[(\lambda|f|_{L^{q^*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}} + (\mu|g|_{L^{q^*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}\Big] \\ &= -C_*\Big[(\lambda|f|_{L^{q^*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}} + (\mu|g|_{L^{q^*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}\Big]. \end{aligned}$$

This completes the proof of Lemma 4.1. □

Lemma 4.2. *Let $\{(u_n,v_n)\} \subset \mathscr{H}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$. Then $\{(u_n,v_n)\}$ is bounded in $\mathscr{H}$.*

Proof. Let $\{(u_n,v_n)\}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ in $\mathscr{H}$. Then we assume by contradiction that $\|(u_n,v_n)\| \to \infty$ as $n\to\infty$. Defining $(\widehat{u}_n,\widehat{v}_n) := \frac{(u_n,v_n)}{\|(u_n,v_n)\|}$, we have that $\{(\widehat{u}_n,\widehat{v}_n)\}$ is a bounded sequence in $\mathscr{H}$. So, up to a subsequence $(\widehat{u}_n,\widehat{v}_n) \rightharpoonup (\widehat{u},\widehat{v})$ weakly in $\mathscr{H}$ and $\widehat{u}_n \to \widehat{u}$, $\widehat{v}_n \to \widehat{v}$ strongly in $L^q(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)$ for $q\in[1,2^*(\beta))$. This implies that

$$Q_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n) = Q_{\lambda,\mu}(\widehat{u},\widehat{v}) + o_n(1). \tag{4.1}$$

Since $\{(u_n,v_n)\}$ is a $(PS)_c$-sequence for $I_{\lambda,\mu}$ and $\|(u_n,v_n)\|\to\infty$ as $n\to\infty$, we obtain

$$\frac{1}{2}\|(\widehat{u}_n,\widehat{v}_n)\|^2 - \frac{\|(u_n,v_n)\|^{q-2}}{q}Q_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n) - \frac{\|(u_n,v_n)\|^{p_1+p_2-2}}{p_1+p_2}H(\widehat{u}_n,\widehat{v}_n) = o_n(1), \tag{4.2}$$

and

$$\|(\widehat{u}_n,\widehat{v}_n)\|^2 - \|(u_n,v_n)\|^{q-2}Q_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n) - \|(u_n,v_n)\|^{p_1+p_2-2}H(\widehat{u}_n,\widehat{v}_n) = o_n(1). \tag{4.3}$$

From (4.1), (4.2), and (4.3), we deduce that

$$\|(\widehat{u}_n,\widehat{v}_n)\|^2 = \frac{2(p_1+p_2-q)}{q(p_1+p_2-2)}\|(u_n,v_n)\|^{q-2}Q_{\lambda,\mu}(\widehat{u},\widehat{v}) + o_n(1). \tag{4.4}$$

For n large enough, we use $\|(u_n,v_n)\|\to\infty$ and (4.4) to obtain $\|(\widehat{u}_n,\widehat{v}_n)\|^2 \to 0$, which is a contradiction to the fact that $\|(\widehat{u}_n,\widehat{v}_n)\| = 1$. □

Lemma 4.3. *Assume $(fg)_1$, $(h)_1$ hold, then $I_{\lambda,\mu}$ satisfies the $(PS)_c$-condition for all $c\in(0,c_\infty)$, where*

$$c_\infty := \frac{2-\alpha}{2(Q-\alpha)}\left(S_{\alpha,p_1,p_2}\right)^{\frac{Q-\alpha}{2-\alpha}}|h|_\infty^{-\frac{Q-2}{2-\alpha}} - C_*\Big((\lambda|f|_{L^{q_*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}+(\mu|g|_{L^{q_*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}\Big)$$

and C_* is given in (3.6).

Proof. Let $\{(u_n,v_n)\}\subset\mathscr{H}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ with $0<c<c_\infty$. By Lemma 4.2, one has that $\{(u_n,v_n)\}$ is a bounded sequence in $\mathscr{H}$. Hence, up to a subsequence, $(u_n,v_n)\rightharpoonup(u,v)$ weakly in $\mathscr{H}$, $u_n\rightharpoonup u$ and $v_n\rightharpoonup v$ weakly in $L^{2^*(\alpha)}(\Omega,\frac{\psi^\alpha}{d(\xi)^\alpha}d\xi))$, $u_n\to u$ and $v_n\to v$ strongly in $L^s(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi))$ for all $s\in[1,2^*(\beta))$, and $u_n(\xi)\to u(\xi)$ and $v_n(\xi)\to v(\xi)$ a.e. in Ω. Thus

$$Q_{\lambda,\mu}(u_n,v_n)=Q_{\lambda,\mu}(u,v)+o_n(1). \tag{4.5}$$

Also, $I'_{\lambda,\mu}(u,v)=0$ follows from Lemma 4.1. Set $\tilde u_n=u_n-u$, $\tilde v_n=v_n-v$. By Brézis-Lieb Lemma [24] and Vitali theorem, we arrive at

$$\|(\tilde u_n,\tilde v_n)\|^2=\|(u_n,v_n)\|^2-\|(u,v)\|^2+o_n(1), \tag{4.6}$$

and

$$H(\tilde u_n,\tilde v_n)=H(u_n,v_n)-H(u,v)+o_n(1). \tag{4.7}$$

Using $I_{\lambda,\mu}(u_n,v_n)=c+o_n(1)$, $I'_{\lambda,\mu}(u_n,v_n)=o_n(1)$, (4.5), (4.6), and (4.7), we obtain

$$\frac{1}{2}\|(\tilde u_n,\tilde v_n)\|^2-\frac{1}{p_1+p_2}H(\tilde u_n,\tilde v_n)=c-I_{\lambda,\mu}(u,v)+o_n(1), \tag{4.8}$$

and $\|(\tilde u_n,\tilde v_n)\|^2-H(\tilde u_n,\tilde v_n)=o_n(1)$. Therefore, we assume that

$$\|(\tilde u_n,\tilde v_n)\|^2\to l,\quad H(\tilde u_n,\tilde v_n)\to l \text{ as } n\to\infty. \tag{4.9}$$

If $l=0$, then proof is complete. If $l>0$, then, by definition of S_{α,p_1,p_2} and (4.9), we obtain

$$\begin{aligned}S_{\alpha,p_1,p_2}\cdot l^{\frac{2}{p_1+p_2}}&\le S_{\alpha,p_1,p_2}\lim_{n\to\infty}\left(|h|_\infty\int_\Omega\frac{\psi^\alpha|u_n|^{p_1}|v_n|^{p_2}}{d(\xi)^\alpha}d\xi\right)^{\frac{2}{p_1+p_2}}\\&\le|h|_\infty^{\frac{2}{p_1+p_2}}\lim_{n\to\infty}\|(\tilde u_n,\tilde v_n)\|^2=|h|_\infty^{\frac{2}{p_1+p_2}}l.\end{aligned}$$

As $p_1+p_2=2^*(\alpha)$, we see that the above relation gives

$$l\ge\left(S_{\alpha,p_1,p_2}\right)^{\frac{Q-\alpha}{2-\alpha}}\cdot|h|_\infty^{-\frac{(Q-2)}{2-\alpha}}.$$

Hence, by (4.8), (4.9), and Lemma 4.1, we obtain

$$\begin{aligned}c&=\Big(\frac{1}{2}-\frac{1}{p_1+p_2}\Big)l+I_{\lambda,\mu}(u,v)+o_n(1)\\&\ge\frac{2-\alpha}{2(Q-\alpha)}\left(S_{\alpha,p_1,p_2}\right)^{\frac{Q-\alpha}{2-\alpha}}|h|_\infty^{-\frac{Q-2}{2-\alpha}}\\&\quad-C_*\Big((\lambda|f|_{L^{q_*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}+(\mu|g|_{L^{q_*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}\Big):=c_\infty,\end{aligned}$$

which is a contradiction to $c < c_\infty$. The proof is complete. □

5. Proof of Theorems 1.1 and 1.2

In this section, we show the existence of Palais-Smale sequence in $\mathcal{N}^+_{\lambda,\mu}$, $\mathcal{N}^-_{\lambda,\mu}$ and give the proof of Theorems 1.1 and 1.2.

Lemma 5.1. *Let $1 < q < 2$, $0 \le \alpha < 2$, $0 \le \beta < 2$, and $2 < p_1 + p_2 \le 2^*(\alpha)$. Then the following results hold.*

(i) *If $(\lambda,\mu) \in \mathscr{C}_{\Lambda_1}$, then there exists a sequence $\{(u_n,v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ such that*

$$I_{\lambda,\mu}(u_n,v_n) = c_{\lambda,\mu} + o_n(1) \ \text{ and } \ I'_{\lambda,\mu}(u_n,v_n) = o_n(1) \ \text{ in } \mathscr{H}^{-1}.$$

(ii) *If $(\lambda,\mu) \in \mathscr{C}_{\Lambda_2}$, then there exists a sequence $\{(u_n,v_n)\} \subset \mathcal{N}^-_{\lambda,\mu}$ such that*

$$I_{\lambda,\mu}(u_n,v_n) = c^-_{\lambda,\mu} + o_n(1) \ \text{ and } \ I'_{\lambda,\mu}(u_n,v_n) = o_n(1) \ \text{ in } \mathscr{H}^{-1}.$$

Proof. (i) It follows from Lemma 3.1 that $I_{\lambda,\mu}$ is coercive and bounded below. Then the Ekeland variational principle implies that there exists a minimizing sequence $\{(u_n,v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(u_n,v_n) < \inf_{\mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(u,v) + \frac{1}{n} = c_{\lambda,\mu} + \frac{1}{n}, \tag{5.1}$$

and

$$I_{\lambda,\mu}(u_n,v_n) < I_{\lambda,\mu}(u,v) + \frac{1}{n}\|(u,v)-(u_n,v_n)\|, \quad \forall (u,v) \in \mathcal{N}_{\lambda,\mu}. \tag{5.2}$$

For n large enough, we use (5.1) and $c_{\lambda,\mu} < 0$ to obtain $I_{\lambda,\mu}(u_n,v_n) < c_{\lambda,\mu} + \frac{1}{n} < \frac{c_{\lambda,\mu}}{2}$. Therefore, for $(u_n,v_n) \in \mathcal{N}_{\lambda,\mu}$, we have

$$I_{\lambda,\mu}(u_n,v_n) \ge -\Big(\frac{p_1+p_2-q}{q(p_1+p_2)}\Big) Q_{\lambda,\mu}(u_n,v_n),$$

which together with $I_{\lambda,\mu}(u_n,v_n) < \frac{c_{\lambda,\mu}}{2}$ implies that

$$Q_{\lambda,\mu}(u_n,v_n) > -\frac{q(p_1+p_2)c_{\lambda,\mu}}{2(p_1+p_2-q)} > 0. \tag{5.3}$$

Thus, (5.3) and (3.4) yield $(u_n,v_n) \ne (0,0)$. In order to finalize the proof, it is sufficient to show that

$$I'_{\lambda,\mu}(u_n,v_n) \to 0 \text{ in } \mathscr{H}^{-1} \text{ as } n \to \infty. \tag{5.4}$$

Indeed, it follows from Lemma 3.6 that there exists a differentiable function $\zeta_n : B_d((0,0),\delta_n) \to \mathbb{R}^+$, for some $\delta_n > 0$, such that $\zeta_n(w_1,w_2)((u_n,v_n)-(w_1,w_2)) \in \mathcal{N}_{\lambda,\mu}$ for all $(w_1,w_2) \in \mathscr{H}$, where $B_d((0,0),\delta_n) := \{(u,v) \in \mathscr{H} : d((u,v),(0,0)) < \delta_n\}$. For any $(u,v) \in \mathscr{H}\backslash\{(0,0)\}$, define

$$(w_{\eta,1},w_{\eta,2}) := \eta \frac{(u,v)}{\|(u,v)\|}, \quad \eta \in (0,\delta_n);$$
$$(\varpi_{\eta,1},\varpi_{\eta,2}) := \zeta_n(w_{\eta,1},w_{\eta,2})((u_n,v_n)-(w_{\eta,1},w_{\eta,2})).$$

Using the fact that $(\varpi_{\eta,1},\varpi_{\eta,2}) \in \mathscr{N}_{\lambda,\mu}$, and (5.2), we obtain that

$$I_{\lambda,\mu}(\varpi_{\eta,1},\varpi_{\eta,2}) - I_{\lambda,\mu}(u_n,v_n) > -\frac{1}{n}\|(\varpi_{\eta,1},\varpi_{\eta,2}) - (u_n,v_n)\|.$$

Now, we apply the mean value theorem to the left hand side of the last inequality to deduce

$$\begin{aligned} I_{\lambda,\mu}(\varpi_{\eta,1},\varpi_{\eta,2}) - I_{\lambda,\mu}(u_n,v_n) = {} & \langle I'_{\lambda,\mu}(u_n,v_n),(\varpi_{\eta,1},\varpi_{\eta,2}) - (u_n,v_n)\rangle \\ & + o_n(\|(\varpi_{\eta,1},\varpi_{\eta,2}) - (u_n,v_n)\|). \end{aligned}$$

Thus,

$$\begin{aligned} & \langle I'_{\lambda,\mu}(u_n,v_n),(\varpi_{\eta,1},\varpi_{\eta,2}) - (u_n,v_n)\rangle + o_n(\|(\varpi_{\eta,1},\varpi_{\eta,2}) - (u_n,v_n)\|) \\ & \quad > -\frac{1}{n}\|(\varpi_{\eta,1},\varpi_{\eta,2}) - (u_n,v_n)\|. \end{aligned} \tag{5.5}$$

Regarding the first term in (5.5), we have that

$$\begin{aligned} & \langle I'_{\lambda,\mu}(u_n,v_n),(\varpi_{\eta,1},\varpi_{\eta,2}) - (u_n,v_n)\rangle \\ = {} & \langle I'_{\lambda,\mu}(u_n,v_n),\zeta_n(w_{\eta,1},w_{\eta,2})((u_n,v_n) - (w_{\eta,1},w_{\eta,2})) - (u_n,v_n)\rangle \\ = {} & \langle I'_{\lambda,\mu}(u_n,v_n),-(w_{\eta,1},w_{\eta,2})\rangle + \langle I'_{\lambda,\mu}(u_n,v_n),(\zeta_n(w_{\eta,1},w_{\eta,2}) - 1)((u_n,v_n) - (w_{\eta,1},w_{\eta,2}))\rangle. \end{aligned} \tag{5.6}$$

Therefore,

$$\begin{aligned} & \langle I'_{\lambda,\mu}(u_n,v_n),-(w_{\eta,1},w_{\eta,2})\rangle + \langle I'_{\lambda,\mu}(u_n,v_n),(\zeta_n(w_{\eta,1},w_{\eta,2}) - 1)((u_n,v_n) - (w_{\eta,1},w_{\eta,2}))\rangle \\ & \quad > -\frac{1}{n}\|(\varpi_{\eta,1},\varpi_{\eta,2}) - (u_n,v_n)\| + o_n(\|(\varpi_{\eta,1},\varpi_{\eta,2}) - (u_n,v_n)\|). \end{aligned}$$

By the definition of $(w_{\eta,1},w_{\eta,2})$ and $(\varpi_{\eta,1},\varpi_{\eta,2})$, we obtain

$$\begin{aligned} & -\eta\langle I'_{\lambda,\mu}(u_n,v_n),\frac{(u,v)}{\|(u,v)\|}\rangle + (\zeta_n(w_{\eta,1},w_{\eta,2}) - 1) \\ & \quad \times \langle I'_{\lambda,\mu}(u_n,v_n) - I'_{\lambda,\mu}(\varpi_{\eta,1},\varpi_{\eta,2}),(u_n,v_n) - (w_{\eta,1},w_{\eta,2})\rangle \\ & > -\frac{1}{n}\|(\varpi_{\eta,1},\varpi_{\eta,2}) - (u_n,v_n)\| + o_n(\|(\varpi_{\eta,1},\varpi_{\eta,2}) - (u_n,v_n)\|). \end{aligned}$$

The last inequality implies that

$$\begin{aligned} & \langle I'_{\lambda,\mu}(u_n,v_n),\frac{(u,v)}{\|(u,v)\|}\rangle \\ & \le \frac{\|(\varpi_{\eta,1},\varpi_{\eta,2}) - (u_n,v_n)\|}{n\eta} + \frac{o_n(\|(\varpi_{\eta,1},\varpi_{\eta,2}) - (u_n,v_n)\|)}{\eta} \\ & + \frac{(\zeta_n(w_{\eta,1},w_{\eta,2}) - 1)}{\eta}\langle I'_{\lambda,\mu}(u_n,v_n) - I'_{\lambda,\mu}(\varpi_{\eta,1},\varpi_{\eta,2}),(u_n,v_n) - (w_{\eta,1},w_{\eta,2})\rangle. \end{aligned} \tag{5.7}$$

From Lemma 3.6, we see that

$$\lim_{\eta\to 0}\frac{|\zeta_n(w_{\eta,1},w_{\eta,2}) - 1|}{\eta} = \lim_{\eta\to 0}\frac{|\zeta_n(w_{\eta,1},w_{\eta,2}) - \zeta_n(0,0)|}{\eta} \le \|\zeta'_n(0,0)\|.$$

Simple calculations yield

$$\begin{aligned}\|(\varpi_{\eta,1},\varpi_{\eta,2})-(u_n,v_n)\| &\le |\zeta_n(w_{\eta,1},w_{\eta,2})|\cdot\|(w_{\eta,1},w_{\eta,2})\|+|\zeta_n(w_{\eta,1},w_{\eta,2})-1|\cdot\|(u_n,v_n)\|\\ &=\eta|\zeta_n(w_{\eta,1},w_{\eta,2})|+|\zeta_n(w_{\eta,1},w_{\eta,2})-1|\cdot\|(u_n,v_n)\|. \qquad (5.8)\end{aligned}$$

Using the last two identities and (5.7), we conclude that

$$\begin{aligned}\langle I'_{\lambda,\mu}(u_n,v_n),\frac{(u,v)}{\|(u,v)\|}\rangle &\le \frac{|\zeta_n(w_{\eta,1},w_{\eta,2})|}{n}+\frac{1}{n}\frac{|\zeta_n(w_{\eta,1},w_{\eta,2})-1|}{\eta}\|(u_n,v_n)\|\\ &+\frac{(\zeta_n(w_{\eta,1},w_{\eta,2})-1)}{\eta}\langle I'_{\lambda,\mu}(u_n,v_n)-I'_{\lambda,\mu}(\varpi_{\eta,1},\varpi_{\eta,2}),(u_n,v_n)-(w_{\eta,1},w_{\eta,2})\rangle. \qquad (5.9)\end{aligned}$$

Taking $\eta\to 0$ in (5.9) for a fixed $n\in\mathbb{N}$, we can find a constant $C>0$ (independent of η) such that

$$\langle I'_{\lambda,\mu}(u_n,v_n),\frac{(u,v)}{\|(u,v)\|}\rangle \le \frac{C}{n}(1+\|\zeta_n'(0,0)\|) \text{ as } \eta\to 0.$$

In order to complete the proof of (5.4), we only need to prove that $\|\zeta_n'(0,0)\|$ is uniformly bounded in n. From Lemma 3.6, for $(w_1,w_2)\in\mathscr{H}$, one has

$$\langle \zeta_n'(0,0),(w_1,w_2)\rangle=\frac{2\mathscr{B}_{(u,v)}(w_1,w_2)-q\mathscr{Q}_{(u,v)}(w_1,w_2)-2\mathscr{P}_{(u,v)}(w_1,w_2)}{(2-q)\|(u,v)\|^2-(p_1+p_2-q)H(u,v)}.$$

By Hölder's inequality, is it easy to see that, for some constant $M_1>0$,

$$2\mathscr{B}_{(u,v)}(w_1,w_2)-q\mathscr{Q}_{(u,v)}(w_1,w_2)-2\mathscr{P}_{(u,v)}(w_1,w_2)\le M_1\|(w_1,w_2)\|.$$

Then, we only need to prove that

$$|(2-q)\|(u_n,v_n)\|^2-(p_1+p_2-q)H(u_n,v_n)|\ge M_2>0 \qquad (5.10)$$

as n large enough. On the contrary, suppose that there exists a subsequence $\{(u_n,v_n)\}$ such that

$$(2-q)\|(u_n,v_n)\|^2-(p_1+p_2-q)H(u_n,v_n)=o_n(1). \qquad (5.11)$$

From (5.11) and $(u_n,v_n)\in\mathscr{N}_{\lambda,\mu}$, we have

$$\|(u_n,v_n)\|^2=\frac{p_1+p_2-q}{2-q}H(u_n,v_n)+o_n(1)$$

and

$$\|(u_n,v_n)\|^2=\frac{p_1+p_2-q}{p_1+p_2-2}Q_{\lambda,\mu}(u_n,v_n)+o_n(1).$$

Using the last two identities and (3.2) and (3.4), we then obtain that

$$\|(u_n,v_n)\|\ge\left(\frac{2-q}{p_1+p_2-q}\frac{S(\alpha,Q)^{\frac{p_1+p_2}{2}}}{|h|_\infty}\right)^{\frac{1}{p_1+p_2-2}}+o_n(1),$$

and

$$\begin{aligned}\|(u_n,v_n)\|&\le\left(\frac{p_1+p_2-q}{p_1+p_2-2}\right)^{\frac{1}{2-q}}S(\beta,Q)^{-\frac{q}{2(2-q)}}\\ &\times\left((\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}+(\mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}\right)^{\frac{1}{2}}+o_n(1).\end{aligned}$$

This implies that

$$(\lambda|f|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}+(\mu|g|_{L^{q*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)})^{\frac{2}{2-q}}\geq\Lambda_1,$$

which is a contradiction. Therefore, (5.10) holds. Hence, there exists a constant $M>0$ such that

$$\langle I'_{\lambda,\mu}(u_n,v_n),\frac{(u,v)}{\|(u,v)\|}\rangle\leq\frac{M}{n}.$$

This implies (5.4), and completes the proof of (i).

(ii) The proof goes exactly as the first part by using Lemma 3.7. □

In the end of this section, we use results above to prove the existence of positive vector soltuions on $\mathcal{N}^+_{\lambda,\mu}$, as well as on $\mathcal{N}^-_{\lambda,\mu}$. This couples with the fact that $\mathcal{N}^+_{\lambda,\mu}\cap\mathcal{N}^-_{\lambda,\mu}=\emptyset$ yield Theorem 1.2.

Theorem 5.1. *Suppose that $1<q<2$, $0\leq\alpha,\beta<2$, p_1, $p_2>1$ with $2<p_1+p_2\leq 2^*(\alpha)$. If the parameters λ, μ satisfy $(\lambda,\mu)\in\mathscr{C}_{\Lambda_1}$, where Λ_1 is given in (1.9), then system (1.1) has at least one positive vector solution $(u^1_{\lambda,\mu},v^1_{\lambda,\mu})\in\mathcal{N}^+_{\lambda,\mu}$ such that $I_{\lambda,\mu}(u^1_{\lambda,\mu},v^1_{\lambda,\mu})=c^+_{\lambda,\mu}<0$, $u^1_{\lambda,\mu}\neq 0$, and $v^1_{\lambda,\mu}\neq 0$.*

Proof. By Lemma 5.1 (i), there exists a minimizing sequence $\{(u_n,v_n)\}\subset\mathcal{N}_{\lambda,\mu}$ for $I_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(u_n,v_n)=c_{\lambda,\mu}+o_n(1),\quad I'_{\lambda,\mu}(u_n,v_n)=o_n(1)\text{ in }\mathscr{H}^{-1}. \tag{5.12}$$

By coercivity of $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, we obtain that $\{(u_n,v_n)\}$ is bounded in $\mathscr{H}$. Then, passing to a subsequence, still denoted by $\{(u_n,v_n)\}$, there exists $(u^1_{\lambda,\mu},v^1_{\lambda,\mu})\in\mathscr{H}$ such that $u_n\rightharpoonup u^1_{\lambda,\mu}$, $v_n\rightharpoonup v^1_{\lambda,\mu}$ weakly in $S^1_0(\Omega)$, and

$$\begin{aligned}&u_n\to u^1_{\lambda,\mu},\,v_n\to v^1_{\lambda,\mu}\text{ weakly in }L^{2^*(\alpha)}(\Omega,\frac{\psi^\alpha}{d(\xi)^\alpha}d\xi),\\&u_n\to u^1_{\lambda,\mu},\,v_n\to v^1_{\lambda,\mu}\text{ strongly in }L^s(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi),\,\forall s\in(1,2^*(\beta)),\\&u_n(\xi)\to u^1_{\lambda,\mu}(\xi),\,v_n(\xi)\to v^1_{\lambda,\mu}(\xi)\text{ a.e. }\Omega.\end{aligned} \tag{5.13}$$

First, we claim that $(u^1_{\lambda,\mu},v^1_{\lambda,\mu})$ is a nontrivial solution to (1.1). From (5.12) and (5.13), one can easily verify that

$$Q_{\lambda,\mu}(u_n,v_n)=Q_{\lambda,\mu}(u^1_{\lambda,\mu},v^1_{\lambda,\mu})+o_n(1), \tag{5.14}$$

and $(u^1_{\lambda,\mu},v^1_{\lambda,\mu})$ is a weak solution to system (1.1). From $(u_n,v_n)\in\mathcal{N}_{\lambda,\mu}$ and the definition of $I_{\lambda,\mu}$, we have

$$Q_{\lambda,\mu}(u_n,v_n)=\frac{q(p_1+p_2-2)}{2(p_1+p_2-q)}\|(u_n,v_n)\|^2-\frac{q(p_1+p_2)}{p_1+p_2-q}I_{\lambda,\mu}(u_n,v_n). \tag{5.15}$$

Then, letting $n\to\infty$ in (5.15) and using (5.12) and (5.14) with $c_{\lambda,\mu}<0$, we obtain

$$Q_{\lambda,\mu}(u^1_{\lambda,\mu},v^1_{\lambda,\mu})\geq-\frac{q(p_1+p_2)}{(p_1+p_2-q)}c_{\lambda,\mu}>0,$$

which implies that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \neq (0,0)$. Therefore $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is a nontrivial solution to (1.1) for $(\lambda,\mu) \in \mathscr{C}_{\Lambda_1}$.

Now, we show that $(u_n, v_n) \to (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $\mathscr{H}$ and $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = c_{\lambda,\mu}$. If $(u,v) \in \mathscr{N}_{\lambda,\mu}$, then

$$I_{\lambda,\mu}(u,v) = \frac{p_1+p_2-2}{2(p_1+p_2)}\|(u,v)\|^2 - \frac{p_1+p_2-q}{q(p_1+p_2)}Q_{\lambda,\mu}(u,v). \tag{5.16}$$

To prove that $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = c_{\lambda,\mu}$, in view of $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathscr{N}_{\lambda,\mu}$, (5.16), and Fatou's lemma, we obtain

$$\begin{aligned} c_{\lambda,\mu} \le I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) &= \frac{p_1+p_2-2}{2(p_1+p_2)}\|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^2 - \frac{p_1+p_2-q}{q(p_1+p_2)}Q_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \\ &\le \liminf_{n\to\infty}\Big(\frac{p_1+p_2-2}{2(p_1+p_2)}\|(u_n,v_n)\|^2 - \frac{(p_1+p_2-q)}{q(p_1+p_2)}Q_{\lambda,\mu}(u_n,v_n)\Big) \\ &\le \liminf_{n\to\infty} I_{\lambda,\mu}(u_n,v_n) = c_{\lambda,\mu}. \end{aligned} \tag{5.17}$$

So, (5.17) and (5.13) imply that $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = c_{\lambda,\mu}$ and $(u_n, v_n) \to (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $\mathscr{H}$.

The next step is to prove that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathscr{N}^+_{\lambda,\mu}$. On the contrary, if $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathscr{N}^-_{\lambda,\mu}$, by using (3.20) and (5.17), we have that $H(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) > 0$ and $Q_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) > 0$. It then follows from Lemma 3.5 (ii) that there exist unique t_1^+ and $t_1^- > 0$ such that $(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) \in \mathscr{N}^+_{\lambda,\mu}$, $(t_1^- u^1_{\lambda,\mu}, t_1^- v^1_{\lambda,\mu}) \in \mathscr{N}^-_{\lambda,\mu}$, and $t_1^+ < t_1^- = 1$. Following the proof of Lemma 3.5, we have that

$$\frac{d}{dt}I_{\lambda,\mu}(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) = 0, \quad \frac{d^2}{dt^2}I_{\lambda,\mu}(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) > 0.$$

Thus, there exists a $\bar t$ such that $t_1^+ < \bar t < t_1^- = 1$ and $I_{\lambda,\mu}(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) < I_{\lambda,\mu}(\bar t u^1_{\lambda,\mu}, \bar t v^1_{\lambda,\mu})$. We again use Lemma 3.5 to obtain

$$\begin{aligned} c^+_{\lambda,\mu} &\le I_{\lambda,\mu}(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) < I_{\lambda,\mu}(\bar t u^1_{\lambda,\mu}, \bar t v^1_{\lambda,\mu}) \\ &\le I_{\lambda,\mu}(t_1^- u^1_{\lambda,\mu}, t_1^- v^1_{\lambda,\mu}) = I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = c_{\lambda,\mu}, \end{aligned}$$

which is a contradiction, that is, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \notin \mathscr{N}^-_{\lambda,\mu}$. Since $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = I_{\lambda,\mu}(|u^1_{\lambda,\mu}|, |v^1_{\lambda,\mu}|)$, and $(|u^1_{\lambda,\mu}|, |v^1_{\lambda,\mu}|) \in \mathscr{N}^+_{\lambda,\mu}$ is a solution to (1.1), we may assume, without loss of generality, that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is a non-negative solution to (1.1), and Bony's maximum principle [25] implies that $u^1_{\lambda,\mu}(\xi) > 0$ and $v^1_{\lambda,\mu}(\xi) > 0$ in Ω.

To complete the proof of Theorem 5.1, we need to show that the solutions $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is not semi-trivial. Without loss of generality, we may assume that $v^1_{\lambda,\mu} \equiv 0$. Then $u^1_{\lambda,\mu}$ is a non-trivial solution of

$$-\Delta_{\mathbb{G}} u = \lambda f(\xi)\frac{\psi^\beta |u|^{q-2}u}{d(\xi)^\beta} \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega,$$

and satisfies

$$\|u^1_{\lambda,\mu}\|^2_{S^1_0(\Omega)} = \lambda \int_\Omega f(\xi)\frac{\psi^\beta |u^1_{\lambda,\mu}|^q}{d(\xi)^\beta}d\xi > 0.$$

Taking $w \in S_0^1(\Omega) \setminus \{0\}$ such that

$$\|w\|^2_{S_0^1(\Omega)} = \mu \int_\Omega g(\xi) \frac{\psi^\beta |w|^q}{d(\xi)^\beta} d\xi, \tag{5.18}$$

we find from Lemma 3.5 that there exists a unique $t_1 \in (0, t_{\max}(u^1_{\lambda,\mu}, w))$ such that $(t_1 u^1_{\lambda,\mu}, t_1 w) \in \mathscr{N}^+_{\lambda,\mu}$ and $I_{\lambda,\mu}(t_1 u^1_{\lambda,\mu}, t_1 w) = \inf_{0 \le t \le t_{\max}} I_{\lambda,\mu}(t u^1_{\lambda,\mu}, t w)$, where

$$\begin{aligned} t_{\max}(u^1_{\lambda,\mu}, w) &= \left(\frac{(p_1+p_2-q)(\int_\Omega \lambda f(\xi) \frac{\psi^\beta |u^1_{\lambda,\mu}|^q}{d(\xi)^\beta} d\xi + \int_\Omega \mu g(\xi) \frac{\psi^\beta |w|^q}{d(\xi)^\beta} d\xi)}{(p_1+p_2-2)\|(u^1_{\lambda,\mu}, w)\|^2} \right)^{\frac{1}{2-q}} \\ &= \left(\frac{p_1+p_2-q}{p_1+p_2-2} \right)^{\frac{1}{2-q}} > 1. \end{aligned}$$

On the other hand, by (5.18), we have

$$\begin{aligned} & I_{\lambda,\mu}(u^1_{\lambda,\mu}, 0) - I_{\lambda,\mu}(u^1_{\lambda,\mu}, w) \\ &= -\frac{1}{2}\|w\|^2_{S_0^1(\Omega)} + \frac{\mu}{q} \int_\Omega g(\xi) \frac{\psi^\beta |w|^q}{d(\xi)^\beta} d\xi + \int_\Omega h(\xi) \frac{\psi^\alpha |u^1_{\lambda,\mu}|^{p_1} |w|^{p_2}}{d(\xi)^\alpha} d\xi \\ &= \frac{2-q}{2q}\|w\|^2_{S_0^1(\Omega)} + \int_\Omega h(\xi) \frac{\psi^\alpha |u^1_{\lambda,\mu}|^{p_1} |w|^{p_2}}{d(\xi)^\alpha} d\xi > 0. \end{aligned}$$

This and the fact that $(u^1_{\lambda,\mu}, 0) \in \mathscr{N}^+_{\lambda,\mu}$ imply that

$$c^+_{\lambda,\mu} \le I_{\lambda,\mu}(t_1 u^1_{\lambda,\mu}, t_1 w) \le I_{\lambda,\mu}(u^1_{\lambda,\mu}, w) < I_{\lambda,\mu}(u^1_{\lambda,\mu}, 0) = c^+_{\lambda,\mu},$$

which is a contradiction. Hence, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is not semi-trivial. This completes the proof. □

Remark 5.1. From the definition of $C_{\lambda,\mu}$ (see (3.8)), we have $C_{\lambda,\mu} \to 0$ as $\lambda, \mu \to 0^+$. On the other hand, using Theorem 3.1 (i) and (3.7), for $(\lambda, \mu) \in \mathscr{C}_{\Lambda_1}$, we have

$$0 > c^+_{\lambda,\mu} \ge c_{\lambda,\mu} = I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) > -\frac{p_1+p_2-q}{q(p_1+p_2)} C_{\lambda,\mu} \|(u,v)\|^q \to 0.$$

This implies that $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \to 0$ as $\lambda \to 0^+$ and $\mu \to 0^+$.

Theorem 5.2. *Let* $1 < q < 2$, $0 \le \alpha < 2$, $0 \le \beta < 2$, $2 < p_1 + p_2 < 2^*(\alpha)$, *and* λ *and* μ *satisfy* $(\lambda, \mu) \in \mathscr{C}_{\Lambda_2}$, *where* $\Lambda_2 = (\frac{q}{2})^{\frac{2}{2-q}} \Lambda_1$. *Then system* (1.1) *has at least one positive vector solution* $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathscr{N}^-_{\lambda,\mu}$ *such that* $I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = c^-_{\lambda,\mu} > 0$ *and* $u^2_{\lambda,\mu} \neq 0$, $v^2_{\lambda,\mu} \neq 0$.

Proof. Let $\{(u_n, v_n)\}$ be a minimizing sequence for $I_{\lambda,\mu}$ on $\mathscr{N}^-_{\lambda,\mu}$ such that $I_{\lambda,\mu}(u_n, v_n) = c^-_{\lambda,\mu} + o_n(1)$ and $I'_{\lambda,\mu}(u_n, v_n) = o_n(1)$ in $\mathscr{H}^{-1}$, given in the second part of Lemma 5.1. It then follows from Lemma 3.1 and the compact imbedding theorem that there exist a subsequence $\{(u_n, v_n)\}$, still denote by $\{(u_n, v_n)\}$, and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathscr{H}$ such that $u_n \rightharpoonup u^2_{\lambda,\mu}$ and $v_n \rightharpoonup v^2_{\lambda,\mu}$ weakly in $S_0^1(\Omega)$, $u_n \to u^2_{\lambda,\mu}$ and $v_n \to v^2_{\lambda,\mu}$ strongly in $L^q(\Omega, \frac{\psi^\beta}{d(\xi)^\beta} d\xi)$, and $L^{p_1+p_2}(\Omega, \frac{\psi^\alpha}{d(\xi)^\alpha} d\xi)$ for

$p_1+p_2<2^*(\alpha)$. This implies

$$Q_{\lambda,\mu}(u_n,v_n)=Q_{\lambda,\mu}(u^2_{\lambda,\mu},v^2_{\lambda,\mu})+o_n(1),$$

and

$$H(u_n,v_n)=H(u^2_{\lambda,\mu},v^2_{\lambda,\mu})+o_n(1).$$

Using (3.20) and (3.21), we find that there exists $C_3>0$ such that $H(u_n,v_n)>C_3$ for $(u_n,v_n)\in\mathcal{N}^-_{\lambda,\mu}$, so $H(u^2_{\lambda,\mu},v^2_{\lambda,\mu})\geq C_3>0$.

Now, we prove that $(u_n,v_n)\to(u^2_{\lambda,\mu},v^2_{\lambda,\mu})$ in $\mathcal{H}$. Indeed, if not, then we have $\|(u^2_{\lambda,\mu},v^2_{\lambda,\mu})\|<\liminf_{n\to\infty}\|(u_n,v_n)\|$. Then using Lemma 3.5, there exists a unique $t_2^->0$ such that $(t_2^-u^2_{\lambda,\mu},t_2^-v^2_{\lambda,\mu})\in\mathcal{N}^-_{\lambda,\mu}$. Since $(u_n,v_n)\in\mathcal{N}^-_{\lambda,\mu}$, $I_{\lambda,\mu}(u_n,v_n)\geq I_{\lambda,\mu}(tu_n,tv_n)$ for all $t\geq 0$, we have

$$\begin{aligned}c^-_{\lambda,\mu}&\leq I_{\lambda,\mu}(t^-u^2_{\lambda,\mu},t^-v^2_{\lambda,\mu})<\lim_{n\to\infty}I_{\lambda,\mu}(t^-u_n,t^-v_n)\\&\leq\lim_{n\to\infty}I_{\lambda,\mu}(u_n,v_n)=c^-_{\lambda,\mu}.\end{aligned}$$

Hence, $(u_n,v_n)\to(u^2_{\lambda,\mu},v^2_{\lambda,\mu})$ strongly in $\mathcal{H}$. This implies

$$I_{\lambda,\mu}(u^2_{\lambda,\mu},v^2_{\lambda,\mu})=\lim_{n\to\infty}I_{\lambda,\mu}(u_n,v_n)=c^-_{\lambda,\mu}.$$

By using the same arguments as in the proof of Theorem 5.1 for all $(\lambda,\mu)\in\mathscr{C}_{\Lambda_2}$, we have that $(u^2_{\lambda,\mu},v^2_{\lambda,\mu})$ is a positive solution to system (1.1).

Finally, we show that $(u^2_{\lambda,\mu},v^2_{\lambda,\mu})$ is not semi-trivial. Using Theorem 3.1 (ii), we obtain

$$I_{\lambda,\mu}(u^2_{\lambda,\mu},v^2_{\lambda,\mu})=c^-_{\lambda,\mu}>0. \tag{5.19}$$

If $(u^2_{\lambda,\mu},v^2_{\lambda,\mu})$ is a semi-trivial solution to (1.1), then $(u^2_{\lambda,\mu},0)$ (or $(0,v^2_{\lambda,\mu})$) is nontrivial solution to the following equation

$$-\Delta_{\mathbb{G}}u=\lambda f(\xi)\frac{\psi^\nu|u|^{q-2}u}{d(\xi)^\nu}\ \text{ in }\Omega\ \text{ and }\ u=0\text{ on }\partial\Omega.$$

Then

$$\begin{aligned}I_{\lambda,\mu}(u^2_{\lambda,\mu},0)&=\frac{1}{2}\|u^2_{\lambda,\mu}\|^2_{S^1_0(\Omega)}-\frac{\lambda}{q}\int_\Omega f(\xi)\frac{\psi^\nu|u^2_{\lambda,\mu}|^q}{d(\xi)^\nu}d\xi\\&=-\frac{2-q}{2q}\|u^2_{\lambda,\mu}\|^2_{S^1_0(\Omega)}<0.\end{aligned} \tag{5.20}$$

From (5.19) and (5.20), we obtain that $(u^2_{\lambda,\mu},v^2_{\lambda,\mu})$ is not semi-trivial. This completes the proof of Theorem 5.2. □

Proof of Theorems 1.1 and 1.2. Theorems 1.1 and 1.2 follow from Theorems 5.1 and 5.2, respectively. Also from Theorem 5.1 and 5.2, we obtain that, for all $1<q<2$, $0\leq\alpha<2$, $0\leq\beta<2$, $2<p_1+p_2<2^*(\alpha)$, and $\lambda,\mu>0$ with $(\lambda,\mu)\in\mathscr{C}_{\Lambda_2}$ (where $\Lambda_2<\Lambda_1$), system (1.1) has two positive solutions $(u^1_{\lambda,\mu},v^1_{\lambda,\mu})\in\mathcal{N}^+_{\lambda,\mu}$ and $(u^2_{\lambda,\mu},v^2_{\lambda,\mu})\in\mathcal{N}^-_{\lambda,\mu}$. Since $\mathcal{N}^+_{\lambda,\mu}\cap\mathcal{N}^-_{\lambda,\mu}=\emptyset$, we can conclude that $(u^1_{\lambda,\mu},v^1_{\lambda,\mu})$ and $(u^2_{\lambda,\mu},v^2_{\lambda,\mu})$ are distinct. □

6. Proof of Theorem 1.3

In this section, we show the existence of a second weak solution in the critical case $p_1+p_2=2^*(\alpha)$ as a limit of Palais-Smale sequence, which is obtained by minimizing sequence for $I_{\lambda,\mu}$ in $\mathcal{N}^-_{\lambda,\mu}$.

Lemma 6.1. *Under the assumptions of Theorem 1.3, there exist a nonnegative function* $(u_{\lambda,\mu},v_{\lambda,\mu})\in\mathscr{H}\setminus\{(0,0)\}$ *and* $\Lambda_*>0$ *such that*

$$\sup_{t\ge 0} I_{\lambda,\mu}(tu_{\lambda,\mu},tv_{\lambda,\mu})<c_\infty \tag{6.1}$$

for all $\lambda,\ \mu>0$ *with* $(\lambda,\mu)\in\mathscr{C}_{\Lambda_*}$*, where* c_∞ *is the constant given in Lemma 4.3. In particular,* $c^-_{\lambda,\mu}<c_\infty$ *for all* $(\lambda,\mu)\in\mathscr{C}_{\Lambda_*}$.

Proof. We first consider the functional $J:\mathscr{H}\to\mathbb{R}$ given by

$$J(u,v)=\frac{1}{2}\|(u,v)\|^2-\frac{1}{2^*(\alpha)}\int_\Omega\frac{\psi^\alpha|u|^{p_1}|v|^{p_2}}{d(\xi)^\alpha}d\xi,\quad \forall(u,v)\in\mathscr{H}.$$

Let u_ε be given in (2.3), and let $(\overline{u}_\varepsilon,\overline{v}_\varepsilon)=(\sqrt{p_1}u_\varepsilon,\sqrt{p_2}u_\varepsilon)$. Then

$$\begin{aligned}J(t\overline{u}_\varepsilon,t\overline{v}_\varepsilon)&=\frac{2^*(\alpha)}{2}t^2\|\overline{u}_\varepsilon\|^2_{S^1_0(\Omega)}-\frac{p_1^{\frac{p_1}{2}}p_2^{\frac{p_2}{2}}}{2^*(\alpha)}t^{2^*(\alpha)}\int_\Omega\frac{\psi^\alpha|u_\varepsilon|^{2^*(\alpha)}}{d(\xi)^\alpha}d\xi\\&=\frac{2^*(\alpha)}{2}\Big(S(\alpha,Q)^{\frac{Q-\alpha}{2-\alpha}}+\varepsilon^{Q-2}\Big)t^2\le Ct^2,\end{aligned}$$

where $C>0$. So, there exist $t_0\in(0,1)$ and $\Lambda_3>0$ such that

$$\sup_{t\in[0,t_0]} I_{\lambda,\mu}(t\overline{u}_\varepsilon,t\overline{v}_\varepsilon)<c_\infty\ \text{ for all }(\lambda,\mu)\in\mathscr{C}_{\Lambda_3}. \tag{6.2}$$

Next, we prove that $\sup_{t\in[t_0,\infty]}I_{\lambda,\mu}(t\overline{u}_\varepsilon,t\overline{v}_\varepsilon)<c_\infty$. Let $\theta(t)=J(t\overline{u}_\varepsilon,t\overline{v}_\varepsilon)$. Then, $\theta(0)=0$, $\theta(t)>0$ for $t>0$ small, $\theta(t)<0$ for $t>0$ large and θ attains its maximum at

$$t_0=\left(\frac{\|(\overline{u}_\varepsilon,\overline{v}_\varepsilon)\|^2}{\int_\Omega\frac{\psi^\alpha|\overline{u}_\varepsilon|^{p_1}|\overline{v}_\varepsilon|^{p_2}}{d(\xi)^\alpha}d\xi}\right)^{\frac{1}{2^*(\alpha)-2}}.$$

Then, using (2.4), (2.5), and (2.15), we have

$$\begin{aligned}\sup_{t\ge0}\theta(t)=\theta(t_0)&=\left(\frac{1}{2}-\frac{1}{2^*(\alpha)}\right)\frac{\|(\overline{u}_\varepsilon,\overline{v}_\varepsilon)\|^{\frac{2\cdot2^*(\alpha)}{2^*(\alpha)-2}}}{\left(\int_\Omega\frac{\psi^\alpha|\overline{u}_\varepsilon|^{p_1}|\overline{v}_\varepsilon|^{p_2}}{d(\xi)^\alpha}d\xi\right)^{\frac{2}{2^*(\alpha)-2}}}\\&=\frac{2-\alpha}{2(Q-\alpha)}\left[\left(\frac{p_1}{p_2}\right)^{\frac{p_2}{2^*(\alpha)}}+\left(\frac{p_2}{p_1}\right)^{\frac{p_1}{2^*(\alpha)}}\right]^{\frac{2^*(\alpha)}{2^*(\alpha)-2}}\left[\frac{\|u_\varepsilon\|^2_{S^1_0(\Omega)}}{(\int_\Omega\frac{\psi^\alpha|u_\varepsilon|^{2^*(\alpha)}}{d(\xi)^\alpha}d\xi)^{\frac{2}{2^*(\alpha)}}}\right]^{\frac{2^*(\alpha)}{2^*(\alpha)-2}}\\&=\frac{2-\alpha}{2(Q-\alpha)}\left[\left(\frac{p_1}{p_2}\right)^{\frac{p_2}{2^*(\alpha)}}+\left(\frac{p_2}{p_1}\right)^{\frac{p_1}{2^*(\alpha)}}\right]^{\frac{Q-\alpha}{2-\alpha}}\left[\frac{S(\alpha,Q)^{\frac{Q-\alpha}{2-\alpha}}+O(\varepsilon^{Q-2})}{[S(\alpha,Q)^{\frac{Q-\alpha}{2-\alpha}}+O(\varepsilon^{Q-\alpha})]^{\frac{2}{2^*(\alpha)}}}\right]^{\frac{Q-\alpha}{2-\alpha}}\\&\le\frac{2-\alpha}{2(Q-\alpha)}\cdot(S_{\alpha,p_1,p_2})^{\frac{Q-\alpha}{2-\alpha}}+O(\varepsilon^{Q-2}).\end{aligned}\tag{6.3}$$

In addition, it follows from $f(\xi)\geq a_0$, $g(\xi)\geq b_0$ for all $\xi\in B_d(0,2r_0)\subset\Omega$ and $p_1,\,p_2>1$ that

$$\begin{aligned}Q_{\lambda,\mu}(\overline{u}_\varepsilon,\overline{v}_\varepsilon)&=\lambda p_1^{\frac{q}{2}}\int_\Omega f(\xi)\frac{\psi^\beta|\overline{u}_\varepsilon|^q}{d(\xi)^\beta}d\xi+\mu p_2^{\frac{q}{2}}\int_\Omega g(\xi)\frac{\psi^\beta|\overline{u}_\varepsilon|^q}{d(\xi)^\beta}d\xi\\&\geq(a_0\lambda p_1^{\frac{q}{2}}+b_0\mu p_2^{\frac{q}{2}})\int_{B_d(0,2r_0)}\frac{\psi^\beta|u_\varepsilon|^q}{d(\xi)^\beta}d\xi\\&\geq\min\{a_0,b_0\}(\lambda+\mu)\int_{B_d(0,2r_0)}\frac{\psi^\beta|u_\varepsilon|^q}{d(\xi)^\beta}d\xi.\end{aligned}\tag{6.4}$$

Combining (6.3) and (6.4), we have

$$\begin{aligned}\sup_{t\geq t_0}I_{\lambda,\mu}(t\overline{u}_\varepsilon,t\overline{v}_\varepsilon)&=\sup_{t\geq t_0}\Big(\theta(t)-\frac{1}{q}Q_{\lambda,\mu}(t\overline{u}_\varepsilon,t\overline{v}_\varepsilon)\Big)\\&\leq\frac{2-\alpha}{2(Q-\alpha)}(h(0))^{-\frac{Q-2}{2-\alpha}}(S_{\alpha,p_1,p_2})^{\frac{Q-\alpha}{2-\alpha}}+O(\varepsilon^{Q-2})\\&\quad-\frac{1}{q}t_0^q\min\{a_0,b_0\}(\lambda+\mu)\int_\Omega\frac{\psi^\beta|u_\varepsilon|^q}{d(\xi)^\beta}d\xi\\&\leq\frac{2-\alpha}{2(Q-\alpha)}(h(0))^{-\frac{Q-2}{2-\alpha}}(S_{\alpha,p_1,p_2})^{\frac{Q-\alpha}{2-\alpha}}+O(\varepsilon^{Q-2})\\&\quad-\frac{\min\{a_0,b_0\}t_0^q}{q}(\lambda+\mu)\begin{cases}C\varepsilon^{Q-\beta-\frac{(Q-2)q}{2}}&\text{if } q>\frac{Q-\beta}{Q-2}\\C\varepsilon^{Q-\beta-\frac{(Q-2)q}{2}}|\ln\varepsilon|&\text{if } q=\frac{Q-\beta}{Q-2}\\C\varepsilon^{\frac{(Q-2)q}{2}}&\text{if } q<\frac{Q-\beta}{Q-2},\end{cases}\end{aligned}\tag{6.5}$$

Now, we need to consider two cases:

Case (i) $1\leq q<\frac{Q-\beta}{Q-2}$. It follows from $q<2$ that $Q-2>\frac{q(Q-2)}{2}$. Then, choosing ε small enough, we can deduce that there exists a $\Lambda_4>0$ such that

$$\begin{aligned}O(\varepsilon^{Q-2})&-\frac{\min\{a_0,b_0\}t_0^q}{q}(\lambda+\mu)\varepsilon^{\frac{q(Q-2)}{2}}\\&<-C_*\Big[\Big(\lambda|f|_{L^{q_*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\Big)^{\frac{q}{2-q}}+\Big(\mu|g|_{L^{q_*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\Big)^{\frac{q}{2-q}}\Big]\end{aligned}\tag{6.6}$$

for all $(\lambda,\mu)\in\mathscr{C}_{\Lambda_4}$. Setting $\Lambda_5=\min\{\Lambda_3,\Lambda_4\}$, we see that (6.2), (6.5), and (6.6) yield that

$$\sup_{t\geq0}I_{\lambda,\mu}(tu_0,tv_0)<c_\infty\ \text{for all } (\lambda,\mu)\in\mathscr{C}_{\Lambda_5}.$$

Case (ii) $\frac{Q-\beta}{Q-2}\leq q<2$. It follows from $\frac{Q-\beta}{Q-2}\leq q$ that $Q-2>q\frac{Q-2}{2}\geq Q-\beta-\frac{q(Q-2)}{2}$. Then, for ε small enough, there exists a $\Lambda_6>0$ such that

$$\begin{aligned}O(\varepsilon^{Q-2})&-\frac{\min\{a_0,b_0\}t_0^q}{q}(\lambda+\mu)\varepsilon^{Q-\beta-q\frac{Q-2}{2}}\\&<-C_*\Big[\Big(\lambda|f|_{L^{q_*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\Big)^{\frac{q}{2-q}}+\Big(\mu|g|_{L^{q_*}(\Omega,\frac{\psi^\beta}{d(\xi)^\beta}d\xi)}\Big)^{\frac{q}{2-q}}\Big]\end{aligned}\tag{6.7}$$

for all $(\lambda,\mu) \in \mathscr{C}_{\Lambda_6}$. Therefore, taking $\Lambda_7 = \min\{\Lambda_3, \Lambda_6\}$ and following (6.2), (6.5), and (6.7) one can show that $\sup_{t\geq 0} I_{\lambda,\mu}(tu_0, tv_0) < c_\infty$ for all $(\lambda,\mu) \in \mathscr{C}_{\Lambda_7}$.

Set $\Lambda_* = \min\{\Lambda_2, \Lambda_5, \Lambda_7\}$. From cases (i) and (ii), we see that (6.1) holds by taking $(\bar{u}_\varepsilon, \bar{v}_\varepsilon) = (\sqrt{p_1}\,u_\varepsilon, \sqrt{p_2}\,u_\varepsilon)$. Moreover, from Lemma 3.5, the definition of $c^-_{\lambda,\mu}$ and (6.1), for all $(\lambda,\mu) \in \mathscr{D}_{\Lambda_*}$, we obtain that there exists $t^- > 0$ such that $(t^- u_0, t^- v_0) \in \mathscr{N}^-_{\lambda,\mu}$ and

$$c^-_{\lambda,\mu} \leq I_{\lambda,\mu}(t^- \bar{u}_\varepsilon, t^- \bar{v}_\varepsilon) \leq \sup_{t\geq 0} I_{\lambda,\mu}(t\bar{u}_\varepsilon, t\bar{v}_\varepsilon)) < c_\infty.$$

The proof is thus complete. □

Theorem 6.1. *Under the assumptions of Theorem 1.3, the functional $I_{\lambda,\mu}$ satisfies the $(PS)_{c^-_{\lambda,\mu}}$ condition for all $(\lambda,\mu) \in \mathscr{C}_{\Lambda_*}$. Moreover, system (1.12) has at least one positive vector solution $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ in $\mathscr{N}^-_{\lambda,\mu}$ such that $I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = c^-_{\lambda,\mu} > 0$ and $u^2_{\lambda,\mu} \neq 0$, $v^2_{\lambda,\mu} \neq 0$.*

Proof. By Lemma 5.1 (ii), for $(\lambda,\mu) \in \mathscr{C}_{\Lambda_*}$, there exists a $(PS)_{c^-_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset \mathscr{N}^-_{\lambda,\mu}$ for $I_{\lambda,\mu}$. From Lemma 4.2, we find that $\{(u_n, v_n)\}$ is bounded in $\mathscr{H}$. Using Lemmas 6.1 and 4.3, we have that $I_{\lambda,\mu}$ satisfies the $(PS)_{c^-_{\lambda,\mu}}$-condition. Then, there exists $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathscr{H}$ such that, up to subsequence, $(u_n, v_n) \to (u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ in $\mathscr{H}$. Moreover, $I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = c^-_{\lambda,\mu} > 0$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathscr{N}^-_{\lambda,\mu}$, since $\mathscr{N}^-_{\lambda,\mu}$ is a closed set. Using the argument as in Theorem 5.1, one can easily obtain that $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is a positive solution to system (1.12) for $(\lambda,\mu) \in \mathscr{C}_{\Lambda_*}$. Finally, by using the same arguments as in the proof of Theorem 5.2, for $p_1 + p_2 = 2^*(\alpha)$, we have that $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is not semi-trivial solution to system (1.12). The proof is completed. □

Proof of Theorem 1.3. It follows from Theorems 5.1 and 6.1 that there exist two positive vector solution $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ such that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathscr{N}^+_{\lambda,\mu}$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathscr{N}^-_{\lambda,\mu}$. In addition, we have $\mathscr{N}^+_{\lambda,\mu} \cap \mathscr{N}^-_{\lambda,\mu} = \emptyset$. Thus, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ are two distinct positive solutions to (1.12). □

REFERENCES

[1] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, Stratified Lie Groups and Potential Theory for their Sub-Laplacians, Springer, Berlin, Heidelberg, 2007.

[2] D. Danielli, N. Garofalo, N. C. Phuc, Hardy-Sobolev type inequalities with sharp constants in Carnot-Carathéodory spaces, Potential Anal. 34 (2011), 223-242.

[3] F. Ferrari, B. Franchi, Harnack inequality for fractional sub-Laplacians in Carnot groups, Math. Zeitschrift 279 (2015), 435-458.

[4] N. Garofalo, E. Lanconelli, Existence and nonexistence results for semilinear equations on the Heisenberg group, Indiana Univ. Math. J. 41 (1992), 71-98.

[5] N. Garofalo, D. Vassilev, Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups, Math. Ann. 318 (2000), 453-516.

[6] A. Loiudice, Semilinear subelliptic problems with critical growth on Carnot groups, Manuscript Math. 124 (2007), 247-259.

[7] A. Loiudice, Local behavior of solutions to sunelliptic problems with Hardy potential on Carnot groups, Mediterr. J. Math. 15 (2018), 81.

[8] A. Loiudice, Critical growth problems with singular nonlinearities on Carnot groups, Nonlinear Anal. 126 (2015), 415-436.

[9] A. Loiudice, Critical problems with Hardy potential on Stratified Lie group, Adv. Differential Equations 28 (2023), 1-33.

[10] M. Ruzhansky, D. Suragan, Hardy Inequalities on Homogeneous Groups, Birkhäuser, Cham, 2019.
[11] J. Zhang, On the existence and multiplicity of solutions for a class of sub-Laplacian problems involving critical Sobolev-Hardy exponents on Carnot groups, Appl. Anal. 102 (2023), 4209-4229.
[12] J. Zhang, Existence and multiplicity of positive solutions to sub-elliptic systems with multiple critical exponents on Carnot groups, Proceedings-Mathematical Sciences 133 (2023), 10.
[13] J. Zhang, Sub-elliptic problems with multiple critical Sobolev-Hardy exponents on Carnot groups, Manuscripta Math. 172 (2023), 1-29.
[14] J. Zhang, D. Yang, Fractional p-Sub-Laplacian operator problem with concave-convex nonlinearities on Homogeneous groups, Electronic Research Archive 29 (2021), 3243-3260.
[15] A. Bonfiglioli, F. Uguzzoni, Nonlinear Liouville theorems for some critical problems on H-type groups, J. Functional Anal. 207 (2004), 161-215.
[16] G. B. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups, Ark. Mat. 13 (1975), 161-207.
[17] G. B. Folland, E. M. Stein, Hardy Spaces on Homogeneous Groups. Volume 28 of Mathematical Notes, Princeton University Press, Princeton, NJ, 1982.
[18] Y. Han, P. Niu, Hardy-Sobolev type inequalities on the H-type group, Manuscripta Math. 118 (2005), 235-252.
[19] A. Loiudice, Optimal decay of p-Sobolev extremals on Carnot groups, J. Math. Anal. Appl. 470 (2019), 619-631.
[20] A. Loiudice, A multiplicity result for a non-homogeneous subelliptic problem with Sobolev exponent. In: Georgiev, V., Ozawa, T., Ruzhansky, M., Wirth, J. (eds) Advances in Harmonic Analysis and Partial Differential Equations. Trends in Mathematics, pp. 99-120, Birkhäuser, Cham. 2020.
[21] J. Zhang, S. Zhu, On criticality coupled sub-Laplacian systems with Hardy type potentials on stratified Lie groups, Comm. Anal. Mech. 15 (2023), 70-90.
[22] S. Bordoni, R. Filippucci, P. Pucci, Existence problems on Heisenberg groups involving Hardy and critical terms, J. Geom. Anal. 30 (2020), 1887-1917.
[23] C. O. Alves, D. C. de Morais, M. A. S. Souto, On systems of elliptic equations involving subcritical or critical Sobolev exponents, Nonlinear Anal. 42 (2000), 771-787.
[24] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (1982), 486-490.
[25] J. M. Bony, Principle du maximum, inégalité de Harnack et unicité du probléme de Cauchy pour les opérateurs elliptiques dégénérés, Ann. Inst. Fourier Grenobles 19 (1969), 277-304.