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VARIATIONAL INEQUALITIES, KY FAN MINIMAX INEQUALITY, AND STRONG NASH EQUILIBRIA IN GENERALIZED GAMES

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Abstract. In this paper, we provide necessary and sufficient conditions for the existence of solutions for variational inequalities and sufficient conditions for the existence of solutions for quasi-Ky Fan minimax inequality, quasi-variational inequalities, and generalized variational inequalities. As applications, we apply these results to derive existence results for strong Nash equilibria in generalized games which generalize some existence theorems for Nash equilibria in generalized games in the literature and the existence theorem for strong Nash equilibria in normal-form games by Nessah and Tian [J. Math. Anal. Appl. 414 (2014), 871-885]. We also provide sufficient conditions for the uniqueness of strong Nash equilibria in certain generalized games.
Keywords. Generalized games; Ky Fan minimax inequality; Nash equilibrium; Quasi-variational inequalities.

1. Introduction

The Ky Fan minimax inequality [1] is one of the most important results in mathematical sciences which is equivalent to many important mathematical theorems, such as the classical Brouwer's fixed point theorem, Kakutani fixed point theorem, Fan-Knaster-Kuratowski-Mazurkiewicz (FKKM) Theorem, and variational inequalities. So far, there have been many known variations and generalizations of Ky Fan's original minimax inequality in [1]. For example, Lin and Tian [2] provided generalizations of the Ky Fan minimax inequality by relaxing the compactness and convexity of sets, Scalzo [3] derived an existence theorem for solutions of quasi-Ky Fan minimax inequality, and Chbani and Riahi [4] used an inertial proximal method for solving Ky Fan minimax inequalities.

Variational inequality theory is a very powerful mathematical technology. In recent years, classical variational inequalities and quasi-variational inequalities have been extended and generalized to a wide class of problems arising in economics, finance, game theory, mechanics, nonlinear programming, optimization and control, and so on. For more on variational inequalities and quasi-variational inequalities, we refer to [5]-[14].

Nash equilibrium is one of the most important concepts in game theory which has been studied extensively in the literature. Since Nash [15] established the famous existence theorem for equilibrium of an n-person non-cooperative game in 1950, there have been many variations

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and generalizations of Nash's theorem appeared in the literature; see, e.g., [16]-[24]. Despite of the fact that Nash equilibrium is an important central behavioral solution in noncooperative games, a Nash equilibrium does not necessarily entail efficiency or stability with respect to coalitional deviations. The solution concept of strong Nash equilibrium introduced by Aumann (see [25]) overcomes this shortcoming. The existence of Strong Nash equilibria was studied in [25] - [29]. Unlike standard Nash equilibria, strong Nash equilibria rarely exist. Duo to its usefulness, studying the existence of strong Nash equilibria is desirable.

The main contributions of this paper include: (1) we provide necessary and sufficient conditions for the existence of solutions for variational inequalities and sufficient conditions for the existence of solutions for quasi-Ky Fan minimax inequality, quasi-variational inequalities, and generalized variational inequalities; (2) we apply these results to derive existence results for strong Nash equilibria in generalized games which generalize some existence theorems for Nash equilibria in generalized games, including [9, Theorem 1], and the existence theorem for strong Nash equilibria in normal-form games in [25]; and (3) we provide sufficient conditions for the uniqueness of strong Nash equilibria in certain generalized games.

2. Solutions of (Quasi-)Ky Fan Minimax Inequalities

Let X and C be topological spaces, and let $\varphi(x,z) : X \times C \longrightarrow \mathbb{R}$ be a function. The Ky Fan minimax inequality is the following problem:

$$\begin{cases} \text{Find } x^* \in C & \text{such that} \\ \varphi(x,x^*) \leq 0 & \forall x \in X. \end{cases} \tag{2.1}$$

When $C = X$, under the assumptions that $\varphi(\cdot,y)$ is upper semicontinuous, $\varphi(x,\cdot)$ is semistrictly quasiconvex and lower semicontinuous, and $\varphi(x,y)$ is pseudo-monotone, the Ky Fan minimax inequality is equivalent to the next problem (see [12, Proposition 8]):

$$\begin{cases} \text{Find } x^* \in X & \text{such that} \\ \varphi(x^*,y) \geq 0 & \forall y \in X. \end{cases} \tag{2.2}$$

Let X be a non-empty subset of a topological space, and let $f : X \longrightarrow \mathbb{R}$ be a real-valued function. We say that f is upper semicontinuous (u.s.c.) on X if ,for each $t \in \mathbb{R}$, $\{x \in X | f(x) < t\}$ is open. We say that f is lower semicontinuous (l.s.c.) if $-f$ is upper semicontinuous, that is, $\{x \in X | f(x) > t\}$ is open for each $t \in \mathbb{R}$. f is continuous if it is both u.s.c. and l.s.c..

A correspondence $F : X \mapsto 2^Y$ is said to have *open lower sections* if $F^{-1}(y) = \{x \in X | y \in F(x)\}$ is open in X for every $y \in Y$. A correspondence $\phi : X \mapsto 2^Y$ is said to be *upper semicontinuous* (u.s.c.) if, for any open set V of Y, $\{x \in X | \phi(x) \subseteq V\}$ is open in X.

The following concepts are given in [16] with $\varphi(x,z) = U(x,z) - U(z,z)$.

Definition 2.1. Let X be a non-empty subset of a topological vector space and let C be a non-empty convex subset of X. The function $\varphi(x,z) : X \times X \longrightarrow \mathbb{R}$ is *diagonally transfer lower continuous* in z on C if $\varphi(x,z) > 0$ for $z \in C$ implies that there exists an open neighborhood $O_z \subseteq C$ of z and $x' \in X$ such that $\varphi(x',z') > 0$ for all $z' \in O_z$.

Definition 2.2. Let X be a non-empty convex subset of a topological vector space and C be a non-empty convex subset of X. A function $\varphi(x,z) : X \times C \longrightarrow \mathbb{R}$ is *diagonally transfer quasi-concave* in x if, for any $\{x_1,\ldots,x_k\} \subseteq X$, there exists $\{z_1,\ldots,z_k\} \subseteq C$, where $x_h \longrightarrow z_h$, such that, for each $z \in co\{z_1,\dots,z_k\}$ and $z = \sum_{j=1}^{l} \lambda_j z_{h_j}$ with each $\lambda_j > 0$ and $\sum_{j=1}^{l} \lambda_j = 1$, one has $\varphi(x,z) \le 0$ for some $x \in \{x_{h_1},\dots,x_{h_l}\}$.

In particular, $\varphi(x,z)$ is said to be 0-*diagonally quasi-concave* if $C = X$ and $\{z_1,\dots,z_k\} = \{x_1,\dots,x_k\}$ in Definition 2.2 (see [2] and [3]).

The next concept introduced in [21] is weaker than diagonally transfer quasi-concavity. Let

$$\Delta_n = \{(\lambda_0,\lambda_1,\dots,\lambda_n) | \lambda_i \ge 0 \text{ for } 0 \le i \le n \text{ and } \sum_{i=0}^{n} \lambda_i = 1\}.$$

Definition 2.3. Let X be a topological space, and $A, Y \subseteq X$. A function $\varphi : X \times Y \longrightarrow R$ is called $\mathscr{C}$*-quasi-concave* on A if, for any finite subset $\{x^0,x^1,\dots,x^n\} \subseteq A$, there exists a continuous mapping $\phi_n : \Delta_n \longrightarrow Y$ such that $\min\{\varphi(x^i,\phi_n(\lambda_0,\lambda_1,\dots,\lambda_n)) | i \in J\} \le 0$ for all $(\lambda_0,\lambda_1,\dots,\lambda_n) \in \Delta_n$, where $J = \{j \in \{0,1,\dots,n\} | \lambda_j \ne 0\}$.

Note that a function $\varphi(x,z)$ is diagonally transfer quasi-concave on X implies that φ is $\mathscr{C}$-quasi-concave on X: For any $\{x^0,x^1,\dots,x^n\} \subseteq X$, there exists $\{z^0,z^1,\dots,z^n\} \subseteq C$ such that, for each $z \in co\{z^0,z^1,\dots,z^k\}$, $z = \sum_{j=0}^{n} \lambda_j z^j$, there exists $x^i \in \{x^0,\dots,x^n\}$ satisfying $\varphi(x^i,z) \le 0$. Then take $\phi_n(\lambda_0,\lambda_1,\dots,\lambda_n) = \sum_{j=0}^{n} \lambda_j z^j$ for $\mathscr{C}$-quasi-concavity.

Remark 2.1. Given a bifunction $\varphi(x,y) : X \times X \longrightarrow \mathbb{R}$, we say that $\varphi(x,y)$ is *diagonally transfer upper continuous* in y if $-\varphi(x,y)$ is diagonally transfer lower continuous in y; $\varphi(x,y)$ is *diagonally transfer quasi-convex* in x if $-\varphi(x,y)$ is diagonally transfer quasi-concave in x; and $\varphi(x,y)$ is $\mathscr{C}$*-quasi-convex* in x if $-\varphi(x,y)$ is $\mathscr{C}$-quasi-concave in x.

The following characterization for the existence of solutions of the Ky Fan minimax inequality (2.1) follows easily from [30, Lemma 1].

Theorem 2.1. *Assume that X is a subset of a Hausdorff topological vector space and C is a non-empty convex subset of X. Then the Ky Fan minimax inequality (2.1) has a solution if and only if there exists a non-empty convex compact subset D of C such that the restricted mapping $\varphi|_{X\times D} : X \times D \longrightarrow \mathbb{R}$ is diagonally transfer lower continuous on D and diagonally transfer quasi-concave on X.*

By using a similar approach as the proof of [21, Theorem 1], one can derive the next characterization for the existence of solutions for problem (2.2) which implies a characterization for solutions of variational inequalities in the next section. We will give a proof in Appendix 1.

Theorem 2.2. *Assume that X is a non-empty convex subset of a Hausdorff topological vector space, and let $\varphi : X \times X \longrightarrow \mathbb{R}$ be a function. The problem (2.2) has a solution if and only if there exists a non-empty convex compact subset D of X such that*

(i) D has the fixed point property (continuous functions $f : D \longrightarrow D$ have fixed points);

(ii) the restricted mapping $\varphi|_{D\times X} : D \times X \longrightarrow R$ is diagonally transfer upper continuous on D and $\mathscr{C}$-quasi-convex on X.

Recently, Scalzo [3] introduced the following quasi-Ky Fan minimax inequality. We assume that X is a non-empty convex subset of a Hausdorff topological vector space. Suppose that $\varphi(x,z) : X \times X \longrightarrow \mathbb{R}$ is a function, and let $K : X \mapsto X$ be a set-valued mapping. The quasi-Ky

Fan minimax inequality is the following problem:

$$\begin{cases} \text{Find } x^* \in X & \text{such that} \\ x^* \in K(x^*) & \text{and} \\ \varphi(x,x^*) \leq 0 & \forall x \in K(x^*). \end{cases} \tag{2.3}$$

The next theorem is different from [3, Theorem 2] and [31, Theorem 3.1]. In particular, we assume that X is a non-empty convex subset of a Hausdorff topological vector space (instead of a locally convex Hausdorff topological vector space required in [3] and [31]).

Theorem 2.3. *Assume that X is a non-empty convex compact subset of a Hausdorff topological vector space and*

(i) $\varphi(x,z)$ is 0-diagonally quasi-concave in x;

(ii) $K : X \mapsto 2^X$ is non-empty convex valued and has open lower sections;

(iii) the mapping $P : X \mapsto 2^X$ defined by $P(z) = \{x \in K(z) | \varphi(x,z) > 0\}$ for each $z \in X$ has open lower sections;

(iv) $\Delta = \{x \in X | x \in K(x))\}$ is closed.

Then, the quasi-Ky Fan minimax inequality (2.3) has a solution.

Proof. Define the correspondence $G : X \mapsto 2^X$ by setting

$$G(x) = \begin{cases} K(x) & \text{if } x \in X \setminus \Delta \\ coP(x) & \text{if } x \in \Delta. \end{cases}$$

Then it is clear that G is convex valued. Since P has open lower sections by assumption (iii), it follows from [32, Lemma 5.1] that coP has open lower sections. We claim that G has open lower sections. In fact, since $K(x)$ is convex by assumption (ii) and $P(x) \subseteq K(x)$, we have $coP(x) \subseteq K(x)$ for each $x \in X$, which implies that $coP^{-1}(y) \subseteq K^{-1}(y)$ for each $y \in X$. For any $y \in X$, if $y \in coP(x)$, then $G^{-1}(y) = coP^{-1}(y) \cup [K^{-1}(y) \cap (X \setminus \Delta)]$; if $y \in F(x) \setminus coP(x)$, then $G^{-1}(y) = K^{-1}(y) \cap (X \setminus \Delta)$. Since Δ is closed by assumption (iv), we have that $X \setminus \Delta$ is open in X. Recall that K and coP have open lower sections which imply that $coP^{-1}(y)$ and $K^{-1}(y)$ are open. It follows that $G^{-1}(y)$ is open for each $y \in X$, that is, G has open lower sections.

Now, suppose that $G(x) \neq \emptyset$ for each $x \in X$. Then it follows from the Fan-Browder fixed point theorem that there exists $x^* \in X$ such that $x^* \in G(x^*)$. Since $coP(x) \subseteq K(x)$, we have $G(x) \subseteq K(x)$ for each $x \in X$. It follows that $x^* \in G(x^*) \subseteq K(x^*)$. Thus, $x^* \in \Delta$, which implies that $x^* \in coP(x^*)$ by the definition of $G(x)$. As $coP(x) \subseteq K(x)$ for all $x \in X$, we have $x^* \in coP(x^*) \subseteq K(x^*)$, which implies that there exist $x_1, \dots, x_k \in P(x^*)$ such that $x^* \in sco\{x_1, \dots, x_k\}$, so it follows from 0-diagonally quasi-concavity of φ in assumption (i) that there exists $x \in \{x_1, \dots, x_k\}$ such that $\varphi(x,x^*) \leq 0$. However, $\{x_1, \dots, x_k\} \subseteq P(x^*)$ implies that $\varphi(x_i,x^*) > 0$ for all $1 \leq i \leq k$, a contradiction. Thus, there exists $x' \in X$ such that $G(x') = \emptyset$. Since $K(x) \neq \emptyset$ for all $x \in X$ by assumption (ii), we must have $x' \in \Delta$, that is, $x' \in K(x')$ and $coP(x') = \emptyset$, which implies that $\varphi(x,x') \leq 0$ for each $x \in K(x')$. Therefore x' is a solution to (2.3). □

The compactness condition on X in Theorem 2.3 can be relaxed as follows.

Theorem 2.4. *Let X be a non-empty convex subset of a Hausdorff topological vector space and assume that*

(i) $\varphi(x,z)$ is 0-diagonally quasi-concave in x;

(ii) $K : X \mapsto 2^X$ is non-empty convex valued and has open lower sections;

(iii) the mapping $P : X \mapsto 2^X$ defined by $P(z) = \{x \in K(z) | \varphi(x,z) > 0\}$ for each $z \in X$ has open lower sections;
(iv) $\Delta = \{x \in X | x \in K(x))\}$ is closed and compact;
(v) there exists a non-empty $X_0 \subseteq X$ such that X_0 is contained in a compact convex subset X' of X and the set $D = \cap_{x\in X_0}(K^{-1}(x))^c$ is compact.
Then, the quasi-Ky Fan minimax inequality (2.3) has a solution.

Proof. Define the correspondence $G : X \mapsto 2^X$ as in the proof of Theorem 2.3. The proof is almost the same as the proof of Theorem 2.3 except we need to verify that the condition (3) in [33, Theorem 2.9] holds for $G(x)$ as follows: For each $y \in X$, $(G^{-1}(y))^c = (coP^{-1}(y))^c \cap ((K^{-1}(y))^c \cup \Delta)$ or $(K^{-1}(y))^c \cup \Delta$, which implies that $(G^{-1}(y))^c \subseteq (K^{-1}(y))^c \cup \Delta$. It follows that

$$\cap_{y\in X_0}(G^{-1}(y))^c \subseteq \cap_{y\in X_0}[(K^{-1}(y))^c \cup \Delta] = [\cap_{y\in X_0}(K^{-1}(y))^c] \cup \Delta.$$

Since $\cap_{y\in X_0}(K^{-1}(y))^c$ and Δ are compact by assumptions (iv) and (v), $[\cap_{y\in X_0}(K^{-1}(y))^c] \cup \Delta$ is compact. Since G has open lower sections (as shown in the proof of Theorem 2.3), we have that $G^{-1}(y)$ is open, so $(G^{-1}(y))^c$ is closed for each $y \in X$, which implies that $\cap_{y\in X_0}(G^{-1}(y))^c$ is closed. Thus, $\cap_{y\in X_0}(G^{-1}(y))^c$ is compact as it is a closed subset of a compact set.

Suppose that $G(x) \neq \emptyset$ for all $x \in X$. Then it follows from [33, Theorem 2.9] that G has a fixed point $x^* \in X$, i.e., $x^* \in G(x^*)$. The rest of the proof is the same as that in the proof of Theorem 2.3. □

3. Existence of Solutions for Variational Inequalities

Variational inequalities and complementarity problems are first defined on Euclidean spaces $\mathbb{R}^n$ in [8] using inner products (dot products). One can extend variational inequalities and complementary problems to any real topological vector space V with topological dual V^*: Denote the duality pairing between X and X^* by $\langle\cdot,\cdot\rangle$, let X be a non-empty convex subset of V, and let $F : V \longrightarrow V^*$ be an operator; see, for example, [6], [11], and [14] (Euclidean spaces $\mathbb{R}^n$ and Hilbert spaces with inner products $\langle\cdot,\cdot\rangle$, Banach spaces, and Hausdorff topological vector spaces are special cases).

Problem (2.2) and Ky Fan minimax inequality (2.1) are closely related to the following problems:

(I) *Variational Inequality*: Let X be a non-empty convex subset of a real topological vector space V. For a mapping $F : X \longrightarrow V^*$, the variational inequality $VI(X,F)$ is:

$$\text{Find } x^* \in X \text{ such that } \langle F(x^*), (y - x^*)\rangle \geq 0 \text{ for all } y \in X. \tag{3.1}$$

By taking $\varphi(x,y) = \langle F(x), (y-x)\rangle$, one has that (2.2) and (3.1) are equivalent.

(II) *Complementarity Problem*: Given a convex cone X of a real topological vector space V, let $F : X \longrightarrow V^*$ be a mapping and $X^* = \{x \in \mathscr{H} | \langle x,y\rangle \geq 0 \text{ for all } y \in X\}$. The complementarity problem $CP(X,F)$ is:

$$\text{Find } x^* \in X \text{ such that } F(x^*) \in X^* \text{ and } \langle F(x^*), x^*\rangle = 0. \tag{3.2}$$

By taking $\varphi(x,y) = \langle F(x), (y-x)\rangle$, one sees that (2.2), (3.1), and (3.2) are equivalent ([8, Proposition 1.1.3]).

The next definition for monotonicity of a bifunction $f(x,y)$ is equivalent to Definition 2.3.1 for $F(x)$ in [8] with $f(x,y) = \langle F(x), (y-x)\rangle$.

Definition 3.1. Let X be a subset of a topological vector space and $f : X \times X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a bifunction. Then f is said to be

(i) *strongly monotone* on X with parameter $\mu > 0$ if, for all $x, y \in X$, $f(x,y) + f(y,x) \leq -\mu ||y - x||^2$;

(ii) *monotone* on X if, for all $x, y \in X$, $f(x,y) + f(y,x) \leq 0$;

(iii) *pseudo-monotone* on X if $f(x,y) \geq 0$ implies $f(y,x) \leq 0$ for all $x, y \in X$;

(iv) a mapping $F : X \longrightarrow V^*$ is said to be *Lipschitz continuous* on X with constant $L > 0$ if $||F(x) - F(y|| \leq L||x - y||$ for all $x, y \in X$.

Clearly, the pseudo-monotonicity implies that every solution for problem (2.2) is a solution to Ky Fan minimax inequality (2.1). By [34, Proposition 3.2], one sees that Ky Fan minimax inequality (2.1) is equivalent to problem (2.2) under the conditions that $\varphi(x,y)$ is upper semi-continuous in x for all $y \in X$, semistrictly quasiconvex, and lower semicontinuous in y for all $x \in X$, and pseudo-monotone.

Note that an $x^* \in X$ is a solution to (2.2) with $\varphi(x,y) = \phi(x,y)$ if and only if x^* is a solution to (2.1) with $\varphi(x,y) = -\phi(y,x)$. Through the equivalence (I), Theorems 2.1 and 2.2 imply the following two characterizations for solutions of variational inequalities $VI(X,F)$ which imply existing existence theorems, including those in [8], [10], and [34].

Theorem 3.1. *Assume that X is a non-empty convex subset of a Hausdorff topological vector space, and let $\varphi(x,y) = \langle F(x), (y - x)\rangle$. The variational inequality $VI(X,F)$ has a solution if and only if there exists a non-empty convex compact subset D of X such that the restricted mapping $\varphi|_{D\times X} : D \times X \longrightarrow \mathbb{R}$ is diagonally transfer upper continuous on D and diagonally transfer quasi-convex on X.*

Theorem 3.2. *Assume that X is a non-empty convex subset of a Hausdorff topological vector space, and let $\varphi(x,y) = \langle F(x), (y - x)\rangle$. The variational inequality $VI(X,F)$ has a solution if and only if there exists a non-empty convex compact subset D of X such that*

(i) D has the fixed point property (continuous functions $f : D \longrightarrow D$ have fixed points);

(ii) the restricted mapping $\varphi|_{D\times X} : D \times X \longrightarrow R$ is diagonally transfer upper continuous on D and $\mathscr{C}$-quasi-convex on X.

Recall that the well-known Schauder's fixed point theorem states that any continuous mapping $f : X \longrightarrow X$ on a non-empty convex compact subset X of a normed vector space has a fixed point in X. Note that Banach spaces and normed spaces are special Hausdorff topological vector spaces. Theorem 3.2 has the next corollary.

Corollary 3.1. *Assume that X is a non-empty closed convex subset of a normed vector space and $\varphi(x,y) = \langle F(x), (y - x)\rangle$. The variational inequality $VI(X,F)$ has a solution if and only if there exists a non-empty convex compact subset D of X such that the restricted mapping $\varphi|_{D\times X} : D \times X \longrightarrow R$ is diagonally transfer upper continuous on D and $\mathscr{C}$-quasi-convex on X.*

We denote by V_c^* the subspace of the dual space V^* which consists of all continuous linear functionals on V. Theorem 3.1 implies the next existence theorem for solutions of $VI(X,F)$ which implies existing existence theorems, including those in [8], [10], and [34].

Theorem 3.3. *Assume that X is a non-empty convex compact subset of a normed vector space V, $F : X \longrightarrow V_c^*$ is a continuous mapping, and $\varphi(x,y) = \langle F(x), (y - x)\rangle$. Then the variational inequality $VI(X,F)$ has a solution.*

Proof. By applying Theorem 3.1 with $D = X$, we only need to show that $\varphi(x,y) = \langle F(x),(y-x)\rangle$ is diagonally transfer upper continuous in x on X and diagonally transfer quasi-convex in y on X. Denote the linear continuous functional $f_x = F(x) \in V_c^*$. By the definition, one sees that $\varphi(x,y) = \langle F(x),(y-x)\rangle = f_x(y-x) = f_x(y) - f_x(x)$. We first show that $\varphi(x,y) = f_x(y) - f_x(x)$ is diagonally transfer quasi-convex in y (i.e. $-\varphi(x,y)$ is diagonally transfer quasi-concave in y; see Definition 2.2). For any $\{y_1,\dots,y_k\} \subseteq X$, take $\{z_1,\dots,z_k\} = \{y_1,\dots,y_k\}$. Let $x \in co\{y_1,\dots,y_k\}$ and $x = \lambda_1 y_1 + \cdots + \lambda_k y_k$ with each $\lambda_i \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$ (without loss of generality, assume $\lambda_i > 0$ for all $1 \leq i \leq k$). Since f_x is linear, we have

$$
\begin{aligned}
0 &= \langle F(x),(x-x)\rangle = \varphi(x,x) = \varphi(x,\lambda_1 y_1 + \cdots + \lambda_k y_k) = \langle F(x),(\lambda_1 y_1 + \cdots + \lambda_k y_k - x)\rangle \\
&= \langle F(x),(\lambda_1(y_1-x) + \cdots + \lambda_k(y_k-x))\rangle = f_x(\lambda_1(y_1-x) + \cdots + \lambda_k(y_k-x)) \\
&= \sum_{i=1}^k \lambda_i f_x(y_i - x) \leq \left(\sum_{i=1}^k \lambda_i\right) \max_{1\leq i\leq k} f_x(y_i-x) = \max_{1\leq i\leq k} f_x(y_i-x).
\end{aligned}
$$

Thus, there exists $y_j \in \{y_1,\dots,y_k\}$ such that $f_x(y_j - x) \geq 0$, i.e., $\varphi(x,y_j) = \langle F(x),(y_j-x)\rangle \geq 0$. Thus $\varphi(x,y) = f_x(y) - f_x(x)$ is diagonally transfer quasi-convex in y .

Now, we show that $\varphi(x,y) = f_x(y) - f_x(x)$ is diagonally transfer upper continuous in x on X (i.e., $-\varphi(x,y)$ is diagonally transfer lower continuous in x; see Definition 2.1). Assume that $\varphi(x,y) < 0$ for $x \in X$ and denote $-b = \varphi(x,y)$ with $b > 0$. Since a linear functional in a normed vector space is continuous if and only if it is bounded, f_x is continuous implies that there exists $M_1 > 0$ such that $|f_x(y)| \leq M_1||y||$ for every $y \in X$. Recall that, in a normed vector space V, $|f(x)| \leq ||f||||x||$ for any $x \in V$ and any continuous linear function $f \in V^*$, and the compact subset X is bounded, i.e., there exists $M_2 > 0$ such that $||x|| \leq M_2$ for all $x \in X$. It follows that

$$
\begin{aligned}
|\varphi(x,y) - \varphi(x',y)| &= |f_x(y) - f_x(x) - (f_{x'}(y) - f_{x'}(x'))| \\
&\leq |f_x(y) - f_{x'}(y)| + |f_x(x) - f_{x'}(x'))| \\
&\leq |f_x(y) - f_{x'}(y)| + |f_x(x) - f_x(x'))| + |f_x(x') - f_{x'}(x'))| \\
&= |[f_x - f_{x'}](y)| + |f_x((x) - (x'))| + |[f_x - f_{x'}](x'))| \\
&\leq ||f_x - f_{x'}||||y|| + M_1||x - x'|| + ||f_x - f_{x'}||||x'|| \\
&\leq 2M_2||f_x - f_{x'}|| + M_1||x - x'||.
\end{aligned}
\tag{3.3}
$$

Take $M = \max\{M_1,M_2\}$ and $\varepsilon = \frac{b}{3M}$. Since $f_x = F(x)$ is continuous at x, one sees that there exists an open neighborhood O_x of x in X such that $||x-x'|| < \varepsilon$ and $||f_x - f_{x'}|| < \varepsilon$ for all $x' \in O_x$. By (3.3), we obtain

$$
|\varphi(x,y) - \varphi(x',y)| = |f_x(y) - f_x(x) - (f_{x'}(y) - f_{x'}(x'))| < 2M\varepsilon + M\varepsilon = 3M\varepsilon = b,
$$

which implies that $\varphi(x',y) < 0$ for all $x' \in O_x$, that is, $\varphi(x,y) = f_x(y) - f_x(x)$ is diagonally transfer upper continuous in x on X. Therefore, it follows from Theorem 3.1 that variational inequality $VI(X,F)$ has a solution. □

Let X be a non-empty convex subset of a real topological vector space V. Let $F : X \mapsto 2^{V^*}$ be a set-valued mapping. The *generalized variational inequality problem* $GVI(X,F)$ for the mapping F (see [6]) is:

$$
\text{Find } x^* \in X \text{ and } y^* \in F(x^*) \text{ such that } \langle y^*,(y-x^*)\rangle \geq 0 \text{ for all } y \in X. \tag{3.4}
$$

Clearly, when F is a single-valued mapping (a function), then $y^* = F(x^*)$ and (3.4) is reduced to variational inequality $VI(X,F)$ in (3.1).

The existence of solutions for generalized variational inequalities $GVI(X,F)$ has been studied in the literature; see, e.g., [6]. Recall that a normed vector space is a Hausdorff topological vector space and the Theorem 3.1 in [32] states: Given a paracompact subset X of a Hausdorff space and a topological space Y, any mapping $\varphi : X \mapsto 2^Y$ that is non-empty convex valued and has open lower sections admits a continuous selection, i.e., there exists a continuous function $f : X \longrightarrow Y$ such that $f(x) \in \varphi(x)$ for all $x \in X$. Theorem 3.3 gives directly the next existence result for solutions of $GVI(X,F)$ which implies existing existence theorems for $GVI(X,F)$, including those in [6].

Theorem 3.4. *Assume that X is a non-empty convex compact subset of a normed vector space V, and let $F : X \mapsto 2^{V_c^*}$ be a set-valued mapping, which is non-empty convex valued and has open lower sections. Then the generalized variational inequality $GVI(X,F)$ has a solution.*

The following quasi-variational inequality (QVI) was introduced in [9] to formulate the generalized Nash equilibrium game (see also [8] and [13]).

Definition 3.2. For any convex subset X of a topological vector space V, a quasi-variational inequality $QVI(K,F)$ is defined by a set-valued function $K : X \mapsto 2^X$ and a mapping $F : X \longrightarrow V^*$ as the following problem:

$$\text{Find } x^* \in K(x^*) \text{ such that } \langle F(x^*), (y - x^*) \rangle \geq 0 \text{ for all } y \in K(x^*). \tag{3.5}$$

Let X be a non-empty convex subset of a topological vector space. Assume that $\varphi(x,y) : X \times X \longrightarrow \mathbb{R}$ is a function, and let $K : X \mapsto X$ be a set-valued mapping. Similar to (I), one can see that quasi-variational inequality $QVI(K,F)$ is equivalent to the problem:

$$\begin{cases} \text{Find } x^* \in X & \text{such that} \\ x^* \in K(x^*) & \text{and} \\ \varphi(x^*, y) \geq 0 & \forall y \in K(x^*). \end{cases} \tag{3.6}$$

Clearly, an $x^* \in X$ is a solution to (3.6) with $\varphi(x,y) = \phi(x,y)$ if and only if x^* is a solution to (2.5) with $\varphi(x,y) = -\phi(y,x)$. By the equivalence of $QVI(K,F)$ in (3.5) and problem (3.6), one has the next two existence theorems for solutions of $QVI(K,F)$ from Theorems 2.3 and 2.4, which generalize those existence theorems for QVI in [9] and [11].

Theorem 3.5. *Assume that X is a non-empty convex compact subset of a Hausdorff topological vector space V, $F : X \longrightarrow V^*$ is a mapping, and $\varphi(x,y) = \langle F(x), (y - x) \rangle$. Suppose that*

(i) $\varphi(x,y)$ is 0-diagonally quasi-convex in y;

(ii) $K : X \mapsto 2^X$ is non-empty convex valued and has open lower sections;

(iii) the mapping $P : X \mapsto 2^X$ defined by $P(x) = \{y \in K(x) | \varphi(x,y) < 0\}$ for each $x \in X$ has open lower sections;

(iv) $\Delta = \{x \in X | x \in K(x))\}$ is closed.

Then, the quasi-variational inequality $QVI(K,F)$ has a solution.

Theorem 3.6. *Assume that X is a non-empty convex subset of a Hausdorff topological vector space V, $F : X \longrightarrow V^*$ is a mapping, and $\varphi(x,y) = \langle F(x), (y - x) \rangle$. Suppose that*

(i) $\varphi(x,y)$ is 0-diagonally quasi-convex in y;

(ii) $K : X \mapsto 2^X$ is non-empty convex valued and has open lower sections;

(iii) the mapping $P : X \mapsto 2^X$ defined by $P(x) = \{y \in K(x) | \varphi(x,y) < 0\}$ for each $x \in X$ has open lower sections;

(iv) $\Delta = \{x \in X | x \in K(x))\}$ is closed and compact;

(v) there exists a non-empty $X_0 \subseteq X$ such that X_0 is contained in a compact convex subset X' of X and the set $D = \cap_{x \in X_0} (K^{-1}(x))^c$ is compact.

Then, the quasi-variational inequality $QVI(K,F)$ has a solution.

4. Existence of Strong Nash Equilibria in Generalized Games

In this section, we apply Theorems 2.3 and 2.4 to derive the existence of strong Nash equilibria in generalized games. Let $N = \{1,2,\dots,n\}$ be the set of n players. For each $i \in N$, X_i is the strategy space for player i and let $X = \prod_{i \in N} X_i$ be the set of strategy profiles. A strategy profile $x \in X$ is denoted by (x_i, x_{-i}), where $x_i \in X_i$ and $x_{-i} \in X_{-i} = \prod_{j \neq i} X_j$. A *game* G on player set N is $(X_i, u_i)_{i \in N}$ with the strategy space X_i and the payoff function $u_i : X \longrightarrow \mathbb{R}$ for each $i \in N$.

A *generalized game* with payoff functions $\Gamma = (X_i, F_i, u_i)_{i \in N}$ is a game with player set N such that each player i has strategy space X_i, which is a nonempty subset of a Hausdorff toplogical vector space; each player i has a payoff function $u_i : X \longrightarrow \mathbb{R}$ that depends on his or her own variables x_i as well as on the variables x_{-i} of all other players; and each player i's strategy must belong to a set $F_i(x) = F_i(x_{-i}) \subseteq X_i$ that depends on the rival players' strategies x_{-i}. Clearly, a game $(X_i, u_i)_{i \in N}$ is a special generalized game with $F_i(x) = X_i$ for all $x \in X$ and each $i \in N$. A Nash equilibrium of a generalized game is defined as follows (see [9] and [17]).

Definition 4.1. A vector $x^* = (x_i^*)_{i \in N} \in X = \prod_{i \in N} X_i$ is called a *Nash equilibrium* of a generalized game Γ if $x^* \in F(x^*) = \prod_{i \in N} F_i(x^*)$ and

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*) \text{ for all } x_i \in F_i(x^*) \text{ and for every } i \in N. \tag{4.1}$$

A generalized game Γ on player set N with preferences is a tuple $\Gamma = (X_i, F_i, P_i)_{i \in N}$, where, for each $i \in N$, X_i is non-empty strategy space for player i, which is a subset of a Hausdorff topological vector space, $F_i : X \mapsto X_i$ is the feasible strategy mapping, and $P_i : X \mapsto X$ is the preference mapping for player i.

Remark 4.1. A generalized game $\Gamma = (X_i, F_i, u_i)_{i \in N}$ with payoff functions u_i is a special generalized game with preferences $P_i : X \longrightarrow X$ defined as follows: $P_i(x) = \{y | u_i(y) > u_i(x)\}$ for every $x \in X$ and for each $i \in N$. Clearly, $\Gamma = (X_i, F_i, u_i)_{i \in N}$ is equivalent to $\Gamma = (X_i, F_i, P_i)_{i \in N}$.

Note that a normal-form game $\Gamma = (X_i, P_i)_{i \in N}$ is a special generalized game $\Gamma = (X_i, F_i, P_i)_{i \in N}$ with $F_i(x) = X_i$ for all $x \in X$ and each $i \in N$. The next concept of strong Nash equilibrium for generalized games extends naturally the corresponding concept for normal-form games given in [25] and [35].

Definition 4.2. An $x^* = (x_i^*)_{i \in N} \in X$ is said to be a *strong Nash equilibrium* for a generalized game $\Gamma = (X_i, F_i, P_i)_{i \in N}$ if $x^* \in F(x^*) = \prod_{i \in N} F_i(x^*)$ and for every $S \in \mathscr{N}$, there exists no $x_S \in F_S(x^*) = \prod_{i \in S} F_i(x^*)$ satisfying

$$(x_S, x_{-S}^*) \in P_i(x^*) \text{ for every } i \in S. \tag{4.2}$$

Clearly, a strong Nash equilibrium is a Nash equilibrium in a generalized game.

As in [35], we define the function $\Phi : X \times X \longrightarrow \mathbb{R}$ as follows:

$$\Phi(x,z) = \sum_{S\in\mathcal{N}} \min_{i\in S} \widetilde{P}_i((x_S, z_{-S}), z), \tag{4.3}$$

where $\widetilde{P}_i((x_S, z_{-S}), z) = 1$ if $(x_S, z_{-S}) \in P_i(z)$, and $\widetilde{P}_i((x_S, z_{-S}), z) = 0$ otherwise.

Proposition 4.1. *An $x^* \in X$ is a strong Nash equilibrium of the generalized game $\Gamma = (X_i, F_i, P_i)_{i\in N}$ if and only if $x^* \in F(x^*)$ and $\Phi(x, x^*) \le 0$ for all $x \in F(x^*)$.*

Proof. The necessity follows directly from the definition. For the sufficiency, let $x^* \in F(x^*)$ and $\Phi(x, x^*) \le 0$ for all $x \in F(x^*)$. Note that, for each $S \in \mathcal{N}$ and any $x \in X$ satisfying $x_S \in F_S(x^*)$, we have $(x_S, x^*_{-S}) \in F(x^*)$, so $\Phi((x_S, x^*_{-S}), x^*) \le 0$. It follows from (4.3) that for every $S \in \mathcal{N}$ and all $x_S \in F_S(x^*)$, $(x_S, x^*_{-S}) \notin P_i(x^*)$ for at least one $i \in S$, that is, x^* is a strong Nash equilibrium of Γ. □

The following existence theorem for strong Nash equilibria in generalized games follows from Theorem 2.3 and Proposition 4.1.

Theorem 4.1. *Assume that $\Gamma = (X_i, F_i, P_i)_{i\in N}$ is a generalized game such that, for each $i \in N$,*
(i) X_i is a non-empty convex and compact subset of a Hausdorff topological vector space;
(ii) the mapping $F_i : X \mapsto X_i$ is non-empty convex valued and has open lower sections;
(iii) the mapping $T : X \mapsto 2^X$ defined by $T(z) = \{x \in F(z) | \Phi(x,z) > 0\}$ for each $z \in X$ has open lower sections and $\Phi(x,z)$ is 0-diagonally quasi-concave;
(iv) $\Delta = \{x \in X | x \in F(x)\}$ is closed.
Then Γ has a strong Nash equilibrium.

Proof. Since X_i is compact for each $i \in N$, it follows from Tychonoff theorem that $X = \prod_{i\in N} X_i$ is compact. Moreover, X_i is non-empty convex for each $i \in N$ implies that X is non-empty convex. By assumption (ii), for each $i \in N$, F_i is non-empty convex valued and has open lower sections, i.e., $F_i^{-1}(y_i) = \{x \in X | y_i \in F_i(x)\}$ is open for each $y_i \in X_i$. It follows that $F(x) = \prod_{i\in N} F_i(x)$ is non-empty convex for each $x \in X$ and $F^{-1}(y) = \{x \in X | y \in F(x)\} = \cap_{i\in N}\{x \in X | y_i \in F_i(x)\}$ is open for each $y \in X$, that is, F has open lower sections. Since $T(z) = \{x \in F(z) | \Phi(x,z) > 0\}$ has open lower sections and $\Phi(x,z)$ is 0-diagonally quasi-concave by assumption (iii), it follows from Theorem 2.3 and Proposition 4.1 that Γ has a strong Nash equilibrium. □

As a consequence of Theorem 4.1, together with Remark 4.1, we have the following existence theorem for strong Nash equilibria in generalized games $(X_i, F_i, u_i)_{i\in N}$.

Theorem 4.2. *Assume that $\Gamma = (X_i, F_i, u_i)_{i\in N}$ is a generalized game such that, for each $i \in N$,*
(i) X_i is a non-empty convex and compact subset of a Hausdorff topological vector space;
(ii) the mapping $F_i : X \mapsto X_i$ is non-empty convex valued and has open lower sections;
(iii) u_i is continuous in x and quasi-concave;
(iv) $\Delta = \{x \in X | x \in F(x)\}$ is closed.
Then Γ has a strong Nash equilibrium.

Proof. Similar to the proof of Theorem 4.1, we have that X is non-empty convex and compact, and F is non-empty convex valued and has open lower sections. By Remark 4.1, one can define preference mapping $P_i : X \mapsto 2^X$ for each $i \in N$ by

$$P_i(x) = \{y \in X | u_i(y) > u_i(x)\} \text{ for every } x \in X.$$

Then $\Gamma = (X_i, F_i, u_i)_{i\in N}$ is equivalent to $\Gamma = (X_i, F_i, P_i)_{i\in N}$. Since u_i is continuous by assumption (iii), $u_i(y) - u_i(x)$ is continuous which implies that P_i has open lower sections for each $i \in N$.

Next, we show that $\Phi(x,z)$ is 0-diagonally quasi-concave. Let $\{x_1,\dots,x_k\} \subseteq X$ and $z \in co\{x_1,\dots,x_k\}$. Then $z = \sum_{j=1}^{l} \lambda_j x_{h_j}$ with each $\lambda_j > 0$ and $\sum_{j=1}^{l} \lambda_j = 1$. We need to show that $\Phi(x,z) \le 0$ for some $x \in \{x_{h_1},\dots,x_{h_l}\}$. Let $\phi_i^S(x,z) = u_i(x_S, z_{-S})$ for each $S \in \mathscr{N}$. For each $i \in N$, since u_i is quasi-concave, $\phi_i^S(x,z)$ is quasi-concave in x. It follows that, for each $i \in N$ and every $S \in \mathscr{N}$,

$$\phi_i^S(z,z) = \phi_i^S\left(\sum_{j=1}^{l} \lambda_j x_{h_j}, z\right) \ge \min_{1\le j\le l} \phi_i^S(x_{h_j}, z) = \phi_i^S(x_i^S, z),$$

for some $x_i^S \in \{x_{h_{j_1}},\dots,x_{h_{j_n}}\}$. Let $x \in \{x_{h_{j_1}},\dots,x_{h_{j_n}}\}$ be such that $\phi_i^S(x,z) \le \phi_i^S(x_i^S,z)$ for all $i \in N$ and all $S \in \mathscr{N}$. Then it follows that, for all $S \in \mathscr{N}$ and every $i \in N$,

$$\phi_i^S(x,z) \le \phi_i^S(z,z) \text{ or } u_i(x_S, z_{-S}) \le u_i(z),$$

which implies that $\Phi(x,z) \le 0$ by (4.3). Thus, $\Phi(x,z)$ is 0-diagonally quasi-concave.

Define the mapping $T : X \mapsto 2^X$ by $T(z) = \{x \in F(z) | \Phi(x,z) > 0\}$ for each $z \in X$. We claim that T has open lower sections. In fact, for each $x \in X$, one can show that $T^{-1}(x) = \{z \in X | x \in T(z)\}$ is open as follows: Let $z \in T^{-1}(x)$. Then $x \in T(z)$ means that $x \in F(z)$ and $\Phi(x,z) > 0$. Since F has open lower sections, $F^{-1}(x)$ is open and $z \in F^{-1}(x)$ implies that there exists an open neighborhood O'_z of z such that $O'_z \subseteq F^{-1}(x)$.

On the other hand, by (4.3), $\Phi(x,z) > 0$ implies that there exists $S \in \mathscr{N}$ such that $(x_S, z_{-S}) \in P_i(z)$ for all $i \in S$. Since $P_i(z)$ has open lower sections for each $i \in S$, $z \in \{y \in X | (x_S, z_{-S}) \in P_i(y)\}$ means that there exists an open neighborhood O_z^i of z such that $O_z^i \subseteq \{y \in X | (x_S, z_{-S}) \in P_i(y)\}$ for each $i \in S$. Take $O_z = O'_z \cap (\cap_{i\in S} O_z^i)$. Then O_z is an open neighborhood of z such that $O_z \subseteq F^{-1}(x)$ and $O_z \subseteq \{y \in X | (x_S, z_{-S}) \in P_i(y)\}$ for every $i \in S$. Thus, we have that, for each $z' \in O_z$, $z' \in F^{-1}(x)$ which implies that $x \in F(z')$, and $(x_S, z_{-S}) \in P_i(z')\}$ which implies that $u_i(x_S, z_{-S}) > u_i(z')$ for each $i \in S$. Since u_i is continuous for each $i \in N$, by making the open neighborhood O_z of z small enough if necessary, one may assume that $u_i(x_S, z'_{-S}) > u_i(z')$ for all $z' \in O_z$ and for each $i \in S$, which implies that $\Phi(x,z') > 0$ for all $z' \in O_z$. It follows that $O_z \subseteq T^{-1}(x)$, that is, $T^{-1}(x)$ is open. Thus, T has open lower sections. Now, it follows from Theorem 4.1 that Γ has a strong Nash equilibrium. □

Since a strong Nash equilibrium is a Nash equilibrium in a generalized game, Theorem 4.2 implies [9, Theorem 1] (note that the upper semicontinuity of F_i's implies condition (iv)). Recall that a game $\Gamma = (X_i, u_i)_{i\in N}$ is a special generalized game $\Gamma = (X_i, F_i, u_i)_{i\in N}$ with $F_i(x) = X_i$ for each $i \in N$ and all $x \in X$ which satisfies assumption (ii) and has $\Delta = X$. Theorem 4.2 has the next corollary which strengthens [25, Theorem 3.1].

Corollary 4.1. *Assume that $\Gamma = (X_i, u_i)_{i\in N}$ is a game such that, for each $i \in N$, X_i is a non-empty convex and compact subset of a Hausdorff topological vector space and u_i is continuous in x and quasi-concave. Then Γ has a strong Nash equilibrium.*

Similar to the proof of Theorem 4.2, by applying Theorem 2.4 instead of Theorem 2.3, one obtains the following theorem, which relaxes the compactness of X_i in Theorem 4.2.

Theorem 4.3. *Assume that $\Gamma = (X_i, F_i, u_i)_{i\in N}$ is a generalized game such that, for each $i \in N$,*

(i) X_i is a non-empty convex subset of a Hausdorff topological vector space;

(ii) the mapping $F_i : X \mapsto X_i$ is non-empty convex valued and has open lower sections;
(iii) u_i is continuous in x and quasi-concave;
(iv) $\Delta = \{x \in X | x \in F(x)\}$ is closed and compact;
(v) there exists a non-empty $X_0 \subseteq X$ such that X_0 is contained in a compact convex subset of X and the set $D = \cap_{x\in X_0}(F^{-1}(x))^c$ is compact.

Then Γ has a strong Nash equilibrium.

5. Uniqueness of Strong Nash Equilibria in Generalized Games

Theorem 4.2 provides the existence for strong Nash equilibrium in certain generalized games. In this section, we give sufficient conditions for the uniqueness of the strong Nash equilibrium in certain generalized games with strategy space in $\mathbb{R}^m$. Recall that a strong Nash equilibrium is a Nash equilibrium in a generalized game. The uniqueness of the Nash equilibrium together with the existence of the strong Nash equilibrium imply the uniqueness of the strong Nash equilibrium in a generalized game. Recall from [8, Theorem 2.3.3(b)] that the strong monotonicity of $F(x)$ ensures the uniqueness of solutions for variational inequality $VI(X,F)$.

Let $\Gamma = (X_i, F_i, u_i)_{i\in N}$ be a generalized game such that each u_i is differentiable, and denote $X = \prod_{i\in N} X_i$ and $F = \prod_{i\in N} F_i$. Define

$$\Phi(x) \equiv (-\bigtriangledown_{x_i} u_i(x))_{i\in N} \text{ for each } x \in X, \tag{5.1}$$

where $\bigtriangledown_{x_i} u_i(x)$ is the gradient of the payoff function u_i with respect to x_i. Then it is easy to check that x^* is a Nash equilibrium of a generalized game $\Gamma = (X_i, F_i, u_i)_{i\in N}$ if and only if x^* solves the quasi-variational inequality $QVI(F,\Phi)$ defined by (3.4) (see [8] and [9]), where $\langle \Phi(x), (y-x)\rangle$ is the inner product.

The next uniqueness theorem for quasi-variational inequality $QVI(F,\Phi)$ is [7, Theorem 1].

Theorem 5.1. *(Dreves, 2016). Let X is a non-empty convex subset of a Euclidean space. Suppose that the following assumptions hold:*
(i) Φ defined by (5.1) is Lipschitz continuous with constant $L > 0$ and strongly monotone with modulus $\mu > 0$ and $\gamma = \frac{L}{\mu} \geq 1$;
(ii) for some constant $0 < \alpha < \frac{1}{\gamma(\gamma+\sqrt{\gamma^2-1})}$

$$||Proj_{F(x)}[z] - Proj_{F(y)}[z]|| \leq \alpha ||x-y|| \ \forall x, y, x \in X,$$

where $Proj_{F(x)}[z]$ is the Euclidean projection of point z onto $F(x)$. Then there is a unique solution to the quasi-variational inequality $QVI(F,\Phi)$.

Now, Theorems 4.2 and 5.1 together imply the following uniqueness of strong Nash equilibria in generalized games.

Theorem 5.2. *Assume that $\Gamma = (X_i, F_i, u_i)_{i\in N}$ is a generalized game such that, for each $i \in N$,*
(i) X_i is a non-empty convex and compact subset of a Euclidean space;
(ii) the mapping $F_i : X \longrightarrow 2^{X_i}$ is non-empty convex valued and lower semicontinuous;
(iii) u_i is continuous in x and quasi-concave;
(iv) $\Delta = \{x \in X | x \in F(x)\}$ is closed;
(v) Φ defined by (5.1) is Lipschitz continuous with constant $L > 0$ and strongly monotone with modulus $\mu > 0$ and $\gamma = \frac{L}{\mu} \geq 1$;

(vi) for some constant $0 < \alpha < \frac{1}{\gamma(\gamma+\sqrt{\gamma^2-1})}$

$$||Proj_{F(x)}[z] - Proj_{F(y)}[z]|| \leq \alpha||x-y|| \ \forall x, y, x \in X,$$

where $Proj_{F(x)}[z]$ *is the Euclidean projection of point* z *onto* $F(x)$.

Then Γ *has a unique strong Nash equilibrium.*

Since a game $\Gamma = (X_i, u_i)_{i \in N}$ is a special generalized game $\Gamma = (X_i, F_i, u_i)_{i \in N}$ with $F_i(x) = X_i$ for each $i \in N$ and all $x \in X$, we have $F(x) = \prod_{i \in N} F_i(x) = \prod_{i \in N} X_i = X$ for every $x \in X$, which implies that $\Delta = X$ and $QVI(F, \Phi)$ is the same as $VI(F, \Phi)$ in this case. Thus, Corollary 4.1 and [8, Theorem 2.3.3 (b)] imply the next consequence for the uniqueness of strong Nash equilibria in standard normal-form games.

Theorem 5.3. *Assume that* $\Gamma = (X_i, u_i)_{i \in N}$ *is a game such that, for each* $i \in N$,

(i) X_i *is a non-empty convex and compact subset of a Euclidean space;*

(ii) u_i *is continuous in* x *and quasi-concave;*

(iii) Φ *defined by (5.1) is strongly monotone.*

Then Γ *has a unique strong Nash equilibrium.*

6. Concluding Remarks

In Sections 2 and 3, we provided necessary and sufficient conditions for the existence of solutions for variational inequalities, and sufficient conditions for the existence of solutions for quasi-Ky Fan minimax inequalities, quasi-variational inequalities, and generalized variational inequalities. By applying these existence theorems, we established sufficient conditions for the existence of strong Nash equilibria in generalized games in Section 4. We provide sufficient conditions for the uniqueness of strong Nash equilibria in certain games and generalized games in Section 5.

By [8, Theorem 2.3.3(b)], the strong monotonicity of $F(x)$ ensures the uniqueness of solutions to variational inequality $VI(X, F)$. For quasi-variational inequalities $QVI(K, F)$, many do not have unique solutions. For example, QVIs coming from reformulations of jointly convex generalized Nash equilibrium problems typically have non-unique solutions; see [36]. There are also classes of problems for which uniqueness can be shown; see [7] and [14]. It is interesting to see which uniqueness results for variational inequalities (VIs) can be generalized for QVIs.

Appendix 1

Proof of Theorem 2.2. Necessity: Suppose that problem (2.2) has a solution $x^* \in X$. Then we have $\varphi(x^*,x) \geq 0$ for all $x \in X$. Take $D = \{x^*\}$. Then it is clear that D has the fixed point property and $\varphi|_{D\times X} : D \times X \longrightarrow \mathbb{R}$ is diagonally transfer upper continuous on D. We now prove that $\varphi|_{D\times X}$ is $\mathscr{C}$-quasi-convex on X. Let $\{x^1,\dots,x^k\}$ be any subset of X. Define the continuous mapping $\phi_n : \Delta_n \longrightarrow D$ by

$$\phi_n(\lambda_0,\lambda_1,\dots,\lambda_n) = x^* \text{ for all } (\lambda_0,\lambda_1,\dots,\lambda_n) \in \Delta_n.$$

Then $\varphi(\phi_n(\lambda_0,\lambda_1,\dots,\lambda_n),x) = \varphi(x^*,x) \geq 0$ for all $x \in X$. Thus $\varphi|_{D\times X}$ is $\mathscr{C}$-quasi-convex on X.

Sufficiency: Suppose that there exists a non-empty compact subset D of X satisfying that $\varphi|_{D\times X}$ is diagonally transfer upper continuous on D and $\mathscr{C}$-quasi-convex on X. For simplicity, we denote $\Phi = \varphi|_{D\times X}$. Define the correspondence $G : X \longrightarrow 2^D$ by

$$G(y) = \{x \in D | \Phi(x,y) \geq 0\} \text{ for all } y \in X. \tag{6.1}$$

Since Φ is diagonally transfer upper continuous on D, we have that $\cap_{y\in X} cl_D G(y) = \cap_{y\in X} G(y)$: Clearly, $\cap_{y\in X} G(y) \subseteq \cap_{y\in X} cl_D G(y)$. For the other inclusion, suppose, to the contrary, that there exists some $z \in \cap_{y\in X} cl_D G(y)$ but $z \notin G(y')$ for some $y' \in X$. Then $\Phi(z,y') < 0$. By the diagonally transfer upper continuity of Φ on D, there exists some $y'' \in X$ such that $z \notin cl_D G(y'')$, a contradiction.

We now show that $\{cl_D G(y) | y \in X\}$ has finite intersection property. To the contrary, suppose that there exists a finite set $\{x^0,x^1,\dots,x^n\} \subseteq X$ such that $\cap_{i=0}^n cl_D G(x^i) = \emptyset$. Then $\cup_{i=0}^n (D \setminus cl_D G(x^i)) = D$. Since D is compact and $D \setminus cl_D G(x^i) = (cl_D G(x^i))^c$ is open in D, by [37, Proposition 2], there exists a continuous partition of unity $\{f_0, f_1,\dots, f_n\}$ subordinated to the above covering, i.e., each $f_i : D \longrightarrow [0,1]$ is continuous, $f_i(x) \neq 0$ if and only if $x \in D \setminus cl_D G(x^i)$, and $\sum_{i=0}^n f_i(x) = 1$ for all $x \in D$. Since Φ is $\mathscr{C}$-quasi-convex on X, there exists a continuous mapping $\phi_n : \Delta_n \longrightarrow D$ such that, for $J = \{j \in \{0,1,\dots,n\} | \lambda_j \neq 0\}$,

$$\max\{\Phi(\phi_n(\lambda_0,\lambda_1,\dots,\lambda_n),x^i) | i \in J\} \geq 0 \tag{6.2}$$

for all $(\lambda_0,\lambda_1,\dots,\lambda_n) \in \Delta_n$. Define the mapping $g : D \longrightarrow D$ by

$$g(x) = \phi_n(f_0(x), f_1(x),\dots, f_n(x)) \text{ for all } x \in D.$$

Then $g(x)$ is continuous as ϕ_n and f_i, $i = 0,1,\dots,n$, are continuous. By assumption (i), g has a fixed point $x^* \in D$, that is, $x^* = g(x^*) = \phi_n(f_0(x^*), f_1(x^*),\dots, f_n(x^*))$. Let $J = \{j \in \{0,1,\dots,n\} | f_j(x^*) \neq 0\}$. Then $J \neq \emptyset$. If follows from (6.2) that

$$\max\{\Phi(x^*,x^i) | i \in J\} = \max\{\Phi(\phi_n(f_0(x^*), f_1(x^*),\dots, f_n(x^*)),x^i) | i \in J\} \geq 0.$$

On the other hand, $x^* \notin cl_D G(x^j)$ for each $j \in J$. It follows from (6.1) that $\Phi(x^*,x^j) < 0$ for each $j \in J$, which implies that $\max\{\Phi(x^*,x^i) | i \in J\} < 0$, a contradiction. Thus, $\{cl_D G(y) | y \in X\}$ has finite intersection property. Since D is compact, it follows that $\cap_{y\in X} cl_D G(y) \neq \emptyset$. Therefore, there exists $x^* \in \cap_{y\in X} cl_D G(y) = \cap_{y\in X} G(y) \subseteq X$, which implies that $\varphi(x^*,y) \geq 0$ for all $y \in X$, i.e., x^* is a solution to problem (2.2). □

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