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EXISTENCE OF MULTI-BUMP SOLUTIONS FOR A NONLINEAR KIRCHHOFF EQUATION

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Abstract. We consider the following Kirchhoff problem

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+(1+\varepsilon V(x))u=|u|^{p-2}u,$$

where $a,b>0$, and $2<p<6$. Under suitable assumptions on V, by using the Lyapunov-Schmidt reduction method, we obtain the existence of multi-bump solutions.

Keywords. Kirchhoff equation; Lyapunov-Schmidt reduction method; Multi-bump solutions.

1. Introduction and Main Results

In this paper, we are concerned with the existence of multi-bump solutions for the following nonlinear Kirchhoff equation:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+(1+\varepsilon V(x)u=|u|^{p-2}u, x\in\mathbb{R}^3, \tag{1.1}$$

where $a,b>0, 2<p<6$, and $V(x)\in C\left(\mathbb{R}^3,\mathbb{R}^+\right)$. Eq. (1.1) is related to the stationary solutions of equation, which was derived from the classical D'Alembert wave equation obtained by Kirchhoff [14] in 1877 when considering the changes in the length of the string during vibrations,

$$u_{tt}-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\right)\Delta u=f(x,u), \tag{1.2}$$

where $f(x,u)$ is a general nonlinearity, and u describes a process, which depends on the average of itself. It is worth pointing out in [1] that Eq. (1.2) models several physical systems. For more physical backgrounds, we refer the readers to [3] and the references therein.

Owing to the appearance of the terms $\left(\int|\nabla u|^2dx\right)\Delta u$, problem (1.1) is nonlocal. Consequently, (1.1) is no longer a pointwise identity. This leads to some mathematical difficulties and makes studying such problems more interesting. After the pioneering work of [19], it has received much attention. The existence and qualitative properties of solutions for (1.1)

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have been studied a lot; see [8, 10, 11, 24, 27] for the existence of ground state solutions and [6, 7, 9, 21, 22, 25, 26, 28] for the existence of sign-changing solutions.

Now, the construction of specific forms of multi-bump solutions to Kirchhoff problem (1.1) is under the spotlight. In contrast with the single Schrödinger problem, the Kirchhoff problem contains the non-local term. Hence, we have to prove some new estimates. In 2020, Li et al. [15] focused on the problem

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+u=u^p,\quad u>0,\quad \text{in } \mathbb{R}^3 \tag{1.3}$$

for $1<p<5$. They first established a uniqueness and non-degeneracy result of positive solutions to (1.3), and they proved the existence of positive single-peak solutions to the related perturbed problem

$$-\left(\varepsilon^2 a+\varepsilon b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+V(x)u=u^p,\quad u>0,\quad \text{in}\quad \mathbb{R}^3. \tag{1.4}$$

In [23], Luo, Peng, Wang, and Xiang proved the existence of positive multi-peak positive solutions of (1.4) when $V(x)$ satisfies some suitable assumptions. In [13], Hu and Shuai also obtained multiple positive solutions to this type of perturbation problem with general nonlinearity under some precise hypotheses. Recently, Liu [20] investigated the existence of multi-bump solutions for the following Kirchhoff equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+u=(1-\varepsilon q(x))|u|^{p-2}u, x\in\mathbb{R}^3,$$

where $a,b>0, 2<p<6$, and $q(x)\in C\left(\mathbb{R}^3,\mathbb{R}^+\right)$ satisfies some suitable conditions. By using the Lyapunov-Schmidt reduction method, he extended the results in [17] to the Kirchhoff problem.

Motivated by [16, 18, 20], the present paper is devoted to the existence of multi-bump solutions to Kirchhoff problem (1.1). We use the positive radical solution W_k of

$$-\left(a+kb\int_{\mathbb{R}^3}|\nabla w|^2\right)\Delta w+w=w^{p-1},\quad \text{in } \mathbb{R}^3$$

as the building block of our approximate solutions. From [15], we have the following results about W_k.

Let u be the unique radical ground state to the equation: $-a\Delta u+u=u^{p-1}$. Then, $W_k(x)=u\left(\frac{x}{\mu_k}\right)$, where μ_k is the positive root to equation $\mu^2-kb|\nabla u|_2^2\mu-a=0$. There exists $C_1,C_2>0$ such that

$$\lim_{|x|\to+\infty} D^i W_k(x)|x|e^{\frac{|x|}{\mu_k}}=C_i\mu_k^{1-i}, i=0,1.$$

Moreover, W_k is nondegenerate in $H^1\left(\mathbb{R}^3\right)$ in the sense that there holds

$$\ker L=\operatorname{span}\left\{\partial_{x_1}W_k,\partial_{x_2}W_k,\partial_{x_3}W_k\right\},$$

where L is defined as

$$L\varphi=-\left(a+kb\int_{\mathbb{R}^3}|\nabla W_k|^2\right)\Delta\varphi+\varphi-pW_k^{p-1}\varphi-2kb\left(\int_{\mathbb{R}^3}\nabla W_k\cdot\nabla\varphi\right)\Delta W_k,$$

acting on $L^2\left(\mathbb{R}^3\right)$ with domain $H^1\left(\mathbb{R}^3\right)$.

In the Hilbert space $H^1(\mathbb{R}^3)$, we use the following inner space

$$(u,v)_\varepsilon := \int_{\mathbb{R}^3} a\nabla u\cdot\nabla v + (1+\varepsilon V(x))uv$$

and the induced norm $\|u\|_\varepsilon := \sqrt{(u,u)_\varepsilon}$. The usual inner in $H^1(\mathbb{R}^3)$ is denoted by $(u,v) := \int_{\mathbb{R}^3} a\nabla u\cdot\nabla v + uv$ and the corresponding norm is $\|u\| := \sqrt{(u,u)}$. Let

$$I_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p.$$

Then I_ε is well defined in $H^1(\mathbb{R}^3)$ and belongs to C^1 class.

In order to state our main results, we assume that the potential $V(x)$ satisfies the following restrictions:

(V_1) : $V(x)\in C(\mathbb{R}^3,\mathbb{R}^+)$ and $\lim_{|x|\to\infty} V(x) = 0$.

(V_2) : $\lim_{|x|\to\infty} \frac{\ln V(x)}{|x|} = 0$.

Now we have following theorem.

Theorem 1.1. *Let (V_1) and (V_2) hold. Then, for any positive integer k, there exists $\varepsilon(k) > 0$ such that, for $0<\varepsilon<\varepsilon(k)$, Eq. (1.1) has a k-bump positive solution.*

Our paper is organized as follows. In Section 2, we carry out the reduction procedure. In Section 3, the last section, we construct the multi-bump solution to (1.1).

Notation. In this paper, we make use of the following notations.

- For any $R>0$ and $x\in\mathbb{R}^3$, $B_R(x)$ denotes the open ball of radius R centered at x.
- The letter C and C_i stand for positive constants (possibly different from line to line).
- $|u|_q = (\int_{\mathbb{R}^3}|u|^q dx)^{\frac{1}{q}}$ denotes the norm of u in $L^q(\mathbb{R}^3)$ for $2\le q\le 6$.
- $\int_{\mathbb{R}^3} f$ means the Lebesgue integral of $f(x)$ in $\mathbb{R}^3$.
- The ordinary inner product between two vectors $a,b\in\mathbb{R}^3$ is be denoted by $a\cdot b$.

2. PRELIMINARIES

For $\lambda>0$ and $k\ge 2$, define

$$\Omega_\lambda = \left\{(y_1,\ldots,y_k)\in\left(\mathbb{R}^3\right)^k, \left|y_i-y_j\right|>\lambda \text{ for } i\neq j\right\},$$

and $\Omega_\lambda = \mathbb{R}^3$ for $k=1$. For $y=(y_1,\ldots,y_k)\in\Omega_\lambda$, denote $W_y(x)=\sum_{i=1}^k W_{k,y_i}$, where $W_{k,y_i} = W_k(x-y_i)$. Let $y\in\Omega_\lambda$, and define

$$\mathscr{H}_y = \left\{\varphi\in H^1\left(\mathbb{R}^3\right) \mid \int_{\mathbb{R}^3} W_{k,y_j}^{p-2}\frac{\partial W_{k,y_j}}{\partial x_\alpha}\varphi = 0, \alpha=1,2,3; j=1,2,\ldots,k\right\}.$$

Let $J(\varphi) = I_\varepsilon(W_y+\varphi), \varphi\in\mathscr{H}_y$. We expand $J(\varphi)$ as follows:

$$J(\varphi) =: J(0) + l_y(\varphi) + \frac{1}{2}\langle L_y\varphi,\varphi\rangle - R_y(\varphi),\ \varphi\in\mathscr{H}_y,$$

where $J(0) = I_\varepsilon(W_y)$ and l_y, L_y, and R_y are defined for $\varphi,\psi\in\mathscr{H}_y$ as follows:

$$l_y(\varphi) = (W_y,\varphi)_\varepsilon + b\int_{\mathbb{R}^3}\left|\nabla W_y\right|^2\int_{\mathbb{R}^3}\nabla W_y\cdot\nabla\varphi - \int_{\mathbb{R}^3} W_y^{p-1}\varphi,$$

and L_y is a bounded linear operator from $\mathscr{H}_y$ to $\mathscr{H}_y$ defined by

$$\begin{aligned}\left\langle L_y\varphi,\psi\right\rangle =&(\varphi,\psi)_\varepsilon+2b\left(\int_{\mathbb{R}^3}\nabla W_y\cdot\nabla\varphi\right)\left(\int_{\mathbb{R}^3}\nabla W_y\cdot\nabla\psi\right)\\&+b\int_{\mathbb{R}^3}\left|\nabla W_y\right|^2\int_{\mathbb{R}^3}\nabla\varphi\cdot\nabla\psi-(p-1)\int_{\mathbb{R}^3}W_y^{p-2}\varphi\psi,\end{aligned}$$

and

$$\begin{aligned}R_y(\varphi)=&\frac{1}{p}\int_{\mathbb{R}^3}\left(\left|W_y+\varphi\right|^p-W_y^p-W_y^{p-1}\varphi-\frac{p(p-1)}{2}W_y^{p-2}\varphi^2\right)\\&-\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla\varphi|^2\right)^2-b\int_{\mathbb{R}^3}|\nabla\varphi|^2\int_{\mathbb{R}^3}\nabla W_y\cdot\nabla\varphi.\end{aligned}$$

Now, we demonstrate that L_y is invertible in $\mathscr{H}_y$.

Lemma 2.1. *There are constants $\lambda_0>0,\varepsilon_0>0$, and $C_0>0$ such that, for any $\lambda>\lambda_0, 0<\varepsilon<\varepsilon_0, y\in\Omega_\lambda$, and $\varphi\in\mathscr{H}_y$, $\left\|L_y\varphi\right\|_\varepsilon\geq C_0\|\varphi\|_\varepsilon$.*

Proof. We make a contradiction argument. Assume that there exist $\varepsilon_n\to0$, $\left\{y_{l,n}\right\}_{n=1}^\infty\subset\mathbb{R}^3, l=1,\ldots,k$, with $\left|y_{j,n}-y_{l,n}\right|\to\infty(j\neq l)$, and $\varphi_n\in\mathscr{H}_{y_n}$ with $\|\varphi_n\|_{\varepsilon_n}=1$ such that

$$\left\|L_{y_n}\varphi_n\right\|_{\varepsilon_n}=o(1)\left\|\varphi_n\right\|_{\varepsilon_n}=o(1),$$

where $y_n=\left(y_{1,n},\ldots,y_{k,n}\right)$. Up to a subsequence, we may assume that $\varphi_n\left(\cdot+y_{j,n}\right)\rightharpoonup\varphi_j^*$ in $H^1\left(\mathbb{R}^3\right), j=1,2,\ldots,k$, as $n\to\infty$ and $\varphi_n\left(\cdot+y_{j,n}\right)\to\varphi_j^*$ strongly in $L_{loc}^2\left(\mathbb{R}^3\right), j=1,2,\ldots,k$, as $n\to\infty$. From

$$\int_{\mathbb{R}^3}W_{k,y_{j,n}}^{p-2}\frac{\partial W_{k,y_{j,n}}}{\partial x_\alpha}\varphi_n=0,\ \alpha=1,2,3;j=1,2,\ldots,k,$$

we obtain

$$\int_{\mathbb{R}^3}W_k^{p-2}\frac{\partial W_k}{\partial x_\alpha}\varphi_n(x+y_{j,n})=0,\ \alpha=1,2,3;j=1,2,\ldots,k.$$

Thus φ_j^* satisfies

$$\int_{\mathbb{R}^3}W_k^{p-2}\frac{\partial W_k}{\partial x_\alpha}\varphi_j^*=0,\ \alpha=1,2,3;j=1,2,\ldots,k. \tag{2.1}$$

Define

$$\tilde{H}=\left\{\phi:\phi\in H^1\left(\mathbb{R}^3\right),\int_{\mathbb{R}^3}W_k^{p-2}\frac{\partial W_k}{\partial x_\alpha}\phi=0,\alpha=1,2,3\right\}.$$

Note that

$$\begin{aligned}o(1)\|\phi\|=&\left\langle L_{y_n}\varphi_n,\phi\right\rangle\\=&\int_{\mathbb{R}^3}\left(a\nabla\varphi_n\cdot\nabla\phi+(1+\varepsilon_nV(x))\varphi_n\phi\right)+2b\int_{\mathbb{R}^3}\nabla W_{y_n}\cdot\nabla\varphi_n\int_{\mathbb{R}^3}\nabla W_{y_n}\cdot\nabla\phi\\&+b\int_{\mathbb{R}^3}\left|\nabla W_{y_n}\right|^2\int_{\mathbb{R}^3}\nabla\varphi_n\cdot\nabla\phi-(p-1)\int_{\mathbb{R}^3}W_{y_n}^{p-2}\varphi_n\phi.\end{aligned} \tag{2.2}$$

Let $\phi\in C_0^\infty\left(\mathbb{R}^3\right)\cap\tilde{H}$. Then $\phi_n(x)=:\phi\left(x-y_{j,n}\right)\in C_0^\infty\left(\mathbb{R}^3\right)$. Inserting $\phi_n(x)$ into (2.2) and letting $n\to\infty$, we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(a\nabla\varphi_j^*\cdot\nabla\phi+\varphi_j^*\phi\right)+2kb\int_{\mathbb{R}^3}\nabla W_k\cdot\nabla\varphi_j^*\int_{\mathbb{R}^3}\nabla W_k\cdot\nabla\phi\\&\quad+kb\int_{\mathbb{R}^3}|\nabla W_k|^2\int_{\mathbb{R}^3}\nabla\varphi_j^*\cdot\nabla\phi-(p-1)\int_{\mathbb{R}^3}W_k^{p-2}\varphi_j^*\phi=0.\end{aligned} \tag{2.3}$$

By the density of $C_0^\infty\left(\mathbb{R}^3\right)$ in $H^1\left(\mathbb{R}^3\right)$, we see that (2.3) also holds for any $\phi \in \tilde{H}$.

On the other hand, (2.3) is true for $\frac{\partial W_k}{\partial x_\alpha}$, $\alpha = 1,2,3$. Thus (2.3) is true for any $\varphi \in H^1\left(\mathbb{R}^3\right)$. Since W_k is non-degenerate, we can obtain

$$\varphi_j^* = \sum_{\alpha=1}^{3} c_{\alpha,j}\frac{\partial W_k}{\partial x_\alpha},\ j = 1,2,\ldots,k.$$

It follows from (2.1) that $c_{\alpha,j} = 0, \alpha = 1,2,3; j = 1,2,\ldots,k$. Consequently, $\varphi_j^* = 0,\ j = 1,2,\ldots,k$. Therefore, for any $R > 0$, $\int_{B_R(0)} \varphi_n(x+y_{j,n})^2 = o(1)$. It follows that

$$\begin{aligned}
o(1) =& o(1)\|\varphi_n\|_{\varepsilon_n} = \left\langle L_{y_n}\varphi_n, \varphi_n\right\rangle \\
=& \|\varphi_n\|_{\varepsilon_n}^2 + 2b\left(\int_{\mathbb{R}^3} \nabla W_{y_n}\cdot\nabla\varphi_n\right)^2 + b\int_{\mathbb{R}^3}\left|\nabla W_{y_n}\right|^2\int_{\mathbb{R}^3}|\nabla\varphi_n|^2 - (p-1)\int_{\mathbb{R}^3} W_{y_n}^{p-2}\varphi_n^2 \\
\geq& \|\varphi_n\|_{\varepsilon_n}^2 - (p-1)\int_{\mathbb{R}^3} W_{y_n}^{p-2}\varphi_n^2 \\
\geq& 1 - Ce^{-\frac{(p-2)R}{\mu_k}}\sum_{j=1}^{k}\int_{B_R^c(0)}\varphi_n^2(x+y_{j,n}) - C\sum_{j=1}^{k}\int_{B_R(0)}\varphi_n^2(x+y_{j,n}) \\
\geq& \frac{1}{2} + o_R(1) + o(1),
\end{aligned}$$

which reaches a contradiction. This completes the proof. □

Lemma 2.2. *For any $y \in \Omega_\lambda$, there exists constant $C > 0$ such that*

$$\left|l_y(\varphi)\right| \leq C\left(\sum_{i\neq j} e^{-\frac{p-1}{p\mu_k}|y_i - y_j|} + \varepsilon\right)||\varphi\|_\varepsilon,$$

for large λ.

Proof. Since W_{k,y_i} is the weak solution to the equation

$$-\left(a + kb\int_{\mathbb{R}^3}|\nabla w|^2 dx\right)\Delta w + w = w^{p-1},$$

we have

$$a\int_{\mathbb{R}^3}\nabla W_{k,y_i}\cdot\nabla\varphi + kb\int_{\mathbb{R}^3}\left|\nabla W_{k,y_i}\right|^2\int_{\mathbb{R}^3}\nabla W_{k,y_i}\cdot\nabla\varphi + \int_{\mathbb{R}^3} W_{k,y_i}\varphi = \int_{\mathbb{R}^3} W_{k,y_i}^{p-1}\varphi.$$

Thus

$$\begin{aligned}
l_y(\varphi) =& \int_{\mathbb{R}^3}(a\nabla W_y\cdot\nabla\varphi + (1+\varepsilon V(x))W_y\varphi) + b\int_{\mathbb{R}^3}\left|\nabla W_y\right|^2\int_{\mathbb{R}^3}\nabla W_y\cdot\nabla\varphi - \int_{\mathbb{R}^3} W_y^{p-1}\varphi \\
=& \int_{\mathbb{R}^3}\nabla W_y\cdot\nabla\varphi\left(b\int_{\mathbb{R}^3}\left|\nabla W_y\right|^2 - kb\int_{\mathbb{R}^3}|\nabla W_k|^2\right) \\
&+ \sum_{i=1}^{k}\int_{\mathbb{R}^3} W_{k,y_i}^{p-1}\varphi - \int_{\mathbb{R}^3} W_y^{p-1}\varphi + \varepsilon\int_{\mathbb{R}^3} V(x)W_y\varphi.
\end{aligned}$$

By Lemma 2.5, for $i \neq j$, one has

$$\int_{\mathbb{R}^3}\left|\nabla W_{k,y_i}\cdot\nabla W_{k,y_j}\right| \leq Ce^{-\frac{|y_i - y_j|}{\mu_k}}. \tag{2.4}$$

By Lemmas 2.7 and 2.10, one has

$$\begin{aligned}\left|\int_{\mathbb{R}^3} W_y^{p-1}\varphi-\sum_{i=1}^{k}\int_{\mathbb{R}^3} W_{k,y_i}^{p-1}\varphi\right| &\leq \left(\int_{\mathbb{R}^3}(W_y^{p-1}-\sum_{i=1}^{k}W_{k,y_i}^{p-1})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^3}|\varphi|^p\right)^{\frac{1}{p}}\\ &\leq C\left(\sum_{i\neq j}\int_{\mathbb{R}^3} W_{k,y_i}^{p-1}W_{k,y_j}\right)^{\frac{p-1}{p}}\|\varphi\|_{\varepsilon} \\ &\leq C\left(\sum_{i\neq j} e^{-\frac{p-1}{p\mu_k}|y_i-y_j|}\right)\|\varphi\|_{\varepsilon}\end{aligned} \tag{2.5}$$

Then, it follows from (2.4) and (2.5) that

$$\begin{aligned}|l_y(\varphi)| \leq{}& C\int_{\mathbb{R}^3}|\nabla W_y\cdot\nabla\varphi|\left(\sum_{i\neq j}\int_{\mathbb{R}^3}\left|\nabla W_{k,y_i}\cdot\nabla W_{k,y_j}\right|\right)\\ &+C\left(\sum_{i\neq j} e^{-\frac{p-1}{p\mu_k}|y_i-y_j|}\right)\|\varphi\|_{\varepsilon}+\varepsilon\int_{\mathbb{R}^3}V(x)W_y|\varphi|\\ \leq{}& Ck|\nabla W_k|_2\|\varphi\|_{\varepsilon}\left(\sum_{i\neq j} e^{-\frac{|y_i-y_j|}{\mu_k}}\right)+C\left(\sum_{i\neq j} e^{-\frac{p-1}{p\mu_k}|y_i-y_j|}\right)\|\varphi\|_{\varepsilon}+C\varepsilon\|\varphi\|_{\varepsilon}\\ \leq{}& Ck\left(\sum_{i\neq j} e^{-\frac{|y_i-y_j|}{\mu_k}}\right)\|\varphi\|_{\varepsilon}+C\left(\sum_{i\neq j} e^{-\frac{p-1}{p\mu_k}|y_i-y_j|}\right)\|\varphi\|_{\varepsilon}+C\varepsilon\|\varphi\|_{\varepsilon}\\ \leq{}& C\left(\sum_{i\neq j} e^{-\frac{p-1}{p\mu_k}|y_i-y_j|}\right)\|\varphi\|_{\varepsilon}+C\varepsilon\|\varphi\|_{\varepsilon}.\end{aligned}$$

The result follows immediately. □

Lemma 2.3. *If $\|\varphi\|_{\varepsilon}\leq 1$, then there exists a constant $C>0$, independent of y, such that $\left\|R_y^{(i)}(\varphi)\right\|\leq C\|\varphi\|_{\varepsilon}^{p^*-i}$, $i=0,1,2$, where $p^*=\min\{3,p\}$.*

Proof. The proof of this lemma is the same as the proof of [12, Lemma 3.3], so we omit the details here. □

Proposition 2.1. *There exist $\varepsilon_0>0$ and $\lambda_0>0$ such that, for all $0<\varepsilon<\varepsilon_0$ and $\lambda>\lambda_0$, there exists a C^1 map $v_{\lambda,\varepsilon}:\Omega_\lambda\to H^1(\mathbb{R}^3)$ satisfying*

(i) for any $y\in\Omega_\lambda, v_{\lambda,\varepsilon,y}\in\mathscr{H}_y$ and $\left\langle\frac{\partial J(v_{\lambda,\varepsilon,y})}{\partial v_{\lambda,\varepsilon,y}},\varphi\right\rangle=0$ for all $\varphi\in\mathscr{H}_y$,

(ii) $\left\|v_{\lambda,\varepsilon,y}\right\|_{\varepsilon}\leq\sum_{i\neq j}e^{-\frac{p-1}{p\mu_k}|y_i-y_j|(1-\tau)}+\varepsilon^{1-\tau}$, where $\tau>0$ is a sufficiently small number.

Proof. By Lemma 2.2, we see that l_y is a bounded linear functional in $\mathscr{H}_y$, so there exists an $l_{y,k}\in\mathscr{H}_y$ such that $l_y(v_{\lambda,\varepsilon,y})=(l_{y,k},v_{\lambda,\varepsilon,y})_{\varepsilon}$. Thus, finding a critical point of $J(v_{\lambda,\varepsilon,y})$ is equivalent to solving $l_{y,k}+L_yv_{\lambda,\varepsilon,y}-R_y'(v_{\lambda,\varepsilon,y})=0$. From Lemma 2.1, we only need to solve

$$v_{\lambda,\varepsilon,y}=T(v_{\lambda,\varepsilon,y})=:-L_y^{-1}l_{y,k}+L_y^{-1}R_y'(v_{\lambda,\varepsilon,y}).$$

Let

$$\mathscr{N}=\left\{v_{\lambda,\varepsilon,y}: v_{\lambda,\varepsilon,y}\in\mathscr{H}_y, ||v_{\lambda,\varepsilon,y}||_{\varepsilon}\leq\sum_{i\neq j}e^{-\frac{p-1}{p\mu_k}|y_i-y_j|(1-\tau)}+\varepsilon^{1-\tau}\right\}$$

where $\tau>0$ is a small constant. It follows from Lemma 2.3 that

$$\left\|R_y^{(i)}\left(v_{\lambda,\varepsilon,y}\right)\right\|\leq C\left\|v_{\lambda,\varepsilon,y}\right\|_{\varepsilon}^{p^*-i},\ i=0,1,2,$$

where $p^*=\min\{3,p\}$. In view of Lemmas 2.1 and 2.2, we can obtain

$$\begin{aligned}\left\|T\left(v_{\lambda,\varepsilon,y}\right)\right\|_{\varepsilon}&\leq C\left\|l_{y,k}\right\|+C\left\|R_y'\left(v_{\lambda,\varepsilon,y}\right)\right\|\\&\leq C\left(\sum_{i\neq j}e^{-\frac{p-1}{p\mu_k}|y_i-y_j|}+\varepsilon\right)+C\left(\sum_{i\neq j}e^{-\frac{p-1}{p\mu_k}|y_i-y_j|(1-\tau)}+\varepsilon^{1-\tau}\right)^{p^*-1}\\&\leq\sum_{i\neq j}e^{-\frac{p-1}{p\mu_k}|y_i-y_j|(1-\tau)}+\varepsilon^{1-\tau}.\end{aligned}$$

This proves that $T(\mathscr{N})\subset\mathscr{N}$. Since $p^*-2>0$, we have

$$\begin{aligned}\|T\left(v^1_{\lambda,\varepsilon,y}\right)-T\left(v^2_{\lambda,\varepsilon,y}\right)\|_{\varepsilon}&\leq C\left\|R_y'\left(v^1_{\lambda,\varepsilon,y}\right)-R_y'\left(v^2_{\lambda,\varepsilon,y}\right)\right\|\\&\leq C\left(\left\|v^1_{\lambda,\varepsilon,y}\right\|_{\varepsilon}^{p^*-2}+\left\|v^2_{\lambda,\varepsilon,y}\right\|_{\varepsilon}^{p^*-2}\right)\left\|v^1_{\lambda,\varepsilon,y}-v^2_{\lambda,\varepsilon,y}\right\|_{\varepsilon}\\&\leq\frac{1}{2}\left\|v^1_{\lambda,\varepsilon,y}-v^2_{\lambda,\varepsilon,y}\right\|_{\varepsilon}.\end{aligned}$$

This shows that T is a contraction map. Thus, by contraction mapping theorem, we see that there exists $v_{\lambda,\varepsilon,y}\in\mathscr{N}$ such that $v_{\lambda,\varepsilon,y}=T\left(v_{\lambda,\varepsilon,y}\right)$. Moreover, similar to the proof in [5], we have that $v_{\lambda,\varepsilon}$ is a C^1 map with respect y. The proof is finished. □

For any $y=(y_1,y_2,\dots,y_k)\in\Omega_\lambda$, define $f_{k,\varepsilon}(y)=f_{k,\varepsilon}(y_1,y_2,\dots,y_k)=I_\varepsilon(W_y+v_{\lambda,\varepsilon,y})$. From Proposition 2.1, we derive the following result, whose proof is standard and thus is omitted (see, e.g., [4, 18])

Lemma 2.4. *For large λ and small ε, if $y^0=(y_1^0,\dots,y_k^0)\in\Omega_\lambda$ is a critical point to $f_{k,\varepsilon}$, then $W_{y_0}+v_{\lambda,\varepsilon,y^0}$ is a critical point to I_ε.*

We also give some technical lemmas which are useful in our proof, and some of them can be founded in [2, 15, 17, 18].

Lemma 2.5. *Let $u,u':\mathbb{R}^3\to\mathbb{R}$ be two positive continuous radical function such that $u(x)\sim|x|^ae^{-b|x|}$ and $u'(x)\sim|x|^{a'}e^{-b'|x|}(x\to\infty)$, where $a,a'\in\mathbb{R}$ and $b,b'>0$. If $\xi\in\mathbb{R}^3$ tend to infinity, then the following asymptotic estimates hold. (1) If $b<b'$, then $\int_{\mathbb{R}^3}u_\xi u'\sim|\xi|^ae^{-b|\xi|}$. (2) If $b=b'$ (suppose, for simplicity, that $a>a'$), then*

$$\int_{\mathbb{R}^3}u_\xi u'\sim\begin{cases}|\xi|^{a+a'+2}e^{-b|\xi|},a'>-2,\\|\xi|^ae^{-b|\xi|}\log|\xi|,a'=-2,\\|\xi|^ae^{-b|\xi|},a'<-2.\end{cases}$$

Lemma 2.6. *For $p>1$, there exists $C>0$ such that, for any $a,b\in\mathbb{R}$,*

$$||a+b|^p-|a|^p-|b|^p|\leq C|a|^{p-1}|b|+C|a||b|^{p-1}.$$

Lemma 2.7. *For $p \geq 2$ and $k \in \mathbb{N}$, there exists $C > 0$ such that, for any $a_j \geq 0,\ j = 1, 2, \ldots, k$,*

$$\left(\left(\sum_{j=1}^{k} a_j\right)^{p-1} - \sum_{j=1}^{k} a_j^{p-1}\right)^{\frac{p}{p-1}} \leq C \sum_{i \neq j} a_i^{p-1} a_j.$$

Lemma 2.8. *For $p \geq 2, k \in \mathbb{N}$, and $a_j \geq 0, j = 1, 2, \ldots, k$,*

$$\left(\sum_{j=1}^{k} a_j\right)^{p} \geq \sum_{j=1}^{k} a_j^p + 2(p-1) \sum_{1 \leq l < j \leq k} a_l^{p-1} a_j.$$

Lemma 2.9. *For $p \geq 2$, $k \in \mathbb{N}$, and $a_j \geq 0, j = 1, 2, \ldots, k$,*

$$\left(\sum_{j=1}^{k} a_j\right)^{p} \geq \sum_{j=1}^{k} a_j^p + p \sum_{1 \leq l < j \leq k} a_l^{p-1} a_j.$$

Lemma 2.10. *There exists a positive constant $C > 0$ such that, as $\left|y_i - y_j\right| \to \infty$,*

$$\int_{\mathbb{R}^3} W_{k,y_i}^{p-1} W_{k,y_j} \sim C \left|y_i - y_j\right|^{-1} e^{-\frac{\left|y_i - y_j\right|}{\mu_k}}.$$

3. Proof of the Main Results

We are now in a position to prove Theorem 1.1. We first consider the case $k \geq 2$. Define

$$d = \sup_{y \in \left(\mathbb{R}^3\right)^k} \int_{\mathbb{R}^3} V(x) W_y^2.$$

Choose a number m such that $m > \max\{1, \frac{3pd}{p-2}\}$, and set

$$e = \min\left\{\varepsilon_0, \left(\frac{m(p-2)}{2pC_3}\right)^{\frac{1}{\frac{2(p-1)}{p}(1-2\tau)-1}}, \frac{1}{m}|W_k|_p^p\right\},$$

where C_3 is the positive constant in Lemma 3.1, ε_0 is the number in Lemma 2.1, and τ is the small number in Proposition 2.1 and can be chosen such that $\frac{1}{\frac{2(p-1)}{p}(1-2\tau)-1} > 0$. Then, for any ε satisfying $0 < \varepsilon < e$, there exist $\lambda^* = \lambda^*(\varepsilon) > \widetilde{\lambda} = \widetilde{\lambda}(\varepsilon) > 0$ such that, for $z \in \mathbb{R}^3$ with $|z| \in \left[\widetilde{\lambda}(\varepsilon), \lambda^*(\varepsilon)\right]$,

$$m\varepsilon \leq \int_{\mathbb{R}^3} W_k^{p-1} W_{k,z} \leq 2m\varepsilon. \tag{3.1}$$

Define $F_\varepsilon := \sup\left\{f_{k,\varepsilon}(y) \mid y \in \Omega_{\widetilde{\lambda}(\varepsilon)}\right\}$. In order to obtain a k-bump solution of (1.1), it suffices to prove that F_ε is achieved in the interior of $\Omega_{\widetilde{\lambda}(\varepsilon)}$

Lemma 3.1. . *Let $k \geq 2$. Then, for $\varepsilon > 0$ sufficiently small,*

$$F_\varepsilon > \sup\left\{f_{k,\varepsilon}(y) \mid y \in \Omega_{\widetilde{\lambda}(\varepsilon)} \text{ and } \left|y_i - y_j\right| \in \left[\widetilde{\lambda}(\varepsilon), \lambda^*(\varepsilon)\right] \text{ for some } i \neq j\right\}.$$

Proof. From (3.1) and Lemma 2.10, we can obtain $\widetilde{\lambda}(\varepsilon) = O(\ln \frac{1}{\varepsilon}) \to \infty$ as $\varepsilon \to 0$. Then, for $y = (y_1, \dots, y_k) \in \Omega_{\widetilde{\lambda}(\varepsilon)}$, we have

$$|y_i - y_j|^{-1} e^{-\frac{|y_i - y_j|}{\mu_k}} \le C\varepsilon.$$

Thus, for τ small enough,

$$e^{-\frac{|y_i - y_j|}{\mu_k}(1-\tau)} \le C\varepsilon^{1-2\tau}.$$

Then, by Proposotion 2.1, for $y = (y_1, \dots, y_k) \in \Omega_{\widetilde{\lambda}(\varepsilon)}$, we have

$$\left\| v_{\widetilde{\lambda},\varepsilon,y} \right\|_{\varepsilon} \le \sum_{i \ne j} e^{-\frac{p-1}{p\mu_k}|y_i - y_j|(1-\tau)} + \varepsilon^{1-\tau} \le C\varepsilon^{\frac{p-1}{p}(1-2\tau)}.$$

It is easy to see that

$$\frac{1}{2}\left\langle L_y v_{\widetilde{\lambda},\varepsilon,y}, v_{\widetilde{\lambda},\varepsilon,y} \right\rangle \le C \left\| v_{\widetilde{\lambda},\varepsilon,y} \right\|_{\varepsilon}^2$$

and

$$\left| R_y\left(v_{\widetilde{\lambda},\varepsilon,y} \right) \right| \le C \left\| v_{\widetilde{\lambda},\varepsilon,y} \right\|_{\varepsilon}^{p^*},$$

where $p^* = \min\{3, p\} > 2$. By direct computation, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left(a\left|\nabla W_y\right|^2 + \left|W_y\right|^2 \right) \\
&= k \int_{\mathbb{R}^3} \left(a|\nabla W_k|^2 + |W_k|^2 \right) + 2\sum_{j<l} \int_{\mathbb{R}^3} \left(a\nabla W_{k,y_j} \cdot \nabla W_{k,y_l} + W_{k,y_j} W_{k,y_l} \right) \\
&= k \int_{\mathbb{R}^3} \left(a|\nabla W_k|^2 + |W_k|^2 \right) + 2\sum_{j<l} \int_{\mathbb{R}^3} W_{k,y_j}^{p-1} W_{k,y_l} - 2\sum_{j<l} kb \int_{\mathbb{R}^3} |\nabla W_k|^2 \int_{\mathbb{R}^3} \nabla W_{k,y_j} \cdot \nabla W_{k,y_l},
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\left(\int_{\mathbb{R}^3} \left|\nabla W_y\right|^2 \right)^2 =& k^2 \left(\int_{\mathbb{R}^3} |\nabla W_k|^2 \right)^2 + 4k \int_{\mathbb{R}^3} |\nabla W_k|^2 \sum_{j<l} \int_{\mathbb{R}^3} \nabla W_{k,y_j} \cdot \nabla W_{k,y_l} \\
&+ 4\left(\sum_{j<l} \int_{\mathbb{R}^3} \nabla W_{k,y_j} \cdot \nabla W_{k,y_l} \right)^2.
\end{aligned} \tag{3.3}$$

Letting τ small enough, we have

$$\left(\sum_{j<l} \int_{\mathbb{R}^3} \nabla W_{k,y_j} \cdot \nabla W_{k,y_l} \right)^2 \le Ce^{-\frac{2|y_i - y_j|}{\mu_k}} \le C\varepsilon^{\frac{2(1-2\tau)}{1-\tau}} \le C\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)} \tag{3.4}$$

Then, from (3.2), (3.3), and (3.4), we have

$$\begin{aligned}
& I_\varepsilon(W_y+v_{\widetilde{\lambda},\varepsilon,y}) \\
=& I_\varepsilon(W_y)+l_y\left(v_{\widetilde{\lambda},\varepsilon,y}\right)+\frac{1}{2}\left\langle L_y v_{\widetilde{\lambda},\varepsilon,y}, v_{\widetilde{\lambda},\varepsilon,y}\right\rangle-R_y\left(v_{\widetilde{\lambda},\varepsilon,y}\right) \\
=& I_\varepsilon(W_y)+O\left(\|l_y\|\|v_{\widetilde{\lambda},\varepsilon,y}\|_\varepsilon+\left\|v_{\widetilde{\lambda},\varepsilon,y}\right\|_\varepsilon^2\right) \\
=& \frac{1}{2}\|W_y\|^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla W_y|^2\right)^2+\frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)W_y^2-\frac{1}{p}\int_{\mathbb{R}^3}W_y^p+O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right) \\
=& c_k+\sum_{j<l}\int_{\mathbb{R}^3}W_{k,y_j}^{p-1}W_{k,y_l}+\sum_{j=1}^k\frac{1}{p}\int_{\mathbb{R}^3}W_{k,y_j}^p-\frac{1}{p}\int_{\mathbb{R}^3}W_y^p \\
&+\frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)W_y^2+O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right) \\
=& c_k-L_y+\frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)W_y^2,
\end{aligned}$$

where

$$c_k=\frac{k}{2}\int_{\mathbb{R}^3}\left(a|\nabla W_k|^2+W_k^2\right)+\frac{bk^2}{4}\left(\int_{\mathbb{R}^3}|\nabla W_k|^2\right)^2-\frac{k}{p}\int_{\mathbb{R}^3}W_k^p$$

and

$$L_y=-\sum_{j<l}\int_{\mathbb{R}^3}W_{k,y_j}^{p-1}W_{k,y_l}-\sum_{j=1}^k\frac{1}{p}\int_{\mathbb{R}^3}W_{k,y_j}^p+\frac{1}{p}\int_{\mathbb{R}^3}W_y^p+O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right).$$

Assume that $y=(y_1,\dots,y_k)\in\Omega_{\widetilde{\lambda}(\varepsilon)}$ and $\left|y_j-y_l\right|\in\left[\widetilde{\lambda}(\varepsilon),\lambda^*(\varepsilon)\right]$ for some $j\neq l$. Then, by (3.1) and Lemma 2.8, we obtain

$$\begin{aligned}
L_y=&-\sum_{j<l}\int_{\mathbb{R}^3}W_{k,y_j}^{p-1}W_{k,y_l}-\sum_{j=1}^k\frac{1}{p}\int_{\mathbb{R}^3}W_{k,y_j}^p+\frac{1}{p}\int_{\mathbb{R}^3}W_y^p+O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right) \\
\geq&\frac{p-2}{p}\sum_{j<l}\int_{\mathbb{R}^3}W_{k,y_j}^{p-1}W_{k,y_l}-C_3\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)} \\
\geq&\frac{p-2}{p}m\varepsilon-C_3\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)} \\
\geq&\frac{3}{2}d\varepsilon.
\end{aligned}$$

So,

$$f_{k,\varepsilon}(y)=I_\varepsilon\left(W_y+v_{\widetilde{\lambda},\varepsilon,y}\right)\leq c_k-\frac{3d}{2}\varepsilon+\frac{d}{2}\varepsilon=c_k-d\varepsilon. \tag{3.5}$$

On the other hand, if $y=(y_1,\dots,y_k)\in\Omega_{\tilde\lambda}$ with $\left|y_j-y_l\right|\to\infty$ for all $j\neq l$, then we find from Lemma 2.6 that

$$\begin{aligned}L_y=&-\sum_{j<l}\int_{\mathbb{R}^3}W_{k,y_j}^{p-1}W_{k,y_l}-\sum_{j=1}^{k}\frac{1}{p}\int_{\mathbb{R}^3}W_{k,y_j}^{p}+\frac{1}{p}\int_{\mathbb{R}^3}W_y^p+O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right)\\ \le& C\sum_{j<l}\int_{\mathbb{R}^3}W_{k,y_j}^{p-1}W_{k,y_l}+O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right)\\ =&o(1)+O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right),\end{aligned}$$

where $o(1)$ denotes some quantities depend only on y and converge to 0 as $\left|y_l-y_j\right|\to\infty$. Hence,

$$\begin{aligned}f_{k,\varepsilon}(y)=I_\varepsilon\left(W_y+v_{\tilde\lambda,\varepsilon,y}\right)=&c_k-L_y+\frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)W_y^2\\ \ge& c_k+\frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)W_y^2-C\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}+o(1)\end{aligned}$$

Therefore, for $\varepsilon>0$ small, $\liminf_{\left|y_i-y_j\right|\to\infty}f_{k,\varepsilon}(y)\ge c_k$. This together with (3.5) obtains the desired result immediately. □

Choose $y^{(h)}(\varepsilon)=\left(y_1^{(h)}(\varepsilon),\dots,y_k^{(h)}(\varepsilon)\right)\in\Omega_{\tilde\lambda(\varepsilon)}$ such that $\lim_{h\to\infty}f_{k,\varepsilon}\left(y_1^{(h)}(\varepsilon),\dots,y_k^{(h)}(\varepsilon)\right)=F_\varepsilon$. By Lemma 3.1, we can obtain $\inf_h\min_{l\neq j}\left|y_l^{(h)}(\varepsilon)-y_j^{(h)}(\varepsilon)\right|\ge\lambda^*$. Then, for any $1\le l\le k$, after passing to a subsequence if necessary, we may assume either $\lim_{h\to\infty}y_l^{(h)}(\varepsilon)=y_l^{(0)}(\varepsilon)\in\mathbb{R}^3$ with $\left|y_l^{(0)}(\varepsilon)-y_j^{(0)}(\varepsilon)\right|\ge\lambda^*$ for $l\neq j$ or $\lim_{h\to\infty}\left|y_l^{(h)}(\varepsilon)\right|=\infty$. Let

$$\Pi(\varepsilon)=\left\{1\le l\le k:\ |y_l^{(h)}(\varepsilon)|\to\infty,\text{ as }h\to\infty\right\}.$$

We shall prove that $\Pi(\varepsilon)=\emptyset$ for $\varepsilon>0$ small enough and hence $f_{k,\varepsilon}$ achieves its maximum at

$$\left(y_1^{(0)}(\varepsilon),\dots,y_k^{(0)}(\varepsilon)\right)\in\operatorname{int}\left(\Omega_{\tilde\lambda(\varepsilon)}\right).$$

Lemma 3.2. *Let $k\ge2$. If conditions (V_1) and (V_2) hold, then there exists $\varepsilon(k)>0$ such that, for $\varepsilon\in(0,\varepsilon(k))$, $\Pi(\varepsilon)=\emptyset$.*

Proof. Assume that $\Pi(\varepsilon)\neq\emptyset$ along a sequence $\varepsilon_n\to0$. Without loss of generality, we may assume $\Pi(\varepsilon_n)=\{1,2,\dots,l_k\}$ for all $n\in\mathbb{N}$ and for some $1\le l_k<k$. The case $l_k=k$ can be handled similarly. For simplicity, denote $\varepsilon=\varepsilon_n$ and $\left(y_1^{(h)},\dots,y_k^{(h)}\right)=\left(y_1^{(h)}(\varepsilon_n),\dots,y_k^{(h)}(\varepsilon_n)\right)$ for $h=0,1,2,\dots$. As $h\to\infty$, one has

$$\left|y_1^{(h)}\right|\to\infty,\dots,\left|y_{l_k}^{(h)}\right|\to\infty\text{ and }y_{l_k+1}^{(h)}\to y_{l_k+1}^{(0)},\dots,y_k^{(h)}\to y_k^{(0)}.$$

Let

$$y^{(h)}=\left(y_1^{(h)},\dots,y_k^{(h)}\right),\ y_*^{(h)}=\left(y_{l_k+1}^{(h)},\dots,y_k^{(h)}\right),$$

and define

$$W_h=\sum_{l=1}^{k}W_{k,y_l^{(h)}},W_{h,1}=\sum_{l=1}^{l_k}W_{k,y_l^{(h)}},W_{h,2}=\sum_{l=l_k+1}^{k}W_{k,y_l^{(h)}}.$$

Similar to the computation in Lemma 3.1, we have

$$\begin{aligned}
&f_{k,\varepsilon}\left(y_1^{(h)},\ldots,y_k^{(h)}\right)\\
=&I_\varepsilon\left(W_h+v_{\widetilde{\lambda},\varepsilon,y^{(h)}}\right)\\
=&kE_k-L_{y^{(h)}}+\frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)W_h^2+\frac{bk^2}{4}\left(\int_{\mathbb{R}^3}|\nabla W_k|^2\right)^2\\
=&l_kE_k+(k-l_k)E_k-L_{y_*^{(h)}}+L_{y_*^{(h)}}-L_{y^{(h)}}\\
&+\frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)W_{h,2}^2-\frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)W_{h,2}^2+\frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)W_h^2\\
&+\frac{b(k-l_k)^2}{4}\int_{\mathbb{R}^3}|\nabla W_k|^2\Big)^2+\frac{bk^2}{4}\left(\int_{\mathbb{R}^3}|\nabla W_k|^2\right)^2-\frac{b(k-l_k)^2}{4}\left(\int_{\mathbb{R}^3}|\nabla W_k|^2\right)^2\\
=&l_kE_k+I_\varepsilon\left(W_{h,2}+v_{\widetilde{\lambda},\varepsilon,y_*^{(h)}}\right)+L_{y_*^{(h)}}-L_{y^{(h)}}+\frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)W_h^2-\frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)W_{h,2}^2\\
&+\frac{bl_k(2k-l_k)}{4}\left(\int_{\mathbb{R}^3}|\nabla W_k|^2\right)^2,
\end{aligned}\tag{3.6}$$

where

$$E_k=\frac{k}{2}\int_{\mathbb{R}^3}\left(a|\nabla W_k|^2+W_k^2\right)-\frac{k}{p}\int_{\mathbb{R}^3}W_k^p,$$

$$L_{y^{(h)}}=-\sum_{j<l}\int_{\mathbb{R}^3}W^{p-1}_{k,y_j^{(h)}}W_{k,y_l^{(h)}}-\sum_{j=1}^{k}\frac{1}{p}\int_{\mathbb{R}^3}W^p_{k,y_j^{(h)}}+\frac{1}{p}\int_{\mathbb{R}^3}W_h^p+O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right),$$

and

$$L_{y_*^{(h)}}=-\sum_{l_k<j<l}\int_{\mathbb{R}^3}W^{p-1}_{k,y_j^{(h)}}W_{k,y_l^{(h)}}-\sum_{j=l_k+1}^{k}\frac{1}{p}\int_{\mathbb{R}^3}W^p_{k,y_j^{(h)}}+\frac{1}{p}\int_{\mathbb{R}^3}W_{h,2}^p+O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right).$$

Then, by Lemma 2.9, we have

$$\begin{aligned}
L_{y_*^{(h)}}-L_{y^{(h)}}=&\sum_{j<l\le l_k}\int_{\mathbb{R}^3}W^{p-1}_{k,y_j^{(h)}}W_{k,y_l^{(h)}}+\sum_{j=1}^{l_k}\int_{\mathbb{R}^3}W^{p-1}_{k,y_j^{(h)}}W_{h,2}+\sum_{j=1}^{l_k}\frac{1}{p}\int_{\mathbb{R}^3}W^p_{k,y_j^{(h)}}\\
&+\frac{1}{p}\int_{\mathbb{R}^3}W_{h,2}^p-\frac{1}{p}\int_{\mathbb{R}^3}W_h^p+O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right)\\
<&O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right).
\end{aligned}$$

From (V_1) and $y_l^{(h)}\to\infty$, $l=1,2,\ldots,l_k$, we conclude that

$$\frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)W_h^2-\frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)W_{h,2}^2=o(1),$$

where $o(1)$ converge to 0 as $h\to\infty$. Letting $h\to\infty$ in (3.6), we have

$$M_\varepsilon\le l_kE_k+I_\varepsilon\left(W_{y_*^{(0)}}+v_{\widetilde{\lambda},\varepsilon,y_*^{(0)}}\right)+\frac{bl_k(2k-l_k)}{4}\left(\int_{\mathbb{R}^3}|\nabla W_k|^2\right)^2+O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right)\tag{3.7}$$

In view of Lemma 2.10 and (3.1), we have $C_4\varepsilon \leq \widetilde{\lambda}^{-1}e^{-\frac{\widetilde{\lambda}}{\mu_k}} \leq C_5\varepsilon$, which implies that

$$\frac{2}{3}\mu_k \ln\frac{1}{\varepsilon} < \widetilde{\lambda} < 2\mu_k \ln\frac{1}{\varepsilon}, \tag{3.8}$$

for $\varepsilon > 0$ small enough. Choose δ such that

$$0 < \delta < \frac{2(p-1)(1-2\tau)-p}{14kp}.$$

From assumption (V_2), one sees that there exists $T > 0$ such that

$$V(x) \geq e^{-\delta|x|}, \quad |x| \geq T. \tag{3.9}$$

Define

$$\bar{y}_l^{(\varepsilon)} = (14k\ln\frac{1}{\varepsilon} - 6l\widetilde{\lambda} - 1, 0, 0) \in \mathbb{R}^3,\ l = 1,2,\ldots,k.$$

We know that the open balls $B\left(\bar{y}_l^{(\varepsilon)}, 3\widetilde{\lambda}\right)$ $(l = 1,2,\ldots,k)$ are mutually disjoint. Thus there are l_k integers from $\{1,2,\ldots,k\}$, denoted by $t_1 < t_2 < \cdots < t_{l_k}$, such that $\left|\bar{y}_{t_i}^{(\varepsilon)} - \bar{y}_j^{(0)}\right| \geq 3\widetilde{\lambda}$, $i = 1,\ldots,l_k$, $j = l_k+1,\ldots,k$. Denote $\bar{y}_{t_i}^{(\varepsilon)}$ by $y_i^{(\varepsilon)}$, $i = 1,2,\ldots,l_k$. Then, for ε small enough,

$$T+1 \leq \left|y_i^{(\varepsilon)}\right| \leq 14k\ln\frac{1}{\varepsilon} - 1,\ i = 1,\ldots,l_k, \tag{3.10}$$

$$\left|y_i^{(\varepsilon)} - y_j^{(\varepsilon)}\right| \geq 3\widetilde{\lambda},\ 1 \leq i < j \leq l_k, \tag{3.11}$$

and

$$\left|y_i^{(\varepsilon)} - y_j^{(0)}\right| \geq 3\widetilde{\lambda},\ i = 1,\ldots,l_k,\ j = l_k+1,\ldots,k. \tag{3.12}$$

Therefore

$$\left(y_1^{(\varepsilon)},\ldots,y_{l_k}^{(\varepsilon)}, y_{l_k+1}^{(0)},\ldots,y_k^{(0)}\right) \in \Omega_{\widetilde{\lambda}}.$$

Denote

$$y^{(\varepsilon)} = \left(y_1^{(\varepsilon)},\ldots,y_{l_k}^{(\varepsilon)}, y_{l_k+1}^{(0)},\ldots,y_k^{(0)}\right) \quad \text{and} \quad y_*^{(0)} = \left(y_{l_k+1}^{(0)},\ldots,y_k^{(0)}\right).$$

Let $W_{\varepsilon,1} = \sum_{j=1}^{l_k} W_{k,y_j^{(\varepsilon)}}$ and $W_{\varepsilon,2} = \sum_{j=l_k+1}^{k} W_{k,y_j^{(0)}}$. Then

$$\begin{aligned} I_\varepsilon\left(W_{y^{(\varepsilon)}} + v_{\widetilde{\lambda},\varepsilon,y^{(\varepsilon)}}\right) =& l_kE_k + I_\varepsilon\left(W_{y_*^{(0)}} + v_{\widetilde{\lambda},\varepsilon,y_*^{(0)}}\right) + L_{y_*^{(0)}} - L_{y^{(\varepsilon)}} \\ &+ \frac{\varepsilon}{2}\int_{\mathbb{R}^3} V(x)\left(W_{\varepsilon,1}+W_{\varepsilon,2}\right)^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^3} V(x)W_{\varepsilon,2}^2 \\ &+ \frac{bl_k(2k-l_k)}{4}\left(\int_{\mathbb{R}^3}|\nabla W_k|^2\right)^2. \end{aligned} \tag{3.13}$$

From (3.9) and (3.10), we can obtain

$$\begin{aligned} \frac{\varepsilon}{2}\int_{\mathbb{R}^3} V(x)\left(W_{\varepsilon,1}+W_{\varepsilon,2}\right)^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^3} V(x)W_{\varepsilon,2}^2 \geq& \frac{\varepsilon}{2}\int_{\mathbb{R}^3} V(x)W_{k,y_1^{(\varepsilon)}}^2 \geq \frac{\varepsilon}{2}\int_{\left|x-y_1^{(\varepsilon)}\right|\leq 1} V(x)W_{k,y_1^{(\varepsilon)}}^2 \\ \geq& C_6\varepsilon e^{-\delta\left(\left|y_1^{(\varepsilon)}\right|+1\right)} \geq C_6\varepsilon e^{-\delta 14k\ln\frac{1}{\varepsilon}} = C_6\varepsilon^{14k\delta+1}. \end{aligned} \tag{3.14}$$

By Lemma 2.6, we have

$$\begin{aligned}
L_{y_*^{(0)}} - L_{y(\varepsilon)} = &\sum_{j<l\leq l_k} \int_{\mathbb{R}^3} W^{p-1}_{k,y_j^{(\varepsilon)}} W_{k,y_l^{(\varepsilon)}} + \sum_{j=1}^{l_k} \frac{1}{p} \int_{\mathbb{R}^3} W^{p-1}_{k,y_j^{(\varepsilon)}} W_{\varepsilon,2} \\
&+ \frac{1}{p} \sum_{j=1}^{l_k} \int_{\mathbb{R}^3} W^p_{k,y_j^{(\varepsilon)}} + \frac{1}{p}\int_{\mathbb{R}^3} W^p_{\varepsilon,2} - \frac{1}{p}\int_{\mathbb{R}^3} (W_{\varepsilon,1}+W_{\varepsilon,2})^p + O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right) \\
\geq &\sum_{j=1}^{l_k} \frac{1}{p} \int_{\mathbb{R}^3} W^p_{k,y_j^{(\varepsilon)}} + \frac{1}{p}\int_{\mathbb{R}^3} W^p_{\varepsilon,2} - \frac{1}{p}\int_{\mathbb{R}^3} (W_{\varepsilon,1}+W_{\varepsilon,2})^p + O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right) \\
\geq &-C\sum_{j=1}^{l_k} \int_{\mathbb{R}^3} W^{p-1}_{k,y_j^{(\varepsilon)}} W_{\varepsilon,2} - C\sum_{j=1}^{l_k} \int_{\mathbb{R}^3} W^{p-1}_{\varepsilon,2} W_{k,y_j^{(\varepsilon)}} \\
&-C\sum_{l\leq i<j\leq l_k} \int_{\mathbb{R}^3} W^{p-1}_{k,y_i^{(\varepsilon)}} W_{k,y_j^{(\varepsilon)}} + O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right).
\end{aligned}$$

By Lemma 2.10, (3.8) and (3.11), we have

$$\sum_{l\leq i<j\leq l_k} \int_{\mathbb{R}^3} W^{p-1}_{k,y_i^{(\varepsilon)}} W_{k,y_j^{(\varepsilon)}} = o(1)e^{\frac{-3\widetilde{\lambda}}{\mu_k}} = o(1)e^{\frac{-3\mu_k \frac{2}{3}\ln\frac{1}{\varepsilon}}{\mu_k}} = o\left(\varepsilon^2\right), \text{ as } \varepsilon\to 0.$$

According to (3.12), a similar argument shows that

$$\sum_{j=1}^{l_k} \int_{\mathbb{R}^3} W^{p-1}_{k,y_j^{(\varepsilon)}} W_{\varepsilon,2} + \sum_{j=1}^{l_k} \int_{\mathbb{R}^3} W^{p-1}_{\varepsilon,2} W_{k,y_j^{(\varepsilon)}} = o\left(\varepsilon^2\right), \quad \text{as } \varepsilon\to 0.$$

Thus $L_{y_*^{(0)}} - L_{y(\varepsilon)} \geq O\left(\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)}\right)$, which with (3.13) and (3.14) yields

$$\begin{aligned}
I_\varepsilon\left(W_{y^{(\varepsilon)}} + v_{\widetilde{\lambda},\varepsilon,y^{(\varepsilon)}}\right) &\geq l_k E_k + I_\varepsilon\left(W_{y_*^{(0)}} + v_{\widetilde{\lambda},\varepsilon,y_*^{(0)}}\right) + C_6\varepsilon^{14k\delta+1} - C_7\varepsilon^{\frac{2(p-1)}{p}(1-2\tau)} \\
&\geq l_k E_k + I_\varepsilon\left(W_{y_*^{(0)}} + v_{\widetilde{\lambda},\varepsilon,y_*^{(0)}}\right) + \frac{bl_k(2k-l_k)}{4}\left(\int_{\mathbb{R}^3}|\nabla W_k|^2\right)^2 + C_8\varepsilon^{14k\delta+1},
\end{aligned}$$

which contradicts (3.7). Thus, $\Pi(\varepsilon)=\emptyset$ and $f_{k,\varepsilon}$ achieves its maximum at some point $y^0\in \operatorname{int}\left(\Omega_{\widetilde{\lambda}(\varepsilon)}\right)$. □

We are now to prove Theorem 1.1.

Proof of Theorem 1.1. For $k\geq 2$, by Lemma 3.2, if $0<\varepsilon<\varepsilon(k)$, then $f_{k,\varepsilon}$ achieves its maximum at some point $y^0\in \operatorname{int}\left(\Omega_{\widetilde{\lambda}(\varepsilon)}\right)$. Therefore, $W_{y^0}+v_{\widetilde{\lambda},\varepsilon,y^0}$ is a k-bump solution to (1.1).

For $k=1$, by Proposition 2.1, if $\varepsilon\in(0,\varepsilon_0]$, then

$$\lim_{|y|\to\infty} f_{k,\varepsilon}(y) = \lim_{|y|\to\infty} I_\varepsilon\left(W_y + v_{\widetilde{\lambda},\varepsilon,y}\right) = \frac{1}{2}||W_k||^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla W_k|^2\,dx\right)^2 - \frac{1}{p}\int_{\mathbb{R}^3} W_k^p.$$

Since $f_{k,\varepsilon}$ is defined on all $\mathbb{R}^3$, we have that $f_{k,\varepsilon}$ has a critical point $y^0\in\mathbb{R}^3$ and $W_{y^0}+v_{\widetilde{\lambda},\varepsilon,y^0}$ is a 1-bump solution to (1.1). This completes the proof.

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