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HIGHER-ORDER MIXED DUALITY FOR SET-VALUED OPTIMIZATION PROBLEMS

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Abstract. In this paper, we first introduce the notion of (incident) higher-order generalized epiderivative for a set-valued map and obtain a crucial result of the epiderivative. Then we use this result to establish the higher-order sufficient and necessary optimality conditions of a constrained set-valued optimization problem. By virtue of the epiderivatives and the optimality conditions, we establish the higher-order mixed duality problem for the set-valued optimization problem and obtain the corresponding duality theorems.
Keywords. Higher-order generalized epiderivative; Higher-order mixed duality; Optimality condition; Set-valued optimization; Weak efficient point.

1. Introduction

The duality theory for set-valued optimization problems has always been one of the main issues in the fields of economics, operations research, management science, and so on. The most crucial tool to investigate the Wolfe and Mond-Weir dualities for set-valued optimization problems is graphical or/and epigraphical derivatives; see, for example, [4, 5, 8, 10, 11, 21] and the references therein. For a constrained set-valued optimization problem, Chen et al. [10] introduced its higher-order Mond-Weir and Wolfe type dual problems by virtue of the higher order weak adjacent (contingent) epiderivatives, and furthermore obtained the higher-order Kuhn-Tucker type necessary and sufficient optimality conditions. Wang et al. [21] used the higher-order generalized adjacent derivative to study a Mond-Weir type dual problem for a constrained set-valued optimization problem. Anh [4] introduced the higher-order radial epiderivative and established a mixed dual problem in dealing with generalized strict minimizers by virtue of the higher-order radial epiderivative.

Concerning on studying the duality of optimization problems, we first discuss their optimality conditions. Recently, there are several results on the optimality conditions; see, e.g., [1, 2, 3, 9, 12, 13, 14, 15, 16, 17, 20, 22, 23] and the references therein. Now, let us recall some important results in this field. Jahn and Rauh [12] introduced the notion of the contingent

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epiderivative for a set-valued map which modified a concept introduced by Aubin [6] and proved the optimality conditions based on the notion of the contingent epiderivative that are necessary and sufficient for a set-valued optimization problems under appropriate assumptions. Chen and Jahn [9] introduced a generalized contingent epiderivative of a set-valued map and established a unified sufficient and necessary conditions for set-valued optimization problems in terms of the generalized contingent epiderivative. Li and Chen [14] established the optimality conditions for Henig properly efficient solutions by using the higher-order generalized (contingent) adjacent epiderivatives in a set-valued optimization problem.

It is well known that the above higher-order derivatives need to know their lower order directions, such as first order direction, second order direction, and so on. However, it is difficult to obtain their lower order directions (see Definition 5.61 of [7]), which only can be calculated step by step. We are committed to finding a more streamlined direction to deal with the higher-order optimality conditions and duality. The (incident) higher-order derivatives, which introduced by using (incident) tangent sets by Penot in [19], give us a way of thinking. In this paper, we mainly consider the optimality conditions for weakly efficient solutions and the higher-order mixed type dual of a optimization problem by virtue of the (incident) higher-order generalized epiderivatives.

The paper is organized as follows. Section 2 describes some basic definitions and obtains some properties. In Section 3, we establish the higher-order necessary and sufficient optimality conditions for a constrained set-valued optimization problem (P) by virtue of the higher-order generalized epiderivatives, which are defined at the beginning of this section. In Section 4, the last section, we establish the higher-order mixed type dual problem (DP) of (P) and obtain the corresponding weak duality, strong duality, and converse duality theorems.

2. PRELIMINARIES

Throughout this paper, let X,Y, and Z be real normed linear spaces. Let $C\subseteq Y$ (resp. $D\subseteq Z$) be a nontrivial pointed closed convex cone, which introduces the partial order in Y (resp. Z). Let $B(c,r)$ denote the open ball with centered at c and radius r. For a nonempty set $A\subseteq X$, $\operatorname{int}A$ and $\operatorname{cl}A$ stand for the interior and closure of A, respectively. Let $F:A\rightrightarrows Y$ be a set-valued map. The domain, the graph, and the epigraph of F are defined by $\operatorname{dom}F:=\{x\in A: F(x)\neq\emptyset\}$, $\operatorname{graph}F:=\{(x,y)\in X\times Y: x\in A, y\in F(x)\}$, and $\operatorname{epi}F:=\{(x,y)\in X\times Y: x\in A, y\in F(x)+C\}$, respectively. The profile map of F, denoted by F_+, is defined by $F_+(x):=F(x)+C$ for every $x\in\operatorname{dom}F$. The map F is said to be C-convex on a convex set $A\subseteq X$ if and only if, for any $x_1,x_2\in A$ and $\lambda\in[0,1]$,

$$\lambda F(x_1)+(1-\lambda)F(x_2)\subseteq F\big(\lambda x_1+(1-\lambda)x_2\big)+C.$$

Obviously, $\operatorname{epi}F$ is a convex set in $X\times Y$ when F is C-convex on A. The dual cone C^* of C is defined by

$$C^*:=\{\lambda\in Y^*:\lambda(y)\geq 0\},$$

where Y^* is the topological dual of Y and $\lambda(y)=\langle\lambda,y\rangle$ denotes the value λ at y.

Now, we recall the higher-order tangent cone and the incident higher-order tangent cone arise in [19].

Definition 2.1. Let A be a subset of X, $x\in\operatorname{cl}A$, and $v\in X$.

(i) The higher-order tangent cone to A at x in the direction v is the set

$$T^h(A,x,v) := \limsup_{(t,s)\to(0_+,0_+)} \frac{A-x-tv}{st} = \{y\in X : \liminf_{(t,s)\to(0_+,0_+)} d(y,\frac{A-x-tv}{st}) = 0\}.$$

Thus $y\in T^h(A,x,v)$ if and only if there exist sequences $(t_n),(s_n)\to 0_+$ and $(y_n)\to y$ such that $x+t_nv+t_ns_ny_n\in A$ for all n.

(ii) The incident higher-order tangent cone to A at x in the direction v is the set

$$T^{hi}(A,x,v) := \liminf_{(t,s)\to(0_+,0_+)} \frac{A-x-tv}{st} = \{y\in X : \lim_{(t,s)\to(0_+,0_+)} d(y,\frac{A-x-tv}{st}) = 0\}.$$

Thus $y\in T^{hi}(A,x,v)$ if and only if, for any sequences $(t_n),(s_n)\to 0_+$, there exists a sequence $(y_n)\to y$ such that $x+t_nv+t_ns_ny_n\in A$ for all n.

Remark 2.1.

(i) The mth-order contingent set $T_A^{(m)}(x,v_1,\dots,v_{m-1}) := \limsup_{t\to 0_+}\frac{1}{t^m}(A-x-tv_1-\dots-t^{m-1}v_{m-1}) = \limsup_{t\to 0_+}\frac{1}{t^{m-1}t}\big(A-x-t(v_1-\dots-t^{m-2}v_{m-1})\big)$ (resp. mth-order adjacent set $T_A^{b(m)}(x,v_1,\dots,v_{m-1})$) in [7, 14] carries a more precise information than $T^h(A,x,v)$ (resp. $T^{hi}(A,x,v)$) even if it is smaller.

(ii) The following inclusions hold:

$$T^{hi}(A,x_0,v-x_0)\subseteq T^h(A,x_0,v-x_0)\subseteq \mathrm{cl}\,\Big(\bigcup_{t>0,s>0}\frac{A-x_0-t(v-x_0)}{st}\Big).$$

The converse relations hold when A is a convex set and $x_0,v\in A$; see Proposition 2.2.

(iii) If $A\subseteq X$ is closed, then, for any $\lambda>0$,

$$T^h(A,x,\lambda v)=\lambda T^h(A,x,v) \text{ and } T^{hi}(A,x,\lambda v)=\lambda T^{hi}(A,x,v).$$

To verify the above assertions, it suffices to show the first one since the second one is similar. Indeed, take any $y\in T^h(A,x,\lambda v)$. Then there exist sequences $(t_n),(s_n)\to 0_+$ and $(y_n)\to y$ such that $x+t_n\lambda v+t_ns_ny_n\in A$ for all n. Let $t_n':=\lambda t_n$. Then, $x+t_n'v+t_n's_n\frac{y_n}{\lambda}\in A$, so $\frac{y}{\lambda}\in T^h(A,x,v)$ (noting that $t_n'\to 0_+$ and $y_n\to y$).

Conversely, for any $y\in\lambda T^h(A,x,v)$, there exist sequences $(t_n),(s_n)\to 0_+$ and $y_n\to\frac{y}{\lambda}$ such that $x+t_nv+t_ns_ny_n\in A$, $\forall n$. Let $t_n':=\frac{t_n}{\lambda}$. Then, $x+t_n'\lambda v+t_n's_n\lambda y_n\in A$, which together with $t_n'\to 0_+$ implies that $y\in T^h(A,x,\lambda v)$.

Penot [19] proved that $T^{hi}(A,x,v)$ is a convex cone for each $v\in X$ (further more, $v\in T(A,x):=\limsup_{t\to 0_+}\frac{A-x}{t}$, the tangent cone (or contingent cone) to F at x, $T^{hi}(A,x,v)\neq\emptyset$). Similarly, we can show that $T^h(A,x,v)$ is also a convex cone.

Proposition 2.1. *If A is convex and $v\in X$, then $T^h(A,x,v)$ is a convex cone.*

Proof. Since $T^h(A,x,v)$ is a cone, we only need to show that $y+y'\in T^h(A,x,v)$ for each $y,\ y'\in T^h(A,x,v)$. To finish it, we can find sequences (t_n), (s_n), (t'_n), $(s'_n)\to 0_+$, $(y_n)\to y$ and $(y'_n)\to y'$ such that $a_n:=x+t_nv+t_ns_ny_n\in A$ and $a'_n:=x+t'_nv+t'_ns'_ny'_n\in A$ for all n. Note that A is convex. Then

$$\frac{t'_ns'_n}{t_ns_n+t'_ns'_n}a_n+\frac{t_ns_n}{t_ns_n+t'_ns'_n}a'_n=x+\frac{t_nt'_n(s_n+s'_n)}{t_ns_n+t'_ns'_n}v+\frac{t_ns_nt'_ns'_n}{t_ns_n+t'_ns'_n}(y_n+y'_n)\in A,\ \forall n. \tag{2.1}$$

Let $\tilde{t}_n:=\frac{t_nt'_n(s_n+s'_n)}{t_ns_n+t'_ns'_n}$ and $\tilde{s}_n:=\frac{s_ns'_n}{s_n+s'_n}$. We can easily verify that

$$0<\tilde{t}_n=\frac{t_nt'_ns_n}{t_ns_n+t'_ns'_n}+\frac{t_nt'_ns'_n}{t_ns_n+t'_ns'_n}\leq t'_n+t_n\to 0_+$$

and $0<\tilde{s}_n\leq s'_n\to 0_+$. Hence, $\tilde{t}_n\to 0_+$ and $\tilde{s}_n\to 0_+$. Then, from (2.1) it follows that $y+y'\in T^h(A,x,v)$. The proof is complete. □

Proposition 2.2. *If A is a convex set and $x_0,\ v\in A$, then*

$$T^{hi}(A,x_0,v-x_0)=T^h(A,x_0,v-x_0)=\operatorname{cl}\Big(\bigcup_{t>0,s>0}\frac{A-x_0-t(v-x_0)}{st}\Big).$$

Proof. By Remark 2.1 (ii), we only need to prove that, for any $w_0\in\operatorname{cl}\big(\bigcup_{t>0,s>0}\frac{A-x_0-t(v-x_0)}{st}\big)$, $w_0\in T^{hi}(A,x_0,v-x_0)$. Let $\varepsilon>0$ be fixed. Then there exist $y\in A$ and $s,t>0$ such that

$$w_0-\frac{y-x_0-t(v-x_0)}{ts}\in B(0,\varepsilon).$$

We can choose $\hat{t}$ with $0<\hat{t}\leq\min\{t,1\}$ and $\hat{s}$ with $0<\hat{s}\leq s$. Then, take any $h\in]0,\hat{t}[$ and $m\in]0,\hat{s}[$. To simplify the proof, we set $w:=\frac{y-x_0-t(v-x_0)}{ts}$ to see

$$\begin{aligned}&x_0+h(v-x_0)+hmw\\=&x_0+h(v-x_0)+hm\frac{y-x_0-t(v-x_0)}{st}\\=&(1-h-\frac{mh}{st}+\frac{mh}{s})x_0+(h-\frac{mh}{s})v+\frac{mh}{st}y.\end{aligned}$$

We can easily check that

$$1-h-\frac{mh}{st}+\frac{mh}{s}\geq\frac{t(m-s)(h-1)}{st}\geq 0,\ \ h-\frac{mh}{s}=h(1-\frac{m}{s})\geq 0$$

and $\frac{mh}{st}\geq 0$. Note that $x_0,v\in A$. The convexity of A yields that $x_0+h(v-x_0)+hmw\in A$. Thus, $w_0\in T^{hi}(A,x_0,v-x_0)$, and the proof is complete. □

Similar to the proof of [15, Proposition 3.2], it easy to obtain the following proposition.

Proposition 2.3. *If A is convex, then $T^{hi}(A,x,v)$ is convex.*

3. Higher-Order Optimality Conditions

In this section, we first introduce the notions of the higher-order generalized epiderivatives for set-valued mappings. Then, under appropriate conditions, we establish the higher-order necessary and sufficient optimality conditions by virtue of the higher-order generalized epiderivatives.

Definition 3.1. (See [18])

(i) Let $M \subseteq Y$. $Y \in M$ is said to be a minimal point of M if $(M-y)\cap(-C)=\{0_Y\}$. The set of all minimal elements of M is denoted by $Min_C M$.

(ii) The domination property is said to hold for a subset $M \subseteq Y$ if $M \subseteq Min_C M + C$.

Definition 3.2. [19] Let F be a set-valued map from X to Y, $(x_0,y_0) \in \operatorname{graph} F$ and $(u,v) \in X \times Y$.

(i): The higher-order derivative $D^h F\big((x_0,y_0),(u,v)\big)$ of F at (x_0,y_0) in the direction (u,v) is defined as

$$D^h F\big((x_0,y_0),(u,v)\big)(x) := \Big\{ y \in Y : (x,y) \in T^h\big(\operatorname{graph} F,(x_0,y_0),(u,v)\big) \Big\}.$$

(ii): The incident higher-order derivative $D^{hi} F\big((x_0,y_0),(u,v)\big)$ of F at (x_0,y_0) in the direction (u,v) is defined as

$$D^{hi} F\big((x_0,y_0),(u,v)\big)(x) := \Big\{ y \in Y : (x,y) \in T^{hi}\big(\operatorname{graph} F,(x_0,y_0),(u,v)\big) \Big\}.$$

Similar to [14, Definition 3.2], we introduce the higher-order generalized derivatives as follows.

Definition 3.3. Let F be a set-valued map from X to Y, $(x_0,y_0) \in \operatorname{graph} F$ and $(u,v) \in X \times Y$.

(i): The higher-order generalized epiderivative of F at (x_0,y_0) in the direction (u,v), denoted by $D^h_g F\big((x_0,y_0),(u,v)\big)$, is defined as

$$\begin{aligned} D^h_g F\big((x_0,y_0),(u,v)\big)(x) \quad &:= Min_C \Big\{ y \in Y : (x,y) \in T^h\big(\operatorname{epi} F,(x_0,y_0),(u,v)\big) \Big\} \\ &= Min_C \Big\{ y \in Y : y \in D^h F_+\big((x_0,y_0),(u,v)\big)(x) \Big\}, \\ &\qquad \forall x \in \operatorname{dom}\big[D^h F_+\big((x_0,y_0),(u,v)\big)\big] \end{aligned}$$

(ii): The incident higher-order generalized epiderivative of F at (x_0,y_0) in the direction (u,v), denoted by $D^{hi}_g F\big((x_0,y_0),(u,v)\big)$, is defined as

$$\begin{aligned} D^{hi}_g F\big((x_0,y_0),(u,v)\big)(x) \quad &:= Min_C \Big\{ y \in Y : (x,y) \in T^{hi}\big(\operatorname{epi} F,(x_0,y_0),(u,v)\big) \Big\} \\ &= Min_C \Big\{ y \in Y : y \in D^{hi} F_+\big((x_0,y_0),(u,v)\big)(x) \Big\}, \\ &\qquad \forall x \in \operatorname{dom}\big[D^{hi} F_+\big((x_0,y_0),(u,v)\big)\big]. \end{aligned}$$

Remark 3.1. When F reduces to a single-valued mapping, the higher-order generalized epiderivative and the incident higher-order generalized epiderivative reduce to the lower higher-order derivatives, which were defined by Penot in [19, Definition 18].

Assuming that the set-valued mapping is cone-convex, we use higher-order generalized epiderivatives to establish a crucial result, which is motivated by [14, Proposition 3.1] and [15, Theorem 4.1].

Proposition 3.1. *Let F be C-convex on a nonempty convex set $E \subseteq X$. Let $x_0 \in E$, $y_0 \in F(x_0)$, $u \in E$ and $v \in F(u)+C$. If $D_g^{hi}F\big((x_0,y_0),(u-x_0,v-y_0)\big)(x-x_0) \neq \emptyset$ and the set $P(x-x_0) := \big\{y \in Y : (x-x_0,y) \in T^{hi}\big(epi\, F,(x_0,y_0),(u-x_0,v-y_0)\big)\big\}$ fulfills the domination property for all $x \in E$, then*

$$F(x)-y_0 \subseteq D_g^{hi}F\big((x_0,y_0),(u-x_0,v-y_0)\big)(x-x_0)+C.$$

Proof. Take any $x \in E$ and $y \in F(x)$. For arbitrary sequences $\{t_n\}$, $\{s_n\} \subseteq]0,1[$ with $t_n \to 0_+$ and $s_n \to 0_+$, since E is convex and F is C-convex on E, we have $x_0+t_n(u-x_0) \in E$ and $x_0+t_ns_n(x-x_0) \in E$ due to $0 < s_nt_n < 1$ and

$$y_0+t_n(v-y_0) \in F\big(x_0+t_n(u-x_0)\big)+C,$$

and

$$y_0+t_ns_n(y-y_0) \in F\big(x_0+t_ns_n(x-x_0)\big)+C.$$

Consequently, by convexity again, one has

$$x_n := x_0+\frac{1}{2}t_n(u-x_0)+\frac{1}{2}t_ns_n(x-x_0) \in E$$

and

$$y_n := y_0+\frac{1}{2}t_n(v-y_0)+\frac{1}{2}t_ns_n(y-y_0) \in F(x_n)+C.$$

It means $(x_n,y_n) \in \operatorname{epi} F$ and

$$\frac{(x_n,y_n)-(x_0,y_0)-\frac{t_n}{2}(u-x_0,v-y_0)}{\frac{t_ns_n}{2}} = (x-x_0,y-y_0).$$

Take $t_n' := \frac{t_n}{2} \to 0_+$. It follows that

$$(x-x_0,y-y_0) \in T^{hi}\big(\operatorname{epi} F,(x_0,y_0),(u-x_0,v-y_0)\big).$$

Then, by the domination property of $P(x-x_0)$, we have

$$y-y_0 \in P(x-x_0) \subseteq Min_C P(x-x_0)+C = D_g^{hi}F\big((x_0,y_0),(u-x_0,v-y_0)\big)(x-x_0)+C.$$

Since y is arbitrary given, we have $F(x)-y_0 \subseteq D_g^{hi}F\big((x_0,y_0),(u-x_0,v-y_0)\big)(x-x_0)+C$. $\square$

Now we consider the following constrained set-valued optimization problem (P):

$$(\mathrm{P}) \begin{cases} \min & F(x), \\ s.t. & x \in S, G(x) \cap (-D) \neq \emptyset, \end{cases}$$

where S is a nonempty subset of X, $F : X \rightrightarrows Y$ is C-convex, and $G : X \rightrightarrows Z$ is D-convex on S. A triple $(x,y,z) \in S \times Y \times Z$ is said to be feasible if $x \in \operatorname{dom} F \cap \operatorname{dom} G$, $y \in F(x)$, and $z \in G(x) \cap (-D)$. Set

$$A := \{x \in S : G(x) \cap (-D) \neq \emptyset\} \text{ and } F(A) := \bigcup\{F(x) : x \in A\}.$$

The notation $(F,G)(x)$ is used to denote $F(x) \times G(x)$.

Definition 3.4. [18] A pair (x_0,y_0) with $x_0 \in A$ and $y_0 \in F(x_0)$ is called a weak efficient solution to (P) if $\big(F(A)-y_0\big) \cap (-\operatorname{int} C) = \emptyset$.

Theorem 3.1. *Let* $(x_0,y_0) \in \operatorname{graph} F$ *and* $(u, v-y_0, w) \in X \times (-C) \times (-D)$. *If* (x_0,y_0) *is a weak efficient solution to* (P), *then, for any* $z_0 \in G(x_0) \cap (-D)$,

$$\big[D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big)(x) + C \times D + (0_Y, z_0)\big] \cap (-\operatorname{int}(C \times D)) = \emptyset, \tag{3.1}$$

for all $x \in \Omega := \operatorname{dom}\big[D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big)\big]$.

Proof. Since (x_0,y_0) is a weak efficient solution of (P), then

$$(F(A) - y_0) \cap (-\operatorname{int} C) = \emptyset. \tag{3.2}$$

Assume that (3.1) does not hold. Then there exist $z_0 \in G(x_0) \cap (-D)$, $x \in \Omega$, $(y,z) \in Y \times Z$, $c_0 \in C$ and $d_0 \in D$ such that

$$(y,z) \in D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big)(x) \tag{3.3}$$

and

$$(y,z) + (c_0,d_0) + (0,z_0) = (y+c_0, z+d_0+z_0) \in -\operatorname{int}(C \times D). \tag{3.4}$$

It follows from the definition of $D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big)(x)$ and (3.3) that

$$(x,y,z) \in T^{hi}(\operatorname{epi}(F,G),(x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)).$$

Then, for any sequences $\{t_n\}$, $\{s_n\}$ with $t_n, s_n \to 0_+$, there exists $(x_n,y_n,z_n) \in \operatorname{epi}(F,G)$ such that

$$\frac{(x_n,y_n,z_n) - (x_0,y_0,z_0) - t_n(u-x_0,v-y_0,w-z_0)}{t_n s_n} \to (x,y,z). \tag{3.5}$$

From (3.4) and (3.5), there exists a sufficiently large $N > 0$ such that

$$\frac{(y_n,z_n) - (y_0,z_0) - t_n(v-y_0,w-z_0)}{t_n s_n} + (c_0,d_0) + (0,z_0) \in -\operatorname{int}(C \times D),$$

for $n > N$. Since C and D are cones, one has

$$y_n - y_0 - t_n(v-y_0) + t_n s_n c_0 \in -\operatorname{int} C, \text{ for } n > N \tag{3.6}$$

and

$$z_n - z_0 - t_n(w-z_0) + t_n s_n (d_0 + z_0) \in -\operatorname{int} D, \text{ for } n > N.$$

Note that as $v - y_0 \in -C$, we have $t_n(v-y_0) \in -C$. By (3.6), one has

$$y_n - y_0 \in -\operatorname{int} C + t_n(v-y_0) - t_n s_n c_0 \in -\operatorname{int} C - C - C = -\operatorname{int} C, \text{ for } n > N.$$

Similarly, we have for all $n > N$, $t_n + t_n s_n < 1$ and

$$z_n \in -\operatorname{int} D + (1 - t_n - t_n s_n) z_0 + t_n w - t_n s_n d_0 \subseteq -\operatorname{int} D - D - D - D = -\operatorname{int} D.$$

Since $(x_n,y_n,z_n) \in \operatorname{epi}(F,G)$, then there exist $\tilde{y}_n \in F(x_n)$ and $\tilde{z}_n \in G(x_n)$ such that $\tilde{y}_n \leq_C y_n$ and $\tilde{z}_n \leq_D z_n$. Then, for every $n > N$,

$$\tilde{y}_n - y_0 \in y_n - C - y_0 \subseteq -\operatorname{int} C - C = -\operatorname{int} C$$

and $\tilde{z}_n \in z_n - D \subseteq -\operatorname{int} D$. Thus, $(x_n, \tilde{y}_n, \tilde{z}_n)$ is a feasible triple for all $n > N$ and $\tilde{y}_n - y_0 \in -\operatorname{int} C$, which contradicts to (3.2). Thus (3.1) holds and the proof is complete. □

From the proof of Theorem 3.1, we have the following result.

Corollary 3.1. *Let $(x_0,y_0) \in \operatorname{graph} F$ and $(u,v,w) \in X \times (-C) \times (-D)$. If (x_0,y_0) is a weak efficient solution to (P), then, for any $z_0 \in G(x_0) \cap (-D)$,*

$$\left[D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u,v,w)\big)(x) + C \times D + (0_Y,z_0)\right] \cap \big(\text{-int}\,(C \times D)\big) = \emptyset,$$

for all $x \in \Omega := \operatorname{dom}\left[D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u,v,w)\big)\right]$.

Remark 3.2. We can easily see that the results of Theorem 3.1 and Corollary 3.1 also hold whenever $D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u,v,w)\big)(x)$ is replaced by $D_g^{h}(F,G)\big((x_0,y_0,z_0),(u,v,w)\big)(x)$.

Based on Proposition 3.1 and Theorem 3.1, we establish the following higher-order sufficient optimality conditions and necessary optimality conditions of problem (P).

Theorem 3.2. *Assume that all the following conditions are satisfied:*

(i) *$(F \times G)$ is $(C \times D)$-convex on the convex set S;*
(ii) *$(x_0,y_0) \in \operatorname{graph} F$ and $(u,v-y_0,w) \in X \times (-C) \times (-D)$;*
(iii) *$P(x) := \big\{(y,z) \in Y \times Z : (x,y,z) \in T^{hi}\big(\operatorname{epi}(F,G),(x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big)\big\}$ fulfills domination property for all $z_0 \in G(x_0) \cap (-D)$, $x \in S$ and $(0_Y,0_Z) \in P(0_X)$;*
(iv) *(x_0,y_0) is a weak efficient solution of (P).*

Then, for any $z_0 \in G(x_0) \cap (-D)$, there exist $\lambda \in C^* \setminus \{0_{Y^*}\}$ and $\mu \in D^*$ but not both being zero functionals such that $\mu(z_0) = 0$ and $\lambda(y) + \mu(z) \geq 0$ for all $(y,z) \in D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big)(x)$, and $x \in \Omega := \operatorname{dom}\left[D_g^{hi}(F,G)\ \big((x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big)\right]$.

Proof. Let $z_0 \in G(x_0) \cap (-D)$. Define

$$H := \bigcup_{x \in \Omega} D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big)(x) + C \times D + (0_Y,z_0).$$

Now we prove that H is a convex set by showing that $H_0 = H - (0_Y,z_0)$ is convex. Let $(y_i,z_i) \in H_0$, $i = 1,2$. Then there exist $x_i \in \Omega$, $(y_i',z_i') \in D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big)(x_i)$ and $(c_i,d_i) \in C \times D$ such that $(y_i,z_i) = (y_i',z_i') + (c_i,d_i)$ $i = 1,2$. Then, we have

$$(x_i,y_i',z_i') \in T^{hi}\big(\operatorname{epi}(F,G),(x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big),\ i = 1,2.$$

Since F and G are cone-convex, then epigraph $\operatorname{epi}(F,G)$ is convex. By Proposition 2.3, $T^{hi}(\operatorname{epi}(F, G),(x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big)$ is a convex set. Then, for any $\lambda \in [0,1]$,

$$\lambda(x_1,y_1',z_1') + (1-\lambda)(x_2,y_2',z_2') \in T^{hi}(\operatorname{epi}(F,G),(x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big).$$

By the domination property of $P(x)$, we have

$$\begin{aligned}&\lambda(y_1',z_1') + (1-\lambda)(y_2',z_2')\\ &\in D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big)\big(\lambda x_1 + (1-\lambda)x_2\big) + C \times D.\end{aligned}$$

Since C and D are convex cones, we have

$$\lambda(y_1,z_1) + (1-\lambda)(y_2,z_2) = \lambda\big((y_1',z_1') + (c_1,d_1)\big) + (1-\lambda)\big((y_2',z_2') + (c_2,d_2)\big) \in H_0,$$

which means H is a convex set. By Theorem 3.1, one has $H \cap \big(-\operatorname{int}(C \times D)\big) = \emptyset$. By the separation theorem for convex sets, there exist $0 \neq (\lambda,\mu) \in Y^* \times Z^*$ and $\gamma \in \mathbb{R}$ such that

$$\lambda(\bar{y}) + \mu(\bar{z}) < \gamma \leq \lambda(\tilde{y}) + \mu(\tilde{z}), \text{ for all } (\bar{y},\bar{z}) \in -\operatorname{int}(C \times D),\ (\tilde{y},\tilde{z}) \in H.$$

Then we have $\lambda(\bar{y}) + \mu(\bar{z}) \leq 0$, for all $(\bar{y}, \bar{z}) \in -\text{int}(C \times D)$ and

$$0 \leq \lambda(\tilde{y}) + \mu(\tilde{z}), \text{ for all } (\tilde{y}, \tilde{z}) \in H. \tag{3.7}$$

Obviously, we have $\lambda(\bar{y}) \leq 0$, for all $\bar{y} \in -C$ and $\mu(\bar{z}) \leq 0$ for all $\bar{z} \in -D$. Thus, $\lambda \in C^*$ and $\mu \in D^*$. By the domination property of $P(x)$ for all $x \in S$, one has

$$P(x) \subseteq D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big)(x) + C \times D, \text{ for all } x \in \Omega.$$

Then, we have $\bigcup_{x\in\Omega} P(x) \subseteq H_0$. Noting that $(0_Y, 0_Z) \in P(0_X)$, one has $(0_Y, 0_Z) \in H - (0_Y, z_0)$, which means $(0_Y, z_0) \in H$. By (3.7), we have $\mu(z_0) \geq 0$. It follows from $\mu \in D^*$ and $z_0 \in -D$ that $\mu(z_0) \leq 0$. Hence, we obtain $\mu(z_0) = 0$. Thus, is follows from (3.7) that $\lambda(y) + \mu(z) \geq 0$ for all $(y,z) \in D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big)(x)$ and $x \in \Omega$. □

From the proof of Theorem 3.2, we have the following corollary.

Corollary 3.2. *Assume that all the following conditions are satisfied:*

(i) $(F \times G)$ *is* $(C \times D)$*-convex on the convex set* S*;*
(ii) $(x_0, y_0) \in \text{graph} F$ *and* $(u, v, w) \in X \times (-C) \times (-D)$*;*
(iii) $P(x) := \big\{(y,z) \in Y \times Z : (x,y,z) \in T^{hi}\big(epi\,(F,G),(x_0,y_0,z_0),(u,v,w)\big)\big\}$ *fulfills domination property for all* $z_0 \in G(x_0) \cap (-D)$*,* $x \in S$ *and* $(0_Y, 0_Z) \in P(0_X)$*;*
(iv) (x_0, y_0) *is a weak efficient solution of* (P).

Then, for any $z_0 \in G(x_0) \cap (-D)$*, there exist* $\lambda \in C^* \setminus \{0_{Y^*}\}$ *and* $\mu \in D^*$ *but not both being zero functionals such that*

$$\mu(z_0) = 0 \text{ and } \lambda(y) + \mu(z) \geq 0,$$

for all $(y,z) \in D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u,v,w)\big)(x)$ *and* $x \in \text{dom}\left[D_g^{hi}(F,G)\ \big((x_0,y_0,z_0),(u,v,w)\big)\right]$.

Now, we give a example to verify the 2th-order necessary optimality condition of Theorem 3.2.

Example 3.1. Suppose that $X = Y = Z = R$, $S = [-1,1] \subseteq X$, and $C = D = R_+ = [0, +\infty)$. Let $F : X \rightrightarrows Y$ and $G : X \to Z$, respectively, with $F(x) = \{y \in R : x^4 \leq y \leq 4\}$ and $G(x) = 2x - 1$. Naturally, F and G are two R_+-convex functions on the convex set $[-1,1]$, respectively. Consider the following constrained set-valued optimization problem (P):

$$\begin{cases} \min & F(x), \\ s.t. & x \in S, G(x) \cap (-D) \neq \emptyset. \end{cases}$$

Then we have $A := \{x \in S : G(x) \cap (-D) \neq \emptyset\} = [-1, \frac{1}{2}]$ and $F(A) = [0,4]$.

Let $(x_0, y_0) = (0,0) \in \text{gr} F$. It is easy to verify the point (x_0, y_0) is a weak efficient solution of (P). By the definitions of F and G, one sees that

$$\text{epi}\,(F,G) = \{(x,(y,z)) \in R \times R^2 : -1 \leq x \leq 1, y \geq x^4, z \geq 2x\}.$$

Taking $z_0 = -1 \in G(x_0) \cap (-D)$, one has

$$T_{\text{epi}\,(F,G)}(x_0,y_0,z_0) = \{(x,(y,z)) \in R \times R^2 : y \geq 0, z \geq 2x\}$$

and

$$D_g(F,G)(x_0,y_0,z_0)(x) = \{(y,z) \in R^2 : y = 0, z = 2x\}, x \in R.$$

Let $P(x) = \{(y,z) \in R^2 : (x,(y,z)) \in T_{\text{epi}\,(F,G)}(x_0,y_0,z_0)\}$. Obviously, $P(x)$ fulfills the domination property and $(0,0) \in P(0)$. Then, all the conditions of Theorem 3.2 hold. Take $\lambda > 0$ and $\mu = 0$, for any $(y,z) \in D_g(F,G)(x_0,y_0,z_0)(x)$ and $x \in R$, $\mu(z_0) = 0$ and $\lambda(y) + \mu(z) \geq 0$, which shows that the 1th-order necessary optimality condition of Theorem 3.2 holds. Take $u = 1/4, v = 0$, and $w = -1/2 \in -D$. Then, $u - x_0 = 1/4, v - y_0 = 0$ and $w - z_0 = 1/2$. Hence,

$$T^{hi}_{\mathrm{epi}\,(F,G)}(x_0,y_0,z_0,u-x_0,v-y_0,w-z_0) = \{(x,(y,z)) \in R \times R^2 : y \geq 0, z \geq 2x-1\}$$

and

$$D^{hi}_g(F,G)(x_0,y_0,z_0,u-x_0,v-y_0,w-z_0)(x) = \{(y,z) \in R^2 : y = 0, z = 2x-1\}, x \in R.$$

Hence, it follows that the conditions of Theorem 3.2 hold. Simultaneously, taking $\lambda > 0$ and $\mu = 0$, we have that the 2th-order necessary optimality condition of Theorem 3.2 holds.

Theorem 3.3. *Suppose that the following conditions are satisfied:*

(i) $(F \times G)$ *is* $(C \times D)$*-convex on the convex set* $S \subseteq \mathrm{dom}\,F \cap \mathrm{dom}\,G$*;*
(ii) $A = \{x \in S : G(x) \cap (-D) \neq \emptyset\}$, $u \in A$, $v \in F(u) + C$, $w \in G(u) + D$, $(x_0,y_0) \in \mathrm{graph}\,F$*;*
(iii) $P(x-x_0) := \{(y,z) \in Y \times Z : (x-x_0,y,z) \in T^{hi}(\mathrm{epi}\,(F,G),(x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0))\}$ *fulfills domination property for all* $z_0 \in G(x_0) \cap (-D)$ *and* $x \in S$*;*
(iv) *there exist* $z_0 \in G(x_0) \cap (-D)$, $\lambda \in C^* \backslash \{0_{Y^*}\}$ *and* $\mu \in D^*$ *such that* $\mu(z_0) = 0$ *and* $\lambda(y) + \mu(z) \geq 0$*, for all* $(y,z) \in D^{hi}_g(F,G)((x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0))(x)$ *and* $x \in A$*.*

Then (x_0,y_0) *is a weak efficient solution of* (P)*.*

Proof. On the contrary, assume that

$$(F(A) - y_0) \cap (-\mathrm{int}\,C) \neq \emptyset. \tag{3.8}$$

Then, there exist $x' \in A$ and $y' \in F(x')$ such that $y' - y_0 \in -\mathrm{int}\,C$. Since $x' \in A$, there exists a point $z' \in G(x') \cap (-D)$. By Proposition 3.1 and the domination property, we have

$$(y'-y_0, z'-z_0) \subseteq D^{hi}_g(F,G)((x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0))(x'-x_0) + C \times D.$$

Thus

$$\lambda(y'-y_0-c) + \mu(z'-z_0-d) \geq 0, \text{ for any } c \in C, d \in D. \tag{3.9}$$

Since $y' - y_0 \in -\mathrm{int}\,C$, then $y' - y_0 - c \in -\mathrm{int}\,C - C = -\mathrm{int}\,C$, and $\lambda \in C^* \backslash \{0_{Y^*}\}$, then $\lambda(y' - y_0 - c) < 0$. Since $z' \in G(x') \cap (-D)$, $\mu \in D^*$ and $\mu(z_0) = 0$, one has $\mu(z'-z_0-d) = \mu(z') - \mu(z_0) \leq 0$. Thus $\lambda(y'-y_0-c) + \mu(z'-z_0-d) < 0$, which contradicts to (3.9). Then, (3.8) does not hold, namely, $(F(A) - y_0) \cap -\mathrm{int}\,C = \emptyset$. Thus, (x_0,y_0) is a weak efficient solution of (P) and the proof is complete. □

Similar to the Theorem 3.3, we have the following result.

Corollary 3.3. *Suppose that the following conditions are satisfied:*

(i) $(F \times G)$ *is* $(C \times D)$*-convex on the convex set* $S \subseteq \mathrm{dom}\,F \cap \mathrm{dom}\,G$*;*
(ii) $A = \{x \in S : G(x) \cap (-D) \neq \emptyset\}$, $u \in A$, $v \in F(u) + C$, $w \in G(u) + D$, $(x_0,y_0) \in \mathrm{graph}\,F$*;*
(iii) $P(x-x_0) := \{(y,z) \in Y \times Z : (x-x_0,y,z) \in T^{hi}(\mathrm{epi}\,(F,G),(x_0,y_0,z_0),(u,v,w))\}$ *fulfills domination property for all* $z_0 \in G(x_0) \cap (-D)$ *and* $x \in S$*;*
(iv) *There exist* $z_0 \in G(x_0) \cap (-D)$, $\lambda \in C^* \backslash \{0_{Y^*}\}$ *and* $\mu \in D^*$ *such that* $\mu(z_0) = 0$ *and* $\lambda(y) + \mu(z) \geq 0$*, for all* $(y,z) \in D^{hi}_g(F,G)((x_0,y_0,z_0),(u,v,w))(x)$ *and* $x \in A$*.*

Then (x_0,y_0) *is a weak efficient solution to* (P)*.*

Remark 3.3. We can easily see that the results of Theorems 3.2, 3.3 and Corollaries 3.2, 3.3 also hold whenever $D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u,v,w)\big)(x)$ is replaced by $D_g^{h}(F,G)\big((x_0,y_0,z_0),(u,v,w)\big)(x)$.

4. Higher-Order Mixed Duality

In this section, we use the higher-order necessary and sufficient optimality conditions to establish a higher-order mixed dual problem (DP) of (P) inspired by [8]. The higher-order mixed dual problem (DP) of (P) as follows:

$$
(\mathrm{DP})\begin{cases}
\max & h(x_0,y_0,z_0,\lambda,\mu),\\
s.t. & \langle\lambda,y\rangle+\langle\mu,z\rangle\geq 0,\forall(y,z)\in D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u,v,w)\big)(\Omega),\\
& \delta\langle\mu,z_0\rangle\geq 0,\\
& \lambda\in C^*\backslash\{0_{Y^*}\},\\
& \mu\in D^*,
\end{cases}
$$

where $e\in\operatorname{int}C$, $z_0\in G(x_0)\cap(-D)$, $\delta\in\{0,1\}$, $\Omega:=\operatorname{dom}\big[D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u,v,w)\big)\big]$, and $h(x_0,y_0,z_0,\lambda,\mu):=y_0+\frac{e}{\langle\lambda,e\rangle}(1-\delta)\langle\mu,z_0\rangle$.

Let $M:=\big\{(x_0,y_0,z_0,\lambda,\mu):(x_0,y_0)\in\operatorname{graph}F,z_0\in G(x_0)\cap(-D)$, and $(x_0,y_0,z_0,\lambda,\mu)$ holds and all the conditions of $(DP)\big\}$ be the feasible set of (DP). Then a feasible element $(x_0,y_0,z_0,\lambda,\mu)$ is called a weak efficient solution to (DP) if, for all $(x_0,y_0,z_0,\hat{\lambda},\hat{\mu})\in M$,

$$
\big(h(x_0,y_0,z_0,\hat{\lambda},\hat{\mu})-h(x_0,y_0,z_0,\lambda,\mu)\big)\cap\operatorname{int}C=\emptyset.
$$

Some duality theorems of (P) and (DP) are established as follows.

Theorem 4.1. [Weak duality] *Let $(x_0,y_0)\in\operatorname{graph}F$, $z_0\in G(x_0)\cap(-D)$, and Proposition 3.1 hold for $(F,G)(x)$. Suppose that $(\bar{x},\bar{y})$ is a feasible element of (P) and $(x_0,y_0,z_0,\lambda,\mu)$ is a feasible element of (DP). Then,*

$$
y_0+\frac{e}{\langle\lambda,e\rangle}(1-\delta)\langle\mu,z_0\rangle-\bar{y}\notin\operatorname{int}C. \tag{4.1}
$$

Proof. Suppose on the contrary that

$$
y_0+\frac{e}{\langle\lambda,e\rangle}(1-\delta)\langle\mu,z_0\rangle-\bar{y}\in\operatorname{int}C. \tag{4.2}
$$

Since $\bar{x}\in A$, then there exists a point $\bar{z}\in G(\bar{x})\cap(-D)$. By Proposition 3.1, there exist $(a,b)\in D_g^{hi}(F,G)\ \big((x_0,y_0,z_0),(u-x_0,v-y_0,w-z_0)\big)(x-x_0)$ and $(c,d)\in C\times D$ such that

$$
\bar{y}-y_0=a+c,\bar{z}-z_0=b+d. \tag{4.3}
$$

Combining (4.2) and (4.3), we have

$$
m:=\frac{e}{\langle\lambda,e\rangle}(1-\delta)\langle\mu,z_0\rangle-a\in\operatorname{int}C+c\subseteq\operatorname{int}C.
$$

Since $\lambda\in C^*\backslash\{0_{Y^*}\}$ and $\mu\in D^*$, one has $\langle\lambda,m\rangle>0=\langle\mu,b+d-\bar{z}+z_0\rangle$, which implies that

$$
\langle\mu,\bar{z}\rangle-\delta\langle\mu,z_0\rangle-\langle\mu,d\rangle>\langle\lambda,a\rangle+\langle\mu,b\rangle.
$$

Hence $\langle\lambda,a\rangle+\langle\mu,b\rangle<0$, which contradicts to the first constraint of (DP). Consequently, (4.1) holds and the proof is complete. $\square$

Theorem 4.2. [Strong duality] *Assume that all the conditions of Corollary 3.2 are satisfied. Then there exist $\lambda \in C^*\backslash\{0_{Y^*}\}$ and $\mu \in D^*$ such that $(x_0, y_0, z_0, \lambda, \mu)$ is a weak efficient solution to (DP).*

Proof. By Corollary 3.2, for any $z_0 \in G(x_0) \cap (-D)$, there exist $\lambda \in C^*\backslash\{0_{Y^*}\}$ and $\mu \in D^*$ such that $\mu(z_0) = 0$ and $\lambda(y) + \mu(z) \geq 0$, for all $(y,z) \in D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u,v,w)\big)(x)$ and $x \in \Omega$. It means that $(x_0, y_0, z_0, \lambda, \mu)$ is a feasible element of (DP).

Now, we prove that $(x_0, y_0, z_0, \lambda, \mu)$ is a weak efficient solution to (DP). Otherwise, there exists $(\hat{x}, \hat{y}, \hat{z}, \hat{\lambda}, \hat{\mu}) \in M$ such that $h(\hat{x}, \hat{y}, \hat{z}, \hat{\lambda}, \hat{\mu}) - h(x_0, y_0, z_0, \lambda, \mu) \in \operatorname{int} C$, that is,

$$\hat{y} + \frac{e}{\langle \hat{\lambda}, e \rangle}(1-\delta)\langle \hat{\mu}, \hat{z} \rangle - y_0 \in \operatorname{int} C,$$

which contradicts (4.1) and the proof is complete. □

Theorem 4.3. [Converse duality] *Assume that all the conditions of Corollary 3.3 are satisfied, and $(x_0, y_0, z_0, \lambda, \mu)$ is a efficient element of (DP). Then, (x_0, y_0) is a weak efficient solution to (P).*

Proof. By Corollary 3.3, all the constraints of (DP) are satisfied. Then $(x_0, y_0, z_0, \lambda, \mu)$ is a efficient element of (DP) and (x_0, y_0) is a weak efficient solution to (P). □

Remark 4.1. (i). The duality theorems hold whenever $D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u,v,w)\big)(x)$ is replaced by $D_g^{h}(F,G)\big((x_0,y_0,z_0),(u,v,w)\big)(x)$ in dual problem (DP).

(ii). If $\delta = 1$, then (DP) reduces to the higher-order Mond-Weir dual problem:

$$(\text{MDP}) \begin{cases} \max & y_0 \\ s.t. & \langle \lambda, y \rangle + \langle \mu, z \rangle \geq 0, \forall (y,z) \in D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u,v,w)\big)(\Omega), \\ & \langle \mu, z_0 \rangle \geq 0, \\ & \lambda \in C^*\backslash\{0_{Y^*}\}, \\ & \mu \in D^*. \end{cases}$$

If $\delta = 0$, we have $h(x_0, y_0, z_0, \lambda, \mu) = y_0 + \frac{e}{\langle \lambda, e \rangle}(1-0)\langle \mu, z_0 \rangle = \frac{e}{\langle \lambda, e \rangle}\big(\langle \lambda, y_0 \rangle + \langle \mu, z_0 \rangle\big)$. Since $\frac{e}{\langle \lambda, e \rangle} \in \operatorname{int} C$, the (DP) reduces to the higher-order Wolfe dual problem:

$$(\text{WDP}) \begin{cases} \max & \langle \lambda, y_0 \rangle + \langle \mu, z_0 \rangle \\ s.t. & \langle \lambda, y \rangle + \langle \mu, z \rangle \geq 0, \forall (y,z) \in D_g^{hi}(F,G)\big((x_0,y_0,z_0),(u,v,w)\big)(\Omega), \\ & \lambda \in C^*\backslash\{0_{Y^*}\}, \\ & \mu \in D^*. \end{cases}$$

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