J. Nonlinear Var. Anal. 8 (2024), No. 2, pp. 315-332
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.8.2024.2.08

HÖLDER CONTINUITY AND UPPER BOUND RESULTS FOR GENERALIZED PARAMETRIC ELLIPTICAL VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

VO MINH TAM[1,2], JEIN-SHAN CHEN[2,*]

[1]*Department of Mathematics, Dong Thap University, Cao Lanh City 870000, Dong Thap Province, Vietnam*
[2]*Department of Mathematics, National Taiwan Normal University, Taipei 116059, Taiwan*

Abstract. The main purpose of this paper is to investigate the upper bound and Hölder continuity for a general class of parametric elliptical variational-hemivariational inequalities via regularized gap functions. More precisely, we deliver a formulation of the elliptical variational-hemivariational inequalities in the case of the perturbed parameters governed by both the set of constraints and the mappings (for brevity, PEVHI (CM)). Based on the arguments of monotonicity and properties of the Clarke's generalized directional derivative, we establish an upper bound result for the PEVHI (CM) and provide the Hölder continuity of the solution mapping for the PEVHI (CM) under suitable assumptions on the data.
Keywords. Hölder continuity; Parametric elliptical variational–hemivariational inequality; Regularized gap function; Upper bound.

1. Introduction

In the early 1980s, the theory of variational-hemivariational inequality problems has been introduced as a generalization of variational inequality and hemivariational inequality problems to both the convex and the nonconvex potentials based on the Clarke's generalized gradient of locally Lipschitz functions. This study was applied to various fields of engineering and mechanics, especially in optimization and nonsmooth analysis; see e.g., [27, 28]. Many authors have extensively developed the theory of variational-hemivariational inequalities and elliptical variational-hemivariational inequalities (for brevity, EVHI) in various directions, such as the existence of solution sets, the regularity of solutions, the solution method, and the stability in the sense of well-posedness and convergence; see, e.g., [11, 12, 13, 14, 24, 26, 29, 39] and the references therein.

To formulate variational inequality problems by virtue of optimization problems, Auslender [3] introduced a valuable tool called the gap function. However, in general, this gap function is non-differentiable. This disadvantage was improved by Fukushima [10] by introducing a new class of regularized gap functions for variational inequality problems. Using the form of regularized gap functions in [10], Yamashita and Fukushima [37] established an upper estimate of

*Corresponding author.
E-mail address: vmtam@dthu.edu.vn (V.M. Tam), jschen@math.ntnu.edu.tw (J.-S. Chen).
Received 10 August 2023; Accepted 23 January 2024; Published online 16 February 2024

the distance between an arbitrary feasible point and the solution set of their variational inequality problems, so-called error bound or upper bound. In recent years, regularized gap functions have been employed for problems on various interesting topics, such as the well-posedness [23, 35] and the error bound [1, 16, 17, 18, 19, 21, 31, 32, 33, 34] .

On the other hand, the Hölder continuity is known as an important feature of the stability analysis of solution mappings for parametric or perturbed problems related to the fields of optimization and nonlinear analysis. It can provide an error estimate between the exact solution sets and the parametric or perturbed solution sets of concerning problems. Recently, a lot of authors paid attention to developing the Hölder continuity of solution mappings for various kinds of equilibrium problems, variational inequality problems, optimization problems; see, e.g., [2, 4, 7, 22, 36] and the references therein. In specially, using the property of regularized gap functions, Hu and Li [15] established the Hölder continuity of solution mappings for a class of variational inequalities. Tam [30] also developed the Hölder continuity of solution mappings for vector network equilibrium problems with a polyhedral ordering cone by virtue of the regularized gap functions. To the best of our knowledge, there are only a few works devoted to the Hölder continuity of the solution mapping for variational-hemivariational inequalities. In 2021, Hung et al. [20] studied the Hölder continuity for a parametric elliptical variational-hemivariational inequalities with perturbed constraints (for brevity, PEVHI (C)) applying the nice properties of the regularized gap functions. Besides, Chang et al. [6] investigated the existence of an elliptical variational-hemivariational inequality which is formulated by the perturbed parameters in the setting of mappings (for brevity, PEVHI (M)). They also provided an application of PEVHI (M) to a parametric frictional unilateral contact problem.

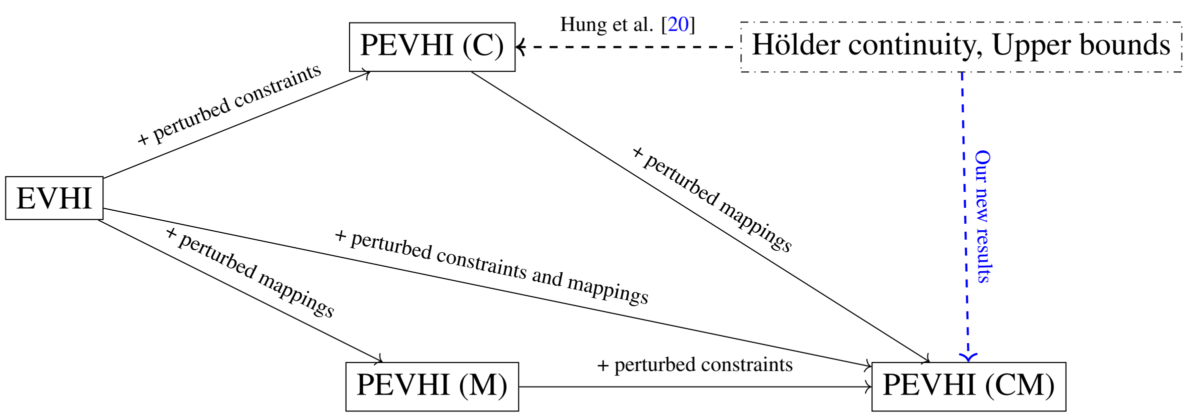

FIGURE 1. Illustration of the development of Hölder continuity and upper bounds results regarding different kinds of PEVHIs based on regularized gap functions.

Motivated essentially by the aforementioned works, in the present paper, we look into the Hölder continuity of solution mappings to a general class of parametric elliptical variational-hemivariational inequalities (for brevity, PEVHI (CM)) which is formulated by the perturbed parameters governed by both the set of constraints and the mappings; see Problem 2.1. In particular, we first establish a regularized gap function of PEVHI (CM) and verify its Hölder continuity. Then, using arguments of monotonicity and properties of the Clarke's directional derivative, an upper bound result depended on parameters for PEVHI is investigated. Finally, we provide the Hölder continuity of the solution mapping for PEVHI (CM) by using the regularized gap function under suitable assumptions on the data. To sum up, Figure 1 illustrates the contribution of this work and how it relates to previous literature on different kinds of PEVHIs.

2. Preliminaries and Formulations

Let $(V, \|\cdot\|_V)$ and $(V^*, \|\cdot\|_{V^*})$ be the real Banach space and its dual space, respectively. Let the duality pairing between V^* and V be denoted by $\langle\cdot,\cdot\rangle_V$. Next, some basic concepts and properties needed in the sequel are recalled. We refer to [8, 9, 25] for more details.

Definition 2.1. Let $\mathscr{R}\colon V \to V^*$ be a single-valued operator. $\mathscr{R}$ is said to be pseudomonotone in the sense of Brézis [5] if it is a bounded operator and for every sequence $\{p_n\} \subset V$ converging weakly to $p \in V$ such that

$$\limsup \langle \mathscr{R} p_n, q_n - p\rangle \leq 0,$$

we have

$$\langle \mathscr{R} p, p-q\rangle \leq \liminf \langle \mathscr{R} p_n, p_n - q\rangle$$

for all $q \in V$.

Definition 2.2. A function $\phi\colon V \to \widehat{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is said to be

(a) proper if $\phi \not\equiv +\infty$;
(b) convex if $\phi(tp + (1-t)q) \leq t\phi(p) + (1-t)\phi(q)$, for all $p, q \in V$ and $t \in [0,1]$;
(c) lower semicontinuous (l.s.c.) at $p_0 \in V$ if for any sequence $\{p_n\} \subset V$ such that $p_n \to p_0$, it holds $\phi(p_0) \leq \liminf \phi(p_n)$;
(d) l.s.c. on V if p is l.s.c. at every $p_0 \in V$.

Definition 2.3. Let $\Psi\colon V \to \widehat{\mathbb{R}}$ be a proper, convex, and l.s.c. function. The convex subdifferential $\partial_c \Psi\colon V \rightrightarrows V^*$ of Ψ is defined by

$$\partial_c \Psi(p) := \left\{ w^* \in V^* \mid \langle w^*, q-p\rangle_V \leq \Psi(q) - \Psi(p), \forall q \in V \right\}$$

for all $p \in V$. An element $w^* \in \partial_c \Psi(p)$ is called a subgradient of Ψ at $p \in V$.

Definition 2.4. A function $\Psi\colon V \to \mathbb{R}$ is said to be locally Lipschitz if, for every $p \in V$, there exist $\mathscr{N}(p)$ and a constant $l_p > 0$ such that

$$|\Psi(p_1) - \Psi(p_2)| \leq l_p \|p_1 - p_2\|_V \text{ for all } p_1, p_2 \in \mathscr{N}(p),$$

where $\mathscr{N}(p)$ is a neighbourhood of p. Given a locally Lipschitz function $\Psi\colon V \to \mathbb{R}$, we denote by $\Psi^0(p;q)$ the Clarke's generalized directional derivative of Ψ at the point $p \in V$ in the direction $q \in V$ defined by

$$\Psi^0(p;q) := \limsup_{v \to p,\ t \to 0^+} \frac{\Psi(v+tq) - \Psi(v)}{t}.$$

The generalized gradient of Ψ at $p \in X$, denoted by $\partial\Psi(p)$, is a subset of V^* given by

$$\partial\Psi(p) := \left\{ w^* \in V^* \mid \Psi^0(p;q) \geq \langle w^*, q\rangle_V \text{ for all } p \in V \right\}.$$

The following lemma describes some fundamental and important characteristics of a locally Lipschitz function's directional derivative; see [8, Proposition 2.1.1].

Lemma 2.1. *Let $\Psi\colon V\to\mathbb{R}$ be a locally Lipschitz function, where V is a real Banach space. Then, for each $q\in V$, $V\ni p\mapsto\Psi^0(p;q)\in\mathbb{R}$ is finite, positively homogeneous, subadditive, and satisfies $|\Psi^0(p;q)|\le l_p\|q\|_V$ for all $q\in X$, where $l_p>0$ is the Lipschitz constant of Ψ near p.*

In the rest of the paper, unless otherwise specified, let $(V,\|\cdot\|_V)$ be a reflexive Banach space and $(E,\|\cdot\|_E)$, $(Z,\|\cdot\|_Z)$, and $(W,\|\cdot\|_W)$ be normed spaces of parameters. Given a set-valued mapping $M\colon E\rightrightarrows V$, a linear control operator $\Phi\colon Z\to V^*$, a nonlinear operator $\mathscr{R}\colon W\times V\times V\to V^*$, a functional $\mathscr{S}\colon W\times V\times V\to\mathbb{R}$, a locally Lipschitz (in general nonconvex) functional $\Psi\colon W\times V\to\mathbb{R}$, and a mapping $f\colon W\to V^*$, we introduce the general class of parametric elliptical variational-hemivariational inequalities as follows:

Problem 2.1. For given $(e,z,w)\in E\times Z\times W$, find $p^*\in M(e)$ such that

$$\begin{aligned}&\langle\mathscr{R}(w,p^*,p^*),q-p^*\rangle_V+\mathscr{S}(w,p^*,q)-\mathscr{S}(w,p^*,p^*)+\Psi^0(w,p^*;q-p^*)\\ \ge\ &\langle f(w)+\Phi(z),q-p^*\rangle_V,\quad\forall q\in M(e),\end{aligned}$$

where $\Psi^0(w,p;q)$ denotes the Clarke's generalized directional derivative of $\Psi(w,\cdot)$ at the point $p\in V$ in direction $q\in V$.

Some special cases of Problem 2.1 are as follows:

(i) Given $w\in W$, if $M(e)\equiv M$, $\Phi(z)\equiv 0$, then Problem 2.1 reduces to the following problem of finding $p^*\in M$ such that

$$\begin{aligned}&\langle\mathscr{R}(w,p^*,p^*),q-p^*\rangle_V+\mathscr{S}(w,p^*,q)-\mathscr{S}(w,p^*,p^*)+\Psi^0(w,p^*;q-p^*)\\ \ge\ &\langle f(w),q-p^*\rangle_V,\end{aligned}$$

for all $q\in M$. This problem was studied in [6].

(ii) Given $(e,z)\in E\times Z$, if $\mathscr{R}(w,p,p)\equiv\mathscr{R}(p)$, $\mathscr{S}(w,p,q)\equiv\mathscr{S}(p,q)$, and $\Psi(w,p)\equiv\Psi(p)$, $f(w)\equiv f$, then Problem 2.1 reduces to the following problem, which was considered in [20] for finding $p^*\in M(e)$ such that

$$\begin{aligned}&\langle\mathscr{R}(p^*),q-p^*\rangle_V+\mathscr{S}(p^*,q)-\mathscr{S}(p^*,p^*)+\Psi^0(p^*;q-p^*)\\ \ge\ &\langle f+\Phi(z),q-p^*\rangle_V,\end{aligned}$$

for all $q\in M(e)$.

In the paper, the following assumptions are imposed on the data of Problem 2.1.

(C1): For each $e\in E, M(e)$ is a nonempty, closed, and convex subset of V.

(C2): $f(w)\in V^*$ and $\Phi(z)\in V^*$ for all $w\in W$ and $z\in Z$.

(C3): $\mathscr{R}\colon W\times V\times V\to V^*$ is such that

(a) there exist $l_{\mathscr{R}},l'_{\mathscr{R}},l''_{\mathscr{R}}>0$ such that for any $w_1,w_2\in W$, $p_1,q_1,p_2,q_2\in M(E)$,

$$\begin{aligned}&\|\mathscr{R}(w_1,p_1,q_1)-\mathscr{R}(w_2,p_2,q_2)\|_{V^*}\\ \le\ &l_{\mathscr{R}}\|w_1-w_2\|_W+l'_{\mathscr{R}}\|p_1-p_2\|_V+l''_{\mathscr{R}}\|q_1-q_2\|_V;\end{aligned}$$

(b) there exists $\alpha_{\mathscr{R}}>0$ such that, for any $w\in W$ and $p_1,p_2,q\in M(E)$,

$$\langle\mathscr{R}(w,p_1,q)-\mathscr{R}(w,p_2,q),p_1-p_2\rangle_V\ge-\alpha_{\mathscr{R}}\|p_1-p_2\|_V^2;$$

(c) there exists $k_{\mathscr{R}}>0$ such that, for any $w\in W$ and $p,q_1,q_2\in M(E)$,

$$\langle\mathscr{R}(w,p,q_1)-\mathscr{R}(w,p,q_2),q_2-q_1\rangle_V\le-k_{\mathscr{R}}\|q_1-q_2\|_V^2.$$

(C4): $\mathscr{S}\colon W\times V\times V\to\mathbb{R}$ is such that

(a) for each $w\in W$ and $p\in V$, $\mathscr{S}(w,p,\cdot)\colon V\to\mathbb{R}$ is convex;
(b) there exist $l_{\mathscr{S}},l'_{\mathscr{S}},l''_{\mathscr{S}}>0$, such that, for any $w_1,w_2\in W$ and $p_1,q_1,p_2,q_2\in M(E)$,

$$\begin{aligned}&|\mathscr{S}(w_1,p_1,q_1)-\mathscr{S}(w_2,p_2,q_2)|\\ \leq\;& l_{\mathscr{S}}\|w_1-w_2\|_W+l'_{\mathscr{S}}\|p_1-p_2\|_V+l''_{\mathscr{S}}\|q_1-q_2\|_V;\end{aligned}$$

(c) there exist $\alpha_{\mathscr{S}}>0$ and $\beta_{\mathscr{S}}\geq 0$, such that, for any $w_1,w_2\in W$ and $p_1,q_1,p_2,q_2\in M(E)$,

$$\begin{aligned}&\mathscr{S}(w_1,p_1,q_2)-\mathscr{S}(w_1,p_1,q_1)+\mathscr{S}(w_2,p_2,q_1)-\mathscr{S}(w_2,p_2,q_2)\\ \leq\;&\alpha_{\mathscr{S}}\|p_1-p_2\|_V\|q_1-q_2\|_V+\beta_{\mathscr{S}}\|w_1-w_2\|_W\|q_1-q_2\|_V.\end{aligned}$$

(C5): $\Psi\colon W\times V\to\mathbb{R}$ is such that

(a) for each $w\in W$, $\Psi(w,\cdot)\colon V\to\mathbb{R}$ is a locally Lipschitz function;
(b) there exist $l_\Psi,l'_\Psi>0$ such that, for any $w_1,w_2\in W$ and $p_1,p_2,q\in M(E)$,

$$|\Psi^0(w_1,p_1;q)-\Psi^0(w_2,p_2;q)|\leq l_\Psi\|w_1-w_2\|_W+l'_\Psi\|p_1-p_2\|_V;$$

(c) there exist $\alpha_\Psi>0$ and $\beta_\Psi\geq 0$ such that, for any $w_1,w_2\in W$ and $p_1,p_2\in M(E)$,

$$\begin{aligned}&\Psi^0(w_1,p_1;p_2-p_1)+\Psi^0(w_2,p_2;p_1-p_2)\\ \leq\;&\alpha_\Psi\|p_1-p_2\|_V^2+\beta_\Psi\|w_1-w_2\|_W\|p_1-p_2\|_V.\end{aligned}$$

(C6): $k_{\mathscr{R}}-\alpha_{\mathscr{R}}-\alpha_{\mathscr{S}}-\alpha_\Psi>0$.

Lemma 2.2. *Let assumptions* **(C3)**(b),(c) *hold. Then, for any* $w\in W,\ p,q\in M(E)$,

$$\langle\mathscr{R}(w,p,p),p-q\rangle_V-\langle\mathscr{R}(w,q,q),p-q\rangle_V\geq(k_{\mathscr{R}}-\alpha_{\mathscr{R}})\|p-q\|_V^2. \tag{2.1}$$

Proof. For any $w\in W,\ p,q\in M(E)$, it follows from hypotheses **(C3)**(b),(c) that

$$\begin{aligned}&\langle\mathscr{R}(w,p,p),p-q\rangle_V-\langle\mathscr{R}(w,q,q),p-q\rangle_V\\ &=\langle\mathscr{R}(w,p,p)-\mathscr{R}(w,q,p),p-q\rangle_V-\langle\mathscr{R}(w,q,q)-\mathscr{R}(w,q,p),p-q\rangle_V\\ &\geq-\alpha_{\mathscr{R}}\|p-q\|_V^2+k_{\mathscr{R}}\|p-q\|_V^2\\ &=(k_{\mathscr{R}}-\alpha_{\mathscr{R}})\|p-q\|_V^2,\end{aligned}$$

which implies that inequality (2.1) holds. □

We now provide the example of operator $\mathscr{R}$ to illustrate the above result.

Example 2.1. Let $V=E=W=\mathbb{R}$ and $M(e)=\left[\frac{1}{2}+\sin^2\left(\frac{e^{\frac{1}{3}}}{2}\right),2\right]$ for all $e\in E$. Let $\mathscr{R}\colon W\times V\times V\to V^*$ be defined by

$$\mathscr{R}(w,p,q)=3\sin^2\left(\frac{w+1}{2}\right)-2p+\frac{7}{2}q^3$$

for all $w\in W,\ p,q\in M(E)$.

(a) For any $w_1, w_2 \in W$, $p_1, q_1, p_2, q_2 \in M(E)$,

$$\begin{aligned}
&|\mathscr{R}(w_1, p_1, q_1) - \mathscr{R}(w_2, p_2, q_2)| \\
\leq & 3\left|\sin^2\left(\frac{w_1+1}{2}\right) - \sin^2\left(\frac{w_2+1}{2}\right)\right| + 2|p_1 - p_2| + \frac{7}{2}|q_1^3 - q_2^3| \\
\leq & 3|w_1 - w_2| + 2|p_1 - p_2| + 42|q_1 - q_2|.
\end{aligned}$$

(b) For any $w \in W$, $p_1, p_2, q \in M(E)$,

$$\langle \mathscr{R}(w, p_1, q) - \mathscr{R}(w, p_2, q), p_1 - p_2 \rangle = 2(p_2 - p_1)(p_1 - p_2) = -2|p_1 - p_2|^2.$$

(c) For any $w \in W$, $p, q_1, q_2 \in M(E)$,

$$\langle \mathscr{R}(w, p, q_1) - \mathscr{R}(w, p, q_2), q_2 - q_1 \rangle = \frac{7}{2}(q_1^3 - q_2^3)(q_2 - q_1) \leq -\frac{21}{8}|q_1 - q_2|^2.$$

Hence, conditions **(C3)**(a–c) hold with $l_{\mathscr{R}} = 3, l'_{\mathscr{R}} = 2, l''_{\mathscr{R}} = 42$, $k_{\mathscr{R}} = \frac{21}{8}$, and $\alpha_{\mathscr{R}} = 2$. Also, for any $w \in W$, $p, q \in M(E)$, we have

$$\begin{aligned}
\langle \mathscr{R}(w, p, p), p - q \rangle - \langle \mathscr{R}(w, q, q), p - q \rangle &= \left[-2(p - q) + \frac{7}{2}(p^3 - q^3)\right](p - q) \\
&\geq -2|p - q|^2 + \frac{21}{8}|p - q|^2 \\
&= (k_{\mathscr{R}} - \alpha_{\mathscr{R}})|p - q|^2.
\end{aligned}$$

Thus inequality (2.1) is valid.

Remark 2.1. (i) Assumption **(C3)**(a) implies that, for each $w \in W$, $\widetilde{\mathscr{R}}(\cdot) := \mathscr{R}(w, \cdot, \cdot)$ is continuous and so it is hemicontinuous. Moreover, it follows from Lemma 2.2 that $\widetilde{\mathscr{R}}(\cdot)$ is monotone. Thus, we can conclude that $\widetilde{\mathscr{R}}(\cdot)$ is pseudomonotone (see [38, Proposition 27.6(a)]).

(ii) It follows immediately from condition **(C4)**(b) that $\mathscr{S}$ is an l.s.c. function in the third argument on $M(E)$.

(iii) By assumption **(C5)**(b), an easy computation proves that there exist $c_0, c_1, c_2 \geq 0$ such that $\|\xi\|_{V^*} \leq c_0 + c_1\|p\|_V + c_2\|w\|_W$ for all $w \in W$, $p \in M(E)$ and $\xi \in \partial\Psi(w, p)$.

(iv) Examples of the functions which satisfy conditions **(C1)–(C6)** can also be founded in [25, 29, 38].

Using assumptions **(C1)–(C6)** and Remark 2.1(i–iii), the existence and uniqueness of solutions for Problem 2.1 considered in [6, Theorem 3.1] are provided in the following result.

Theorem 2.1. *Let assumptions* **(C1)–(C6)** *hold. Then, for every* $(e, z, w) \in E \times Z \times W$, *Problem 2.1 has a unique solution* $p(e, z, w) \in M(e)$.

Remark 2.2. For each $(e, z, w) \in E \times Z \times W$, we denote the solution mapping of Problem 2.1 by $\mho(e, z, w)$, i.e.,

$$\begin{aligned}
\mho(e, z, w) := \Big\{ & p \in M(e) \mid \langle \mathscr{R}(w, p, p) - f(w) - \Phi(z), q - p \rangle_V \\
& + \mathscr{S}(w, p, q) - \mathscr{S}(w, p, p) + \Psi^0(w, p; q - p) \geq 0, \forall q \in M(e) \Big\}.
\end{aligned}$$

It follows from Theorem 2.1 that, for each $(e,z,w) \in E \times Z \times W$, $\mho(e,z,w)$ is a singleton set, i.e., $\mho(e,z,w) = \{p(e,z,w)\}$.

To end this section, we recall the definition of Hölder continuity of set-valued mappings.

Definition 2.5 (Classical notion)**.** A set-valued mapping $M\colon E \rightrightarrows V$ is said to be $l.\nu$-Hölder continuous on $Q \subset E$ for some $l > 0$ and $\alpha > 0$, if, for any $e_1, e_2 \in Q$,

$$M(e_1) \subset M(e_2) + l\|e_1 - e_2\|_E^{\nu} \mathbb{B}_V, \tag{2.2}$$

where $\mathbb{B}_V$ is the closed unit ball of V. If M is a single-valued mapping, then (2.2) reduces to

$$\|M(e_1) - M(e_2)\|_V \le l\|e_1 - e_2\|_E^{\nu}.$$

Remark 2.3. Let $e \in E$ and Q be a neighborhood of e. Then assumption (2.2) also states that M is locally Hölder continuous at e.

3. Main Results

In the rest of paper, let $(\tilde{e}, \tilde{z}, \tilde{w}) \in E \times Z \times W$ be fixed. In this section, we mainly provide an upper bound and the Hölder continuity of the solution mapping $\mho(\cdot,\cdot,\cdot)$ to Problem 2.1 around the considered point $(\tilde{e}, \tilde{z}, \tilde{w})$.

Let $(e,z,w) \in E \times Z \times W$ and $\theta > 0$ be arbitrarily given. We now consider the function $\mathbb{G}_\theta\colon M(e) \times E \times Z \times W \to \mathbb{R}$ defined by

$$\begin{aligned}\mathbb{G}_\theta(p,e,z,w) = \sup_{q \in M(e)} \Big(&\langle \mathscr{R}(w,p,p) - f(w) - \Phi(z), p - q\rangle_V \\ &- \mathscr{S}(w,p,q) + \mathscr{S}(w,p,p) - \Psi^0(w,p;q-p) - \frac{\theta}{2}\|p-q\|_V^2 \Big)\end{aligned} \tag{3.1}$$

for all $p \in M(e)$.

Proposition 3.1. *Suppose that hypotheses* **(C1)–(C6)** *are true. Then, for any* $(e,z,w) \in E \times Z \times W$ *and* $\theta > 0$, *the function* $\mathbb{G}_\theta$ *defined by* (3.1) *satisfies the following properties:*

- (a): $\mathbb{G}_\theta(p,e,z,w) \ge 0$ *for all* $p \in M(e)$;
- (b): $p^* \in M(e)$ *is such that* $\mathbb{G}_\theta(p^*,e,z,w) = 0$ *if and only if* $p^* \in \mho(e,z,w)$, *i.e.,* p^* *is a solution to Problem 2.1.*

Proof. (a) For each $(e,z,w) \in E \times Z \times W$, $\theta > 0$ and $f(w) \in V^*$ fixed, it follows from $p \in M(e)$ and the definition of $\mathbb{G}_\theta$ that

$$\begin{aligned}\mathbb{G}_\theta(p,e,z,w) &\ge \langle \mathscr{R}(w,p,p) - f(w) - \Phi(z), p - p\rangle_V \\ &\quad - \mathscr{S}(w,p,p) + \mathscr{S}(w,p,p) - \Psi^0(w,p;p-p) - \frac{\theta}{2}\|p-p\|_V^2 \\ &= -\Psi^0(w,p;0) \\ &= 0.\end{aligned}$$

(b) Assume that $p^* \in M(e)$ is such that $\mathbb{G}_\theta(p^*,e,z,w) = 0$, namely,

$$\sup_{q\in M(e)} \Big(\langle \mathscr{R}(w,p^*,p^*) - f(w) - \Phi(z), p^* - q\rangle_V \\ -\mathscr{S}(w,p^*,q) + \mathscr{S}(w,p^*,p^*) - \Psi^0(w,p^*;q-p^*) - \frac{\theta}{2}\|p^*-q\|_V^2 \Big) = 0.$$

This means

$$\langle \mathscr{R}(w,p^*,p^*) - f(w) - \Phi(z), p^* - q\rangle_V \\ -\mathscr{S}(w,p^*,q) + \mathscr{S}(w,p^*,p^*) - \Psi^0(w,p^*;q-p^*) \le \frac{\theta}{2}\|p^*-q\|_V^2$$

for all $q \in M(e)$. For any $y \in M(e)$ and $\delta \in (0,1)$, we insert

$$q = q_\delta := (1-\delta)p^* + \delta y \in M(e)$$

into the above inequality to obtain

$$\begin{aligned} &\delta\langle \mathscr{R}(w,p^*,p^*) - f(w) - \Phi(z), p^* - y\rangle_V - \delta\mathscr{S}(w,p^*,y) + \delta\mathscr{S}(w,p^*,p^*) \\ &\qquad -\delta\Psi^0(w,p^*;y-p^*) \\ &\le \langle \mathscr{R}(w,p^*,p^*) - f(w) - \Phi(z), p^* - q_\delta\rangle_V - \mathscr{S}(w,p^*,q_\delta) + \mathscr{S}(w,p^*,p^*) \\ &\qquad -\Psi^0(w,p^*;q_\delta-p^*) \\ &\le \frac{\theta}{2}\|p^*-q_\delta\|_V^2 = \frac{\delta^2\theta}{2}\|p^*-y\|_V^2. \end{aligned}$$

Here, the positive homogeneity of $q \mapsto \Psi^0(w,p^*;q-p^*)$ and convexity of $q \mapsto \mathscr{S}(w,p^*,q)$ are used. Hence, we obtain

$$\langle \mathscr{R}(w,p^*,p^*) - f(w) - \Phi(z), y - p^*\rangle_V \\ +\mathscr{S}(w,p^*,y) - \mathscr{S}(w,p^*,p^*) + \Psi^0(w,p^*;y-p^*) \ge -\frac{\delta^2\theta}{2}\|p^*-y\|_V^2$$

for all $y \in M(e)$. Taking $\delta \to 0^+$ for the above inequality, we find that

$$\langle \mathscr{R}(w,p^*,p^*) - f(w) - \Phi(z), y - p^*\rangle_V \\ +\mathscr{S}(w,p^*,y) - \mathscr{S}(w,p^*,p^*) + \Psi^0(w,p^*;y-p^*) \ge 0$$

for all $y \in M(e)$. Hence, p^* is also a solution to Problem 2.1.

Conversely, let $p^* \in M(e)$ be a solution to Problem 2.1, i.e.,

$$\langle \mathscr{R}(w,p^*,p^*) - f(w) - \Phi(z), q - p^*\rangle_V \\ +\mathscr{S}(w,p^*,q) - \mathscr{S}(w,p^*,p^*) + \Psi^0(w,p^*;q-p^*) \ge 0$$

for all $q \in M(e)$. This implies

$$\begin{aligned}
&\mathbb{G}_\theta(p^*,e,z,w) \\
&= \sup_{q\in M(e)} \Big(\langle \mathscr{R}(w,p^*,p^*) - f(w) - \Phi(z), p^* - q\rangle_V \\
&\quad -\mathscr{S}(w,p^*,q) + \mathscr{S}(w,p^*,p^*) - \Psi^0(w,p^*;q-p^*) - \frac{\theta}{2}\|p^*-q\|_V^2 \Big) \le 0.
\end{aligned}$$

The latter combined with the fact $\mathbb{G}_\theta(p^*,e,z,w) \ge 0$ reveals that $\mathbb{G}_\theta(p^*,e,z,w) = 0$. This completes the proof. □

Remark 3.1. (i) By Fukushima-Yamashita [10, 37] and Proposition 3.1, the function $\mathbb{G}_\theta$ defined by (3.1) is known as a regularized gap function of Problem 2.1.
(ii) For each $(e,z,w) \in E \times Z \times W$ and $\theta > 0$, the close relationship between $\mathbb{G}_\theta$ and $\mho$ is given by

$$\mho(e,z,w) = \{p \in M(e) \mid \mathbb{G}_\theta(p,e,z,w) = 0\}.$$

We now establish an upper bound for Problem 2.1 associated with $\mathbb{G}_\theta$, the regularized gap function.

Theorem 3.1. *Let $(e,z,w) \in \mathscr{N}(\tilde{e}) \times \mathscr{N}(\tilde{z}) \times \mathscr{N}(\tilde{w})$ be fixed and $p^*(e,z,w) \in \mho(e,z,w)$. Assume that all assumptions* **(C1)–(C5)** *hold, and further, for each $\theta > 0$ satisfies*

$$\theta < 2(k_{\mathscr{R}} - \alpha_{\mathscr{R}} - \alpha_{\mathscr{S}} - \alpha_\Psi).$$

Then, for each $p \in M(e)$,

$$\|p - p^*(e,z,w)\|_V \le \frac{1}{\sqrt{k_{\mathscr{R}} - \alpha_{\mathscr{R}} - \alpha_{\mathscr{S}} - \alpha_\Psi - \frac{\theta}{2}}} \sqrt{\mathbb{G}_\theta(p,e,z,w)}. \tag{3.2}$$

Proof. Let $p^*(e,z,w) \in \mho(e,z,w)$, i.e., $p^*(e,z,w) \in M(e)$ and

$$\begin{aligned}
&\langle \mathscr{R}(w,p^*(e,z,w),p^*(e,z,w)) - f(w) - \Phi(z), q - p^*(e,z,w)\rangle_V \\
&\quad + \mathscr{S}(p^*(e,z,w),q) - \mathscr{S}(p^*(e,z,w),p^*(e,z,w)) \\
&\quad + \Psi^0(p^*(e,z,w); q - p^*(e,z,w)) \ge 0,
\end{aligned}$$

for all $q \in M(e)$. For any $p \in M(e)$ fixed, taking $q = p$ in the above inequality, we conclude that

$$\begin{aligned}
&\langle \mathscr{R}(w,p^*(e,z,w),p^*(e,z,w)) - f(w) - \Phi(z), p - p^*(e,z,w)\rangle_V \\
&\quad + \mathscr{S}(p^*(e,z,w),p) - \mathscr{S}(p^*(e,z,w),p^*(e,z,w)) \\
&\quad + \Psi^0(p^*(e,z,w); p - p^*(e,z,w)) \ge 0.
\end{aligned} \tag{3.3}$$

From the definition of $\mathbb{G}_\theta$ in (3.1), we have

$$\begin{aligned}
\mathbb{G}_\theta(p,e,z,w) \ge\ & \langle \mathscr{R}(w,p,p) - f(w) - \Phi(z), p - p^*(e,z,w)\rangle_V \\
& - \mathscr{S}(p,p^*(e,z,w)) + \mathscr{S}(p,p) - \Psi^0(p; p^*(e,z,w) - p) \\
& - \frac{\theta}{2}\|p - p^*(e,z,w)\|_V^2.
\end{aligned} \tag{3.4}$$

It follows from assumptions **(C4)**(c), **(C5)**(c) and Lemma 2.2 that

$$\begin{aligned}
&\langle \mathscr{R}(w,p,p) - f(w) - \Phi(z), p - p^*(e,z,w)\rangle_V \\
&\quad - \mathscr{S}(p,p^*(e,z,w)) + \mathscr{S}(p,p) - \Psi^0(p;p^*(e,z,w)-p) \\
&\geq \langle \mathscr{R}(w,p^*(e,z,w),p^*(e,z,w)) - f(w) - \Phi(z), p - p^*(e,z,w)\rangle_V \\
&\quad + \mathscr{S}(p^*(e,z,w),p) - \mathscr{S}(p^*(e,z,w),p^*(e,z,w)) \\
&\quad + \Psi^0(p^*(e,z,w); p - p^*(e,z,w)) \\
&\quad + (k_{\mathscr{R}} - \alpha_{\mathscr{R}} - \alpha_{\mathscr{S}} - \alpha_\Psi)\|p - p^*(e,z,w)\|_V^2. \qquad (3.5)
\end{aligned}$$

Combining inequalities (3.3)–(3.5), one has

$$\mathbb{G}_\theta(p,e,z,w) \geq \left(k_{\mathscr{R}} - \alpha_{\mathscr{R}} - \alpha_{\mathscr{S}} - \alpha_\Psi - \frac{\theta}{2}\right)\|p - p^*(e,z,w)\|_V^2.$$

Hence, inequality (3.2) holds. □

Let $\mathscr{N}(\tilde{e})$ (resp., $\mathscr{N}(\tilde{w})$) be a bounded neighborhood of the considered point $\tilde{e}$ (resp., $\tilde{w}$). Then we impose the following assumptions for Problem 2.1.

(C7): $M\colon E \rightrightarrows V$ is such that

(a) M is $l_M.\nu$-Hölder continuous on $\mathscr{N}(\tilde{e})$;
(b) for each $e \in \mathscr{N}(\tilde{e})$, $p \in M(e)$, there exists $b_M > 0, \|p\|_V \leq b_M$.

(C8): $\Phi\colon Z \to V^*$ is such that

(a) Φ is $l_\Phi.\mu$-Hölder continuous on $\mathscr{N}(\tilde{z})$;
(b) for each $z \in \mathscr{N}(\tilde{z})$, there exists $b_\Phi > 0$, $\|\Phi(z)\|_{V^*} \leq b_\Phi$.

(C9): $f\colon W \to V^*$ is such that

(a) f is $l_f.1$-Hölder continuous on $\mathscr{N}(\tilde{w})$;
(b) for each $w \in \mathscr{N}(\tilde{w})$, there exists $b_f > 0$, $\|f(w)\|_{V^*} \leq b_f$.

Remark 3.2. Combining assumptions **(C3)**(b) and **(C7)**(b) implies that, for each $w \in \mathscr{N}(\tilde{w})$ and $p,q \in M(\mathscr{N}(\tilde{e}))$, $\|\mathscr{R}(w,p,q)\|_{V^*} \leq b_{\mathscr{R}}$, where

$$b_{\mathscr{R}} = \|\mathscr{R}(0_W,0,0)\|_{V^*} + l_{\mathscr{R}}\|w\|_W + (l'_{\mathscr{R}} + l''_{\mathscr{R}})b_M.$$

Using the imposed assumptions above, we now provide the Hölder property of $\mathbb{G}_\theta$, which will be used to investigate the Hölder continuity of the solution mapping $\mho(\cdot,\cdot,\cdot)$ to Problem 2.1.

Proposition 3.2. *Suppose that hypotheses* **(C1)–(C9)** *hold. Then, for each* $\theta > 0$, *for any* (p_1,e_1,z_1,w_1), $(p_2,e_2,z_2,w_2) \in M(\mathscr{N}(\tilde{e})) \times \mathscr{N}(\tilde{e}) \times \mathscr{N}(\tilde{z}) \times \mathscr{N}(\tilde{w})$,

$$\begin{aligned}
&|\mathbb{G}_\theta(p_1,e_1,z_1,w_1) - \mathbb{G}_\theta(p_2,e_2,z_2,w_2)| \\
\leq\ &\tilde{\mathfrak{L}}(\|p_1 - p_2\|_V + \|w_1 - w_2\|_W + \|e_1 - e_2\|_E^\nu + \|z_1 - z_2\|_Z^\mu), \qquad (3.6)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mathfrak{L}} :=\ &\max\big\{2b_M(l'_{\mathscr{R}} + l''_{\mathscr{R}}) + 2b_M\theta + b_{\mathscr{R}} + b_\Phi + b_f + 2l'_{\mathscr{S}} + l''_{\mathscr{S}} + l'_\Psi + l^*, \\
&2b_M(l_{\mathscr{R}} + l_f) + 2l_{\mathscr{S}} + l_\Psi, l_M\left(2b_M\theta + b_{\mathscr{R}} + b_\Phi + b_f + l'_{\mathscr{S}} + l^*\right), 2b_M l_\Phi\big\}, \qquad (3.7)\\
l^* :=\ &\sup_{p \in M(\mathscr{N}(\tilde{e}))}\left\{l_p > 0 \mid |\Psi^0(w,p;q)| \leq l_p\|q\|_V,\ \forall w \in W, q \in M(\mathscr{N}(\tilde{e}))\right\}.
\end{aligned}$$

Proof. Let $(e_1,z_1,w_1),(e_2,z_2,w_2) \in \mathscr{N}(\tilde{e}) \times \mathscr{N}(\tilde{w})$ and $(p_1,p_2) \in M(e_1) \times M(e_2)$ be fixed. Note that the regularized gap function $\mathbb{G}_\theta$ is defined by (3.1). We have the following assertion: for any $\varepsilon > 0$, there exists $q_\varepsilon \in M(e_1)$ such that

$$\begin{aligned}\mathbb{G}_\theta(p_1,e_1,z_1,w_1) \leq \langle \mathscr{R}(w_1,p_1,p_1) - f(w_1) - \Phi(z_1), p_1 - q_\varepsilon \rangle_V - \mathscr{S}(w_1,p_1,q_\varepsilon)\\ + \mathscr{S}(w_1,p_1,p_1) - \Psi^0(w_1,p_1;q_\varepsilon - p_1) - \frac{\theta}{2}\|p_1 - q_\varepsilon\|_V^2 + \varepsilon.\end{aligned} \tag{3.8}$$

Since $M(\cdot)$ is $l_M.\nu$-Hölder continuous on $\mathscr{N}(\tilde{e})$, we see that there exist $l_M > 0$ and $\alpha > 0$ such that

$$M(e_1) \subset M(e_2) + l_M \|e_1 - e_2\|_E^\nu \mathbb{B}_V, \tag{3.9}$$

which implies that there exists $q_2 \in M(e_2)$ such that

$$\|q_\varepsilon - q_2\|_V \leq l_M \|e_1 - e_2\|_E^\nu. \tag{3.10}$$

Moreover, we also have

$$\begin{aligned}&\mathbb{G}_\theta(p_2,e_2,z_2,w_2)\\ &\geq \langle \mathscr{R}(p_2,p_2,w_2) - f(w_2) - \Phi(z_2), p_2 - q_2 \rangle_V\\ &\quad + \mathscr{S}(w_2,p_2,p_2) - \mathscr{S}(w_2,p_2,q_2) - \Psi^0(p_2;q_2 - p_2) - \frac{\theta}{2}\|p_2 - q_2\|_V^2.\end{aligned} \tag{3.11}$$

From (3.8) and (3.11), we obtain

$$\begin{aligned}&\mathbb{G}_\theta(p_1,e_1,z_1,w_1) - \mathbb{G}_\theta(p_2,e_2,z_2,w_2)\\
&\leq \langle \mathscr{R}(w_1,p_1,p_1) - \mathscr{R}(w_2,p_2,p_2) + \Phi(z_2) - \Phi(z_1) + f(w_2) - f(w_1), p_1 - q_\varepsilon \rangle_V\\
&\quad + \langle \mathscr{R}(w_2,p_2,p_2) - f(w_2) - \Phi(z_2), p_1 - p_2 + q_2 - q_\varepsilon \rangle_V\\
&\quad + \mathscr{S}(w_1,p_1,p_1) - \mathscr{S}(w_2,p_2,p_2) + \mathscr{S}(w_2,p_2,q_2) - \mathscr{S}(w_1,p_1,q_\varepsilon)\\
&\quad + \Psi^0(w_2,p_2;q_2 - p_2) - \Psi^0(w_1,p_1;q_\varepsilon - p_1)\\
&\quad + \frac{\theta}{2}\left(\|p_2 - q_2\|_V^2 - \|p_1 - q_\varepsilon\|_V^2\right) + \varepsilon\\
&\leq (\|\mathscr{R}(w_1,p_1,p_1) - \mathscr{R}(w_2,p_2,p_2)\|_{V^*} + \|\Phi(z_2) - \Phi(z_1)\|_{V^*})(\|p_1\|_V + \|q_\varepsilon\|_V)\\
&\quad + \|f(w_2) - f(w_1)\|_{V^*}(\|p_1\|_V + \|q_\varepsilon\|_V)\\
&\quad + (\|\mathscr{R}(w_2,p_2,p_2)\|_{V^*} + \|f(w_2)\|_{V^*} + \|\Phi(z_2)\|_{V^*})(\|p_1 - p_2\|_V + \|q_2 - q_\varepsilon\|_V)\\
&\quad + |\mathscr{S}(w_1,p_1,p_1) - \mathscr{S}(w_2,p_2,p_2)| + |\mathscr{S}(w_2,p_2,q_2) - \mathscr{S}(w_1,p_1,q_\varepsilon)|\\
&\quad + |\Psi^0(w_2,p_2;q_2 - q_\varepsilon)| + |\Psi^0(w_2,p_2;p_1 - p_2)|\\
&\quad + |\Psi^0(w_2,p_2;q_\varepsilon - p_1) - \Psi^0(w_1,p_1;q_\varepsilon - p_1)|\\
&\quad + \frac{\theta}{2}(\|p_2\|_V + \|q_2\|_V + \|p_1\|_V + \|q_\varepsilon\|_V)(\|p_1 - p_2\|_V + \|q_2 - q_\varepsilon\|_V) + \varepsilon.\end{aligned} \tag{3.12}$$

Hence, from Lemma 2.1, assumptions **(C3)**(a), **(C4)**(b), **(C5)**(b), **(C7)–(C9)**, Remark 3.2, inequality (3.10) and the arbitrariness of ε, we have

$$\begin{aligned}
&\mathbb{G}_\theta(p_1,e_1,z_1,w_1)-\mathbb{G}_\theta(p_2,e_2,z_2,w_2)\\
&\le 2b_M\left(l_{\mathscr{R}}\|w_1-w_2\|_W+(l'_{\mathscr{R}}+l''_{\mathscr{R}})\|p_1-p_2\|_V+l_\Phi\|z_1-z_2\|_Z^\mu\right)\\
&\quad+2b_Ml_f\|w_1-w_2\|_W+(b_{\mathscr{R}}+b_f+b_\Phi)(\|p_1-p_2\|_V+l_M\|e_1-e_2\|_E^\nu)\\
&\quad+l_{\mathscr{S}}\|w_1-w_2\|_W+(l'_{\mathscr{S}}+l''_{\mathscr{S}})\|p_1-p_2\|_V+l_{\mathscr{S}}\|w_1-w_2\|_W+l'_{\mathscr{S}}\|p_1-p_2\|_V\\
&\quad+l_Ml''_{\mathscr{S}}\|e_1-e_2\|_E^\nu+l_Ml_{p_2}\|e_1-e_2\|_E^\nu+l_{p_2}\|p_1-p_2\|_V+l_\Psi\|w_1-w_2\|_W\\
&\quad+l'_\Psi\|p_1-p_2\|_V+\frac{\theta}{2}4b_M\left(\|p_1-p_2\|_V+l_M\|e_1-e_2\|_E^\nu\right)
\end{aligned}$$

and then

$$\begin{aligned}
&\mathbb{G}_\theta(p_1,e_1,z_1,w_1)-\mathbb{G}_\theta(p_2,e_2,z_2,w_2)\\
&\le\left(2b_M(l'_{\mathscr{R}}+l''_{\mathscr{R}})+2b_M\theta+b_{\mathscr{R}}+b_\Phi+b_f+2l'_{\mathscr{S}}+l''_{\mathscr{S}}+l'_\Psi+l^*\right)\|p_1-p_2\|_V\\
&\quad+(2b_M(l_{\mathscr{R}}+l_f)+2l_{\mathscr{S}}+l_\Psi)\|w_1-w_2\|_W\\
&\quad+l_M\left(2b_M\theta+b_{\mathscr{R}}+b_\Phi+b_f+l''_{\mathscr{S}}+l^*\right)\|e_1-e_2\|_E^\nu\\
&\quad+2b_Ml_\Phi\|z_1-z_2\|_Z^\mu\\
&\le\tilde{\mathfrak{L}}(\|p_1-p_2\|_V+\|w_1-w_2\|_W+\|e_1-e_2\|_E^\nu+\|z_1-z_2\|_Z^\mu),
\end{aligned}$$

where $\tilde{\mathfrak{L}}$ is given by (3.7). Therefore, using the symmetry between (p_1,e_1,z_1,w_1) and (p_2,e_2,z_2,w_2), the conclusion of Proposition 3.2 holds. □

Finally, we deduce the Hölder continuity of the solution mapping $\mho(\cdot,\cdot,\cdot)$ to Problem 2.1 around the point $(\tilde{e},\tilde{z},\tilde{w})$ by using the nice properties of the gap function $\mathbb{G}_\theta$ in Proposition 3.2.

Theorem 3.2. *Assume that hypotheses* **(C1)–(C5)** *and* **(C7)–(C9)** *hold. Then, for* $\theta>0$ *with* $\theta<2(k_{\mathscr{R}}-\alpha_{\mathscr{R}}-\alpha_{\mathscr{S}}-\alpha_\Psi)$, *for any* $(e_1,z_1,w_1),(e_2,z_2,w_2)\in\mathscr{N}(\tilde{e})\times\mathscr{N}(\tilde{z})\times\mathscr{N}(\tilde{w})$,

$$\begin{aligned}
&\|p^*(e_1,z_1,w_1)-p^*(e_2,z_2,w_2)\|_V\\
\le\;& l_M\|e_1-e_2\|_E^\nu+\left(\frac{\tilde{\mathfrak{L}}}{k_{\mathscr{R}}-\alpha_{\mathscr{R}}-\alpha_{\mathscr{S}}-\alpha_\Psi-\frac{\theta}{2}}\right)^{\frac{1}{2}}\\
&\times\left[\|w_1-w_2\|_W+(1+l_M)\|e_1-e_2\|_E^\nu+\|z_1-z_2\|_Z^\mu\right]^{\frac{1}{2}},
\end{aligned}\tag{3.13}$$

where $p^*(e_1,z_1,w_1)\in\mho(e_1,z_1,w_1)$, $p^*(e_2,z_2,w_2)\in\mho(e_2,z_2,w_2)$, *and* $\tilde{\mathfrak{L}}$ *is defined by* (3.7).

Proof. Let $(e_1,z_1,w_1),(e_2,z_2,w_2)\in\mathscr{N}(\tilde{e})\times\mathscr{N}(\tilde{z})\times\mathscr{N}(\tilde{w})$ be fixed and

$$p^*(e_1,z_1,w_1)\in\mho(e_1,z_1,w_1),p^*(e_2,z_2,w_2)\in\mho(e_2,z_2,w_2).$$

Then, we have $p^*(e_1,z_1,w_1)\in M(e_1)$. Thus it follows from (3.9) that there exists $p_2\in M(e_2)$ such that

$$\|p^*(e_1,z_1,w_1)-p_2\|_V\le l_M\|e_1-e_2\|_E^\nu.\tag{3.14}$$

Applying (3.2), (3.6), (3.14), and $p^*(e_1,z_1,w_1) \in \mho(e_1,z_1,w_1)$, that is,

$$\mathbb{G}_\theta(p^*(e_1,z_1,w_1),e_1,z_1,w_1) = 0,$$

one has

$$\begin{aligned}
&\|p^*(e_1,z_1,w_1) - p^*(e_2,z_2,w_2)\|_V \\
&\leq \|p^*(e_1,z_1,w_1) - p_2\|_V + \|p_2 - p^*(e_2,z_2,w_2)\|_V \\
&\leq l_M \|e_1 - e_2\|_E^\nu + \frac{1}{\sqrt{k_{\mathscr{R}} - \alpha_{\mathscr{R}} - \alpha_{\mathscr{S}} - \alpha_\Psi - \frac{\theta}{2}}} \sqrt{\mathbb{G}_\theta(p_2,e_2,z_2,w_2)} \\
&= l_M \|e_1 - e_2\|_E^\nu + \frac{1}{\sqrt{k_{\mathscr{R}} - \alpha_{\mathscr{R}} - \alpha_{\mathscr{S}} - \alpha_\Psi - \frac{\theta}{2}}} \times \\
&\qquad \sqrt{\mathbb{G}_\theta(p_2,e_2,z_2,w_2) - \mathbb{G}_\theta(p^*(e_1,z_1,w_1),e_1,z_1,w_1)} \\
&\leq l_M \|e_1 - e_2\|_E^\nu \\
&\quad + \sqrt{\frac{\tilde{\mathfrak{L}}(\|p^*(e_1,z_1,w_1) - p_2\|_V + \|w_1 - w_2\|_W + \|e_1 - e_2\|_E^\nu + \|z_1 - z_2\|_Z^\mu)}{k_{\mathscr{R}} - \alpha_{\mathscr{R}} - \alpha_{\mathscr{S}} - \alpha_\Psi - \frac{\theta}{2}}} \\
&\leq l_M \|e_1 - e_2\|_E^\nu + \left(\frac{\tilde{\mathfrak{L}}}{k_{\mathscr{R}} - \alpha_{\mathscr{R}} - \alpha_{\mathscr{S}} - \alpha_\Psi - \frac{\theta}{2}}\right)^{\frac{1}{2}} \times \\
&\qquad \left[\|w_1 - w_2\|_W + (1 + l_M)\|e_1 - e_2\|_E^\nu + \|z_1 - z_2\|_Z^\mu\right]^{\frac{1}{2}}.
\end{aligned}$$

Therefore, inequality (3.14) holds. □

Remark 3.3. There are some comments regarding Theorem 3.1 and Theorem 3.2.

(i) The upper bound established in Theorem 3.1 derives an upper estimate of the distance from an arbitrary point in the admissible set to the unique solution of PEVHI. Computing the upper estimate in (3.2) is controlled by the regularized gap function $\mathbb{G}_\theta$ depending on the perturbed parameters of PEVHI.

(ii) Let us consider the special case (ii) of Problem 2.1, studied in [20]. Then our Problem 2.1 is a generalization to elliptical variational-hemivariational inequalities associated with perturbed parameters governed by both the set of constraints and the mappings. The Hölder continuous behavior of the solution mapping for Problem 2.1 in Theorem 3.2 depends on the perturbed properties of the set of constraints and the mappings. Thus Theorem 3.2 extends remarkably the corresponding result in [20].

We now present an example to illustrate the main results in the paper.

Example 3.1. Let $V = E = W = Z = \mathbb{R}$ and the functions M and $\mathscr{R}$ be given in Example 2.1. Let $\mathscr{S}\colon W \times V \times V \to \mathbb{R}$, $f\colon W \to V^*$, $\Phi\colon Z \to V^*$, and $\Psi\colon W \times V \to \mathbb{R}$ be defined by

$$\mathscr{S}(w,p,q) = \frac{1}{36}\sin^2\left(\frac{w+1}{2}\right)pq^2, f(w) = \sin^2\left(\frac{w+1}{2}\right) - 7, \Phi(z) = \cos\left(\frac{z^{\frac{1}{3}}+1}{2}\right)$$

and

$$\Psi(w,p)=\begin{cases}\frac{\sin^2\left(\frac{w+1}{2}\right)p^2}{12} & \text{if } p>0\\ (w^2+1)p & \text{if } p\le 0,\end{cases}$$

respectively for all $w,z,p,q\in\mathbb{R}$.

It is clear that $\Psi(w,\cdot)$ is a locally Lipschitz function. Moreover, the Clarke's generalized directional derivative $\Psi^0(w,p;q)$ is given by

$$\Psi^0(w,p;q)=\begin{cases}\frac{\sin^2\left(\frac{w+1}{2}\right)pq}{6} & \text{if } p>0,\\ \max\{0,(w^2+1)q\} & \text{if } p=0,\\ (w^2+1)q & \text{if } p<0,\end{cases}$$

for all $w,p,q\in\mathbb{R}$.

Then Problem 2.1 is equivalent to finding $p\in\left[\frac{1}{2}+\sin^2\left(\frac{e^{\frac{1}{3}}}{2}\right),2\right]$ such that

$$\left[7-2p+\frac{7}{2}q^3-\cos\left(\frac{z^{\frac{1}{3}}+1}{2}\right)+\left(2+\frac{(p+q+6)p}{36}\right)\sin^2\left(\frac{w+1}{2}\right)\right](q-p)\ge 0,$$

for all $q\in\left[\frac{1}{2}+\sin^2\left(\frac{e^{\frac{1}{3}}}{2}\right),2\right]$.

By Example 2.1 and direct checking, it is not difficult to verify that the hypotheses **(C1)–(C3)** hold with $l_{\mathscr{R}}=3, l'_{\mathscr{R}}=2, l''_{\mathscr{R}}=42, k_{\mathscr{R}}=\frac{21}{8}, \alpha_{\mathscr{R}}=2$.

We now check condition **(C4)**. It is clear that $q\mapsto\mathscr{S}(w,p,q)$ is convex for all $w\in W$ and $p\in V$. Moreover, for any $w_1,w_2\in W$, $p_1,q_1,p_2,q_2\in M(E)=\left[\frac{1}{2},2\right]$, we have

$$\begin{aligned}
&|\mathscr{S}(w_1,p_1,q_1)-\mathscr{S}(w_2,p_2,q_2)|\\
&=\left|\frac{1}{36}\sin^2\left(\frac{w_1+1}{2}\right)p_1q_1^2-\frac{1}{36}\sin^2\left(\frac{w_2+1}{2}\right)p_2q_2^2\right|\\
&\le\frac{1}{36}|p_1q_1^2|\left|\sin^2\left(\frac{w_1+1}{2}\right)-\sin^2\left(\frac{w_2+1}{2}\right)\right|+\frac{1}{36}\left|\sin^2\left(\frac{w_2+1}{2}\right)q_1^2\right||p_1-p_2|\\
&\quad+\frac{1}{36}\left|\sin^2\left(\frac{w_2+1}{2}\right)p_2\right||q_1^2-q_2^2|\\
&\le\frac{2}{9}|w_1-w_2|+\frac{1}{9}|p_1-p_2|+\frac{2}{9}|q_1-q_2|;
\end{aligned}$$

and

$$
\begin{aligned}
&\mathscr{S}(w_1,p_1,q_2)-\mathscr{S}(w_1,p_1,q_1)+\mathscr{S}(w_2,p_2,q_1)-\mathscr{S}(w_2,p_2,q_2)\\
&=\frac{1}{36}\sin^2\left(\frac{w_1+1}{2}\right)p_1(q_2^2-q_1^2)+\frac{1}{36}\sin^2\left(\frac{w_2+1}{2}\right)p_2(q_1^2-q_2^2)\\
&=\frac{1}{36}(q_2^2-q_1^2)\left(\sin^2\left(\frac{w_1+1}{2}\right)p_1-\sin^2\left(\frac{w_2+1}{2}\right)p_2\right)\\
&\le\frac{1}{36}|q_1^2-q_2^2|\left(\left|\sin^2\left(\frac{w_1+1}{2}\right)-\sin^2\left(\frac{w_2+1}{2}\right)\right||p_1|+\left|\sin^2\left(\frac{w_2+1}{2}\right)\right||p_1-p_2|\right)\\
&\le\frac{1}{9}|q_1-q_2|\left(2|w_1-w_2|+|p_1-p_2|\right)=\frac{1}{9}|p_1-p_2||q_1-q_2|+\frac{2}{9}|w_1-w_2||q_1-q_2|.
\end{aligned}
$$

This implies that condition **(C4)** is satisfied with $l_{\mathscr{S}}=\frac{2}{9}, l'_{\mathscr{S}}=\frac{1}{9}, l''_{\mathscr{S}}=\frac{2}{9}, \alpha_{\mathscr{S}}=\frac{1}{9}, \beta_{\mathscr{S}}=\frac{2}{9}$. Similarly, we can show that hypothesis **(C5)** hold with $l_\Psi=\frac{2}{3}, l'_\Psi=\frac{1}{3}, \alpha_\Psi=\frac{1}{6}, \beta_\Psi=\frac{1}{3}$. Since $k_{\mathscr{R}}-\alpha_{\mathscr{R}}-\alpha_{\mathscr{S}}-\alpha_\Psi=\frac{25}{72}>0$, the assumption **(C6)** is valid.

Then, by using Theorem 2.1, we obtain that Problem 2.1 has a unique solution and the solution mapping $\mho$ is given by

$$\mho(e,z,w)=\left\{\frac{1}{2}+\sin^2\left(\frac{e^{\frac{1}{3}}}{2}\right)\right\}.$$

Next, let $\theta=\frac{1}{3}$ satisfy the condition $\theta<2(k_{\mathscr{R}}-\alpha_{\mathscr{R}}-\alpha_{\mathscr{S}}-\alpha_\Psi)=\frac{25}{36}$. Then, all the assumptions of Theorem 3.1 are satisfied.

On the other hand, we consider $\tilde{e}=\frac{1}{2}, \tilde{z}=0$, $\tilde{w}=1$ and $\mathscr{N}(\tilde{e})=[0,1]$, $\mathscr{N}(\tilde{z})=[-\frac{1}{2},\frac{1}{2}]$, $\mathscr{N}(\tilde{w})=[\frac{1}{2},\frac{3}{2}]$. Then, we define

$$g(e)=\frac{1}{2}+\sin^2\left(\frac{e^{\frac{1}{3}}}{2}\right),\quad e\in E.$$

Thus $M(e)=[g(e),2]$ for all $e\in E$. For any $e_1,e_2\in\mathscr{N}(\tilde{e})$, we have

$$
\begin{aligned}
|g(e_1)-g(e_2)|&=\left|\sin^2\left(\frac{e_1^{\frac{1}{3}}}{2}\right)-\sin^2\left(\frac{e_1^{\frac{1}{3}}}{2}\right)\right|\\
&\le\left|e_1^{\frac{1}{3}}-e_2^{\frac{1}{3}}\right|\le|e_1-e_2|^{\frac{1}{3}}.
\end{aligned}
$$

Hence,

$$[g(e_1),2]\subset[g(e_2),2]+|e_1-e_2|^{\frac{1}{3}}\mathbb{B}_{\mathbb{R}},\ \forall e_1,e_2\in\mathscr{N}(\tilde{e}),$$

where $\mathbb{B}_{\mathbb{R}}=[-1,1]$. Consequently, for any $e_1,e_2\in\mathscr{N}(\tilde{e})$

$$M(e_1)\subset M(e_2)+|e_1-e_2|^{\frac{1}{3}}\mathbb{B}_{\mathbb{R}},$$

so M is $l_M.\nu$-Hölder continuous on $\mathscr{N}(\tilde{e})$ with $l_M=1, \nu=\frac{1}{3}$. Moreover, for each $e\in\mathscr{N}(\tilde{e})$, $p\in M(e)$, there exists $b_M=2, |p|\le b_M$. Hence, assumption **(C7)** holds. Also, we can verify that conditions **(C8)** and **(C9)** are valid with $\mu=\frac{1}{3}$, $l_\Phi=\frac{1}{2}$, $b_\Phi=1$, $b_{\mathscr{R}}=3\sin^2\left(\frac{1}{2}\right)+\frac{185}{2}$, $l_f=1$ and $b_f=7-\sin^2\left(\frac{3}{4}\right)$. Thus, all the hypotheses in Theorem 3.2 are satisfied.

We now obtain

$$l^* = \sup_{p\in M(\mathscr{N}(\tilde{e}))} \left\{ l_p > 0 \mid |\Psi^0(w,p;q)| \le l_p \|q\|_V,\ \forall w \in W, q \in M(\mathscr{N}(\tilde{e})) \right\} = \frac{1}{3}.$$

Then, it follows from (3.7) that

$$\tilde{\mathfrak{L}} = \frac{5021}{18} + 3\sin^2\left(\frac{1}{2}\right) - \sin^2\left(\frac{3}{4}\right).$$

Let $(e_1,z_1,w_1),(e_2,z_2,w_2) \in \mathscr{N}(\tilde{e}) \times \mathscr{N}(\tilde{z}) \times \mathscr{N}(\tilde{w})$ be fixed and

$$p^*(e_1,z_1,w_1) = \frac{1}{2} + \sin^2\left(\frac{e_1^{\frac{1}{3}}}{2}\right) \in \mho(e_1,z_1,w_1),$$

$$p^*(e_2,z_2,w_2) = \frac{1}{2} + \sin^2\left(\frac{e_2^{\frac{1}{3}}}{2}\right) \in \mho(e_2,z_2,w_2).$$

We have

$$\begin{aligned}
& l_M \|e_1 - e_2\|_E^\nu + \left(\frac{\tilde{\mathfrak{L}}}{k_{\mathscr{R}} - \alpha_{\mathscr{R}} - \alpha_{\mathscr{S}} - \alpha_\Psi - \frac{\theta}{2}} \right)^{\frac{1}{2}} \times \\
& \qquad \left[\|w_1 - w_2\|_W + (1 + l_M)\|e_1 - e_2\|_E^\nu + \|z_1 - z_2\|_Z^\mu \right]^{\frac{1}{2}} \\
& = |e_1 - e_2|^{\frac{1}{3}} + \sqrt{\frac{\frac{5021}{18} + 3\sin^2\left(\frac{1}{2}\right) - \sin^2\left(\frac{3}{4}\right)}{\frac{13}{72}}} \sqrt{|w_1 - w_2| + 2|e_1 - e_2|^{\frac{1}{3}} + |z_1 - z_2|^{\frac{1}{3}}} \\
& \ge |e_1 - e_2|^{\frac{1}{3}} \ge \left| \sin^2\left(\frac{e_1^{\frac{1}{3}}}{2}\right) - \sin^2\left(\frac{e_1^{\frac{1}{3}}}{2}\right) \right| = |p^*(e_1,z_1,w_1) - p^*(e_2,z_2,w_2)| \\
& = \|p^*(e_1,z_1,w_1) - p^*(e_2,z_2,w_2)\|_V.
\end{aligned}$$

This demonstrates the Hölder continuous behavior of the solution mapping $\mho(\cdot,\cdot,\cdot)$ for Problem 2.1 around the point $(\tilde{e},\tilde{z},\tilde{w}) = \left(\frac{1}{2},0,1\right)$ in Theorem 3.2.

4. Conclusion

In this work, we focus on considering a class of the elliptical variational-hemivariational inequalities in the case of the perturbed parameters governed by both the set of constraints and the mappings (for brevity, PEVHI (CM)); see Problem 2.1. Based on the arguments of monotonicity and properties of the Clarke's generalized directional derivative, we provide an upper bound result for PEVHI (CM) (Theorem 3.1) and the Hölder continuity of the solution mapping for PEVHI (CM) (Theorem 3.2) by using regularized gap functions under suitable assumptions on the data.

Acknowledgments

The authors would like to thank the anonymous referees for their helpful comments which have improved the presentation of the paper.

REFERENCES

[1] L.Q. Anh, N.V. Hung, V.M. Tam, Regularized gap functions and error bounds for generalized mixed strong vector quasi equilibrium problems, Comput. Appl. Math. 37 (2018), 5935-5950.

[2] L.Q. Anh, P.Q. Khanh, Uniqueness and Hölder continuity of the solution to multivalued equilibrium problems in metric spaces, J. Glob. Optim. 37 (2007), 449-465.

[3] A. Auslender, Optimisation: Méthodes Numériques, Masson, Paris, 1976. (in French)

[4] M. Bianchi, R. Pini, Sensitivity for parametric vector equilibria, Optimization 55 (2006), 221-230.

[5] H. Brézis, Equations et inéquations non-linéaires dans les espaces vectoriel en dualité, Ann. Inst. Fourier 18 (1968), 115-175.

[6] S.S. Chang, Salahuddin, L. Wang, C.F. Wen, On the parametric elliptical variational-hemivariational inequality problem with applications, Appl. Anal. 101 (2021), 6645-6667.

[7] C.R. Chen, L.L. Li, M.H. Li, Hölder continuity results for nonconvex parametric generalized vector quasiequilibrium problems via nonlinear scalarizing functions, Optimization 65 (2016), 35-51.

[8] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.

[9] Z. Denkowski, S. Migórski, N.S. Papageorgiou, An Introduction to Nonlinear Analysis: Theory, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.

[10] M. Fukushima, Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems, Math. Program. 53 (1992), 99-110.

[11] D. Han, W. Han, Numerical analysis of an evolutionary variational-hemivariational inequality with application to a dynamic contact problem, J. Comput. Appl. Math. 358 (2019), 163-178.

[12] W. Han, Numerical analysis of stationary variational-hemivariational inequalities with applications in contact mechanics, Math. Mech. Solids 23 (2018), 279-293.

[13] W. Han, S. Migórski, M. Sofonea, A class of variational-hemivariational inequalities with applications to frictional contact problems, SIAM J. Math. Anal. 46 (2014), 3891-3912.

[14] W. Han, M. Sofonea, D. Danan, Numerical analysis of stationary variational-hemivariational inequalities, Numer. Math. 139 (2018), 563-592.

[15] X.F. Hu, X.B. Li, Hölder continuity of a parametric variational inequality, Bull. Iranian Math. Soc. 43 (2017), 2371-2381.

[16] N.V. Hung, S. Migórski, V.M. Tam, S.D. Zeng, Gap functions and error bounds for variational-hemivariational inequalities, Acta. Appl. Math. 169 (2020), 691-709.

[17] N.V. Hung, V. Novo, V.M. Tam, Error bound analysis for vector equilibrium problems with partial order provided by a polyhedral cone, J. Glob. Optim. 82 (2022), 139-159.

[18] N.V. Hung, V.M. Tam, Error bound analysis of the D-gap functions for a class of elliptic variational inequalities with applications to frictional contact mechanics, Z. Angew. Math. Phys. 72 (2021), 173.

[19] N.V. Hung, V.M. Tam, Y. Zhou, A new class of strong mixed vector GQVIP-generalized quasi-variational inequality problems in fuzzy environment with regularized gap functions based error bounds, J. Comput. Appl. Math. 381 (2021), 113055.

[20] N.V. Hung, V.M. Tam, Z. Liu, J.C. Yao, A novel approach to Hölder continuity of a class of parametric variational-hemivariational inequalities, Oper. Res. Lett. 49 (2021), 283-289.

[21] S.A. Khan, J.W. Chen, Gap functions and error bounds for generalized mixed vector equilibrium problems, J. Optim. Theory Appl. 166 (2015), 767-776.

[22] X.B. Li, S.J. Li, Hölder continuity of perturbed solution set for convex optimization problems, Appl. Math. Comput. 232 (2014), 908-918.

[23] M.B. Lignola, Regularized gap functions for variational problems, Oper. Res. Lett. 36 (2008), 710-714.

[24] O.T. Mewomo, C.C. Okeke, F.U. Ogbuisi, Iterative solutions of split fixed point and monotone inclusion problems in Hilbert spaces, J. Appl. Numer. Optim. 5 (2023), 271-285.

[25] S. Migórski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Advances in Mechanics and Mathematics 26, Springer, New York, 2013.

[26] A.O.E. Owolabi, O.T. Mewomo, J.C. Yao, X. Qin, Strong convergence analysis for solving quasi-monotone variational inequalities and fixed point problems in reflexive Banach spaces, J. Nonlinear Funct. Anal. 2023 (2023), 30.

[27] P.D. Panagiotopoulos, Nonconvex problems of semipermeable media and related topics, Z. Angew. Math. Mech. (ZAMM) 65 (1985), 29-36.
[28] P.D. Panagiotopoulos, Hemivariational Inequalities, Applications in Mechanics and Engineering. Springer-Verlag, Berlin, 1993.
[29] M. Sofonea, S. Migórski, Variational-Hemivariational Inequalities with Applications, Pure and Applied Mathematics, Chapman & Hall/CRC Press, Boca Raton, 2018.
[30] V.M. Tam, Sharp Hölder continuous behaviour of solutions to vector network equilibrium problems with a polyhedral ordering cone, Filomat 36 (2022), 4563-4573.
[31] V.M. Tam, Upper-bound error estimates for double phase obstacle problems with Clarke's subdifferential, Numer. Funct. Anal. Optim. 43 (2022), 463-485.
[32] V.M. Tam, J.-S. Chen, Upper error bounds of DG-functions for history-dependent variational-hemivariational inequalities, Appl. Set-Valued Anal. Optim. 5 (2023), 347-367.
[33] V.M. Tam, J.-S. Chen, Upper bounds for vector equilibrium problems associated with a p-order cone on Hadamard manifolds, J. Nonlinear Convex Anal. 24 (2023), 2593-2609.
[34] V.M. Tam, J.-S. Chen, On the D-gap functions for variational-hemivariational inequalities with an application to contact mechanics, Pacific J. Optim. in press (2024).
[35] G. Virmani, M. Srivastava, Levitin-Polyak well-posedness of constrained inverse quasivariational inequality, Numer. Funct. Anal. Optim. 38 (2017), 91-109.
[36] Y.D. Xu, C.R. Chen, C.J. Fang, Hölder continuity for solution mappings of parametric non-convex strong generalized Ky Fan inequalities, Numer. Funct. Anal. Optim. 41 (2020), 344-360.
[37] N. Yamashita, M. Fukushima, Equivalent unconstrained minimization and global error bounds for variational inequality problems, SIAM J. Control Optim. 35 (1997), 273-284.
[38] E. Zeidler, Nonlinear Functional Analysis and Its Applications, II/B: Nonlinear Monotone Operators, Springer, New York, 1990.
[39] S.D. Zeng, S. Migórski, Z. Liu, J.C. Yao, Convergence of a generalized penalty method for variational-hemivariational inequalities, Commun. Nonlinear Sci. Numer. Simul. 92 (2020), 105476.