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DIFFERENCE OF COMPOSITION OPERATORS ON THE BÉKOLLÉ WEIGHTED BERGMAN SPACES IN THE UNIT BALL

HUIJIE LIU[1], CEZHONG TONG[1,*], ZIXING YUAN[2]

[1]*Department of Mathematics, Hebei University of Technology, Tianjin, China*
[2]*School of Mathematics and Statistics, Wuhan University, Wuhan, China*

Abstract. In this paper, we completely characterize the boundedness and compactness of the differences for composition operators on the weighted Bergman spaces with Békollé weights in the open unit ball and estimate the norm and essential norm of weighted composition operators.
Keywords. Bergman spaces; Békollé weights; Carleson measures; Composition operators.

1. Introduction

Let $\mathbb{B}_n$ denote the open unit ball in the n dimensional Euclidean complex space $\mathbb{C}^n$ and $H(\mathbb{B}_n)$ the space of analytic functions on $\mathbb{B}_n$. If $n = 1$, then $\mathbb{B}_n$ is the unit disk in the complex plane. Every analytic self mapping φ of $\mathbb{B}_n$ can induce a *composition operator* C_φ on $H(\mathbb{B}_n)$ defined by

$$C_\varphi f = f \circ \varphi$$

for $f \in H(\mathbb{B}_n)$. Furthermore, if v is a function defined on $\mathbb{B}_n$, functions v and φ can induce a *weighted composition operator* vC_φ for which

$$vC_\varphi f = v \cdot f \circ \varphi$$

where $f \in H(\mathbb{B}_n)$.

It is followed by the Littlewood subordination principle that every composition operator acts continuously on the classic Hardy space and standard weighted Bergman spaces of the unit disk. The boundedness and compactness of weighted composition operators on standard weighed Bergman spaces are characterized by Čučković and Zhao [6] in terms of generalized Berezin transforms. Later, in 2007, they applied the Bergman projection to generalize those results to the higher dimensional case in [7]. Efforts have been expended on characterizing those analytic maps which induce bounded composition operators on various analytic function spaces. Readers interested in this topic can refer to the books [3] by Cowen and MacCluer, [22] by Shapiro, [27] by Zhao and Zhu, and [28, 29] by Zhu, which are excellent sources for the development of the theory on composition operators and functions spaces.

*Corresponding author.
E-mail address: liuhuijie_1@163.com (H. Liu), ctong@hebut.edu.cn, cezhongtong@hotmail.com (C. Tong), yuan980127@163.com (Z. Yuan).
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Another active topic in the past decades is the differences of two composition operators acting between two analytic function spaces. In 1989, Shapiro and Sundberg [23] characterized the compactness of the differences $C_\varphi - C_\psi$ by the boundary conditions of φ and ψ. In 2005, Moorhouse [16] used pseudohyperbolic distance between $\varphi(z)$ and $\psi(z)$ to study the difference on the standard weighted Bergman spaces. In 2007, Kriete and Moorhouse [13] extended those results to general linear combinations of composition operators. In 2011, Saukko [20, 21] studied the differences of composition operators between standard weighted Bergman spaces. The key tools are Carleson measures and interpolation sequences, and the proof of their main theorem depends highly on the density of the polynomials in the standard Bergman spaces.

Muckenhoupt weights and weighted inequalities arise naturally in Fourier analysis, but their use is best justified by the variety of applications in which they appear. In the 1970s, Muckenhoupt characterized those positive functions u for which the Hardy-Littlewood maximal operator maps $L^p(\mathbb{R}^n, u(x)\mathrm{d}x)$ to itself. This characterization led to the introduction of the Muckenhoupt class A_p and the development of weighted inequalities. Its extensions to complex valued spaces and vector valued spaces become active in the past decades. The weighted Bergman spaces considered in this paper are equipped with the so-called *Békollé weights* (see more details in Preliminaries). These weights are the Bergman space analogues of the Muckenhoupt class A_p used in harmonic analysis. Békollé and Bonami introduced these weights in [1, 2], and characterized the boundedness of the Bergman projection. The Sharp dependence of the operator norm on the $B_{p,b}$ characteristic were given by Pott and Reguera in [17] and Rahm, Tchoundja and Wick in [19]. Constantin [5] proved Carleson-type embedding theorems for weighted Bergman spaces with Békollé weights on the unit disk, and characterized the boundedness, compactness and Schatten class of Toeplitz type operators, integral operators, and composition operators.

The motivation of this paper is to characterize the boundedness and compactness of weighted composition operators and then the differences on the Bergman spaces with Békollé weights. Our main results can be regarded as a generalization of [6] and [20, 21] in the Békollé weights settings. Inspired by the ideas of [7] and [26], we mainly apply the Bergman projection in [1] and employ some results on Carleson measures in [4, 5, 19] to overcome the obstacles. In [21], Saukko's proof of difference when $0 < q < p < \infty$ is base on the atomic decomposition and interpolation. Unfortunately, these keys are not available for the high dimensional cases. Inspired by the ideas of [24], we mainly apply joint pull-back measure, Carleson embedding theorem, and Khinchine inequality to overcome it. At the same time, Saukko pointed out that, by Pitt's theorem, the operator $C_\varphi - C_\psi : A_b^p \to L^q(\mu)$ is compact whenever it is bounded for $1 < q < p < \infty$. In fact, we can prove that it is true for full range $0 < q < p < \infty$ on the weighted Bergman spaces with Békollé weights in the open unit ball.

Now, we introduce our main results first. All the notations will be specified in Sections 2 and 3. The first two results are on the essential norms of weighted composition operators.

Theorem 1.1. *Suppose $1 < p \le q < \infty$, $0 < r < 1$, $1 < p_0 \le q$, $b > -1$, and $u \in B_{p_0,b}$. Let v be an analytic function on $\mathbb{B}_n$ and φ an analytic self mapping of $\mathbb{B}_n$ such that $vC_\varphi : A_b^p(u) \to A_b^q(u)$ is bounded. Then*

$$\|vC_\varphi : A_b^p(u) \to A_b^q(u)\|_e^q \lesssim \left\| \left(\mu_{v,\varphi}^{q,u,b} \right)_\delta \right\|_{r,Geo}.$$

Theorem 1.2. *Let $1 < p \le q < \infty$, $1 < p_0 \le p$, $b > -1$, and $u \in B_{p_0,b}$. Suppose that v is a measurable function on $\mathbb{B}_n$ and vC_φ is bounded from $A_b^p(u)$ into $L_b^q(u)$. Then*

$$\|vC_\varphi : A_b^p(u) \to L_b^q(u)\|_e \simeq \limsup_{|w|\to 1} \|vC_\varphi(g_w^s)\|_{L_b^q(u)},$$

where $s \ge (n+1+b)p_0/p$ and g_w^s is the test function as (2.5).

We next characterize the differences of composition operators by the following two theorems.

Theorem 1.3. *Let $0 < p \le q < \infty$, $0 < r < 1$, $p_0 > 1$, $b > -1$, $u \in B_{p_0,b}$, and positive r small enough. Suppose that φ and ψ are holomorphic self mappings of $\mathbb{B}_n$. Then*

$$C_\varphi - C_\psi : A_b^p(u) \to A_b^q(u)$$

is bounded if and only if σC_φ and σC_ψ are bounded from $A_b^p(u)$ into $L_b^q(u)$. Furthermore,

(i):
$$\|C_\varphi - C_\psi\|_{A_b^p(u)\to A_b^q(u)} \simeq \max\left\{\left\|\mu_{\sigma,\varphi}^{q,u,b}\right\|_{r,Geo}, \left\|\mu_{\sigma,\psi}^{q,u,b}\right\|_{r,Geo}\right\};$$

(ii): *if $1 < p_0 \le p$, then*
$$\|C_\varphi - C_\psi\|_{A_b^p(u)\to A_b^q(u),e} \simeq \max\left\{\lim_{\delta\to 1}\left\|\left(\mu_{\sigma,\varphi}^{q,u,b}\right)_\delta\right\|_{r,Geo}, \lim_{\delta\to 1}\left\|\left(\mu_{\sigma,\psi}^{q,u,b}\right)_\delta\right\|_{r,Geo}\right\}.$$

We now introduce the joint pull-back measure $\omega_{\mu,q}$ on $\mathbb{B}_n$ defined by

$$\omega_{\mu,q}(E) = \int_{\varphi^{-1}(E)} \sigma(z)^q d\mu(z) + \int_{\psi^{-1}(E)} \sigma(z)^q d\mu(z)$$

for all Borel sets E in $\mathbb{B}_n$. This measure, which first appeared in [12, Theorem 1.1], satisfies the key property that

$$\left\|\sigma C_\varphi f\right\|_{L^q(\mu)}^q + \left\|\sigma C_\psi f\right\|_{L^q(\mu)}^q = \int_{\mathbb{B}_n} |f(z)|^q d\omega_{\mu,q}(z). \tag{1.1}$$

The above equality is from [11, p.163].

Theorem 1.4. *Let $0 < q < p < \infty$, $0 < r < 1$, $p_0 > 1$, $b > -1$, and $u \in B_{p_0,b}$. Let μ be a finite positive Borel measure on $\mathbb{B}_n$, and let φ and ψ be holomorphic self mappings of $\mathbb{B}_n$. Then the following are equivalent:*

(i): *The operator $C_\varphi - C_\psi : A_b^p(u) \to L^q(\mu)$ is bounded;*
(ii): *The operator $C_\varphi - C_\psi : A_b^p(u) \to L^q(\mu)$ is compact;*
(iii): *The joint pull-back measure $\omega_{\mu,q}$ is a q-Carleson measure.*

The paper is organized as follows. the Békollé weights condition and related weighted Bergman spaces are introduced in Section 2. We also give some key lemmas, which will be used frequently in the later sections. In Section 3, we apply Carleson embedding theorems to estimate the essential norms of weighted composition operators in terms of both geometric norms of pullback measures and test functions. In Section 4, we investigate the differences of composition operators when $0 < p \le q < \infty$. Our main result, Theorem 1.3, illustrates the equivalence between the norms of the differences and the norms of the pullback measures. In Section 5, we investigate the boundedness and compactness of differences for composition operators when $0 < q < p < \infty$. The methods are different from the standard weighted Bergman spaces.

Throughout the paper, for real positive quantities Q_1 and Q_2, we write $Q_1 \lesssim Q_2$ (or $Q_2 \gtrsim Q_1$) if there is a positive constant C such that $Q_1 \le C \cdot Q_2$. And we write $Q_1 \simeq Q_2$ if $Q_1 \lesssim Q_2$ and $Q_1 \gtrsim Q_2$.

2. PRELIMINARIES

If μ is a positive measure on $\mathbb{B}_n$ and $p > 0$, we denote $L^p(\mu)$ the Lebesgue space over $\mathbb{B}_n$ with respect to μ. That is, $L^p(\mu)$ consists of all functions f defined on $\mathbb{B}_n$ for which

$$\|f\|_{L^p(\mu)} := \left[\int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}\mu(z)\right]^{1/p} < \infty.$$

When $p \ge 1$, $\|\cdot\|_{L^p(\mu)}$ defines a norm and $L^p(\mu)$ becomes a Banach space.

Let $\mathrm{d}V$ denotes the standard Lebesgue measure on $\mathbb{B}_n$. For $b > -1$, the constant c_b is chosen so that $\int_{\mathbb{B}_n} c_b(1-|z|^2)^b \mathrm{d}V(z) = 1$. We define $\mathrm{d}\nu_b(z) = c_b(1-|z|^2)^b \mathrm{d}V(z)$. If u is a positive locally integrable function on $\mathbb{B}_n$, i.e. positive $u \in L^1_{loc}(\mathrm{d}\nu_b)$, we let $L^p_b(u)$ denote the space of measurable functions on $\mathbb{B}_n$ that are pth power integrable with respect to $u\mathrm{d}\nu_b$. That is,

$$\|f\|_{L^p_b(u)} := \left(\int_{\mathbb{B}_n} |f(z)|^p u(z) \mathrm{d}\nu_b(z)\right)^{1/p} < \infty.$$

The Bergman space $A^p_b(u)$ is defined to be a subspace of analytic functions in $L^p_b(u)$ with $L^p_b(u)$-norm. We write $A^p(u) = A^p_0(u)$ for short. The most common reproducing kernel for the unit ball has the form

$$K^s_w(z) = \frac{1}{(1-\langle z,w\rangle)^{n+1+s}},$$

where $\langle z,w\rangle = \sum_{i=1}^n \bar{w}_i z_i$ for $w = (w_1,\dots,w_n) \in \mathbb{B}_n$ and $z = (z_1,\dots,z_n) \in \mathbb{B}_n$, and it corresponds to the space $A^2(1)$.

The following notations are used throughout the paper. For a weight u and a Borel subset $E \subset \mathbb{B}_n$, we set $u_b(E) = \int_E u\mathrm{d}\nu_b$, $\nu_b(E) = \int_E \mathrm{d}\nu_b$, and $u(E) = u_0(E)$, $\nu(E) = \nu_0(E)$ for short. We use $\mathbb{1}_E$ to represent the characteristic function of E. We denote by

$$\langle f\rangle_E^{\mathrm{d}\mu} := \frac{\int_E f(z)\mathrm{d}\mu(z)}{\mu(E)}$$

for integrable f and measure μ.

If we define P_b by

$$P_b f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1-\langle z,w\rangle)^{n+1+b}} \mathrm{d}\nu_b(w).$$

The problem of characterizing the weights for which the Bergman projection P_b is a bounded orthogonal projection from $L^{p_0}_b(u)$ to $A^{p_0}_b(u)$ was solved by Békollé [1] who found that these weights are precisely $u \in B_{p_0,b}$.

$B_{p_0,b}$ **condition.** The Carleson tent over nonzero $a \in \mathbb{B}_n$ is defined to be the set:

$$T_a := \left\{ z = (z_1,\dots,z_n) \in \mathbb{B}_n : \left|1 - \frac{\langle z,a\rangle}{|a|}\right| < 1 - |a| \right\}$$

where $a=(a_1,\dots,a_n)$. The Carleson tent over 0 is $\mathbb{B}_n$. We say u satisfies $B_{p_0,b}$ condition, or $u\in B_{p_0,b}$, if

$$[u]_{B_{p_0,b}} := \sup_{a\in\mathbb{B}_n} \langle u\rangle_{T_a}^{\mathrm{d}v_b}\left(\langle u^{-p_0'/p_0}\rangle_{T_a}^{\mathrm{d}v_b}\right)^{p_0-1} \lesssim 1,$$

where $1/p_0+1/p_0'=1$.

Let Φ_a be the involution of $\mathbb{B}_n$, that is,

$$\Phi_a(w)=\frac{a-P_a(w)-s_aQ_a(w)}{1-w\bar{a}},$$

where $P_a(w)=\frac{\langle w,a\rangle}{|a|^2}z$, $Q_a(w)=w-P_a(w)$, and $s_a=\sqrt{1-|a|^2}$. It is well known that Φ_a is a biholomorphic mapping of $\mathbb{B}_n$ onto itself, also called an *involution* of $\mathbb{B}_n$, with the following properties (see [18]):

(i): $\Phi_a(0)=a$, $\Phi_a(a)=0$;
(ii): $\Phi_a(\Phi_a(z))=z$;
(iii): $1-|\Phi_a(z)|^2=(1-|a|^2)(1-|z|^2)/|1-\langle z,a\rangle|^2$;
(iv): $1-\langle\Phi_a(z),\Phi_a(w)\rangle=(1-|a|^2)(1-\langle z,w\rangle)/(1-\langle z,a\rangle)(1-\langle a,w\rangle)$.

Recall that the pseudohyperbolic metric $\rho:\mathbb{B}_n\times\mathbb{B}_n\to[0,1)$ is defined by $\rho(z,w)=|\Phi_z(w)|$ for $z,w\in\mathbb{B}_n$. We denote the pseudohyperbolic ball by $B_\rho(a,r)=\{z\in\mathbb{B}_n:\rho(z,a)<r\}$. It is also well known that the pseudohyperbolic metric of $\mathbb{B}_n$ has the following properties (see [9]): for $z,w,a\in\mathbb{B}_n$ and the unitary matrix U, we have

$$\rho(U(z),U(w))=\rho(z,w),\quad \text{and}$$

$$\rho(\Phi_a(z),\Phi_a(w))=\rho(z,w),\quad \text{and}$$

$$\frac{|\rho(z,a)-\rho(a,w)|}{1-\rho(z,a)\rho(a,w)}\le\rho(z,w)\le\frac{\rho(z,a)+\rho(a,w)}{1+\rho(z,a)\rho(a,w)}. \tag{2.1}$$

We can define the so-called *Bergman metric*, β on $\mathbb{B}_n$, by:

$$\beta(z,w)=\frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)}.$$

Let $B_\beta(z,r)$ be the ball in the Bergman metric of radius r centered at z. It is well known that for $w\in B_\beta(z,\mathrm{arctanh}r)$ (equivalently $w\in B_\rho(z,r)$) there holds:

$$\mathrm{vol}_b B_\beta(z,\mathrm{arctanh}r)\simeq|1-z\bar{w}|^{n+1+b}\simeq(1-|z|^2)^{n+1+b}\simeq(1-|w|^2)^{n+1+b}, \tag{2.2}$$

where the constants depend only on r. (See [29].) We will make heavy use of these estimates.

We also need the following covering lemma in the proofs of the embedding theorem.

Lemma 2.1 (Theorem 2.23 in [29])**.** *There exists a positive N such that for any $0<r\le1$ we can find a sequence $\{a_k\}$ in $\mathbb{B}_n$ with the following properties.*

(1): $\mathbb{B}_n=\cup_k B_\beta(a_k,r)$;
(2): *The set $B_\beta(a_k,r/4)$ are mutually disjoint;*
(3): *Each point $z\in\mathbb{B}_n$ belongs to at most N of the sets $B_\beta(a_k,2r)$.*

We will also use the following class of weights, which is denoted by $C_{p,b}$. For $p>1$ and $b>-1$, a positive locally integrable weights u belongs to $C_{p,b}$, or say u satisfies $C_{p,b}$ condition if

$$[u]_{C_{p,b}} := \sup_{z\in\mathbb{B}_n, r>0} \langle u\rangle^{\mathrm{d}\nu_b}_{B_\rho(z,r)} \left(\langle u^{-p'/p}\rangle^{\mathrm{d}\nu_b}_{B_\rho(z,r)}\right)^{p-1} \lesssim 1$$

where $1/p+1/p'=1$. Condition $C_{p,b}$ seems to depend on a choice of $r<1$, but it is known that the same class of weights is obtained for any $r\in(0,1)$ and $B_{p,b}\subset C_{p,b}$ for every $b>-1$. To see this, we note that, for $a\in\mathbb{B}_n$ and a given $r>0$, there is a $a'\in\mathbb{B}_n$ such that $B_\rho(a,r)\subset T_{a'}$ with comparable volumes; see more details in [14].

In the Békollé setting, two Bergman metric balls have comparable weighted volumes when their centers are close enough.

Lemma 2.2. *Let $u\in C_{p,b}$ for some $p>1$, and let $t,s\in(0,1)$ and $z,w\in\mathbb{B}_n$ with $\rho(z,w)<r$ for some $r>0$. Then*

$$u_b(B_\rho(z,t)) \simeq u_b(B_\rho(w,s)),$$

where the constant is independent of z and w.

Proof. Notice that if $B_\rho(z,t)\subset B_\rho(w,s)$, then $u\in C_{p,b}$ and $\rho(z,w)<r$ imply that

$$\begin{aligned} u_b(B_\rho(z,t))^{1/p} &\le u_b(B_\rho(w,s))^{1/p} \lesssim \mathrm{vol}_b(B_\rho(w,s))\left[\left(u^{-p'/p}\right)_b(B_\rho(w,s))\right]^{-\frac{1}{p'}} \\ &\le \mathrm{vol}_b(B_\rho(w,s))\left[\left(u^{-p'/p}\right)_b(B_\rho(z,t))\right]^{-\frac{1}{p'}} \lesssim \frac{\mathrm{vol}_b(B_\rho(w,s))}{\mathrm{vol}_b(B_\rho(z,t))} u_b(B_\rho(z,t))^{1/p} \\ &\simeq u_b(B_\rho(z,t))^{1/p}. \end{aligned}$$

We can easily see that both $B_\rho(z,t)$ and $B_\rho(w,s)$ are subsets of $B_\rho(w,t+s+r)$, and hence

$$u_b(B_\rho(z,t) \simeq u_b(B_\rho(w,t+s+r)) \simeq u_b(B_\rho(w,s)).$$

□

Similarly, if $u\in B_{p_0,b}$, it is worthy to be noted that

$$u_b(B_\rho(a,r)) \simeq u_b(T_{a'})$$

whenever $B_\rho(a,r)\subset T_{a'}$ with comparable volumes. Interested readers can refer to [14] and [29, Lemma 5.23] for details.

The point evaluations on $A^p_b(u)$ are bounded linear functionals for $p>0$. To be more precisely, we have the following estimate. The proof is included for the completeness.

Lemma 2.3. *If $p>0$, $p_0>1, 0<r<1$, and a weight $u\in C_{p_0,b}$, $\sigma := u^{-p_0'/p_0}$, we have the following estimate*

$$|f(z)|^p \lesssim u_b(B_\rho(z,r))^{-1}\int_{B_\rho(z,r)} |f(w)|^p u(w)d\nu_b(w) \lesssim \frac{\|f\|^p_{L^p_b(u)}}{u_b(B_\rho(z,r))},$$

where the constant involved is independent of $z\in\mathbb{B}_n$.

Proof. For any $f \in A_b^p(u)$, by the subharmonicity of $|f(z)|$, one can obtain that

$$
\begin{aligned}
|f(z)|^{p/p_0} &\lesssim \frac{1}{\mathrm{vol}_b(B_\rho(z,r))} \int_{B_\rho(z,r)} |f(w)|^{p/p_0} \mathrm{d}\nu_b(w) \\
=& \frac{1}{\mathrm{vol}_b(B_\rho(z,r))} \int_{B_\rho(z,r)} |f(w)|^{p/p_0} u^{1/p_0}(w) \cdot u^{-1/p_0}(w) \mathrm{d}\nu_b(w) \\
\leq& \frac{\left[\left(u^{-p_0'/p_0}\right)_b (B_\rho(z,r))\right]^{1/p_0'}}{\mathrm{vol}_b(B_\rho(z,r))} \left(\int_{B_\rho(z,r)} |f(w)|^p u(w) \mathrm{d}\nu_b(w)\right)^{1/p_0} \\
\lesssim& \frac{1}{u_b(B_\rho(z,r))^{\frac{1}{p_0}}} \left(\int_{B_\rho(z,r)} |f(w)|^p u(w) \mathrm{d}\nu_b(w)\right)^{1/p_0},
\end{aligned}
$$

where the last inequality follows by the fact that $u \in C_{p_0,b}$. That completes the proof. $\square$

The following lemma will be used to connect the difference of composition operators with weighted composition operators, which is analogues to [15, Lemma 3.5].

Lemma 2.4. *Let $0 < p \leq q < \infty$, $p_0 > 1$, $b > -1$, $0 < r < 1$, and the weight $u \in C_{p_0,b}$. Then there exist constants $C = C(b,r,p)$ and $R' = R'(r)$ such that*

$$
|f(z) - f(a)|^q \leq C\rho(z,a)^q \frac{\int_{B_\rho(a,R')} |f(w)|^p u(w) d\nu_b(w)}{u_b(B_\rho(a,R'))^{q/p}}
$$

for z, a such that $\rho(z,a) < r$, where $f \in A_b^p(u)$ with $\|f\|_{A_b^p(u)} \leq 1$.

Proof. We firstly prove the case when $q = p$. Let $g = f \circ \Phi_a$, where Φ_a is the involution interchanging the origin and a. Hence $f = g \circ \Phi_a$, and

$$
|f(z) - f(a)| = |g(\Phi_a(z)) - g(0)| \leq |\Phi_a(z)| \sup_{|\xi| < |\Phi_a(z)|} |\nabla g(\xi)|,
$$

where

$$
\nabla g(\xi) = \left(\frac{\partial}{\partial z_1} g, \ldots, \frac{\partial}{\partial z_n} g\right)(\xi).
$$

Let $R = \frac{1+r}{2}$. According to Theorem 2.2 in [29], one has

$$
g(R\xi) = \int_{\mathbb{B}_n} \frac{g(R\eta)}{(1 - \langle \xi, \eta \rangle)^{n+1}} \mathrm{d}\nu(\eta), \quad \forall \xi \in \mathbb{B}_n.
$$

Changing variables gives

$$
g(\xi) = \int_{\mathbb{B}_n} \frac{g(R\eta)}{\left(1 - \frac{\langle \xi, \eta \rangle}{R}\right)^{n+1}} \mathrm{d}\nu(\eta) = R^2 \int_{R\mathbb{B}_n} \frac{g(\eta)}{(R^2 - \langle \xi, \eta \rangle)^{n+1}} \mathrm{d}\nu(\eta).
$$

Then we have

$$\begin{aligned}\left|\frac{\partial}{\partial z_j}g(\xi)\right| &= \left|R^2\int_{R\mathbb{B}_n}\frac{g(\eta)(n+1)\bar{\eta}_j}{(R^2-\langle\xi,\eta\rangle)^{n+2}}\mathrm{d}\nu(\eta)\right| \\ \le&(n+1)\left(\frac{1+r}{2}\right)^2\left(\frac{4}{1-r^2}\right)^{n+2}\int_{R\mathbb{B}_n}|g(\eta)|\mathrm{d}\nu(\eta) \\ \le&C_r\sup_{|\eta|<R}|g(\eta)|.\end{aligned}$$

It follows that

$$|\nabla g(\xi)|^p \le n^{p/2}C_r\sup_{|\eta|<R}|g(\eta)|^p$$

for every $|\xi|<r$. Hence

$$\begin{aligned}|f(z)-f(a)|^p &\le |\Phi_a(z)|^p\sup_{|\xi|<|\Phi_a(z)|}|\nabla g(\xi)|^p \\ \le& n^{p/2}C_r|\Phi_a(z)|^p\sup_{|\eta|<R}|g(\eta)|^p = n^{p/2}C_r|\Phi_a(z)|^p\sup_{|\Phi_a(\zeta)|<R}|f(\zeta)|^p.\end{aligned} \tag{2.3}$$

For every $\zeta\in\mathbb{B}_n$ with $\rho(a,\zeta)<R=(1+r)/2$, we let $R'=\frac{1+3r}{2+r+r^2}$. According to the strong triangle inequality (2.1), it is easy to find that $\rho(\zeta,a)<(1+r)/2$ and $\rho(\zeta,\omega)<r$ imply that $\rho(a,\omega)<\frac{(1+r)/2+r}{1+r(1+r)/2}=R'$. By Lemma 2.3 and Lemma 2.2 we have

$$\begin{aligned}|f(\zeta)|^p &\lesssim \frac{1}{u_b(B_\rho(\zeta,r))}\int_{\rho(\zeta,\omega)<r}|f(\omega)|^p u(\omega)\mathrm{d}\nu_b(\omega) \\ \lesssim&\frac{1}{u_b(B_\rho(a,R'))}\int_{\rho(a,\omega)<R'}|f(\omega)|^p u(\omega)\mathrm{d}\nu_b(\omega).\end{aligned} \tag{2.4}$$

The inequality that we need can be obtained by plugging (2.4) into (2.3). The case when $q>p$ can be proved by the fact that $|f(z)-f(a)|^q=(|f(z)-f(a)|^p)^{q/p}$ and $\|f\|_{A^p_b(u)}\le 1$. □

2.1. **Carleson measures.** Carleson measure plays a role in the study of composition operators. Let $p,q>0$. A positive Borel measure μ on $\mathbb{B}_n$ is called a *q-Carleson measure* for $A^p_b(u)$ if the embedding $I:A^p_b(u)\to L^q(\mathrm{d}\mu)$ is bounded. If $s>0$, we denote

$$G^s_w(z)=(1-\langle z,w\rangle)^{-s}\quad z,w\in\mathbb{B}_n.$$

The following Lemma is from [25, lemma 2.4].

Lemma 2.5. *Let $p>0$, $p_0>1$, $b>-1$, and the weight $u\in B_{p_0,b}$. Then*

$$\frac{u_b(T_w)^{\frac{1}{p}}}{(1-|w|)^s}\lesssim\|G^s_w\|_{L^p_b(u)}\lesssim\frac{u_b(T_w)^{\frac{1}{p}}}{(1-|w|)^{\max\{(n+1+b)p_0/p,s\}}}$$

where the constant involved is independent of $w\in\mathbb{B}_n$.

Let $u\in B_{p_0,b}$. If $s\ge(n+1+b)p_0/p$, we denote by

$$g^s_w=\frac{(1-|w|)^s}{u_b(B_\rho(w,r))^{1/p}}G^s_w, \tag{2.5}$$

and then

$$\|g_w^s\|_{L_b^p(u)} \simeq \left\| \frac{(1-|w|)^s}{u_b(T_{w'})^{1/p}} G_{w'}^s \right\|_{L_b^p(u)} \simeq 1$$

for some w' with $1-|w| \simeq 1-|w'|$, which can be derived from [29, Lemma 5.23].

Boundedness and compactness of the embedding $I : A_b^p(u) \to L^q(\mathrm{d}\mu)$ when $0 < p, q < \infty$ are characterized in [25] and we summarize them as below lemmas.

Lemma 2.6. *Let $q \geq p > 0$, $p_0 > 1$, $u \in B_{p_0,b}$ is a weight, and μ is a positive Borel measure on $\mathbb{B}_n$. Then the following conditions are equivalent*

(a): *$I : A_b^p(u) \to L^q(d\mu)$ is bounded, that is,*

$$\left(\int_{\mathbb{B}_n} |f(z)|^q d\mu(z)\right)^{1/q} \lesssim \left(\int_{\mathbb{B}_n} |f(z)|^p u(z) d\nu_b(z)\right)^{1/p}$$

for all holomorphic f in $\mathbb{B}_n$;

(b): *$\mu(T_a) \lesssim u_b(T_a)^{q/p}$ for all $a \in \mathbb{B}_n$;*

(c): *there is a $r > 0$ such that $\mu(B_\beta(a,r)) \lesssim u_b(B_\beta(a,r))^{q/p}$ for all $a \in \mathbb{B}_n$;*

(d): *there is a $r > 0$ such that $\mu(B_\beta(a_k,r)) \lesssim u_b(B_\beta(a_k,r))^{q/p}$ for the sequence $\{a_k\}$ described in Lemma 2.1;*

(e): *whenever $s \geq (n+1+b)p_0/p$,*

$$\sup_{w \in \mathbb{B}_n} \int_{\mathbb{B}_n} \left| \frac{1-|w|^2}{1-\langle z,w\rangle} \right|^{qs} u_b(B_\beta(w,r))^{-q/p} d\mu(z) \lesssim 1.$$

Lemma 2.7. *Let $u \in B_{p_0,b}$ for some $p_0 > 1$, and let μ be a positive finite Borel measure on $\mathbb{B}_n$. If $p > q > 0$, then the embedding from $A_b^p(u)$ into $L^q(\,\mathrm{d}\mu)$ is bounded. To be more precisely,*

$$\int_{\mathbb{B}_n} |f(z)|^q \,\mathrm{d}\mu(z) \lesssim \|f\|_{A_b^p(u)}^q$$

if and only if the function

$$\mathbb{B}_n \ni z \mapsto \frac{\mu\left(B_\rho(z,r)\right)}{u_b\left(B_\rho(z,r)\right)}$$

belongs to $L_b^{\frac{p}{p-a}}(u)$ for some $r \in (0,1)$.

Lemma 2.8. *Let $0 < p \leq q < \infty$, $0 < r < 1$, $p_0 > 1$, $b > -1$, and $u \in B_{p_0,b}$. Let μ be a positive Borel measure on $\mathbb{B}_n$. Then the following assertions are equivalent:*

(a): *the embedding $I : A_b^p(u) \to L^q(d\mu)$ is compact, that is,*

$$\lim_{k\to\infty} \int_{\mathbb{D}} |f_k(z)|^q d\mu(z) = 0$$

whenever $\{f_k\}$ is bounded in $A_b^p(u)$ that converges to 0 uniformly on compact subsets of $\mathbb{B}_n$;

(b): *let $T_a = \{z \in \mathbb{B}_n : |1-\langle z, a/|a|\rangle| < 1-|a|\}$,*

$$\lim_{|a|\to 1} \frac{\mu(T_a)}{u_b(T_a)^{q/p}} = 0;$$

(c): *let $B_\beta(a,r)$ be the Bergman metric ball,*

$$\lim_{|a|\to 1} \frac{\mu(B_\beta(a,r))}{u_b(B_\beta(a,r))^{q/p}} = 0;$$

(d):

$$\lim_{k\to\infty} \frac{\mu(B_\beta(a_k,r))}{u_b(B_\beta(a_k,r))^{q/p}} = 0,$$

where $\{a_k\}$ is the sequence described in Lemma 2.1.

Lemma 2.9. *Let $p > q > 0$, $r > 0$, $u \in B_{p_0,b}$ be a weight and μ be a positive Borel measure on $\mathbb{B}_n$. Then the embedding $I : A_b^p(u) \to L^q(d\mu)$ is compact if and only if I is bounded.*

3. Weighted Composition Operators

The embedding theorem can be applied to weighted composition operators. For a q-Carleson measure on $A_b^p(u)$, we denote by

$$\|\mu\|_{Oper}^q := \sup_{\|f\|_{A_b^p(u)}\le 1} \int_{\mathbb{B}_n} |f(z)|^q \mathrm{d}\mu(z).$$

Suppose that $v : \mathbb{B}_n \to \mathbb{C}$ is a measurable function and φ is an analytic self mapping of $\mathbb{B}_n$. We define the *pullback measure* $\mu_{v,\varphi}^{q,u,b}$ of v and φ by

$$\mu_{v,\varphi}^{q,u,b}(E) := \int_{\varphi^{-1}(E)} |v(z)|^q u(z) \mathrm{d}v_b(z)$$

for a Borel set $E \subset \mathbb{B}_n$. The *geometric norm* of the pullback measure $\mu_{v,\varphi}^{q,u,b}$ is defined by

$$\left\|\mu_{v,\varphi}^{q,u,b}\right\|_{r,Geo} := \sup_{w\in\mathbb{B}_n} \frac{\mu_{v,\varphi}^{q,u,b}(B_\beta(w,r))}{u_b(B_\beta(w,r))^{q/p}},$$

where $B_\beta(w,r)$ is the Bergman metric ball of radius r centered at w.

The following result is a direct consequence of Carleson embedding theorem.

Corollary 3.1. *Let $0 < p \le q < \infty$, $0 < r < 1$, $p_0 > 1$, $b > -1$, and $u \in B_{p_0,b}$. Let v be a measurable function on $\mathbb{B}_n$ and φ an analytic self mapping of $\mathbb{B}_n$. Then the following are equivalent:*

(i): *The weighted composition operator $vC_\varphi : A_b^p(u) \to L_b^q(u)$ is bounded;*

(ii): *the* geometric norm *of the pullback measure $\mu_{v,\varphi}^{q,u,b}$ is finite, that is,*

$$\left\|\mu_{v,\varphi}^{q,u,b}\right\|_{r,Geo} < \infty;$$

(iii): *let g_w^s be defined as equation (2.5), then*

$$\sup_{w\in\mathbb{D}} \left\|vC_\varphi(g_w^s)\right\|_{L_b^q(u)}^q < \infty.$$

*Furthermore, $\|vC_\varphi\|_{A_b^p(u)\to L_b^q(u)}^q$ and $\left\|\mu_{v,\varphi}^{q,u,b}\right\|_{r,Geo}$, and the quantity in **(iii)** are comparable.*

We denote $\mu_\delta(E) := \mu((\mathbb{B}_n \setminus \delta\mathbb{B}_n) \cap E)$ for $\delta \in (0,1)$ and positive Borel measure μ. It is clear that μ_δ is a Carleson measure if μ is a Carleson measure. We are going to estimate the essential norm of weighted composition operators between Bergman spaces with Békollé weights by the pullback measures. That is to prove Theorem 1.1.

Proof of Theorem 1.1. For $0 < \delta < 1$, we define $M_\delta^\varphi : A_b^p(u) \to L_b^p(u)$ by

$$M_\delta^\varphi f(z) = \mathbb{1}_{\varphi^{-1}(\delta\mathbb{B}_n)}(z) f(z).$$

It is easy to see that every M_δ^φ is compact since φ is analytic on $\mathbb{B}_n$. Recall that P_b is a projection from $L_b^q(u)$ onto $A_b^q(u)$ when $u \in B_{p_0,b} \subset B_{q,b}$ for $q \geq p_0$. Now we can estimate the essential norm of vC_φ as follows

$$\begin{aligned}
&\|vC_\varphi : A_b^p(u) \to A_b^q(u)\|_e^q \\
=&\inf\{\|vC_\varphi - K : A_b^p(u) \to A_b^q(u)\| : K \text{ is compact}\}^q \\
\leq&\left\|vC_\varphi - P_b M_\delta^\varphi(vC_\varphi) : A_b^p(u) \to A_b^q(u)\right\|^q \\
=&\sup_{\|f\|_{A_b^p(u)} \leq 1} \|P_b(I - M_\delta^\varphi) vC_\varphi(f)\|_{A_b^q(u)}^q \\
\leq&\|P_b : L_b^q(u) \to A_b^q(u)\|^q \sup_{\|f\|_{A_b^p(u)} \leq 1} \int_{\mathbb{B}_n \setminus \varphi^{-1}(\delta\mathbb{B}_n)} |v(z) f(\varphi(z))|^q u(z) \mathrm{d}v_b(z) \\
\lesssim&\sup_{\|f\|_{A_b^p(u)} \leq 1} \int_{\varphi^{-1}(\mathbb{B}_n \setminus \delta\mathbb{B}_n)} |v(z) f(\varphi(z))|^q u(z) \mathrm{d}v_b(z).
\end{aligned}$$

Since vC_φ is bounded from $A_b^p(u)$ to $L_b^q(u)$, the pullback measure $\mu_{v,\varphi}^{q,u,b}$ is a Carleson measure, so is $\left(\mu_{v,\varphi}^{q,u,b}\right)_\delta$ for any $\delta \in (0,1)$. Combining this fact with Lemma 2.3 and Fubini's Theorem, we have

$$\begin{aligned}
&\int_{\varphi^{-1}(\mathbb{B}_n \setminus \delta\mathbb{B}_n)} |v(z) f(\varphi(z))|^q u(z) \mathrm{d}v_b(z) \\
\lesssim&\int_{\mathbb{B}_n} \mathbb{1}_{\varphi^{-1}(\mathbb{B}_n \setminus \delta\mathbb{B}_n)} |v(z)|^q \frac{\int_{B_\beta(\varphi(z),r)} |f(w)|^q u(w) \mathrm{d}v_b(w)}{u_b(B_\beta(\varphi(z),r))} u(z) \mathrm{d}v_b(z) \\
\simeq&\int_{\mathbb{B}_n} |f(w)|^q \frac{\int_{\varphi^{-1}(B_\beta(w,r) \cap (\mathbb{B}_n \setminus \delta\mathbb{B}_n))} |v(z)|^q u(z) \mathrm{d}v_b(z)}{u_b(B_\beta(w,r))} u(w) \mathrm{d}v_b(w) \\
=&\int_{\mathbb{B}_n} |f(w)|^q \frac{\left(\mu_{v,\varphi}^{q,u,b}\right)_\delta (B_\beta(w,r))}{u_b(B_\beta(w,r))^{q/p}} u_b(B_\beta(w,r))^{\frac{q-p}{p}} u(w) \mathrm{d}v_b(w) \\
\leq&\left\|\left(\mu_{v,\varphi}^{q,u,b}\right)_\delta\right\|_{r,Geo} \int_{\mathbb{B}_n} |f(w)|^p \left[|f(w)| u_b(B_\beta(w,r))^{1/p}\right]^{q-p} u(w) \mathrm{d}v_b(w) \\
\leq&\left\|\left(\mu_{v,\varphi}^{q,u,b}\right)_\delta\right\|_{r,Geo} \|f\|_{A_b^p(u)}^p \|f\|_{A_b^p(u)}^{q-p} \\
=&\left\|\left(\mu_{v,\varphi}^{q,u,b}\right)_\delta\right\|_{r,Geo} \|f\|_{A_b^p(u)}^q,
\end{aligned}$$

which completes the proof. □

In the rest of this section, we are going to characterize the essential norm of weighted composition operators in terms of test functions. We firstly prove two key lemmas which will be used to get the main result of this section.

Lemma 3.1. *Let* $0 < p \le q < \infty$*,* $p_0 > 1$*,* $u \in B_{p_0,b}$*, and* $\delta \in (0,1)$*. Suppose that a positive measure* μ *on* $\mathbb{B}_n$ *is a q-Carleson measure for* $A_b^p(u)$*. Then* μ_δ *is also a q-Carleson measure for* $A_b^p(u)$*. Moreover, for any fixed* $0 < \varepsilon < 1$ *and* $s \ge (n+1+b)p_0/p$*, we have*

$$\|\mu_\delta\|_{Oper}^q \lesssim \sup_{w \in \mathbb{B}_n \setminus (1-\varepsilon)\delta\mathbb{B}_n} \int_{\mathbb{B}_n} |g_w^s(z)|^q d\mu(z).$$

Proof. For $z \in \mathbb{B}_n \setminus \delta\mathbb{B}_n$, we have $\rho(z,0) = |z| \ge \delta$. For $0 < \varepsilon < 1$ fixed, $\rho(w,z) < \varepsilon\delta$ implies that

$$\rho(w,0) \ge \rho(z,0) - \rho(w,z) > \delta(1-\varepsilon).$$

Keeping this fact in mind, we have that

$$
\begin{aligned}
&\|\mu_\delta\|_{Oper}^q = \sup_{\|f\|_{A_b^p(u)} \le 1} \int_{\mathbb{B}_n} |f(z)|^q \mathrm{d}\mu_\delta(z) \\
&\lesssim \sup_{\|f\|_{A_b^p(u)} \le 1} \int_{\mathbb{B}_n} \left(\frac{1}{u_b(B_\rho(z,\delta\varepsilon))} \int_{B_\rho(z,\delta\varepsilon)} |f(w)|^p u(w) \mathrm{d}\nu_b(w) \right)^{q/p} \mathrm{d}\mu_\delta(z) \\
&= \sup_{\|f\|_{A_b^p(u)} \le 1} \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{\mathbb{1}_{B_\rho(z,\delta\varepsilon)}(w)}{u_b(B_\rho(z,\delta\varepsilon))} |f(w)|^p u(w) \mathrm{d}\nu_b(w) \right)^{q/p} \mathrm{d}\mu_\delta(z) \\
&\lesssim \sup_{\|f\|_{A_b^p(u)} \le 1} \left[\int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{\mathbb{1}_{B_\rho(z,\delta\varepsilon)}(w)}{u_b(B_\rho(z,\delta\varepsilon))^{q/p}} |f(w)|^q \mathrm{d}\mu_\delta(z) \right)^{p/q} u(w) \mathrm{d}\nu_b(w) \right]^{q/p} \\
&\simeq \sup_{\|f\|_{A_b^p(u)} \le 1} \left[\int_{\mathbb{B}_n} \left(\int_{B_\rho(w,\delta\varepsilon)} \frac{1}{u_b(B_\rho(w,\delta\varepsilon))^{q/p}} \mathrm{d}\mu_\delta(z) \right)^{p/q} |f(w)|^p u(w) \mathrm{d}\nu_b(w) \right]^{q/p} \\
&\simeq \sup_{\|f\|_{A_b^p(u)} \le 1} \left[\int_{\mathbb{B}_n} \left(\int_{B_\rho(w,\delta\varepsilon)} \frac{(1-|w|)^{qs} |1-\langle z,w\rangle|^{-qs}}{u_b(B_\rho(w,\delta\varepsilon))^{q/p}} \mathrm{d}\mu_\delta(z) \right)^{p/q} |f(w)|^p u(w) \mathrm{d}\nu_b(w) \right]^{q/p} \\
&\le \sup_{\|f\|_{A_b^p(u)} \le 1} \left(\sup_{w \in \mathbb{B}_n \setminus (1-\varepsilon)\delta\mathbb{B}_n} \int_{\mathbb{B}_n} |g_w^s(z)|^q \mathrm{d}\mu(z) \right) \cdot \|f\|_{A_b^p(u)}^q,
\end{aligned}
$$

where the first inequality follows from Lemma 2.3; the second inequality follows from Minkowski's inequality for integrals; the fifth line follows from the fact that $\mathbb{1}_{B_\rho(z,r)}(w) = \mathbb{1}_{B_\rho(w,r)}(z)$ and Lemma 2.2; the sixth line follows from the equation (2.2), and the last inequality follows from the observation in the beginning of the proof. □

Lemma 3.2. *Let $1 < p_0 \le p < \infty$, $b > -1$, and $u \in B_{p_0,b}$. Suppose that P_b is the Bergman projection from $L_b^p(u)$ into $A_b^p(u)$. Then, for $f \in A_b^p(u)$,*

$$\lim_{\delta\to 1^-} \sup_{\|f\|_{A_b^p(u)}\le 1} |P_b(f - \mathbb{1}_{\delta\mathbb{B}_n} f)(w)| = 0$$

uniformly on compact subsets of $\mathbb{B}_n$.

Proof. Using Hölder's inequality for $f \in A_b^p(u)$ with $\|f\|_{A_b^p(u)} \le 1$ and noting that w is in a compact subset of $\mathbb{B}_n$, we have

$$\begin{aligned} |P_b(f - \mathbb{1}_{\delta\mathbb{B}_n} f)(w) &= \left| \int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n} \frac{f(z)}{(1-\langle z,w\rangle)^{n+1+b}} \mathrm{d}\nu_b(z) \right| \\ &\le \left(\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n} |f(z)|^p u(z) \mathrm{d}\nu_b(z) \right)^{1/p} \cdot \left(\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n} \frac{u^{-p'/p}(z)\mathrm{d}\nu_b(z)}{|1-\langle z,w\rangle|^{p'(n+1+b)}} \right)^{1/p'} \\ &\lesssim \left(\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n} u^{-p'/p}(z)\mathrm{d}\nu_b(z) \right)^{1/p'} \cdot \|f\|_{A_b^p(u)} \end{aligned}$$

where $p' = p/(p-1)$. By Lemma 4 in [1], the conjugate weight $u^{-p'/p}$ is integrable. Hence,

$$\sup_{\|f\|_{A_b^p(u)}\le 1} |P_b(f - \mathbb{1}_{\delta\mathbb{B}_n})(w)| \lesssim \left(u^{-p'/p}\right)_b (\mathbb{B}_n \setminus \delta\mathbb{B}_n)^{1/p'} \to 0$$

as $\delta \to 1$, which completes the proof. □

Now we can estimate the essential norm in terms of test functions. That is prove Theorem 1.2.

Proof of Theorem 1.2. We firstly note that g_w^s converges to 0 uniformly on compact subsets of $\mathbb{B}_n$ as $|w| \to 1$ whenever $s \ge (n+1+b)p_0/p$. For any compact operator $\mathscr{K} : A_b^p(u) \to L_b^q(u)$, one has $\|\mathscr{K} g_w^s\|_{L_b^q(u)} \to 0$ as $|w| \to 1$. Therefore,

$$\begin{aligned} \|vC_\varphi - \mathscr{K}\|_{A_b^p(u)\to L_b^q(u)} &\gtrsim \limsup_{|w|\to 1} \|(vC_\varphi - \mathscr{K})g_w^s\|_{L_b^q(u)} \\ &\ge \limsup_{|w|\to 1} \left(\|(vC_\varphi)g_w^s\|_{L_b^q(u)} - \|\mathscr{K} g_w^s\|_{L_b^q(u)} \right) \\ &= \limsup_{|w|\to 1} \|(vC_\varphi)g_w^s\|_{L_b^q(u)}. \end{aligned}$$

Hence the essential norm $\|vC_\varphi\|_{A_b^p(u)\to L_b^q(u),e}$ equals to

$$\inf\left\{ \|vC_\varphi - \mathscr{K}\|_{A_b^p(u)\to L_b^q(u)} : \mathscr{K} \text{ is compact} \right\} \ge \limsup_{|w|\to 1} \|(vC_\varphi)g_w^s\|_{L_b^q(u)}.$$

To prove the contrary inequality, it is easy to see that $(\mathbb{1}_{\kappa\mathbb{B}_n}\cdot) : A_b^p(u) \to L_b^p(u)$ is compact for any $0 < \kappa < 1$. And we define $T_\kappa : A_b^p(u) \to L_b^q(u)$ by

$$T_\kappa(f) = vC_\varphi P_b(\mathbb{1}_{\kappa\mathbb{B}_n} f),$$

where $\kappa \in (0,1)$ and the Bergman projection $P_b : L_b^p(u) \to A_b^p(u)$. Hence T_κ is compact. For any $f \in A_b^p(u)$ with norm 1, we have

$$\begin{aligned}
\|(vC_\varphi - T_\kappa)f\|_{L_b^q(u)}^q &= \int_{\mathbb{B}_n} |vC_\varphi f(w) - vC_\varphi P_b(\mathbb{1}_{\kappa\mathbb{B}_n}(w)f(w))|^q u(w)\mathrm{d}v_b(w) \\
&\le \int_{\mathbb{B}_n} |f(w) - P_b(\mathbb{1}_{\kappa\mathbb{B}_n} f)(w)|^q \mathrm{d}\mu_{v,\varphi}^{q,u,b}(w) \\
&= \left(\int_{\delta\mathbb{B}_n} + \int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n}\right) |P_b(f - \mathbb{1}_{\kappa\mathbb{B}_n} f)(w)|^q \mathrm{d}\mu_{v,\varphi}^{q,u,b}(w) \\
&:= I_1 + I_2
\end{aligned}$$

where $0 < \delta < 1$. Item I_1 can be estimated by Lemma 3.2. Indeed, for any $\varepsilon > 0$, there exists a $0 < \kappa_0 < 1$ so that, for any $\kappa \in (\kappa_0, 1)$, $\sup_{\|f\|_{A_b^p(u)} \le 1} I_1 < \varepsilon$.

To estimate I_2, noting that $\left(\mu_{v,\varphi}^{q,u,b}\right)_\delta$ is also an q-Carleson measure for $A_b^p(u)$, we have

$$\begin{aligned}
I_2 &= \int_{\mathbb{B}_n} |P_b(f - \mathbb{1}_{\kappa\mathbb{B}_n} f)(w)|^q \mathrm{d}\left(\mu_{v,\varphi}^{q,u,b}\right)_\delta(w) \\
&\le \left\|\left(\mu_{v,\varphi}^{q,u,b}\right)_\delta\right\|_{Oper}^q \cdot \|P_b(f - \mathbb{1}_{\kappa\mathbb{B}_n} f)\|_{A_b^p(u)}^q \\
&\le \left\|\left(\mu_{v,\varphi}^{q,u,b}\right)_\delta\right\|_{Oper}^q \cdot \|(1 - \mathbb{1}_{\kappa\mathbb{B}_n}) f\|_{A_b^p(u)}^q \cdot \|P_b\|_{L_b^p(u)\to A_b^p(u)}^q \\
&\le \left\|\left(\mu_{v,\varphi}^{q,u,b}\right)_\delta\right\|_{Oper}^q \cdot \|P_b\|_{L_b^p(u)\to A_b^p(u)}^q.
\end{aligned}$$

Combining the estimate above, for any fixed $0 < \delta < 1$, whenever $\kappa_0 < \kappa < 1$, we see that

$$\begin{aligned}
\|vC_\varphi - T_\kappa\|_{A_b^p(u)\to L_b^q(u)}^q &= \sup_{\|f\|_{A_b^p(u)} \le 1} \|(vC_\varphi - T_\kappa)f\|_{L_b^q(u)}^q \\
&\le \varepsilon + \left\|\left(\mu_{v,\varphi}^{q,u,b}\right)_\delta\right\|_{Oper}^q \|P_b\|_{L_b^p(u)\to A_b^p(u)}^q.
\end{aligned}$$

Since ε is arbitrary, the essential norm of $vC_\varphi : A_b^p(u) \to L_b^q(u)$ can be controlled by

$$\inf_\kappa \|vC_\varphi - T_\kappa\|_{A_b^p(u)\to L_b^q(u)} \le \left\|\left(\mu_{v,\varphi}^{q,u,b}\right)_\delta\right\|_{Oper}^q \|P_b\|_{A_b^p(u)\to L_b^q(u)}^q$$

for any $0 < \delta < 1$. By letting $\delta \to 1^-$, we obtain that

$$\|vC_\varphi\|_{A_b^p(u)\to L_b^q(u),e} \le \|P_b\|_{L_b^p(u)\to A_b^p(u)} \cdot \limsup_{\delta\to 1^-} \left\|\left(\mu_{v,\varphi}^{q,u,b}\right)_\delta\right\|_{Oper}^q.$$

To complete the proof, we use Lemma 3.1 to conclude that

$$\begin{aligned}
\|vC_\varphi\|_{A_b^p(u)\to L_b^q(u),e}^q &\lesssim \limsup_{|w|\to 1^-} \int_{\mathbb{B}_n} |g_w^s(z)|^q \mathrm{d}\mu_{v,\varphi}^{q,u,b}(z) \\
&= \limsup_{|w|\to 1^-} \int_{\mathbb{B}_n} |v(z)|^q |g_w^s(\varphi(z))|^q u(z)\mathrm{d}v_b(z) \\
&= \limsup_{|w|\to 1^-} \|vC_\varphi(g_w^s)\|_{L_b^q(u)}^q.
\end{aligned}$$

$\square$

4. The proof of Theorem 1.3

In this section, we characterize the boundedness, compactness, the operator norm, and the essential norm of the difference $C_\varphi - C_\psi$ when $0 < p \leq q < \infty$. That is to prove Theorem 1.3. We divide the proof into the following 4 propositions: Proposition 4.1 to Proposition 4.4. We first look into the upper bound of the operator norm and the essential norm.

Proposition 4.1. *Let $0 < p \leq q < \infty$, $0 < r < 1$, $p_0 > 1$, $b > -1$, and $u \in B_{p_0,b}$. Suppose that φ and ψ are analytic self mappings of $\mathbb{B}_n$. We denote by $\sigma := \rho(\varphi,\psi)$. If the operators σC_φ and σC_ψ map $A_b^p(u)$ into $L_b^q(u)$, then the difference operator $C_\varphi - C_\psi$ is bounded from $A_b^p(u)$ into $A_b^q(u)$ with*

$$\|C_\varphi - C_\psi\|^q_{A_b^p(u)\to A_b^q(u)} \lesssim \max\left\{\left\|\mu^{q,u,b}_{\sigma,\varphi}\right\|_{R',Geo}, \left\|\mu^{q,u,b}_{\sigma,\psi}\right\|_{R',Geo}\right\},$$

where the involved constants depend only on R' and r.

Proof. Denote by $E = \{z \in \mathbb{B}_n : \sigma(z) \geq r\}$ and $E' = \mathbb{B}_n \setminus E$. Then

$$\begin{aligned}
&\|(C_\varphi - C_\psi)\|^q_{A_b^p(u)\to A_b^q(u)} \\
&= \int_{\mathbb{B}_n} |(C_\varphi - C_\psi)f(z)|^q u(z) \mathrm{d}v_b(z) \\
&= \int_E |(C_\varphi - C_\psi)f(z)|^q u(z) \mathrm{d}v_b(z) + \int_{E'} |(C_\varphi - C_\psi)f(z)|^q u(z) \mathrm{d}v_b(z) := \mathscr{E} + \mathscr{E}'.
\end{aligned}$$

According to the triangle inequality, we have

$$\begin{aligned}
\mathscr{E} &= \int_E |(C_\varphi - C_\psi)f(z)|^q u(z) \mathrm{d}v_b(z) \\
&\lesssim \int_E |(\sigma C_\varphi)f(z)|^q u(z) \mathrm{d}v_b(z) + \int_E |(\sigma C_\psi)f(z)|^q u(z) \mathrm{d}v_b(z) \\
&\leq \|\sigma C_\varphi\|^q_{A_b^p(u)\to L_b^q(u)} + \|\sigma C_\psi\|^q_{A_b^p(u)\to L_b^q(u)}
\end{aligned}$$

whenever $\|f\|_{A_b^p(u)} \leq 1$. By Corollary 3.1, item $\mathscr{E}$ can be controlled by

$$\left\|\mu^{q,u,b}_{\sigma,\varphi}\right\|_{R',Geo} + \left\|\mu^{q,u,b}_{\sigma,\psi}\right\|_{R',Geo}.$$

By Lemma 2.4 and Fubini's theorem we have

$$\begin{aligned}
\mathscr{E}' &= \int_{E'} |f(\varphi(z)) - f(\psi(z))|^q u(z) \mathrm{d}v_b(z) \\
&\lesssim \int_{E'} \sigma(z)^q \frac{\int_{\rho(\varphi(z),\omega)<R'} |f(\omega)|^p u(\omega) \mathrm{d}v_b(\omega)}{u_b(B_\rho(\varphi(z),R'))^{q/p}} u(z) \mathrm{d}v_b(z) \\
&\lesssim \int_{\mathbb{B}_n} |f(\omega)|^p \frac{\int_{\varphi^{-1}(\{\rho(z,\omega)<R'\})\cap E'} \sigma(z)^q u(z) \mathrm{d}v_b(z)}{u_b(B_\rho(\omega,R'))^{q/p}} u(\omega) \mathrm{d}v_b(\omega) \\
&\leq \|f\|^p_{A_b^p(u)} \left\|\mu^{q,u,b}_{\sigma,\varphi}\right\|_{R',Geo}.
\end{aligned}$$

Combining the above estimates for $\mathscr{E}$ and $\mathscr{E}'$, we see that the upper bound for the norm of $C_\varphi - C_\psi$

$$\|C_\varphi - C_\psi\|^q_{A_b^p(u)\to A_b^q(u)} \lesssim \max\left\{\left\|\mu^{q,u,b}_{\sigma,\varphi}\right\|_{R',Geo}, \left\|\mu^{q,u,b}_{\sigma,\psi}\right\|_{R',Geo}\right\}.$$

□

To see the upper bound of the essential norm of the difference, the method for the standard weighted Bergman spaces in [20] does not work any longer.

Proposition 4.2. *Let $1 < p_0 \le p \le q < \infty$, $b > -1$, and $u \in B_{p_0,b}$. Suppose that φ, ψ and σ are the same as in Proposition 4.1. Then*

$$\|C_\varphi - C_\psi\|^q_{A^p_b(u)\to A^q_b(u),e} \lesssim \max\left\{ \lim_{\delta\to 1} \left\| \left(\mu^{q,u,b}_{\sigma,\varphi}\right)_\delta \right\|_{r,Geo}, \lim_{\delta\to 1} \left\| \left(\mu^{q,u,b}_{\sigma,\varphi}\right)_\delta \right\|_{r,Geo} \right\}.$$

Proof. Let P_b be the Bergman projection from $L^p_b(u)$ to $A^p_b(u)$. We have

$$\begin{aligned}\|C_\varphi - C_\psi\|^q_{A^p_b(u)\to A^q_b(u),e} &= \inf\left\{ \|C_\varphi - C_\psi - \mathscr{K}\|^q_{A^p_b(u)\to A^q_b(u)} : \mathscr{K} \text{ is compact} \right\} \\ &\le \|(C_\varphi - C_\psi) - (C_\varphi - C_\psi)P_b(\mathbb{1}_{\kappa\mathbb{B}_n}\cdot)\|^q_{A^p_b(u)\to A^q_b(u)} \\ &= \sup_{\|f\|_{A^p_b(u)}\le 1} \int_{\mathbb{B}_n} |(C_\varphi - C_\psi)P_b[(1-\mathbb{1}_{\kappa\mathbb{B}_n})f](z)|^q u(z)\mathrm{d}\nu_b(z).\end{aligned}$$

Denote by $E = \{z \in \mathbb{B}_n : \sigma(z) \ge r\}$ and $E' = \mathbb{B}_n \setminus E$. Then the integral in the expression above can be divided into the following two parts

$$\left(\int_E + \int_{E'}\right) |(C_\varphi - C_\psi)P_b[(1-\mathbb{1}_{\kappa\mathbb{B}_n})f](z)|^q u(z)\mathrm{d}\nu_b(z) := \mathscr{E} + \mathscr{E}'.$$

We apply the triangle inequality to estimate $\mathscr{E}$. With the analogous arguments in the proof of Theorem 1.2, the first part $\mathscr{E}$ can be dominated by

$$\begin{aligned}&\int_E |(\sigma C_\varphi)P_b[(1-\mathbb{1}_{\kappa\mathbb{B}_n})f](z)|^q u(z)\mathrm{d}\nu_b(z) + \int_E |(\sigma C_\psi)P_b[(1-\mathbb{1}_{\kappa\mathbb{B}_n})f](z)|^q u(z)\mathrm{d}\nu_b(z) \\ &\lesssim \left\| \left(\mu^{q,u,b}_{\sigma,\varphi}\right)_\delta \right\|^q_{Oper} + \left\| \left(\mu^{q,u,b}_{\sigma,\psi}\right)_\delta \right\|^q_{Oper} \simeq \left\| \left(\mu^{q,u,b}_{\sigma,\varphi}\right)_\delta \right\|^q_{r,Geo} + \left\| \left(\mu^{q,u,b}_{\sigma,\psi}\right)_\delta \right\|^q_{r,Geo}.\end{aligned}$$

Now we turn to $\mathscr{E}'$. Let $0 < \delta < 1$ be arbitrary. By Lemma 3.2, we see that

$$\begin{aligned}&\lim_{\kappa\to 1^-} \int_{E'\cap\varphi^{-1}(\delta\mathbb{B}_n)} \left|C_\varphi P_b[(1-\mathbb{1}_{\kappa\mathbb{B}_n})f](z)\right|^q u(z)\mathrm{d}\nu_b(z) \\ &\le \lim_{\kappa\to 1^-} \int_{\delta\mathbb{B}_n} |P_b[(1-\mathbb{1}_{\kappa\mathbb{B}_n})f](w)|^q \mathrm{d}\mu^{q,u,b}_{1,\varphi}(w) = 0.\end{aligned}$$

By the strong triangle inequality (2.1), we find $\delta' \in (0,1)$ such that

$$E' \cap \varphi^{-1}(\delta\mathbb{B}_n) \subset \varphi^{-1}(\delta'\mathbb{B}_n).$$

Applying Lemma 3.2 one more time yields

$$\begin{aligned}&\lim_{\kappa\to 1^-} \int_{E'\cap\varphi^{-1}(\delta\mathbb{B}_n)} \left|C_\varphi P_b[(1-\mathbb{1}_{\kappa\mathbb{B}_n})f](z)\right|^q u(z)\mathrm{d}\nu_b(z) \\ &\le \lim_{\kappa\to 1^-} \int_{\psi^{-1}(\delta\mathbb{B}_n)} \left|C_\varphi P_b[(1-\mathbb{1}_{\kappa\mathbb{B}_n})f](z)\right|^q u(z)\mathrm{d}\nu_b(z).\end{aligned}$$

Consequently, we note that $P_b[(1-\mathbb{1}_{\kappa\mathbb{B}_n})\cdot]: A_b^p(u)\to A_b^p(u)$ are uniformly bounded for κ and then

$$\limsup_{\kappa\to 1^-}\sup_{\|f\|_{A_b^p(u)}\le 1}\mathscr{E}'\lesssim \sup_{\|f\|_{A_b^p(u)}\le 1}\int_F |(C_\varphi - C_\psi)f(z)|^q u(z)\mathrm{d}\nu_b(z),$$

where $F=E'\cap\varphi^{-1}(\mathbb{B}_n\setminus\delta\mathbb{B}_n)$. Following Lemma 2.4 and Fubini's theorem, we have

$$\begin{aligned}
&\int_F |(C_\varphi - C_\psi)f(z)|^q u(z)\mathrm{d}\nu_b(z)\\
\lesssim&\int_F \sigma^q(z)\frac{\int_{\{w:\rho(\varphi(z),w)<R'\}}|f(w)|^p u(w)\mathrm{d}\nu_b(w)}{u_b(B_\rho(\varphi(z),R'))^{q/p}}u(z)\mathrm{d}\nu_b(z)\\
\lesssim&\int_{\mathbb{B}_n}|f(w)|^p\frac{\int_{\varphi^{-1}(\{z:\rho(z,w)<R'\})\cap F}\sigma(z)^q u(z)\mathrm{d}\nu_b(z)}{u_b(B_\rho(w,R'))^{q/p}}u(w)\mathrm{d}\nu_b(w)\\
\le&\int_{\mathbb{B}_n}|f(w)|^p\frac{\int_{\varphi^{-1}(\{z:\rho(z,w)<R'\})\cap(\mathbb{B}_n\setminus\delta\mathbb{B}_n)}\sigma(z)^q u(z)\mathrm{d}\nu_b(z)}{u_b(B_\rho(w,R'))^{q/p}}u(w)\mathrm{d}\nu_b(w)\\
\le&\|f\|^p_{A_b^p(u)}\left\|\left(\mu_{\sigma,\varphi}^{q,u,b}\right)_\delta\right\|_{r,Geo}.
\end{aligned}$$

Letting $\delta\to 1$ and using the above estimates, we arrive at

$$\begin{aligned}
\|C_\varphi - C_\psi\|^q_{A_b^p(u)\to A_b^q(u),e}&\le \lim_{\kappa\to 1^-}\sup_{\|f\|_{A_b^p(u)}\le 1}\mathscr{E}+\lim_{\kappa\to 1^-}\sup_{\|f\|_{A_b^p(u)}\le 1}\mathscr{E}'\\
&\lesssim\max\left\{\lim_{\delta\to 1}\left\|\left(\mu_{\sigma,\varphi}^{q,u,b}\right)_\delta\right\|_{r,Geo},\lim_{\delta\to 1}\left\|\left(\mu_{\sigma,\psi}^{q,u,b}\right)_\delta\right\|_{r,Geo}\right\}.
\end{aligned}$$

That completes the proof. □

In the rest of the section, we prove the lower bound for the norm of the differences $C_\varphi - C_\psi$.

Lemma 4.1. *Suppose* $0<r<1$. *Then, for every* $w\in\mathbb{B}_n$,

$$\left|1-\frac{1-\langle z,a\rangle}{1-\langle w,a\rangle}\right|\gtrsim |a|\rho(z,w)$$

whenever $a\in\mathbb{B}_n$ *and* $z\in B_\rho(a,r)$.

Proof. Let $a,z\in\mathbb{B}_n$ such that $\rho(a,z)<r$. For every $w\in\mathbb{B}_n$, we have that

$$\begin{aligned}
|z-w|^2 &= |z-(P_z(w)+Q_z(w))|^2=|z-P_z(w)|^2+|Q_z(w)|^2\\
&\ge |z-P_z(w)|^2+|s_zQ_z(w)|^2=|z-P_z(w)-s_zQ_z(w)|^2
\end{aligned}$$

where $s_z=(1-|z|^2)^{1/2}<1$. It folows that

$$\left|1-\frac{1-\langle z,a\rangle}{1-\langle w,a\rangle}\right|=|a|\frac{|z-w|}{|1-\langle z,w\rangle|}\left|\frac{1-\langle w,z\rangle}{1-\langle w,a\rangle}\right|\gtrsim |a|\rho(z,w)$$

where the last inequality follows from the estimate above. □

We need the following lemma which is an analogue to [20, Lemma 4.4].

Lemma 4.2. *Let $0 < r < 1$ and $s > 0$. Then, for $a \in \mathbb{B}_n$, $z \in B_\rho(a,r)$, and $w \in \mathbb{B}_n$,*

$$\left| \left(\frac{1-|a|^2}{1-\langle z,a\rangle} \right)^s - \left(\frac{1-|a|^2}{1-\langle w,a\rangle} \right)^s \right| \gtrsim |a|\rho(z,w),$$

where the constant involved depends only on r and s.

Proof. For $a,w \in \mathbb{B}_n$ and $z \in B_\rho(a,r)$, we note that

$$\left| \left(\frac{1-|a|^2}{1-\langle z,a\rangle} \right)^s - \left(\frac{1-|a|^2}{1-\langle w,a\rangle} \right)^s \right| = \left| \frac{1-|a|^2}{1-\langle z,a\rangle} \right|^s \left| 1 - \left(\frac{1-\langle z,a\rangle}{1-\langle w,a\rangle} \right)^s \right|.$$

By the same arguments as those in [20, Lemma 4.4], we can obtain the desired inequality. Hence we omit the details here. □

Now we are ready to give the lower bound for the operator norm of the differences $C_\varphi - C_\psi$.

Proposition 4.3. *Let $0 < p \le q < \infty$, $0 < r < 1/8$, $p_0 > 1$, $b > -1$, and $u \in B_{p_0,b}$. Suppose that φ and ψ are analytic self mappings of $\mathbb{B}_n$ such that $C_\varphi - C_\psi$ is bounded from $A_b^p(u)$ into $A_b^q(u)$. Then the operator norm of $C_\varphi - C_\psi$ is bounded below by*

$$\|C_\varphi - C_\psi\|_{A_b^p(u)\to A_b^q(u)} \gtrsim \max\left\{ \left\| \mu_{\sigma,\varphi}^{q,u,b} \right\|_{r,Geo}, \left\| \mu_{\sigma,\psi}^{q,u,b} \right\|_{r,Geo} \right\}.$$

Proof. For $s \ge (n+1+b)p_0/p$ and fixed $0 < r < 1/8$, let $g_{w,r}^s$ be the test function defined as follows

$$g_{w,r}^s = \frac{(1-|w|)^s}{u_b(B_\rho(w,r))^{1/p}} G_w^s.$$

Notice that $\|g_{w,r}^s\|_{A_b^p(u)} \simeq 1$ by Lemma 2.5. It suffices to prove that

$$\|(C_\varphi - C_\psi)\|_{A_u^p(u)\to A_b^q(u)}^q \gtrsim \frac{\mu_{\sigma,\varphi}^{q,u,b}(B_\rho(w,r))}{u_b(B_\rho(w,r))^{q/p}}$$

for every $w \in \mathbb{B}_n$. Denote by $g_w^s = g_{w,1/2}^s$. Applying Lemma 4.2, one can calculate that

$$\begin{aligned} &\|(C_\varphi - C_\psi)g_w^s\|_{A_b^q(u)}^q \\ \ge &\int_{\rho(\varphi(z),w)<\frac{1}{2}} \left| \left(\frac{1-|w|^2}{1-\langle \varphi(z),w\rangle} \right)^s - \left(\frac{1-|w|^2}{1-\langle \psi(z),w\rangle} \right)^s \right|^q u_b(B_\rho(w,1/2))^{-q/p} u(z) \mathrm{d}\nu_b(z) \\ \gtrsim &\int_{\rho(\varphi(z),w)<\frac{1}{2}} |w|^q \sigma(z)^q u_b(B_\rho(w,1/2))^{-q/p} u(z) \mathrm{d}\nu_b(z). \end{aligned}$$

Note that, for $0 < r < 1/8$,

$$\begin{aligned}\|C_\varphi - C_\psi\|_{A^p_b(u)\to A^q_b(u)} &\geq \sup_{|w|\geq 2^{-3}} \|(C_\varphi - C_\psi)g^s_w\|^q_{A^q_b(u)} \\ &\gtrsim \sup_{|w|\geq 2^{-3}} \int_{\varphi^{-1}(B_\rho(w,1/2))} \sigma(z)^q u(z)\mathrm{d}\nu_b(z)\cdot u_b(B_\rho(w,1/2))^{-q/p} \\ &= \sup_{|w|\geq 2^{-3}} \frac{\mu^{q,u,b}_{\sigma,\varphi}(B_\rho(w,1/2))}{u_b(B_\rho(w,1/2))^{q/p}} \gtrsim \sup_{|w|\geq 2^{-3}} \frac{\mu^{q,u,b}_{\sigma,\varphi}(B_\rho(w,r))}{u_b(B_\rho(w,r))^{q/p}},\end{aligned}$$

where the last inequality follows from Lemma 2.2 and the constant involved depends on r but not on w. For every $w \in B_\rho(0,1/8)$, one can use triangle inequality (2.1) to conclude $B_\rho(w,r) \subset B_\rho(w,1/8) \subset B_\rho(1/4,1/2)$. Again, we use the arguments above to have

$$\sup_{|w|<1/8} \frac{\mu^{q,u,b}_{\sigma,\varphi}(B_\rho(w,r))}{u_b(B_\rho(w,r))^{q/p}} \lesssim \frac{\mu^{q,u,b}_{\sigma,\varphi}(B_\rho(1/4,1/2))}{u_b(B_\rho(1/4,1/2))^{q/p}} \lesssim \|C_\varphi - C_\psi\|_{A^p_b(u)\to A^q_b(u)},$$

which completes the proof. □

Similarly, we can obtain the lower bound for the essential norm of the differences $C_\varphi - C_\psi$.

Proposition 4.4. *Let $0 < p \leq q < \infty$, $0 < r < 1/8$, $p_0 > 1$, $b > -1$, and $u \in B_{p_0,b}$. Suppose that φ and ψ are analytic self mappings of $\mathbb{B}_n$ such that $C_\varphi - C_\psi$ is bounded from $A^p_b(u)$ into $A^q_b(u)$. Then,*

$$\|C_\varphi - C_\psi\|_{A^p_b(u)\to A^q_b(u),e} \gtrsim \max\left\{ \lim_{\delta\to 1} \left\| \left(\mu^{q,u,b}_{\sigma,\varphi}\right)_\delta \right\|_{r,Geo}, \lim_{\delta\to 1} \left\| \left(\mu^{q,u,b}_{\sigma,\psi}\right)_\delta \right\|_{r,Geo} \right\}.$$

Proof. Let g^s_w be the test function defined in (2.5). By the same process as those in Proposition 4.3 with some modifications, we have

$$\limsup_{|w|\to 1} \|(C_\varphi - C_\psi)g^s_w\|_{A^q_b(u)} \gtrsim \max\left\{ \lim_{\delta\to 1} \left\| \left(\mu^{q,u,b}_{\sigma,\varphi}\right)_\delta \right\|_{r,Geo}, \lim_{\delta\to 1} \left\| \left(\mu^{q,u,b}_{\sigma,\psi}\right)_\delta \right\|_{r,Geo} \right\}.$$

Then the desired result follows by

$$\|C_\varphi - C_\psi\|_{A^p_b(u)\to A^q_b(u),e} \geq \limsup_{|w|\to 1} \|(C_\varphi - C_\psi)g^s_w\|_{A^q_b(u)},$$

since g^s_w converges to 0 uniformly on compact subsets of $\mathbb{B}_n$. □

Summarizing Proposition 4.1, Proposition 4.2, Proposition 4.3, and Proposition 4.4, we conclude the proof of Theorem 1.3, which is a generalization of the standard weighted Bergman spaces.

5. The proof of Theorem 1.4

At the end of this paper, we study the boundedness and compactness of the difference $C_\varphi - C_\psi$ when $0 < q < p < \infty$.

Khinchine inequality Consider a sequence of Rademacher functions $r_k(t)$; see [8, Appendix A]. For almost every $t \in (0,1)$, the sequence $\{\gamma_k(t)\}$ consists of signs ± 1. We state first the classical Khinchine's inequality; see [8, Appendix A].

Khinchine's inequality: Let $0 < p < \infty$. Then, for any sequence $\{c_k\}$ of complex numbers, we have

$$\left(\sum_k |c_k|^2\right)^{p/2} \simeq \int_0^1 \left|\sum_k c_k r_k(t)\right|^p dt. \tag{5.1}$$

Recall that if $\{e_k\}$ is an orthonormal basis of $A_b^2(u)$, then the Bergman kernel in $A_b^2(u)$ is given by $K(z,w) = \sum_k e_k(z)\overline{e_k(w)}$, and $K(z,z) = \sum_k |e_k(z)|^2$.

The following Lemma was given in [25, Lemma 4.1].

Lemma 5.1. *Let $p_0 > 1$ and $u \in B_{p_0,b}$. Then there exists an $r \in (0,1)$ such that $K(z,z) \simeq u_b\left(B_\beta(z,r)\right)^{-1}$, $z \in \mathbb{B}_n$.*

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. (ii) $\Rightarrow$ (i) is obviously. First, we prove (i) $\Rightarrow$ (iii). Let $s \geq (n+1+b)p_0/p$. By (2.5), there exists a positive constant $r < 1$ such that the function

$$\widetilde{g}^s(z) := \sum_{k=1}^{\infty} \frac{c_k}{u_b(B_\rho(a_k,2r))^{1/p}} \left(\frac{1-|a_k|}{1-\langle z,a_k\rangle}\right)^s.$$

is in $A_b^p(u)$, where $\{a_k\}_{k=1}^\infty$ is any r-lattice and $\{c_k\}_{k=1}^\infty$ is any sequence in l^p. Moreover,

$$\|\widetilde{g}^s\|_{A_b^p(u)}^p \simeq \|\{c_k\}\|_{l^p}^p = \sum_{k=1}^{\infty} |c_k|^p.$$

Put

$$\widetilde{g}_k^s(z) := \frac{1}{u_b(B_\rho(a_k,2r))^{1/p}} \left(\frac{1-|a_k|}{1-\langle z,a_k\rangle}\right)^s,$$

and let $d = \left\|C_\varphi - C_\psi\right\|$, which is finite by assumption. Then

$$\begin{aligned}\left[\int_{\mathbb{B}_n} \left|\sum_{k=1}^{\infty} c_k \left(\widetilde{g}_k^s(\varphi(z)) - \widetilde{g}_k^s(\psi(z))\right)\right|^q d\mu(z)\right]^{1/q} &= \left\|\left(C_\varphi - C_\psi\right)\widetilde{g}^s\right\|_{L^q(\mu)} \\ &\leq d\|\widetilde{g}^s\|_{A_b^p(u)} \\ &\lesssim d\left(\sum_{k=1}^{\infty}|c_k|^p\right)^{1/p}.\end{aligned}$$

Replacing c_k by $c_k r_k(t)$, it follows that

$$\begin{aligned}&\int_0^1 \int_{\mathbb{B}_n} \left|\sum_{k=1}^{\infty} c_k r_k(t) \left(\widetilde{g}_k^s(\varphi(z)) - \widetilde{g}_k^s(\psi(z))\right)\right|^q d\mu(z)dt \\ &\lesssim d^q \int_0^1 \left[\sum_{k=1}^{\infty} |c_k r_k(t)|^p\right]^{q/p} dt \\ &= d^q \left(\sum_{k=1}^{\infty}|c_k|^p\right)^{q/p}.\end{aligned}$$

Applying (5.1) and Fubini's theorem yields

$$\begin{aligned}&\int_{\mathbb{B}_n}\left[\sum_{k=1}^{\infty}|c_k|^2\left|\widetilde{g}_k^s(\varphi(z))-\widetilde{g}_k^s(\psi(z))\right|^2\right]^{q/2}d\mu(z)\\ \simeq&\int_{\mathbb{B}_n}\int_0^1\left|\sum_{k=1}^{\infty}c_kr_k(t)\left(\widetilde{g}_k^s(\varphi(z))-\widetilde{g}_k^s(\psi(z))\right)\right|^q dt\,d\mu(z)\\ \lesssim& d^q\left(\sum_{k=1}^{\infty}|c_k|^p\right)^{q/p}.\end{aligned}$$

By Lemma 4.2, one has

$$\begin{aligned}&\left|\widetilde{g}_k^s(\varphi(z))-\widetilde{g}_k^s(\psi(z))\right|^2\\ =&\frac{1}{u_b(B_\rho(a_k,2r))^{2/p}}\left|\left(\frac{1-|a_k|}{1-\langle\varphi(z),a_k\rangle}\right)^s-\left(\frac{1-|a_k|}{1-\langle\psi(z),a_k\rangle}\right)^s\right|^2\\ \gtrsim&\frac{|a_k|^2\rho(\varphi(z),\psi(z))^2\mathbb{1}_{\varphi^{-1}(B_\rho(a_k,2r))}(z)}{u_b(B_\rho(a_k,2r))^{2/p}},\end{aligned}\tag{5.2}$$

for all $z\in\mathbb{B}_n$ and $k\in\mathbb{N}$. By Lemma 2.1 (iii), each $B_\rho(\varphi(z),2r)$ $(z\in\mathbb{B}_n)$ contains at most N points of $\{a_k\}_{k=1}^{\infty}$ (this integer N depends on r only). Re-indexing the points of this r-lattice if necessary so that they are of non-decreasing moduli, we also have $|a_k|\geq 2r$ for all $k\geq N+1$. These facts, together with (5.2), imply that

$$\begin{aligned}&\int_{\mathbb{B}_n}\left[\sum_{k=1}^{\infty}|c_k|^2\left|\widetilde{g}_k^s(\varphi(z))-\widetilde{g}_k^s(\psi(z))\right|^2\right]^{q/2}d\mu(z)\\ \gtrsim&\int_{\mathbb{B}_n}\left[\sum_{k=1}^{\infty}\frac{|a_k|^2|c_k|^2\rho(\varphi(z),\psi(z))^2\mathbb{1}_{\varphi^{-1}(B_\rho(a_k,2r))}(z)}{u_b(B_\rho(a_k,2r))^{2/p}}\right]^{q/2}d\mu(z)\\ \geq&\max\{N^{\frac{q}{2}-1},1\}\int_{\mathbb{D}}\sum_{k=1}^{\infty}\frac{|a_k|^q|c_k|^q\rho(\varphi(z),\psi(z))^q\mathbb{1}_{\varphi^{-1}(B_\rho(a_k,2r))}(z)}{u_b(B_\rho(a_k,2r))^{q/p}}d\mu(z)\\ \gtrsim&\int_{\mathbb{B}_n}\sum_{k=N+1}^{\infty}\frac{|c_k|^q\rho(\varphi(z),\psi(z))^q\mathbb{1}_{\varphi^{-1}(B_\rho(a_k,2r))}(z)}{u_b(B_\rho(a_k,2r))^{q/p}}d\mu(z)\\ =&\sum_{k=N+1}^{\infty}|c_k|^q\frac{\int_{\varphi^{-1}(B_\rho(a_k,2r))}\rho(\varphi(z),\psi(z))^q d\mu(z)}{u_b(B_\rho(a_k,2r))^{q/p}}.\end{aligned}$$

Interchanging the roles of φ and ψ, we obtain

$$\begin{aligned}&\int_{\mathbb{B}_n}\left[\sum_{k=1}^{\infty}|c_k|^2\left|\widetilde{g}_k^s(\psi(z))-\widetilde{g}_k^s(\varphi(z))\right|^2\right]^{q/2}d\mu(z)\\ \gtrsim&\sum_{k=N+1}^{\infty}|c_k|^q\frac{\int_{\psi^{-1}(B_\rho(a_k,2r))}\rho(\varphi(z),\psi(z))^q d\mu(z)}{u_b(B_\rho(a_k,2r))^{q/p}}.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{k=N+1}^{\infty}|c_k|^q\frac{\omega_{\mu,q}(B_\rho(a_k,2r))}{u_b(B_\rho(a_k,2r))^{q/p}} &\lesssim \int_{\mathbb{B}_n}\left[\sum_{k=1}^{\infty}|c_k|^2\left|\widetilde{g}_k^s(\psi(z))-\widetilde{g}_k^s(\varphi(z))\right|^2\right]^{q/2}d\mu(z)\\ &\lesssim d^q\left(\sum_{k=1}^{\infty}|c_k|^p\right)^{\frac{q}{p}}.\end{aligned}\tag{5.3}$$

If we now take $c_k = d_k^{1/q}$ in (5.3), where $\{d_k\}_{k=1}^{\infty}$ is an arbitrary sequence of $l^{p/q}$, then

$$\sum_{k=N+1}^{\infty}|d_k|\frac{\omega_{\mu,q}(B_\rho(a_k,2r))}{u_b(B_\rho(a_k,2r))^{q/p}}<\infty.$$

Therefore,

$$\left\{\frac{\omega_{\mu,q}(B_\rho(a_k,2r))}{u_b(B_\rho(a_k,2r))^{q/p}}\right\}_{k=1}^{\infty}\in l^{p/(p-q)},$$

where $l^{p/(p-q)}$ is the dual space of $l^{p/q}$, that is,

$$\sum_{k=1}^{\infty}\left(\frac{\omega_{\mu,q}(B_\rho(a_k,2r))}{u_b(B_\rho(a_k,2r))}\right)^{\frac{p}{p-q}}\cdot u_b\left(B_\rho\left(a_k,2r\right)\right)<\infty.$$

Now we consider the $L^{\frac{p}{p-q}}$ norm of the function

$$z\mapsto\frac{\omega_{\mu,q}(B_\rho(z,r_0))}{u_b(B_\rho(z,r_0))},$$

where $0<r_0<\frac{r}{r+1}$. It is easy to see that $B_\rho(z,r_0)\subset B_\rho\left(w,2r\right)$ for those $z\in B_\rho\left(w,r\right)$. Hence we obtain that

$$\begin{aligned}\int_{\mathbb{B}_n}\left(\frac{\omega_{\mu,q}(B_\rho(z,r_0))}{u_b(B_\rho(z,r_0))}\right)^{\frac{p}{p-q}}u(z)\mathrm{d}v_b(z) &\lesssim \sum_{k=1}^{\infty}\int_{B_\rho(a_k,r)}\left(\frac{\omega_{\mu,q}(B_\rho(z,r_0))}{u_b(B_\rho(z,r_0))}\right)^{\frac{p}{p-q}}u(z)\mathrm{d}v_b(z)\\ &\lesssim \sum_{k=1}^{\infty}\left(\frac{\omega_{\mu,q}(B_\rho(a_k,2r))}{u_b(B_\rho(a_k,2r))}\right)^{\frac{p}{p-q}}\cdot u_b\left(B_\rho\left(a_k,2r\right)\right)<\infty.\end{aligned}$$

By Lemma 2.7, we obtain (iii).

Finally, we prove (iii) $\Rightarrow$ (ii). Let $\{f_n\}_{n=1}^{\infty}$ be any bounded sequence of $A_b^p(u)$ such that $f_n\to 0$ uniformly on compact subsets of $\mathbb{B}_n$. Write

$$\left\|\left(C_\varphi-C_\psi\right)f_n\right\|_{L^q(\mu)}^q:=I_1+I_2,$$

where

$$I_1:=\int_E|f_n(\varphi(z))-f_n(\psi(z))|^q\,d\mu(z),I_2:=\int_{\mathbb{B}_n\setminus E}|f_n(\varphi(z))-f_n(\psi(z))|^q\,d\mu(z)$$

and $E=\{z\in\mathbb{B}_n:\sigma(z)\geq r\}$. We first estimate I_1 by (1.1)

$$\begin{aligned}I_1&=\int_E\frac{1}{|\sigma(z)|^q}|\sigma(z)f_n(\varphi(z))-\sigma(z)f_n(\psi(z))|^q\,d\mu(z)\\&\leq\left(\frac{2}{r}\right)^q\left[\int_E|\sigma(z)|^q|f_n(\varphi(z))|^q d\mu(z)+\int_E|\sigma(z)|^q|f_n(\psi(z))|^q d\mu(z)\right]\\&=\left(\frac{2}{r}\right)^q\int_{\mathbb{B}_n}|f_n(z)|^q\,d\omega_{\mu,q}(z).\end{aligned}$$

Since $\omega_{\mu,q}$ is also a vanishing (p,q)-Carleson measure, we have

$$\int_{\mathbb{B}_n}|f_n(z)|^q\,d\omega_{\mu,q}(z)\to 0,\quad n\to\infty. \tag{5.4}$$

It remains to consider J_2. By Lemma 2.2, Lemma 2.4, and $1\!\!1_{B_\rho(\psi(z),R')}(w)=1\!\!1_{\psi^{-1}(B_\rho(w,R'))}(z)$, we have

$$\begin{aligned}I_2&\lesssim\int_{\mathbb{B}_n\setminus E}\frac{\sigma(z)^q}{u_b(B_\rho(\psi(z),R'))}\int_{B_\rho(\psi(z),R')}|f_n(w)|^q u(w)dA_b(w)d\mu(z)\\&\simeq\int_{\mathbb{B}_n\setminus E}\sigma(z)^q\int_{\mathbb{B}_n}\frac{|f_n(w)|^q 1\!\!1_{B_\rho(\psi(z),R')}(w)}{u_b(B_\rho(w,R'))}u(w)dA_b(w)d\mu(z)\\&=\int_{\mathbb{B}_n\setminus E}\sigma(z)^q\int_{\mathbb{B}_n}\frac{|f_n(w)|^q 1\!\!1_{\psi^{-1}(B_\rho(w,R')}(z)}{u_b(B_\rho(w,R'))}u(w)dA_b(w)d\mu(z)\\&=\int_{\mathbb{B}_n}\frac{|f_n(w)|^q}{u_b(B_\rho(w,R'))}\int_{\psi^{-1}(B_\rho(w,R'))\cap\mathbb{B}_n\setminus E}\sigma(z)^q d\mu(z)u(w)dA_b(w)\\&\leq\int_{\mathbb{B}_n}|f_n(w)|^q\frac{\omega_{\mu,q}(B_\rho(w,R'))}{u_b(B_\rho(w,R'))}u(w)dA_b(w).\end{aligned}$$

Let $\mathbb{B}_r:=\{z\in\mathbb{B}_n:|z|<r\}$. With

$$\frac{\omega_{\mu,q}(B_\rho(w,R'))}{u_b(B_\rho(w,R'))}\in L_b^{p/(p-q)}(u),$$

and the dominated convergence theorem, we have that

$$\int_{\mathbb{B}_n}\left[\frac{\omega_{\mu,q}(B_\rho(w,R'))}{u_b(B_\rho(w,R'))}\right]^{p/(p-q)}1\!\!1_{\mathbb{B}\setminus\mathbb{B}_r}(w)u(w)dA_b(w)\to 0\text{ as }r\to 1^-.$$

Thus, by Hölder's inequality, we obtain

$$\begin{aligned}&\int_{\mathbb{B}\setminus\overline{\mathbb{B}_r}}|f_n(w)|^q\frac{\omega_{\mu,q}(B_\rho(w,R'))}{u_b(B_\rho(w,R'))}u(w)dA_b(w)\\&\leq\|f_n\|^q_{A_b^p(u)}\left\|\frac{\omega_{\mu,q}(B_\rho(w,R'))}{u_b(B_\rho(w,R'))}1\!\!1_{\mathbb{B}\setminus\overline{\mathbb{B}_r}}\right\|_{L_b^{p/(p-q)}(u)}\to 0.\end{aligned} \tag{5.5}$$

Moreover, since $\omega_{\mu,q}(B_\rho(w,R')\leq 2\mu(\mathbb{B}_n))$ and Lemma 5.1, we arrive at

$$\int_{\mathbb{B}_r}|f_n(w)|^q\frac{\omega_{\mu,q}(B_\rho(w,R'))}{u_b(B_\rho(w,R'))}u(w)dA_b(w)\lesssim\int_{\mathbb{B}_r}|f_n(w)|^q u(w)dA_b(w).$$

From the uniform convergence of the sequence $\{f_n\}_{n=1}^{\infty}$ to zero on the compact set $\overline{\mathbb{B}_r}$, we have

$$\int_{\mathbb{B}_r} |f_n(w)|^q u(w) dA_b(w) \to 0, \quad n \to \infty,$$

which together with (5.4) and (5.5) gives $\left\| \left(C_\varphi - C_\psi \right) f_n \right\|_{L^q(\mu)} \to 0$. □

6. Further questions

We completely characterized the boundedness and compactness of the differences for composition operators on the weighted Bergman spaces with Békollé weights. Another topic related to the differences of composition operators is the topological structure of the space of all composition operators $\mathscr{C}(A_b^p(u))$ with the norm topology. This topic was initiated by Berkson in 1981 and studied extensively by Shapiro, Bourdon, Gallardo-Gutiërrez, Moorhouse, and many other mathematicians. Hence, our further questions in the Békollé weighted settings raise naturally as follows:

(1): do all the compact composition operators on $A_b^p(u)$ form a connected component in $\mathscr{C}(A_b^p(u))$?
(2): how to characterize the connected components in $\mathscr{C}(A_b^p(u))$?
(3): how to characterize the isolated elements in $\mathscr{C}(A_b^p(u))$?

An effective method to study those problems on the unweighed Bergman spaces is to use the fractional linear mappings of $\mathbb{D}$ or $\mathbb{B}_n$, but that trick could hardly work on the weighted Bergman spaces with Békollé weights. That is because we do not know the explicit expression of the weights and we can hardly conduct the computations directly. We probably have to use some abstract tools to push our study forward, just like the Aleksandrov measures introduced in [10].

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