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ON CONSTRAINT QUALIFICATIONS FOR AN INFINITE SYSTEM OF QUASICONVEX INEQUALITIES

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Abstract. In this paper, we consider an infinite inequality system defined by a family of proper lower semicontinuous quasiconvex functions in real normed linear spaces. By using the interior-point condition and approximate continuity assumption of the functions, we establish some sufficient conditions for ensuring the basic constraint qualification for quasiconvex programming.
Keywords. Constraint qualification; Interior-point condition; Quasiconvex programming.

1. Introduction

Consider the inequality system of the following quasiconvex optimization problem:

$$(P) \qquad \begin{array}{ll} \inf & h(x) \\ \text{s. t.} & h_i(x) \le 0,\ i \in I, \\ & x \in D, \end{array}$$

where I denotes a nonempty (possibly infinite) index set, D is a nonempty convex subset of a real normed linear space X, $h : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is a proper convex function, and $h_i : X \to \overline{\mathbb{R}}$ is a proper quasiconvex function for each $i \in I$.

Since the constraint qualifications play an important role in studying Farkas lemma, duality theory, and optimality condition for optimization problem, the constraint qualifications for quasiconvex programming and their applications were widely studied and extensively developed; see, e.g., [2, 3, 12, 13, 14, 15, 16, 18, 20] and the references therein. Among these constraint qulifications, the basic constraint qualification for quasiconvex programming (Q-BCQ in short) is a sort of constraint qualification that of much significance. For example, in [15], the authors introduced the Q-BCQ and established the optimality conditions of quasiconvex programming via the Q-BCQ in the case that I is a finite index set and D is the whole space in a locally convex Hausdorff topological space. Later, this condition was extended in [16] to the general case that I is an infinite index set. Recently, in order to generalize the Q-BCQ, in [3], the authors

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introduced the constraint qualification (Q-BCQ)$_h$ and established sufficient and necessary conditions to characterize the total Lagrange duality for quasiconvex programming by applying the (Q-BCQ)$_h$.

In light of the importance of the Q-BCQ, it is not only natural but also beneficial to investigate the sufficient conditions to guarantee it. For this purpose, in [16], the authors introduced the constraint qualification Q-CCCQ and showed that the Q-BCQ is a necessary condition of the Q-CCCQ. While, in [20], the author established some sufficient conditions for the Q-CCCQ by applying an interior-point condition when functions have continuity. Obviously, those conditions in [20] can guarantee that the Q-BCQ is satisfied. But, we hope to find some weaker conditions that makes the Q-BCQ condition hold.

Note that, in the case that $h_i, i \in I$ are proper convex functions on X, problem (P) is transformed into a classical convex programming, and the Q-BCQ is converted into the basic constraint qualification (BCQ in brief) for the convex inequality system. For convex programming, many scholars have introduced a series of constraint conditions to make the BCQ hold; see, e.g., [4, 5, 6, 7, 8, 9, 10, 11] and the references therein. Especially, the authors in [8] introduced the concepts of the Slater condition and the weak Slater condition and established some sufficient conditions to guarantee the BCQ for an infinite convex inequality system, and they also established in [9] some sufficient conditions for the closed convex sets system to satisfy the strong CHIP in terms of the interior-point condition in the special case that constraint functions $h_i, i \in I$ are indicator functions of closed convex sets. While, in [10], the authors established the equivalence of the BCQ and the strong CHIP under certain conditions.

Motivated by the works mentioned above, we in this paper aim to establish some sufficient conditions to ensure that Q-BCQ holds in normed spaces in terms of the interior-point condition. Note that the Q-BCQ for quasiconvex programming is closely related to the strong CHIP for convex programming. By applying these relations, we study the property of the Q-BCQ and give some alternative forms of the Q-BCQ. Then, by applying the sufficient conditions that were originally proposed in [9] to ensure the strong CHIP, we provide some sufficient conditions to ensure the Q-BCQ in terms of the interior-point condition together with the lower semicontinuity or the Kuratowski continuity of the function $i \mapsto h_i(x)$ and some property of some finite subsystems of the constraint system.

The paper is organized as follows. In Section 2, we recall some necessary notations and preliminary results. In Section 3, the last section, the alternative form of the Q-BCQ is given and sufficient conditions to ensure the Q-BCQ in terms of the interior-point condition are provided.

2. NOTATIONS AND PRELIMINARY RESULTS

The notations used in the present paper are standard (see [13] and [19]). In particular, we assume throughout the whole paper that X is a real normed linear space with its dual spaces X^*, endowed with the weak*-topology $w^*(X^*, X)$. By $\langle x^*, x\rangle$, we denote the value of the functional $x^* \in X^*$ at $x \in X$, i.e., $\langle x^*, x\rangle = x^*(x)$. We use $\mathbb{B}(x, \varepsilon)$ to denote the closed ball with center x and radius ε. Let C be a nonempty subset in X. The interior (resp. convex cone hull, affine hull, boundary, relative boundary) of C is denoted by int C (resp. cone C, aff C, bd C, rb C). The indicator function and the distance function of C are defined respectively by

$$\delta_C(x) := \begin{cases} 0, & x \in C, \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$d(x,C) := \inf\{\|x-c\| : \ c \in C\} \text{ for each } x \in X.$$

Let Z be a nonempty, convex, and closed subset in X. The interior and boundary of C relative to Z are denoted by $\operatorname{rint}_Z C$ and $\operatorname{bd}_Z C$, respectively; they are defined to be, respectively, the interior and boundary of the set $\operatorname{aff} Z \cap C$ in the metric space $\operatorname{aff} Z$. Thus, a point $z \in \operatorname{rint}_Z C$ if and only if there exists $\varepsilon > 0$ such that $z \in \operatorname{aff} Z \cap \mathbb{B}(z,\varepsilon) \subseteq C$. Obviously, $\operatorname{ri} C = \operatorname{rint}_C C$. While $z \in \operatorname{bd}_Z C$ if and only if $z \in \operatorname{aff} Z$ and, for any $\varepsilon > 0$, $\operatorname{aff} Z \cap \mathbb{B}(z,\varepsilon)$ intersects Z and its complement. The normal cone $N(x;Z)$ of a convex set $Z \subseteq X$ at the point $x \in Z$ are defined by

$$N_Z(x) := \{x^* \in X^* : \langle x^*, z-x\rangle \le 0 \text{ for each } z \in Z\}.$$

Furthermore, let T be an arbitrary (possibly infinite) index set. We use $\mathbb{R}^{(T)}$ to denote the space of real tuples $\lambda := (\lambda_t)_{t\in T}$ with only finitely many $\lambda_t \neq 0$, and let $\mathbb{R}^{(T)}_+$ denote the nonnegative cone in $\mathbb{R}^{(T)}$, that is,

$$\mathbb{R}^{(T)}_+ := \left\{(\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} : \lambda_t \ge 0 \text{ for each } t \in T\right\}.$$

Let $\{A_t : t \in T\}$ be a family of subsets of X. The set $\sum_{t\in T} A_t$ is defined by

$$\sum_{t\in T} A_t := \begin{cases} \{\sum_{t\in T_0} a_t : a_t \in A_t, T_0 \subseteq T \text{ is finite}\}, & T \neq \emptyset, \\ \{0\}, & T = \emptyset. \end{cases}$$

Let $f : X \to \overline{\mathbb{R}}$ be a proper convex function. The effective domain and epigraph of f are defined, respectively, by $\operatorname{dom} f := \{x \in X : \ f(x) < +\infty\}$ and $\operatorname{epi} f := \{(x,r) \in X \times \mathbb{R} : \ f(x) \le r\}$. The subdifferential of f at $x \in \operatorname{dom} h$ is defined by

$$\partial f(x) := \{x^* \in X^* : f(x) + \langle x^*, y-x\rangle \le f(y) \text{ for all } y \in X\}.$$

In particular,

$$N_Z(x) = \partial \delta_Z(x) \text{ for each } x \in Z. \tag{2.1}$$

If $f,g : X \to \overline{\mathbb{R}}$ are proper convex functions satisfying $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, then

$$\partial f(x) + \partial g(x) \subseteq \partial (f+g)(x) \text{ for each } x \in \operatorname{dom} f \cap \operatorname{dom} g. \tag{2.2}$$

The infimal convolution function $f \square g : X \to \mathbb{R} \cup \{\pm\infty\}$ of f and g is defined by

$$(f \square g)(a) := \inf_{x\in X} \{f(x) + g(a-x)\} \ \text{ for each } a \in X,$$

which is called exact at $a \in X$ when there is an $x \in X$ such that $(f \square g)(a) = f(x) + g(a-x)$.

The following lemma is be used in the sequel (see [19]).

Lemma 2.1. *Let $f,g : X \to \overline{\mathbb{R}}$ be proper convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$*. If f or g is continuous at some point of* $\operatorname{dom} f \cap \operatorname{dom} g$*, then $\partial(f+g)(x) = \partial f(x) + \partial g(x)$ for each* $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

Recall that a function $f : X \to \overline{\mathbb{R}}$ is said to be quasiconvex if, for all $x,y \in X$ and $\alpha \in [0,1]$, the following inequality holds: $f((1-\alpha)x + \alpha y) \le \max\{f(x), f(y)\}$, and f is said to quasiconcave if $-f$ is quasiconvex. Obviously, each convex function is quasiconvex, but the opposite is not true. Moreover, f is said to be quasiaffine iff it is quasiconvex and quasiconcave. By [13], it is known that f is lower semicontinuous (lsc in brief) quasiaffine if and only if there exist $k \in Q$ and $w \in X^*$ such that $f = k \circ w$, where $Q = \{k : \mathbb{R} \to \overline{\mathbb{R}} \ : \ k \text{ is lsc and non-decreasing}\}$. Recall from [14] that a set $G = \{(k_j, w_j) \mid j \in J\} \subseteq Q \times X^*$ is said to be a generator of f if and only if

$f = \sup_{j\in J} k_j \circ w_j$. The following lemma is taken from [13] that shows that each lsc quasiconvex function has at least one generator.

Lemma 2.2. *Let f be a function from X to $\overline{\mathbb{R}}$. Then f is lsc quasiconvex if and only if exist $\{(k_j, w_j) : j \in J\} \subseteq Q \times X^*$ such that $f = \sup_{j\in J} k_j \circ w_j$.*

3. Sufficient Conditions for the Q-BCQ

Throughout this paper, let I be an arbitrary index set, D be a nonempty, convex, and closed subset of the real normed linear space X, and $\{h_i : i \in I\}$ be a family of proper lsc quasiconvex functions on X. Consider the following quasiconvex inequality system

$$x \in D; h_i(x) \le 0, i \in I. \tag{3.1}$$

Let $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subseteq Q \times X^*$ be a generator of h_i for each $i \in I$, and let $T := \{t = (i, j) \mid i \in I, j \in J_i\}$. For each $x \in X$, let $T(x)$ be the active index set of the system (3.1) relative to $\{(k_t, w_t) : t \in T\}$, that is, $T(x) := \{t \in T : \langle w_t, x\rangle = k_t^{-1}(0)\}$, where k_t^{-1} is the hypo-epi-inverse of k_t and defined by

$$k_t^{-1}(r) = \inf\{u \in \mathbb{R} \mid r < k_t(u)\} = \sup\{s \in \mathbb{R} \mid k_t(s) \le r\}.$$

As usual, we use A to denote the solution set of the system (3.1), that is, $A := \{x \in D : h_i(x) \le 0, i \in I\}$, and assume that $A \neq \emptyset$. Note that, since D is convex and $h_i, i \in I$ are quasiconvex, it follows that A is convex.

The following constraint qualification for quasiconvex programming extends the one introduced in [15], where the authors only considered the case when D is the whole space.

Definition 3.1. The system $\{D; h_i : i \in I\}$ is said to satisfy the basic constraint qualification for quasiconvex programming (Q-BCQ in brief) with respect to (w.r.t. in short) $\{(k_t, w_t) \mid t \in T\}$ at $x \in A$ if

$$N_A(x) = N_D(x) + \operatorname{cone} \bigcup_{t\in T(x)} \{w_t\}. \tag{3.2}$$

Moreover, $\{D; h_i : i \in I\}$ is said to satisfy the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ if it satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ at each point $x \in A$.

For each $t \in T$, we define $f_t : X \to \overline{\mathbb{R}}$ by $f_t(x) = \langle w_t, x\rangle - k_t^{-1}(0)$ for each $x \in X$. Then system (3.1) can be rewritten as the following convex inequality system

$$x \in D; f_t(x) \le 0, t \in T, \tag{3.3}$$

and hence the solution set of (3.1) reduces to $A = \{x \in D : f_t(x) \le 0, t \in T\}$. For each $x \in X$, let $\tilde{T}(x)$ be the active index set of system (3.3), that is, $\tilde{T}(x) := \{t \in T : f_t(x) = 0\}$. Then, by definition of the function f_t,

$$\tilde{T}(x) = \{t \in T : \langle w_t, x\rangle = k_t^{-1}(0)\} = T(x). \tag{3.4}$$

The following theorem gives an alternative form of the Q-BCQ.

Theorem 3.1. *Let $x \in A$. The following statements are equivalent.*

(i) *System $\{D; h_i : i \in I\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ at x.*

(ii) *System $\{D; f_t : t \in T\}$ satisfies the BCQ at x, that is,*

$$N_A(x) = N_D(x) + \operatorname{cone} \bigcup_{t\in \tilde{T}(x)} \{\partial f_t(x)\}. \tag{3.5}$$

Proof. Since f_t is an affine function for each $t \in T$, it can be verified by definition that

$$\begin{aligned}\partial f_t(x) &= \{x^* \in X^* : f_t(x) + \langle x^*, y-x\rangle \le f_t(y), \forall y \in X\}\\ &= \{x^* \in X^* : \langle w_t, x\rangle + \langle x^*, y-x\rangle \le \langle w_t, y\rangle, \forall y \in X\}\\ &= \{w_t\}.\end{aligned}$$

Thus, by (3.4), the desired result follows immediately. □

Remark 3.1. Let $x \in A$. In view of (2.1) and (2.2), we see that

$$\begin{aligned}N_D(x) + \text{cone} \bigcup_{t\in T(x)} \{w_t\} &= N_D(x) + \text{cone}\left(\bigcup_{t\in \tilde{T}(x)} \partial f_t(x)\right)\\ &\subseteq N_D(x) + \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \partial\left(\sum_{t\in\tilde{T}(x)} \lambda_t f_t\right)(x)\\ &\subseteq \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \partial\left(\delta_D + \sum_{t\in\tilde{T}(x)} \lambda_t f_t\right)(x)\\ &\subseteq \partial\delta_A(x) = N_A(x).\end{aligned}$$

Thus, (3.2) and (3.5) can be equivalently replaced by

$$N_A(x) \subseteq N_D(x) + \text{cone} \bigcup_{t\in T(x)} \{w_t\}$$

and

$$N_A(x) \subseteq N_D(x) + \text{cone} \bigcup_{t\in \tilde{T}(x)} \{\partial f_t(x)\}.$$

In the remainder of this section, we always assume that I is a compact metric space. The following interior-point conditions were introduced in [9, Definition 3.1].

Definition 3.2. Let D and $C_i, i \in I$ be nonempty, convex, and closed subsets of X. System $\{D, C_i : i \in I\}$ is said to satisfy

(i) the D-interior-point condition if

$$D \cap \left(\bigcap_{i\in I} \text{rint}_D\, C_i\right) \ne \emptyset;$$

(ii) the strong D-interior-point condition if

$$D \cap \left(\text{rint}_D \bigcap_{i\in I} C_i\right) \ne \emptyset.$$

The following notion of semicontinuity of a function was introduced in [20].

Definition 3.3. A function $h : I \to \overline{\mathbb{R}}$ is upper semicontinuous at $i_0 \in I$ if, for any $\varepsilon > 0$, there exists a neighborhood $U(i_0)$ of i_0 such that

$$h(i) < h(i_0) + \varepsilon \text{ for each } i \in U(i_0), \tag{3.6}$$

and that h is upper semicontinuous on I if (3.6) holds at each $i_0 \in I$.

For the following concept of the semicontinuity of a set-valued map, readers may refer to standard texts such as [20].

Definition 3.4. Let $H : I \to 2^X \backslash \{\emptyset\}$ be a set-valued mapping and $i_0 \in I$. The mapping H is said to be lower semicontinuous at i_0 if, for any $x_0 \in H(i_0)$ and any $\varepsilon > 0$, there exists a neighborhood $U(i_0)$ of i_0 such that $\mathbb{B}(x_0, \varepsilon) \cap H(i) \neq \emptyset$, and H is said to be lower semicontinuous on I if it is lower semicontinuous at each $i_0 \in I$.

The following notion of Kuratowski continuity was introduced in [17].

Definition 3.5. Let $H : I \to 2^X \backslash \{\emptyset\}$ be a set-valued function defined on I and $i_0 \in I$. Then H is said to be

(i) upper Kuratowski semicontinuous at i_0 if, for any sequence $\{i_n\} \subseteq I$, the relations $\lim\limits_{n\to\infty} i_n = i_0$, $\lim\limits_{n\to\infty} x_{i_n} = x_{i_0}$, $x_{i_n} \in H(i_n), n = 1, 2, \ldots$ imply $x_{i_0} \in H(i_0)$;

(ii) lower Kuratowski semicontinuous at i_0 if, for any sequence $\{i_n\} \subseteq I$, the relations $\lim\limits_{n\to\infty} i_n = i_0$, $x_0 \in H(i_0)$ imply $\lim\limits_{n\to\infty} d_{H(i_n)}(x_0) = 0$;

(iii) Kuratowski continuous at i_0 if H is both upper Kuratowski semicontinuous and lower Kuratowski semicontinuous at i_0;

(iv) Kuratowski continuous on I if it is Kuratowski continuous at each point of I.

Clearly, by [9],

$$H \text{ is lower semicontinuous } \iff H \text{ is lower Kuratowski semicontinuous.}$$

Let $i \in I$, and define $C_i := \{x \in X : h_i(x) \leq 0\}$. For any proper function $\varphi : X \to \overline{\mathbb{R}}$ and any $x \in X$, define $[\varphi(x)]_+ := \max\{\varphi(x), 0\}$. The following lemmas regarding the lower semicontinuity were given in [9, Proposition 3.1] and [20, Lemma 4.4], respectively.

Lemma 3.1. *Let $H : I \to 2^X \backslash \{\emptyset\}$ be a set-valued function and $i_0 \in I$. Then the following statements are equivalent*

(i) *H is lower semicontinuous at i_0.*

(ii) *For any $x_0 \in H(i_0)$, there exists $x_i \in H(i)$ for each $i \in I$ such that* $\lim\limits_{i\to i_0} \|x_i - x_0\| = 0$.

(iii) *For any $x_0 \in H(i_0)$,* $\lim\limits_{i\to i_0} d_{H(i)}(x_0) = 0$.

Lemma 3.2. *Let $i_0 \in I$. Suppose that the function $i \mapsto h_i(x)$ is upper semicontinuous at i_0 for each $x \in \operatorname{aff} D$. Suppose further that at least one of the following statements is satisfied:*

(i) $\emptyset \neq \operatorname{rint}_D C_{i_0} \subseteq \{x \in X : h_{i_0}(x) < 0\}$.

(ii) *For each $x \in X$, there exists $\tau_x > 0$ such that*

$$d(x, \operatorname{aff} D \cap C_i) \leq \tau_x [h_i(x)]_+ \text{ for each } i \in I.$$

Then the set-valued mapping $i \mapsto \operatorname{aff} D \cap C_i$ is lower semicontinuous at i_0.

By [9], for each $x_0 \in D \cap (\bigcap_{i\in I} C_i)$, let $I_D^{\mathrm{rb}}(x_0) = \{i \in I : x_0 \in \operatorname{bd}_D C_i\}$. Since $\operatorname{bd}_D C_i = \operatorname{bd} C_i \backslash \operatorname{int}_D C$, it follows that $I_D^{\mathrm{rb}}(x_0) \subseteq \{i \in I : x_0 \in \operatorname{bd} C_i\}$. Below we are going to state and prove the main results of this section. The following theorems gives some sufficient conditions for ensuring the Q-BCQ.

Theorem 3.2. *Let $x \in D \cap (\bigcap_{i\in I} C_i)$. Suppose that the following conditions are satisfied:*

(a) *System $\{D, C_i : i \in I\}$ satisfies the strong D-interior-point condition.*

(b) *For each $x_0 \in \text{aff } D$, the function $i \mapsto h_i(x_0)$ is upper semicontinuous on I.*

(c) *Either for each $i \in I$, $\emptyset \neq \text{rint}_D\, C_i \subseteq \{x \in X : h_i(x) < 0\}$ or for each $x \in X$, there exists $\tau_x > 0$ such that*

$$d(x, \text{aff } D \cap C_i) \leq \tau_x [h_i(x)]_+ \text{ for each } i \in I.$$

(d) *The pair $\{\text{aff } D, C_i\}$ has the strong CHIP at x for each $i \in I$, that is,*

$$N_{(\text{aff } D)\cap C_i}(x) = N_{\text{aff } D}(x) + N_{C_i}(x) \text{ for each } i \in I.$$

(e) *Either D is finite dimensional or $I_D^{\text{rb}}(x)$ is a finite set.*

Then system $\{D; h_i : i \in I\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ at x.

Proof. By Lemma 3.2, conditions (b) and (c) ensure that the set-valued mapping $i \mapsto \text{aff } D \cap C_i$ is lsc on I. Furthermore, condition (a) implies that

$$D \cap \left(\text{rint}_D \bigcap_{i\in I} C_i \right) \neq \emptyset.$$

Hence, by applying [9, Theorem 4.1], we know that conditions (a)-(e) imply that system $\{D, C_i : i \in I\}$ satisfies the strong CHIP, that is,

$$N_{D\cap(\bigcap_{i\in I} C_i)}(x) = N_D(x) + \sum\nolimits_{i\in I} N_{C_i}(x) \subseteq N_D(x) + N_{\bigcap_{i\in I} C_i}(x). \tag{3.7}$$

Define $F : X \to \overline{\mathbb{R}}$ by $F(x) := \sup_{t\in T} f_t(x)$ for each $x \in X$. Then F is continuous. Moreover, function $t \to f_t(x_0)$ is upper semicontinuous on T. Thus, one can conclude by [9] that

$$\partial F(x) = \text{cone} \sum_{t\in \widetilde{T}(x)} \partial f_t(x). \tag{3.8}$$

Note that $C_i = \{x \in X : h_i(x) \leq 0\}$ for each $i \in I$. By definition of f_t, one has that $C_i = \{x \in X : f_t(x) \leq 0\}$, where $t = (i, j) \in T$. Then

$$N_{\bigcap_{i\in I} C_i}(x) = N_{\bigcap_{t\in T} f_t^{-1}(\mathbb{R}_-)}(x) = N_{F^{-1}(\mathbb{R}_-)}(x) \subseteq \text{cone } \partial F(x),$$

where the last inclusion holds by [1, Corollary 1]. This together with (3.8) implies that

$$N_{\bigcap_{i\in I} C_i}(x) \subseteq \text{cone} \sum_{t\in \widetilde{T}(x)} \partial f_t(x),$$

and

$$N_A(x) \subseteq N_D(x) + \text{cone} \sum_{t\in \widetilde{T}(x)} \partial f_t(x),$$

thanks to (3.7). This means that system $\{D; f_t : t \in T\}$ satisfies the BCQ at x. Therefore, by Theorem 3.1, we see that system $\{D; h_i : i \in I\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ at x. The proof is complete. □

Theorem 3.3. *Let $x \in D \cap (\bigcap_{i\in I} C_i)$. Suppose that the following conditions are satisfied:*

(a) *D is finite dimensional.*

(b) *The set-valued function $i \mapsto \text{aff } D \cap C_i$ is Kuratowski continuous on I.*

(c) *For any finite subset J of I with $|J| \leq l$, the subsystem $\{D, C_i : i \in J\}$ satisfies the D-interior-point condition, where $|J|$ denotes the cardinality of the set J and $l = \dim D < +\infty$.*

(d) *For each finite subset J of I, the subsystem $\{D; h_i : i \in J\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T_J\}$ at x, where $T_J = \{t = (i, j) \mid i \in J, j \in J_i\}$, that is,*

$$N_{D\cap(\bigcap_{i\in J} C_i)}(x) = N_D(x) + \text{cone} \bigcup_{t\in T_J(x)} \{w_t\}.$$

Then the system $\{D; h_i : i \in I\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ at x.

Proof. Note that condition (c) implies that, for each finite subset J of I,

$$D \cap \left(\bigcap_{i\in J} \text{rint}_D\, C_i \right) \neq \emptyset.$$

Furthermore, one can verify by definition that, for each $i \in I$,

$$\text{cone} \left(\bigcup_{j\in J_i(x)} \{w_{(i,j)}\} \right) \subseteq N_{C_i}(x).$$

Thus, condition (d) implies that, for each finite subset J of I, system $\{D, C_i : i \in J\}$ satisfies the following condition

$$N_{D\cap(\bigcap_{i\in J} C_i)}(x) \subseteq N_D(x) + \sum_{i\in J} N_{C_i}(x).$$

Hence, by applying [9, Theorem 5.1], we know that conditions (a)-(d) assert that

$$N_{D\cap(\bigcap_{i\in I} C_i)}(x) = N_D(x) + \textstyle\sum_{i\in I} N_{C_i}(x) \subseteq N_D(x) + N_{\bigcap_{i\in I} C_i}(x).$$

Then, by the proof of Theorem 3.2, we see that system $\{D; h_i : i \in I\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ at x. The proof is complete. ☐

Below we are devoted to the optimality condition and total Lagrange duality of quasiconvex optimization problem (P) by apply Theorems 3.2 and 3.3. To do this, we always assume that dom $h \cap A \neq \emptyset$. Following [3], we define the dual problem of (P) by

$$(D) \qquad \max_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{x\in C} \left\{ h(x) + \sum_{t\in T} \lambda_t (w_t(x) - k_t^{-1}(0)) \right\}.$$

To establish the total duality between (P) and its dual problem (D), the authors in [3] introduced the following constraint qualification.

Definition 3.6. The family $\{D; h_i : i \in I\}$ satisfies the basic constraint qualification for quasiconvex programming relative to h ((Q-BCQ)$_h$ in brief) w.r.t. $\{(k_t, w_t) \mid t \in T\}$ at $x \in$ dom $h \cap A$ if

$$\partial(h + \delta_A)(x) = \partial h(x) + N_D(x) + \text{cone} \bigcup_{t\in T(x)} \{w_t\}.$$

Further, $\{D; h_i : i \in I\}$ is said to satisfy the (Q-BCQ)$_h$ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ if it satisfies the (Q-BCQ)$_h$ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ at each $x \in$ dom $h \cap A$.

Note that, if h is continuous at some point of A, then, by Lemma 2.1,

$$\text{the Q-BCQ} \implies \text{the (Q-BCQ)}_h.$$

Thus, by Theorem 3.2 and Theorem 3.3, we can obtain the following proposition.

Proposition 3.1. *Let $x \in D \cap (\bigcap_{i \in I} C_i)$. Suppose that h is continuous at some point of A. If the conditions (a)-(e) in Theorem 3.2 or (a)-(d) in Theorem 3.3 hold, then $\{D; h_i : i \in I\}$ satisfies the (Q-BCQ)$_h$ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ at x.*

The following theorem is a direct consequence of Proposition 3.1 and [3, Theorem 5.1].

Theorem 3.4. *Let $x \in D \cap (\bigcap_{i \in I} C_i)$. Suppose that h is continuous at some point of A. If the conditions (a)-(e) in Theorem 3.2 or (a)-(d) in Theorem 3.3 hold, then the stable total Lagrange duality between (P) and (D) holds, that is, for each $p \in X^*$,*

$$\min_{x \in A}\{h(x) - \langle p, x\rangle\} = \max_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C}\{h(x) - \langle p, x\rangle + \sum_{t \in T} \lambda_t (w_t(x) - k_t^{-1}(0))\}.$$

From Theorem 3.2 and Theorem 3.3, we have the following corollary.

Corollary 3.1. *Let $x \in D \cap (\bigcap_{i \in I} C_i)$. Suppose that (a)-(e) in Theorem 3.2 or (a)-(d) in Theorem 3.3 hold. If (h, δ_A) satisfies $(h + \delta_A)^*(0) = (h^* \Box \delta_A^*)(0)$ and $h^* \Box \delta_A^*$ is exact at 0, then x is a minimizer to problem (P) if and only if exists $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ such that*

$$0 \in \partial h(x) + N_D(x) + \sum_{t \in T(x)} \lambda_t w_t. \tag{3.9}$$

In particularly, if h is continuous at some point of A, then x is a minimizer to problem (P) if and only if exists $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ such that (3.9) holds.

Proof. By Theorems 3.2 and 3.3, we see that $\{D; h_i : i \in I\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ at x. Thus, by Theorem 3.1, $\{D; f_t : t \in T\}$ satisfies the BCQ at x. Moreover, by the proof of Theorem 3.1, one has that $\partial f_t(x) = \{w_t\}$ for each $x \in D \cap (\bigcap_{i \in I} C_i)$. Therefore, the results hold directly from [4, Corollary 4.2]. The proof is complete. □

Let $S(P)$ denote the solution set of problem (P), that is, $S(P) := \{x \in A : h(x) = \inf_{y \in A} h(y)\}$. The proof of the following corollary is almost similar to that of Corollary 3.1, so we omit it here.

Corollary 3.2. *Let $x \in D \cap (\bigcap_{i \in I} C_i)$ and $x_0 \in S(P)$. Suppose that (a)-(e) in Theorem 3.2 or (a)-(d) in Theorem 3.3 hold. If (h, δ_A) satisfies $(h + \delta_A)^*(0) = (h^* \Box \delta_A^*)(0)$ and $h^* \Box \delta_A^*$ is exact at 0, then*

$$h(x_0) = \max_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C}\{h(x) + \sum_{t \in T} \lambda_t (w_t(x) - k_t^{-1}(0))\}. \tag{3.10}$$

In particularly, if h is continuous at some point of A, then (3.10) holds.

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REFERENCES

[1] F.H. Clarke, Optimization and Nonsmooth Analysis, SIAM, 1990.
[2] D. Fang, X. Luo, X. Wang, Strong and total Lagrange dualities for quasiconvex programming, J. Appl. Math. 2014 (2014), 453912.
[3] D. Fang, T. Yang, Y.C. Liou, Strong and total lagrange dualities for quasiconvex programming, J. Nonlinear Variational Anal. 6 (2022), 1-15.
[4] D. Fang, C. Li, K.F. Ng, Constraint qualifications for optimality conditions and total Lagrange dualities in convex infinite programming, Nonlinear Anal. 73 (2010), 1143-1159.
[5] D. Fang, Y. Zhang, Extended Farkas's lemmas and strong dualities for conic constraint problem involving composite functions, J. Optim. Theory Appl. 176 (2018), 351-376.
[6] D. Fang, Y. Zhang, Optimality conditions and total dualities for conic programming involving composite function, Optimization 69 (2020), 305-327.
[7] L. Liu, X. Qin, Strong convergence theorems for solving pseudo-monotone variational inequality problems and applications, Optimization, 71 (2022), 3603-3626.
[8] C. Li, K.F. Ng, On constraint qualification for an infinite system of convex inequalities in a Banach space, SIAM J. Optim. 15 (2005), 488-512.
[9] C. Li, K.F. Ng, Strong CHIP for infinite system of closed convex sets in normed linear spaces, SIAM J. Optim. 16 (2005), 311-340.
[10] C. Li, K.F. Ng, T.K. Pong, Constraint qualifications for convex inequality systems with applications in constrained optimization, SIAM J. Optim. 19 (2008), 163-187.
[11] C. Li, K.F. Ng, J.C. Yao, et al, The FM and BCQ qualifications for inequality systems of convex functions in normed linear spaces, SIAM J. Optim. 31 (2021), 1410-1432.
[12] L.V. Nguyen, X. Qin, Some results on strongly pseudomonotone quasi-variational inequalities, Set-Valued Var. Anal. 28 (2020), 239-257.
[13] J.P. Penot, M. Volle, On quasi-convex duality, Math. Oper. Res. 15 (1990), 597-625.
[14] S. Suzuki, D. Kuroiwa, On set containment characterization and constraint qualification for quasiconvex programming, J. Optim. Theory Appl. 149 (2011), 554-563.
[15] S. Suzuki, D. Kuroiwa, Optimality conditions and the basic constraint qualification for quasiconvex programming, Nonlinear Anal. 74 (2011), 1279-1285.
[16] S. Suzuki, D. Kuroiwa, Necessary and sufficient conditions for some constraint qualifications in quasiconvex programming, Nonlinear Anal. 75 (2012), 2851-2858.
[17] I. Singer, The Theory of Best Approximation and Functional Analysis, SIAM, 1974.
[18] B. Tan, X. Qin, S.Y. Cho, Revisiting subgradient extragradient methods for solving variational inequalities, Numer. Algo. 90 (2022), 1593-1615.
[19] C. Zălinescu, Convex Analysis in General Vector Spaces, World Scientific, New Jersey, 2002.
[20] X. Zhao, On constraint qualification for an infinite system of quasiconvex inequalities in normed linear space, Taiwanese J. Math. 20 (2016), 685-697.