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GENERALIZED HUKUHARA WEAK SUBDIFFERENTIAL AND ITS APPLICATION ON IDENTIFYING OPTIMALITY CONDITIONS FOR NONSMOOTH INTERVAL-VALUED FUNCTIONS

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Abstract. In this paper, we introduce the idea of gH-weak subdifferential for interval-valued functions (IVFs) and show how to calculate gH-weak subgradients. It is observed that a nonempty gH-weak subdifferential set is convex and closed. In characterizing the class of functions for which the gH-weak subdifferential set is nonempty, it is identified that this class is the collection of gH-lower Lipschitz IVFs. In checking the validity of the sum rule of gH-weak subdifferential for a pair of IVFs, a counterexample is obtained, which reflects that the sum rule does not hold. However, under a mild restriction on one of the IVFs, one-sided inclusion for the sum rule holds. As applications, we employ gH-weak subdifferential to provide a few optimality conditions for nonsmooth IVFs. Further, a necessary optimality condition for interval optimization problems with a difference of two nonsmooth IVFs as the objective is established. Next, a necessary and sufficient condition via augmented normal cone and gH-weak subdifferential of IVFs for finding weak efficient points is presented. Lastly, in investigating a 'sup-relation' between gH-direction derivative and gH-weak subgradients, we approximately compute gH-weak subgradient at each iterative step. In the sequel, we propose $\mathscr{W}$-gH-weak subgradient method to identify a weak efficient solution of an unconstrained nonsmooth IOP. We apply the proposed method to solve an interval optimization problem by taking a test example. We present a convergence analysis of the proposed method for constant and diminishing step sizes.
Keywords. gH-weak subgradient; gH-Fréchet subdifferential; Interval optimization; Nonsmooth interval-valued functions.

1. Introduction

The interval arithmetic of Moore [19] is the milestone in interval analysis. The realistic applicability of Moore's method is relevant till today. We can currently find several papers in the community of interval-valued optimization problems (IOPs) where Moore's interval analysis is applied extensively. To find optimality conditions for IOPs, ideas of derivatives for interval-valued functions (IVFs) were proposed [5, 13, 18, 21, 27]. In [18], the concept of gH-differentiability for IVFs was introduced. Chalco-Cano et al. [6] addressed the algebraic property of gH-differentiable interval-valued functions. Ghosh et al. [13] proved the existence of gH-directional derivative for convex IVFs and presented optimality conditions for IOPs.

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It is a familiar fact that, in nonsmooth optimization, the classical gradient algorithm fails: even in finding the optimum point, as there is no derivative, the conventional optimality condition $\nabla f(x) = 0$ does not hold. More crucially, it is observed that optima of almost everywhere differentiable function categorically arise at nondifferentiable points—for instance, take the minimization of $f(x) = |x|$. The notion of subdifferential, defined by Rockafellar [24], is a crucial factor in the body of optimization theory that perfectly replaces the role of the gradient to identify optima for convex functions. However, subdifferential is inadequate in developing optimality conditions for nonconvex optimization problems. Due to this insufficiency, the idea of subdifferential has been generalized. The most common of such generalizations is weak subdifferential [3]. Based on this notion, a strong duality theorem for the nonconvex inequality-constrained problems has been found by defining a weak conjugate function [30]. A substantial application of this notion in duality theory with the help of a weak subdifferentiable perturbation function was given in [26].

In the context of the nonsmooth calculus for nondifferentiable convex IVFs, Ghosh et al. [11] recently proposed the idea of gH-subgradient and gH-subdifferential. The same article [11] found that gH-directional derivative is the maximum of all the products of the direction and gH-subgradients. Afterward, Anshika et al. [1] characterized weak efficiency for nonconvex composite optimization problems with the subdifferential sets of convex interval-valued functions. In [1], by formulating the supremum and infimum of an IVF, a Fermat-type, a Fritz-John-type, and a KKT-type weak efficiency condition for nonsmooth IOPs have been derived. Anshika and Ghosh [2] introduced gH-subdifferential of the interval-valued function. Furthermore, Chauhan et al. [8] derived the notion of gH-Clarke derivative for IVFs and IOPs. Under the Clarke subdifferentiablility assumption, Chen and Li [7] provided KKT conditions for efficient solutions. In addition, Karaman [16] presented two subdifferentials for interval-valued functions and some optimality criteria, which were obtained by using subdifferentials.

From the available literature on nonsmooth IOPs, it is found that the study of gH-weak subdifferential notion has not yet been addressed. However, the notion of gH-weak subdifferential might be effective in characterizing and capturing the efficient solutions of IOPs with nonconvex and nonsmooth IVFs. By using a subgradient, one may face difficulties in solving the problem which does not satisfy the convexity assumption because a subgradient refers to the slope of a supporting hyperplane to the graph of convex functions in convex analysis. Thus, in this study, we introduce the notion of a weak subgradient, which does not need any kind of convexity.

In this article, we attempt to show various properties of weak-subdifferential and their use in nonsmooth nonconvex IOPs. As an application of the proposed gH-weak subdifferential, we will give a necessary and sufficient optimality condition for finding weak efficient points of difference of two IVFs. In the last, similar to the conventional weak-subgradient method [9] for real-valued optimization, we show a gH-weak subgradient method to obtain an efficient solution of nonsmooth, nonconvex IOPs.

The rest of the article is presented as follows. Section 2 is devoted to the conventional properties of intervals, followed by the calculus of IVFs. Section 3 introduces the notion of gH-weak subdifferential for IVFs and discusses their properties such as convexity, closedness, and nonemptiness. Additionally, the role of gH-weak subdifferential to derive the necessary condition for weak efficiency for gH-weak subdifferentiable IVFs is presented in Section 3. In Section 4, we analyze the necessary condition for obtaining an efficient solution of the difference of two IVFs. In Section 5, we establish a 'sup-relation' between gH-direction derivative and gH-weak subgradients. Using this relation, in Section 6, we present a $\mathscr{W}$-gH-weak subgradient method to obtain a weak efficient solution to an unconstrained IOP with its algorithmic implementation and convergence analysis. Finally, we draw a conclusion with future directions to extend the present study.

2. Preliminaries and Terminologies

In this section, required terminologies and notions on intervals, including calculus of IVFs are given. Throughout the paper, we extensively use the following notations.

- $\mathbb{R}$ is the set of real numbers.
- $\mathbb{R}_+$ represents the set of nonnegative real numbers.
- $I(\mathbb{R})$ is the collection of all compact intervals.
- $\overline{I(\mathbb{R})} = I(\mathbb{R}) \cup \{-\infty, +\infty\}$.
- $\mathbf{0} = [0,0]$.
- Elements of $I(\mathbb{R})$ are presented by bold capital letters: $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \ldots$.
- $\mathscr{B}(h, \delta)$ represents a ball with center at h and radius δ in $\mathbb{R}^n$.
- $I(\mathbb{R})^n = I(\mathbb{R}) \times I(\mathbb{R}) \times I(\mathbb{R}) \times \cdots \times I(\mathbb{R})$ (n times).
- Interval vectors in $I(\mathbb{R})^n$ are denoted by $\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}, \widehat{\mathbf{Z}}, \ldots$.
- $B_\alpha(\bar{u})$ is the open ball with center at $\bar{u} \in \mathbb{R}^n$ and radius $\alpha \geq 0$.
- $\mathscr{N}(\bar{x})$ is a neighborhood of $\bar{x} \in \mathbb{R}^n$.
- $\|\cdot\|_{I(\mathbb{R})}$ denotes the norm on $I(\mathbb{R})$.

2.1. **Arithmetic and dominance of intervals.** Throughout this subsection, we represent an element $\mathbf{X}$ of $I(\mathbb{R})$ by the corresponding small letter:

$$\mathbf{X} = [\underline{x}, \overline{x}], \text{ where } \underline{x} \text{ and } \overline{x} \text{ are in } \mathbb{R} \text{ with } \underline{x} \leq \overline{x}.$$

Recall that Moore's interval addition ($\oplus$), subtraction ($\ominus$), and multiplication ($\odot$) [19, 20] are given by

$$\mathbf{X} \oplus \mathbf{Y} = \left[\, \underline{x} + \underline{y},\ \overline{x} + \overline{y} \,\right], \ \mathbf{X} \ominus \mathbf{Y} = \left[\, \underline{x} - \overline{y},\ \overline{x} - \underline{y} \,\right], \text{ and}$$
$$\mathbf{X} \odot \mathbf{Y} = \left[\min\left\{\underline{x}\,\underline{y},\ \underline{x}\overline{y},\ \overline{x}\underline{y},\ \overline{x}\overline{y}\right\},\ \max\left\{\underline{x}\,\underline{y},\ \underline{x}\overline{y},\ \overline{x}\underline{y},\ \overline{x}\overline{y}\right\}\right].$$

Definition 2.1. (gH-difference of intervals [25]). *The gH-difference for a pair of intervals $\boldsymbol{P}$ and $\boldsymbol{Q}$, denoted by $\boldsymbol{P} \ominus_{gH} \boldsymbol{Q}$, is the interval $\boldsymbol{Y}$ such that $\boldsymbol{P} = \boldsymbol{Q} \oplus \boldsymbol{Y}$ or $\boldsymbol{Q} = \boldsymbol{P} \ominus \boldsymbol{Y}$. It is well-known that, for $\boldsymbol{P} = \left[\underline{p}, \overline{p}\right]$ and $\boldsymbol{Q} = \left[\underline{q}, \overline{q}\right]$,*

$$\boldsymbol{P} \ominus_{gH} \boldsymbol{Q} = \left[\min\{\underline{p} - \underline{q}, \overline{p} - \overline{q}\}, \max\{\underline{p} - \underline{q}, \overline{p} - \overline{q}\}\right] \text{ and } \boldsymbol{P} \ominus_{gH} \boldsymbol{P} = \boldsymbol{0}.$$

For two elements $\widehat{\boldsymbol{I}} = (\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{I}_n)$ and $\widehat{\boldsymbol{J}} = (\boldsymbol{J}_1, \boldsymbol{J}_2, \ldots, \boldsymbol{J}_n)$ of $I(\mathbb{R})^n$, the algebraic operation $\widehat{\boldsymbol{I}} \star \widehat{\boldsymbol{J}}$ is defined by $\widehat{\boldsymbol{I}} \star \widehat{\boldsymbol{J}} = (\boldsymbol{I}_1 \star \boldsymbol{J}_1, \boldsymbol{I}_2 \star \boldsymbol{J}_2, \ldots, \boldsymbol{I}_n \star \boldsymbol{J}_n)$, where $\star \in \{\oplus, \ominus, \ominus_{gH}\}$.

Definition 2.2. (Dominance of intervals). *Let $\boldsymbol{Z}$ and $\boldsymbol{W}$ be in $I(\mathbb{R})$.*

(i) $\boldsymbol{W}$ is called dominated *by $\boldsymbol{Z}$ if $\underline{z} \leq \underline{w}$ and $\overline{z} \leq \overline{w}$, and then we express it by $\boldsymbol{Z} \preceq \boldsymbol{W}$.*
(ii) $\boldsymbol{W}$ is said to be strictly dominated *by $\boldsymbol{Z}$ if either '$\underline{z} \leq \underline{w}$ and $\overline{z} < \overline{w}$' or '$\underline{z} < \underline{w}$ and $\overline{z} \leq \overline{w}$', and then we express it by $\boldsymbol{Z} \prec \boldsymbol{W}$.*
(iii) If $\boldsymbol{W}$ is not dominated by $\boldsymbol{Z}$, then we write $\boldsymbol{Z} \npreceq \boldsymbol{W}$. If $\boldsymbol{W}$ is not strictly dominated by $\boldsymbol{Z}$, then we write $\boldsymbol{Z} \nprec \boldsymbol{W}$.
(iv) If $\boldsymbol{W} \npreceq \boldsymbol{Z}$ and $\boldsymbol{Z} \npreceq \boldsymbol{W}$, then it is called that none of $\mathbf{W}$ and $\mathbf{Z}$ dominates the other *or $\mathbf{W}$ and $\boldsymbol{Z}$* are not comparable.

For any two elements $\widehat{\mathbf{I}} = (\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n)^\top$ and $\widehat{\mathbf{J}} = (\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_n)^\top$ in $I(\mathbb{R})^n$,

$$\widehat{\mathbf{I}} \preceq \widehat{\mathbf{J}} \iff \mathbf{I}_j \preceq \mathbf{J}_j \text{ for all } j = 1, 2, \ldots, n.$$

2.2. **Concavity and differential calculus of IVFs.** Let $\emptyset \neq \mathscr{Y} \subseteq \mathbb{R}^n$. Let an IVF $\Phi : \mathscr{Y} \to I(\mathbb{R})$ be presented by

$$\Phi(y) = \left[\underline{\phi}(y), \overline{\phi}(y)\right] \ \forall\, y \in \mathscr{Y},$$

where $\underline{\phi}(y) \leq \overline{\phi}(y)$ for all $y \in \mathscr{Y}$ and $\underline{\phi}$ and $\overline{\phi}$ are called lower and upper real-valued functions on $\mathscr{Y}$.

Definition 2.3. (Concave IVF). *If $\mathscr{Y}$ is convex, then an IVF Φ is said to be a* concave *IVF on $\mathscr{Y}$ if, for any $y_1, y_2 \in \mathscr{Y}, \beta_1,\ \beta_2 \in [0,\ 1]$, and $\beta_1 + \beta_2 = 1$,*

$$\beta_1 \odot \Phi(y_1) \oplus \beta_2 \odot \Phi(y_2) \preceq \Phi(\beta_1 y_1 + \beta_2 y_2).$$

Lemma 2.1. *If Φ is a concave IVF on a convex set $\mathscr{Y} \subseteq \mathbb{R}^n$, then $\underline{\phi}$ and $\overline{\phi}$ are concave on $\mathscr{Y}$ and vice-versa.*

Proof. The proof is similar to the proof of [29, Proposition 6.1]. □

Example 2.1. *Let $\mathscr{Y}$ be the Euclidean space $\mathbb{R}^n$. Then, the IVF $\Phi : \mathscr{Y} \to I(\mathbb{R})$ which is defined by*

$$\Phi(y) = \widehat{\boldsymbol{M}}^{\top} \odot y \ominus_{gH} \|y\|, \text{ where } \widehat{\boldsymbol{M}} = (\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots, \boldsymbol{M}_n) \in \boldsymbol{I}(\mathbb{R})^n,$$

and for all $y = (y_1, y_2, \ldots, y_n) \in \mathscr{Y}$ is a concave IVF on $\mathscr{Y}$. The reason is as follows.

Without loss of generality, the first p components of y are assumed to be non-negative, and the rest $n-p$ be negative. Then, letting $\boldsymbol{M}_i = [\underline{m}_i, \overline{m}_i]$ for all $i = 1, 2, \ldots, n$,

$$\Phi(y) = \bigoplus_{i=1}^{p} [\underline{m}_i y_i, \overline{m}_i y_i] \oplus \bigoplus_{j=p+1}^{n} [\overline{m}_j y_j, \underline{m}_j y_j] \ominus_{gH} \|y\|.$$

It is evident that $\sum_{i=1}^{p} \underline{m}_i y_i + \sum_{j=p+1}^{n} \overline{m}_j y_j$ and $\sum_{i=1}^{p} \overline{m}_i y_i + \sum_{j=p+1}^{n} \underline{m}_j y_j$, being linear, are concave functions. Also, $-\|y\|$ is a concave function. Therefore, $\sum_{i=1}^{p} \underline{m}_i y_i + \sum_{j=p+1}^{n} \overline{m}_j y_j - \|y\|$ and $\sum_{i=1}^{p} \overline{m}_i y_i + \sum_{j=p+1}^{n} \underline{m}_j y_j - \|y\|$ are concave functions. Hence, by Lemma 2.1, Φ is a concave IVF.

Definition 2.4. (gH-continuity [12]). *An IVF Φ is said to be gH-*continuous *at $u \in \mathscr{Y}$ if $\lim_{\|d\| \to 0} (\Phi(u+d) \ominus_{gH} \Phi(u)) = \boldsymbol{0}$. If at every $u \in \mathscr{Y}$, Φ is gH-continuous, then Φ is called gH-*continuous *on $\mathscr{Y}$.*

Lemma 2.2. (See [14]). *For a gH-continuous IVF Φ, its $\underline{\phi}$ and $\overline{\phi}$ are continuous and vice-versa.*

Definition 2.5. (gH-derivative [4]). *Let $\mathscr{Y} \subseteq \mathbb{R}^n$. The gH-*derivative *of an IVF $\Phi : \mathscr{Y} \to I(\mathbb{R})$ at $u \in \mathscr{Y}$ is the limit*

$$\Phi'(u) := \lim_{d \to 0} \tfrac{1}{d} \odot \left\{\Phi(u+d) \ominus_{gH} \Phi(u)\right\}.$$

Definition 2.6. (gH-Gáteaux derivative [13]). *Let an IVF Φ be defined on a nonempty open subset $\mathscr{Y}$ of $\mathbb{R}^n$. Then, Φ is known to be gH-*Gáteaux differentiable *with gH-*Gáteaux derivative *$\Phi_{\mathscr{G}}(u)$ at $u \in \mathscr{Y}$ if the following limit*

$$\Phi_{\mathscr{G}}(u)(h) := \lim_{\beta \to 0+} \frac{1}{\beta} \odot (\Phi(u+\beta h) \ominus_{gH} \Phi(u))$$

is finite for all $h \in \mathbb{R}^n$ and $\Phi_{\mathscr{G}}(u)$ is a gH-continuous and linear IVF from $\mathbb{R}^n$ to $I(\mathbb{R})$.

Definition 2.7. (gH-Fréchet derivative [13]). *Let an IVF Φ be defined on a nonempty open subset $\mathscr{Y}$ of $\mathbb{R}^n$. Then, Φ is said to be gH-*Fréchet differentiable *at $u \in \mathscr{Y}$ if there exists a gH-continuous and linear mapping $\boldsymbol{G} : \mathscr{Y} \to I(\mathbb{R})$ such that*

$$\lim_{\|h\| \to 0} \tfrac{1}{\|h\|} \odot \left(\|\Phi(u+h) \ominus_{gH} \Phi(u) \ominus_{gH} \boldsymbol{G}(h)\|_{I(\mathbb{R})}\right) = 0,$$

where $\boldsymbol{G}$ will be referred to as $\Phi_{\mathscr{F}}(u)$.

Definition 2.8. (Efficient point [13]). *Let $\mathscr{Y} \subseteq \mathbb{R}^n$ and $\Phi : \mathbb{R}^n \to I(\mathbb{R})$ be an IVF. A point $u \in \mathscr{Y}$ is said to be an* efficient point *of the IVF $\Phi : \mathscr{Y} \to I(\mathbb{R})$ if $\Phi(y) \not\prec \Phi(u)$ for all $y \in \mathscr{Y}$.*

Definition 2.9. (Weak efficient point [1]). *Let $\mathscr{Y} \subseteq \mathbb{R}^n$ and $\Phi : \mathbb{R}^n \to I(\mathbb{R})$ be an IVF. A point $u \in \mathscr{Y}$ is said to be a* weak efficient point *of the IVF $\Phi : \mathscr{Y} \to I(\mathbb{R})$ if $\Phi(u) \preceq \Phi(y)$ for all $y \in \mathscr{Y}$.*

2.3. **Few properties of the elements in $I(\mathbb{R})$.** Let $\mathbf{Y} = [\underline{y}, \overline{y}]$ and $\widehat{\mathbf{Y}} = (\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n)$ be elements in $I(\mathbb{R})$ and $I(\mathbb{R})^n$, respectively. The following two functions $\|\cdot\|_{I(\mathbb{R})} : I(\mathbb{R}) \to \mathbb{R}_+$ and $\|\cdot\|_{I(\mathbb{R})^n} : I(\mathbb{R})^n \to \mathbb{R}_+$ are referred to as norm [19, 20] on $I(\mathbb{R})$ and $I(\mathbb{R})^n$, respectively:

$$\|\mathbf{Y}\|_{I(\mathbb{R})} = \max\{|\underline{y}|, |\overline{y}|\}, \text{ and } \|\widehat{\mathbf{Y}}\|_{I(\mathbb{R})^n} = \sum_{j=1}^{n} \|\mathbf{Y}_j\|_{I(\mathbb{R})}.$$

Lemma 2.3. *For any $\boldsymbol{W}, \boldsymbol{Y}, \boldsymbol{Z} \in I(\mathbb{R})$ and $\varepsilon \geq 0$, we have*

$$\varepsilon \preceq (\boldsymbol{W} \ominus_{gH} \boldsymbol{Y}) \ominus_{gH} \boldsymbol{Z} \implies \boldsymbol{Z} \oplus \varepsilon \preceq \boldsymbol{W} \ominus_{gH} \boldsymbol{Y}.$$

Proof. See Appendix A. □

Lemma 2.4. *For any $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W} \in I(\mathbb{R})$, we have*

$$(\boldsymbol{X} \oplus \boldsymbol{Y}) \ominus_{gH} (\boldsymbol{Z} \oplus \boldsymbol{W}) \subseteq (\boldsymbol{X} \ominus_{gH} \boldsymbol{Z}) \oplus (\boldsymbol{Y} \ominus_{gH} \boldsymbol{W}).$$

Proof. See Appendix B. □

Lemma 2.5. *For any $\boldsymbol{W}, \boldsymbol{Y}, \boldsymbol{Z} \in I(\mathbb{R})$,*

$$\boldsymbol{0} \ominus_{gH} \{((-1 \odot \boldsymbol{W}) \ominus_{gH} (-1 \odot \boldsymbol{Y})) \ominus_{gH} (-1 \odot \boldsymbol{Z})\} = ((\boldsymbol{W} \ominus_{gH} \boldsymbol{Y}) \ominus_{gH} \boldsymbol{Z}).$$

Proof. See Appendix C.

□

Lemma 2.6. *For all $\boldsymbol{X}, \boldsymbol{Y}$, and $\boldsymbol{Z}$ of $I(\mathbb{R})$,*

(i) if $\boldsymbol{0} \preceq \boldsymbol{X} \ominus_{gH} \boldsymbol{Y}$, then $\boldsymbol{0} \ominus_{gH} \boldsymbol{Z} \preceq (\boldsymbol{X} \ominus_{gH} \boldsymbol{Y}) \ominus_{gH} \boldsymbol{Z}$,
(ii) if $\boldsymbol{Z} \preceq \boldsymbol{X} \ominus_{gH} \boldsymbol{Y}$, then $\boldsymbol{Z} \ominus_{gH} \boldsymbol{W} \preceq (\boldsymbol{X} \ominus_{gH} \boldsymbol{Y}) \ominus_{gH} \boldsymbol{W}$ for all $\boldsymbol{W} \in I(\mathbb{R})$,
(iii) if $\boldsymbol{X} \ominus_{gH} \boldsymbol{Y} \preceq [L, L]$, then $[-L, -L] \preceq \boldsymbol{Y} \ominus_{gH} \boldsymbol{X}$, where $L \in \mathbb{R}$,
(iv) if $[-\gamma, -\gamma] \preceq \boldsymbol{X} \ominus_{gH} \boldsymbol{Y}$, then $\boldsymbol{Y} \ominus_{gH} [\gamma, \gamma] \preceq \boldsymbol{X}$, where $\gamma \in \mathbb{R}$, and
(v) if $\boldsymbol{Z} \preceq \boldsymbol{X} \oplus \boldsymbol{Y}$, then $\boldsymbol{Z} \ominus_{gH} \boldsymbol{Y} \preceq \boldsymbol{X}$.

Proof. See Appendix D.

□

Definition 2.10. (Sequence in $I(\mathbb{R})^n$[11]). *A function $\widehat{\boldsymbol{\Phi}} : \mathbb{N} \to I(\mathbb{R})^n$ is called a sequence in $I(\mathbb{R})^n$, where $\mathbb{N}$ is the set of all natural numbers.*

Definition 2.11. (Closed set in $I(\mathbb{R})^n$[1]). *A nonempty subset $\mathscr{U} \subseteq I(\mathbb{R})^n$ is known to be* closed *if for every convergent sequence $\{\widehat{\boldsymbol{M}}_k\}$ in $\mathscr{U}$ converging to $\widehat{\boldsymbol{M}}$, $\widehat{\boldsymbol{M}}$ must belong to $\mathscr{U}$.*

Definition 2.12. (Closure of a set in $I(\mathbb{R})^n$). *Let $\mathscr{Y} \subseteq I(\mathbb{R})^n$. The intersection of all closed sets containing $\mathscr{Y}$ is called the closure of $\mathscr{Y}$, abbreviated by $cl(\mathscr{Y})$.*

Definition 2.13. (Convergent sequence in $I(\mathbb{R})^n$[11]). *Let $\{\widehat{\boldsymbol{M}}_k\}$ be a sequence in $I(\mathbb{R})^n$. If there exists $\widehat{\boldsymbol{M}} \in I(\mathbb{R})^n$ for which for any $\varepsilon > 0$ there exists $p \in \mathbb{N}$ such that $\|\widehat{\boldsymbol{M}}_k \ominus_{gH} \widehat{\boldsymbol{M}}\|_{I(\mathbb{R})^n} < \varepsilon$ for all $k \geq p$, then $\{\widehat{\boldsymbol{M}}_k\}$ is said to be convergent and converges to $\widehat{\boldsymbol{M}}$.*

Remark 2.1. *It is to note that if a sequence $\{\widehat{\boldsymbol{M}}_k\} = (\boldsymbol{M}_{k1}, \boldsymbol{M}_{k2}, \ldots, \boldsymbol{M}_{kn})^\top$ in $I(\mathbb{R})^n$ converges to $\widehat{\boldsymbol{M}} = (\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots, \boldsymbol{M}_n)^\top \in I(\mathbb{R})^n$, then by the definition of norm on $I(\mathbb{R})^n$, the sequence $\boldsymbol{M}_{kj}$ in $I(\mathbb{R})$ converges to $\boldsymbol{M}_j \in I(\mathbb{R})$ for all $j = 1, 2, \ldots, n$. Also, according to the definition of norm on $I(\mathbb{R})$, the sequences $\{\underline{m}_{kj}\}$ and $\{\overline{m}_{kj}\}$ in $\mathbb{R}$ converge to $\{\underline{m}_j\}$ and $\{\overline{m}_j\}$, respectively, for all j.*

Definition 2.14. (Infimum and supremum of a subset of $\overline{I(\mathbb{R})}$ [17]).

Let $\mathscr{U} \subseteq \overline{I(\mathbb{R})}$. We call an interval $\mathbf{X} \in I(\mathbb{R})$ a lower bound (respectively, an upper bound) of $\mathscr{U}$ if $\boldsymbol{U} \in \mathscr{U}$ implies $\mathbf{X} \preceq \boldsymbol{U}$ (respectively, $\mathbf{U} \preceq \boldsymbol{X}$).
A lower bound $\mathbf{X}$ of $\mathscr{U}$ is called infimum *of $\mathscr{U}$, denoted by* $\inf \mathscr{U}$*, if for any lower bound $\mathbf{Z}$ of $\mathscr{U}$, $\mathbf{Z} \preceq \mathbf{X}$.*
An upper bound $\mathbf{X}$ of $\mathscr{U}$ is called supremum *of $\mathscr{U}$, denoted by* $\sup \mathscr{U}$*, if for any upper bound $\mathbf{Z}$ of $\mathscr{U}$, $\mathbf{X} \preceq \mathbf{Z}$.*

Remark 2.2. [17] *Let $\mathscr{S} = \left\{[a_\mu, b_\mu] \in \overline{I(\mathbb{R})} : \mu \in \Lambda \text{ and } \Lambda \text{ being an index set}\right\}$. Then, by Definition 2.14, it follows that* $\inf \mathscr{S} = \left[\inf_{\mu\in\Lambda} a_\mu,\ \inf_{\mu\in\Lambda} b_\mu\right]$ and $\sup \mathscr{S} = \left[\sup_{\mu\in\Lambda} a_\mu,\ \sup_{\mu\in\Lambda} b_\mu\right]$.

3. gH-Weak Subdifferential Calculus for IVFs

In this section, we introduce the ideas of gH-weak subgradient and gH-weak subdifferential for IVFs. Some properties of gH-weak subdifferential and inclusion for sum rule are provided. Its relation with gH-Fréchet lower subdifferential is also discussed.

Definition 3.1. (gH-weak subdifferential). *Let $\emptyset \neq \mathscr{Y} \subseteq \mathbb{R}^n$ and Φ be an IVF defined on $\mathscr{Y}$. A pair $(\widehat{\boldsymbol{G}^w}, c) \in I(\mathbb{R})^n \times \mathbb{R}_+$ is said to be a gH-weak subgradient of Φ at $u \in \mathscr{Y}$ if, for every $y \in \mathscr{Y}$,*

$$\widehat{\boldsymbol{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \Phi(y) \ominus_{gH} \Phi(u). \tag{3.1}$$

The set of all gH-weak subgradients of Φ at $u \in \mathscr{Y}$, i.e.,

$$\partial^w \Phi(u) = \left\{(\widehat{\boldsymbol{G}^w}, c) \in I(\mathbb{R})^n \times \mathbb{R}_+ : \widehat{\boldsymbol{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \Phi(y) \ominus_{gH} \Phi(u)\ \forall\, y \in \mathscr{Y}\right\}$$

is said to be gH-weak subdifferential of Φ at $u \in \mathscr{Y}$.

Example 3.1. *Let an IVF $\Phi : [-1,1] \to I(\mathbb{R})$ be defined by $\Phi(y) = \left[y^2, |y|\right]$, where $y \in [-1,1]$.*

Let us compute the gH-weak subdifferential of Φ at 0 and 1, i.e., $\partial^w\Phi(0)$ and $\partial^w\Phi(1)$, respectively. Note that

$$\begin{aligned}\partial^w\Phi(0) &= \left\{(\mathbf{G}_1^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : \mathbf{G}_1^w \odot y \ominus_{gH} c|y| \preceq \left[y^2, |y|\right]\ \forall\, y \in [-1,1]\right\}\\ &= \left\{\left([\underline{g_1^w}, \overline{g_1^w}], c\right) \in I(\mathbb{R}) \times \mathbb{R}_+ : [\underline{g_1^w}, \overline{g_1^w}] \odot y \ominus_{gH} c|y| \preceq \left[y^2, |y|\right]\ \forall\, y \in [-1,1]\right\},\end{aligned}$$

which yields the following two cases corresponding to $y \in [0,1]$ and $y \in [-1,0]$.

(i)

$$\begin{aligned}\partial^w\Phi(0) &= \left\{\left([\underline{g_1^w}, \overline{g_1^w}], c\right) \in I(\mathbb{R}) \times \mathbb{R}_+ : [\underline{g_1^w}, \overline{g_1^w}] \odot y \ominus_{gH} c|y| \preceq \left[y^2, |y|\right]\ \forall\, y \in [0,1]\right\}\\ &= \left\{\left([\underline{g_1^w}, \overline{g_1^w}], c\right) \in I(\mathbb{R}) \times \mathbb{R}_+ : \underline{g_1^w} y - cy \leq y^2 \text{ and } \overline{g_1^w} y - cy \leq y\ \forall\, y \in [0,1]\right\}\\ &= \left\{\left([\underline{g_1^w}, \overline{g_1^w}], c\right) \in I(\mathbb{R}) \times \mathbb{R}_+ : \underline{g_1^w} - c \leq 0 \text{ and } \overline{g_1^w} - c \leq 1\right\}.\end{aligned}$$

(ii) Likewise,

$$\partial^w\Phi(0) = \left\{\left([\underline{g_1^w}, \overline{g_1^w}], c\right) \in I(\mathbb{R}) \times \mathbb{R}_+ : -1 \leq \underline{g_1^w} + c \text{ and } 0 \leq \overline{g_1^w} + c\right\}.$$

Hence, by combining Case (i) and Case (ii), we obtain

$$\partial^w \Phi(0) = \left\{ (\mathbf{G}_1^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : [-1-c, -c] \preceq \mathbf{G}_1^w \preceq [c, 1+c] \right\}.$$

Similarly,

$$\partial^w \Phi(1) = \left\{ (\mathbf{G}_2^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : [1-c, 2-c] \preceq \mathbf{G}_2^w \right\}.$$

Remark 3.1. *To understand the geometric interpretation of the gH-weak subdifferential of an IVF Φ, let $(\widehat{\boldsymbol{G}^w}, c) \in \partial^w \Phi(u)$. This means that $(\widehat{\boldsymbol{G}^w}, c) \in I(\mathbb{R})^n \times \mathbb{R}_+$, for every $c \geq 0$, is a gH-weak subgradient of Φ at $u \in \mathscr{Y}$ if and only if there exists a concave and gH-continuous IVF $\boldsymbol{H} : \mathscr{Y} \to I(\mathbb{R})$, which is defined by $\boldsymbol{H}(y) = \Phi(u) \oplus \widehat{\boldsymbol{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \ \forall\, y \in \mathscr{Y}$, that satisfies*

$$(\forall\, y \in \mathscr{Y})\ \boldsymbol{H}(y) \preceq \Phi(y) \text{ and } \boldsymbol{H}(u) = \Phi(u).$$

This condition shows that $\boldsymbol{H}$ must intersect Φ at least at the point $(u, \Phi(u))$ from bottom. Hence, it concludes that if Φ is gH-weak subdifferentiable at u and $(\widehat{\boldsymbol{G}^w}, c) \in \partial^w \Phi(u)$, then the graph of IVF $\boldsymbol{H}$, that is,

$$Gr(\boldsymbol{H}) = \{(y, \boldsymbol{Y}) \in \mathscr{Y} \times I(\mathbb{R}) : \boldsymbol{Y} = \boldsymbol{H}(y)\}$$

always lie below the epigraph of Φ, i.e.,

$$Epi(\Phi) = \{(y, \boldsymbol{Y}) \in \mathscr{Y} \times I(\mathbb{R}) : \Phi(y) \preceq \boldsymbol{Y}\},$$

such that

$$Epi(\Phi) \subset Epi(\boldsymbol{H}) \text{ and } cl(Epi(\Phi)) \bigcap Gr(\boldsymbol{H}) \text{ is nonempty.}$$

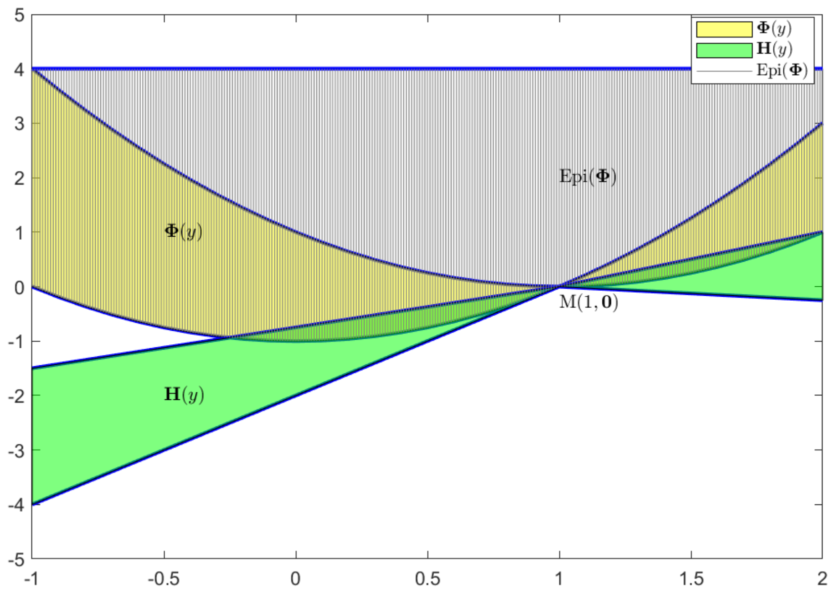


FIGURE 1. Geometrical representation of two possible gH-Dini Hadamard ε-subgradients of Ψ of Example 3.1

For example, Let $\mathscr{Y} = [-1, 2]$. Consider an IVF $\mathbf{\Phi} : \mathscr{Y} \to I(\mathbb{R})$ which is given by

$$\Phi(y) = \begin{cases} \left[y^2 - 1, (y-1)^2\right], & \text{if } y \in [-1, 1] \\ \left[(y-1)^2, y^2 - 1\right], & \text{if } y \in (1, 2]. \end{cases}$$

The gH-weak subdifferential of Φ at $u=1$ is

$$\partial^w\Phi(1)=\{(\mathbf{G}^w,c)\in I(\mathbb{R})\times\mathbb{R}_+:[-c,2-c]\preceq\mathbf{G}^w\preceq[c,2+c]\}.$$

For instance, $(\mathbf{G}^w,c)=([0.25,1.5],0.5)\in\partial^w\Phi(1)$, geometrically indicates that the IVF

$$\mathbf{H}(y)=\Phi(1)\oplus[0.25,1.5]\odot(y-1)\ominus_{gH}0.5|y-1|$$

intersects

$$\text{Epi}(\Phi)=\{(y,\mathbf{4})\in\mathscr{Y}\times\mathbb{R}:\Phi(y)\preceq\mathbf{4}\}$$

at the point $\text{M}(1,\mathbf{0})$ from below as shown in Figure 1. We also observe from the figure that

$$\text{Epi}(\Phi)\subset\text{Epi}(\mathbf{H}),\ \text{and}\ cl(\text{Epi}(\Phi))\bigcap Gr(\mathbf{H})\ \text{is nonempty.}$$

Theorem 3.1. (Convexity of gH-weak subdifferential). *Let $\mathscr{Y}\subset\mathbb{R}^n$. Let the gH-weak subdifferential of $\Phi:\mathscr{Y}\to I(\mathbb{R})$ at u be nonempty. Then, $\partial^w\Phi(u)$ is convex.*

Proof. Let $(\widehat{\mathbf{G}_1^w},c_1)$ and $(\widehat{\mathbf{G}_2^w},c_2)\in\partial^w\Phi(u)$, where

$$\widehat{\mathbf{G}_1^w}=(\mathbf{G}_{11}^w,\mathbf{G}_{12}^w,\ldots,\mathbf{G}_{1n}^w)^\top$$

and

$$\widehat{\mathbf{G}_2^w}=(\mathbf{G}_{21}^w,\mathbf{G}_{22}^w,\ldots,\mathbf{G}_{2n}^w)^\top.$$

Let $\beta\in[0,1]$. From the definition of $\partial^w\Phi(u)$, we have

$$\widehat{\mathbf{G}_1^w}^\top\odot(y-u)\ominus_{gH}c_1\|y-u\|\preceq\Phi(y)\ominus_{gH}\Phi(u)\ \text{and}\tag{3.2}$$

$$\widehat{\mathbf{G}_2^w}^\top\odot(y-u)\ominus_{gH}c_2\|y-u\|\preceq\Phi(y)\ominus_{gH}\Phi(u),\tag{3.3}$$

for all $y\in\mathscr{Y}$. Up to a rearrangement of terms, let the first m components of $(y-u)$ be non-negative, and the rest be negative. Then, from inequalities (3.2) and (3.3), we have

$$\bigoplus_{i=1}^m(y_i-u_i)\odot\mathbf{G}_{1i}^w\bigoplus_{j=m+1}^n(y_j-u_j)\odot\mathbf{G}_{1j}^w\ominus_{gH}c_1\|y-u\|\preceq\Phi(y)\ominus_{gH}\Phi(u)$$

and

$$\bigoplus_{i=1}^m(y_i-u_i)\odot\mathbf{G}_{2i}^w\bigoplus_{j=m+1}^n(y_j-u_j)\odot\mathbf{G}_{2j}^w\ominus_{gH}c_2\|y-u\|\preceq\Phi(y)\ominus_{gH}\Phi(u).$$

Thus,

$$\bigoplus_{i=1}^m\beta\odot((y_i-u_i)\odot\mathbf{G}_{1i}^w)\bigoplus_{j=m+1}^n\beta\odot((y_j-u_j)\odot\mathbf{G}_{1j}^w)\ominus_{gH}\beta c_1\|y-u\|\preceq\beta\odot(\Phi(y)\ominus\Phi(u))\tag{3.4}$$

and

$$\bigoplus_{i=1}^m(1-\beta)\odot((y_i-u_i)\odot\mathbf{G}_{2i}^w)\bigoplus_{j=m+1}^n(1-\beta)\odot((y_j-u_j)\odot\mathbf{G}_{2j}^w)\ominus_{gH}(1-\beta)c_2\|y-u\|$$
$$\preceq(1-\beta)\odot(\Phi(y)\ominus\Phi(u)).\tag{3.5}$$

By adding (3.4) and (3.5), we obtain

$$\bigoplus_{i=1}^m(y_i-u_i)\odot\{\beta\odot\mathbf{G}_{1i}^w\oplus(1-\beta)\odot\mathbf{G}_{2i}^w\}\bigoplus_{j=m+1}^n(y_j-u_j)\odot\{\beta\odot\mathbf{G}_{1j}^w\oplus(1-\beta)\odot\mathbf{G}_{2j}^w\}$$
$$\ominus_{gH}(\beta c_1\oplus(1-\beta)c_2)\|y-u\|\preceq\Phi(y)\ominus_{gH}\Phi(u).\tag{3.6}$$

Therefore, we have

$$\{\beta \odot \widehat{\mathbf{G}_1^w} \oplus (1-\beta) \odot \widehat{\mathbf{G}_2^w}\}^\top \odot (y-u) \ominus_{gH} (\beta c_1 \oplus (1-\beta)c_2)\|y-u\| \preceq \Phi(y) \ominus_{gH} \Phi(u),$$

i.e., $(\beta \odot \widehat{\mathbf{G}_1^w} \oplus (1-\beta) \odot \widehat{\mathbf{G}_2^w},\ \beta c_1 \oplus (1-\beta)c_2) \in \partial^w \Phi(u)$, which proves the convexity of $\partial^w \Phi(u)$. □

Theorem 3.2. (Closedness of gH-weak subdifferential). *Let $\emptyset \neq \mathscr{Y} \subseteq I(\mathbb{R})^n$. If for an IVF $\Psi : \mathscr{Y} \to I(\mathbb{R})$, the set $\partial^w \Psi(u)$ is nonempty at $u \in \mathscr{Y}$, then $\partial^w \Psi(u)$ is closed.*

Proof. Let $\{(\widehat{\mathbf{G}_k^w}, c_k)\}$ be an arbitrary sequence in $\partial^w \Psi(u)$ converging to $(\widehat{\mathbf{G}^w}, c) \in I(\mathbb{R})^n \times \mathbb{R}_+$, where $\widehat{\mathbf{G}_k^w} = (\mathbf{G}_{k1}^w, \mathbf{G}_{k2}^w, \ldots, \mathbf{G}_{kn}^w)^\top$ and $\widehat{\mathbf{G}^w} = (\mathbf{G}_1^w, \mathbf{G}_2^w, \ldots, \mathbf{G}_n^w)^\top$. Since $(\widehat{\mathbf{G}_k^w}, c) \in \partial^w \Psi(u)$ for all $d \in \mathscr{Y}$, we obtain $\widehat{\mathbf{G}_k^w}^\top \odot d \ominus_{gH} c_k\|d\| \preceq \Psi(u+d) \ominus_{gH} \Psi(u)$, which implies

$$\bigoplus_{i=1}^{n} d_i \odot \mathbf{G}_{ki}^w \ominus_{gH} c_k\|d\| \preceq \Psi(u+d) \ominus_{gH} \Psi(u). \tag{3.7}$$

Up to a rearrangement of terms, let the first p components of d be non-negative, and the rest be negative. Then, from (3.7), we have

$$\bigoplus_{i=1}^{p} d_i \odot \mathbf{G}_{ki}^w \bigoplus_{j=p+1}^{n} d_j \odot \mathbf{G}_{kj}^w \ominus_{gH} c_k\|d\| \preceq \Psi(u+d) \ominus_{gH} \Psi(u)$$

$$\Longrightarrow \bigoplus_{i=1}^{p} d_i \odot [\underline{g_{ki}^w}, \overline{g_{ki}^w}] \bigoplus_{j=p+1}^{n} d_j \odot [\underline{g_{kj}^w}, \overline{g_{kj}^w}] \ominus_{gH} c_k\|d\| \preceq \Psi(u+d) \ominus_{gH} \Psi(u).$$

Therefore,

$$\sum_{i=1}^{p} \underline{g_{ki}^w} d_i + \sum_{j=p+1}^{n} \overline{g_{kj}^w} d_j - c_k\|d\| \leq \min\left\{\underline{\Psi}(u+d) - \underline{\Psi}(u), \overline{\Psi}(u+d) - \overline{\Psi}(u)\right\} \tag{3.8}$$

and

$$\sum_{i=1}^{p} \overline{g_{ki}^w} d_i + \sum_{j=p+1}^{n} \underline{g_{kj}^w} d_j - c_k\|d\| \leq \max\left\{\underline{\Psi}(u+d) - \underline{\Psi}(u), \overline{\Psi}(u+d) - \overline{\Psi}(u)\right\}. \tag{3.9}$$

Since sequence $\widehat{\mathbf{G}_k^w}$ converges to $\widehat{\mathbf{G}^w}$, and sequences $\{\underline{g_{ki}^w}\}$ and $\{\overline{g_{ki}^w}\}$ converge to $\{\underline{g_i^w}\}$ and $\{\overline{g_i^w}\}$, respectively for all i. Thus, by (3.8) and (3.9), we have

$$\begin{aligned}\sum_{i=1}^{p} \underline{g_{ki}^w} d_i + \sum_{j=p+1}^{n} \overline{g_{kj}^w} d_j - c_k\|d\| &\to \sum_{i=1}^{p} \underline{g_i^w} d_i + \sum_{j=p+1}^{n} \overline{g_j^w} d_j - c\|d\| \\ &\leq \min\left\{\underline{\Psi}(u+d) - \underline{\Psi}(u), \overline{\Psi}(u+d) - \overline{\Psi}(u)\right\}\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{p} \overline{g_{ki}^w} d_i + \sum_{j=p+1}^{n} \underline{g_{kj}^w} d_j - c_k\|d\| &\to \sum_{i=1}^{p} \overline{g_i^w} d_i + \sum_{j=p+1}^{n} \underline{g_j^w} d_j - c\|d\| \\ &\leq \max\left\{\underline{\Psi}(u+d) - \underline{\Psi}(u), \overline{\Psi}(u+d) - \overline{\Psi}(u)\right\}.\end{aligned}$$

Hence, for any $u \in \mathscr{Y}$,

$$\left[\sum_{i=1}^{p} \underline{g_i^w} d_i + \sum_{j=p+1}^{n} \overline{g_j^w} d_j - c\|d\|, \sum_{i=1}^{p} \overline{g_i^w} d_i + \sum_{j=p+1}^{n} \underline{g_j^w} d_j - c\|d\|\right] \preceq \Psi(u+d) \ominus_{gH} \Psi(u)$$

$$\implies \bigoplus_{i=1}^{p} [\underline{g_i^w} d_i, \overline{g_i^w} d_i] \bigoplus_{j=p+1}^{n} [\overline{g_j^w} d_j, \underline{g_j^w} d_j] \ominus_{gH} c\|d\| \preceq \Psi(u+d) \ominus_{gH} \Psi(u)$$

$$\implies \bigoplus_{i=1}^{p} d_i \odot \mathbf{G}_i^w \bigoplus_{j=p+1}^{n} d_j \odot \mathbf{G}_j^w \ominus_{gH} c\|d\| \preceq \Psi(u+d) \ominus_{gH} \Psi(u)$$

$$\implies \widehat{\mathbf{G}^w}^\top \odot d \ominus_{gH} c\|d\| \preceq \Psi(u+d) \ominus_{gH} \Psi(u).$$

Therefore, $\widehat{\mathbf{G}^w} \in \partial^w \Psi(u)$, and hence $\partial^w \Psi(u)$ is closed. □

Definition 3.2. (gH-Fréchet lower subdifferential). *Let $\Phi : \mathscr{Y} \to I(\mathbb{R}) \cup \{-\infty, +\infty\}$ be an IVF that is finite at an $u \in \mathscr{Y}$. Then, the gH-Fréchet lower subdifferential of Φ at u is defined by*

$$\partial_{\mathscr{F}}^{-} \Phi(u) = \Big\{ \widehat{\boldsymbol{G}} : \boldsymbol{0} \preceq \liminf_{\substack{y \to u \\ y \neq u}} \frac{1}{\|y-u\|} \odot \{\Phi(y) \ominus_{gH} \Phi(u) \ominus_{gH} \widehat{\boldsymbol{G}}^\top \odot (y-u)\},$$
$$\text{where } \widehat{\boldsymbol{G}} : \mathscr{Y} \to I(\mathbb{R}) \text{ is a } gH\text{-continuous and linear IVF} \Big\}.$$

One important fact is that gH-weak subdifferential is an immediate consequence of gH-Fréchet lower subdifferential.

Theorem 3.3. *Let $\emptyset \neq \mathscr{Y} \subseteq \mathbb{R}^n$. If $\Phi : \mathscr{Y} \to I(\mathbb{R})$ has gH-Fréchet lower subdifferential $\widehat{\boldsymbol{G}}$ at the point u, then $(\widehat{\boldsymbol{G}}, \varepsilon)$ is a gH-weak subgradient of Φ at u for any $\varepsilon \in \mathbb{R}_+$.*

Proof. Let $\widehat{\mathbf{G}} \in \partial_{\mathscr{F}}^{-} \Phi(u)$. Due to Definition 3.2, we can write

$$\boldsymbol{0} \preceq \liminf_{\substack{y \to u \\ y \neq u}} \frac{1}{\|y-u\|} \odot \{\Phi(y) \ominus_{gH} \Phi(u) \ominus_{gH} \widehat{\mathbf{G}}^\top \odot (y-u)\}.$$

Then, for the $\varepsilon > 0$ in the hypothesis there exists $\delta > 0$ such that

$$-\varepsilon\|y-u\| \preceq \Phi(y) \ominus_{gH} \Phi(u) \ominus_{gH} \widehat{\mathbf{G}}^\top \odot (y-u) \ \forall\, y \in B_\delta(u),$$

Then, from Lemma 2.3, we have $\widehat{\mathbf{G}}^\top \odot (y-u) \ominus_{gH} \varepsilon\|y-u\| \preceq \Phi(y) \ominus_{gH} \Phi(u)$. By Definition 3.1, $(\widehat{\mathbf{G}}, \varepsilon)$ is a gH-weak subdifferential of Φ at u. □

Lemma 3.1. *For any $y \in \mathbb{R}^n$ and $\widehat{\boldsymbol{C}} = (\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \ldots, \boldsymbol{C}_n) \in I(\mathbb{R})^n$, $-\|y\|\|\widehat{\boldsymbol{C}}\|_{I(\mathbb{R})^n} \preceq \|y^\top \odot \widehat{\boldsymbol{C}}\|_{I(\mathbb{R})}$.*

Proof. See Appendix E. □

To investigate the class of interval-valued functions for which weak subgradients always exist, we need the following definition.

Definition 3.3. (gH-lower Lipschitz IVF). *Let $\emptyset \neq \mathscr{Y} \subseteq \mathbb{R}^n$. An IVF $\Phi : \mathscr{Y} \to \overline{I(\mathbb{R})}$ is called gH-lower locally Lipschitz at $u \in \mathscr{Y}$ if $\exists\, L \geq 0$ and a neighbourhood $\mathscr{N}(u)$ of u such that*

$$-L\|y-u\| \preceq \Phi(y) \ominus_{gH} \Phi(u) \ \forall\, y \in \mathscr{N}(u). \tag{3.10}$$

If the inequality (3.10) satisfies for all $y \in \mathscr{Y}$, then Φ is called gH-lower Lipschitz at $u \in \mathscr{Y}$ with Lipschitz constant L.

Example 3.2. *Let $\Phi : [1,\infty) \to I(\mathbb{R})$ be an IVF, and defined by $\Phi(y) = \ln y \odot \boldsymbol{C}$ for all $y \in [1,\infty)$, where $\boldsymbol{0} \preceq \boldsymbol{C} = [\underline{c}, \overline{c}]$. Let $\delta > 0$. We choose the neighbourhood of u, $\mathcal{N}_\delta(u) = \{y : |y-u| < \delta\}$. If $0 < y-u < \delta$, then $u < y$ and also then $\frac{y}{u} > 1$ and then*

$$0 < \ln\frac{y}{u} < \frac{y}{u} - 1, \text{ since } \ln(1+p) < p \text{ if } p > 0$$
$$\leq y-u. \tag{3.11}$$

Since $\underline{c}, \overline{c} \geq 0$, we have $(\ln y - \ln u)\underline{c} \leq (y-u)\underline{c}$ and $(\ln y - \ln u)\overline{c} \leq (y-u)\overline{c}$. Then,

$$(\ln y - \ln u) \odot \boldsymbol{C} \preceq (y-u) \odot \boldsymbol{C}. \tag{3.12}$$

If $-\delta < y-u < 0$, then $y < u$ and also then $\frac{u}{y} > 1$ and then

$$0 < \ln\frac{u}{y} < \frac{u}{y} - 1, \text{ since } \ln(1+p) < p \text{ if } p > 0$$
$$\leq u-y. \tag{3.13}$$

Then, $(\ln u - \ln y) \odot \boldsymbol{C} \preceq (u-y) \odot \boldsymbol{C}$, which together with (3.12) yields that

$$|\ln y - \ln u| \odot \boldsymbol{C} \preceq |y-u| \odot \boldsymbol{C}$$
$$\implies \ln u \odot \boldsymbol{C} \ominus_{gH} \ln y \odot \boldsymbol{C} \preceq |y-u| \odot \boldsymbol{C}$$
$$\implies -|y-u| \odot \boldsymbol{C} \preceq \ln y \odot \boldsymbol{C} \ominus_{gH} \ln u \odot \boldsymbol{C}$$
$$\implies -\overline{c}|y-u| \preceq \Phi(y) \ominus_{gH} \Phi(u).$$

This shows that Φ is gH-lower locally Lipschitz on $\mathcal{N}_\delta(u)$ with $L = \overline{c}$. From arbitrariness of y, u in $[1,\infty)$, we conclude that Φ is gH-lower Lipschitz on $[1,\infty)$.

Theorem 3.4. *Let $\emptyset \neq \mathscr{Y} \subseteq \mathbb{R}^n$. Let $\Phi : \mathscr{Y} \to \overline{I(\mathbb{R})}$ be an IVF, where $\Phi(u)$ is finite for some $u \in \mathscr{Y}$. Then, the following three statements are equivalent:*

(a) Φ is gH-weak subdifferentiable at u.
(b) Φ is gH-lower Lipschitz at u.
(c) Φ is gH-lower locally Lipschitz at u, and there exists a number $p \geq 0$ and an interval $\boldsymbol{Q}$ such that

$$-p\|y\| \oplus \boldsymbol{Q} \preceq \Phi(y) \ \forall\, y \in \mathscr{Y}. \tag{3.14}$$

Proof. (a) implies (b) : Suppose that Φ is gH-weak subdifferentiable at u. Then, there exists $(\widehat{\mathbf{G}^w}, c) \in I(\mathbb{R})^n \times \mathbb{R}_+$ such that, for any $y \in \mathscr{Y}$,

$$\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \Phi(y) \ominus_{gH} \Phi(u). \tag{3.15}$$

From Lemma 3.1, we have $-\|\widehat{\mathbf{G}^w}\|_{I(\mathbb{R})^n}\|y-u\| - c\|y-u\| \preceq \widehat{\mathbf{G}^w} \odot (y-u) \ominus_{gH} c\|y-u\|$. Hence, the inequality (3.15) yields

$$-(\|\widehat{\mathbf{G}^w}\| + c)\|y-u\| \preceq \Phi(y) \ominus_{gH} \Phi(u) \text{ by Lemma 2.3 (ii) of [1].}$$

By choosing $L = (\|\widehat{\mathbf{G}^w}\| + c)$, we obtain $-L\|y-u\| \preceq \Phi(y) \ominus_{gH} \Phi(u)$ for all $y \in \mathscr{Y}$. So, Φ is gH-lower Lipschitz at u.

(b) implies (c) : Suppose that (b) is satisfied. It needs to prove that the inequality (3.14) holds. Then, there exists an $L \geq 0$ such that

$$-L\|y-u\| \preceq \Phi(y) \ominus_{gH} \Phi(u). \tag{3.16}$$

Note that $-L\|y\|-L\|u\| \leq -L\|y-u\|$. So, inequality (3.16) gives $-L\|y\|-L\|u\| \preceq \Phi(y) \ominus_{gH} \Phi(u)$, which gives $\Phi(u) \ominus_{gH} L\|u\| - L\|y\| \preceq \Phi(y)$ by (iv) of Lemma 2.6. Taking $\mathbf{Q} = \Phi(u) \ominus_{gH} L\|u\|$ and $p = L$, we obtain $-p\|y\| \oplus \mathbf{Q} \preceq \Phi(y)$ for all $y \in \mathscr{Y}$.

(c) implies (a) : Let $\mathscr{N}(u)$ be an ε-neighbourhood of u such that (3.10) holds. Then,

$$-L\|y-u\| \preceq \Phi(y) \ominus_{gH} \Phi(u),\ \forall\, y \in \mathscr{N}(u) \tag{3.17}$$

and

$$-p\|y\| \oplus \mathbf{Q} \preceq \Phi(y),\ \forall\, y \in \mathbb{R}^n. \tag{3.18}$$

Assume to the contrary that Φ is not gH-weak subdifferentiable at u. Then, for any $(\widehat{\mathbf{G}}_n^w, c_n) \in I(\mathbb{R})^n \times \mathbb{R}_+$, there exists y_n such that

$$\Phi(y_n) \ominus_{gH} \Phi(u) \prec \widehat{\mathbf{G}}_n^{w\,\top} \odot (y_n - u) \ominus_{gH} c_n\|y_n - u\|.$$

If the sequence $\{\widehat{\mathbf{G}}_n^w\}$ is assumed to be converging to $\widehat{\mathbf{G}}^w$, then

$$\begin{aligned}\Phi(y_n) \ominus_{gH} \Phi(u) &\preceq \widehat{\mathbf{G}}^{w\,\top} \odot (y_n - u) \ominus_{gH} c_n\|y_n - u\| \\ &\preceq \|\widehat{\mathbf{G}}^w\|\|y_n - u\| - c_n\|y_n - u\|, \text{ by Theorem 3.1 of [11]}.\end{aligned} \tag{3.19}$$

By putting $y = y_n$ in (3.18), we have

$$-p\|y_n - u\| - p\|u\| \oplus \mathbf{Q} \preceq -p\|y_n\| \oplus \mathbf{Q} \preceq \Phi(y_n),$$

which implies

$$(-p\|y_n - u\| - p\|u\| \oplus \mathbf{Q}) \ominus_{gH} \Phi(u) \preceq \Phi(y_n) \ominus_{gH} \Phi(u) \text{ by Note 2 of [1]}. \tag{3.20}$$

From (3.19), (3.20), and [1, Lemma 2.3 (ii)], we deduce that

$$\begin{aligned}(-p\|y_n - u\| - p\|u\| \oplus \mathbf{Q}) \ominus_{gH} \Phi(u) &\preceq \|\widehat{\mathbf{G}}^w\|\|y_n - u\| - c_n\|y_n - u\|, \\ \text{or, } (c_n - p - \|\widehat{\mathbf{G}}^w\|)\|y_n - u\| &\preceq \Phi(u) \oplus p\|u\| \ominus_{gH} \mathbf{Q} \text{ by (iii) of Lemma 2.6}.\end{aligned} \tag{3.21}$$

Assume, without loss of generality, that $c_n - p - \|\widehat{\mathbf{G}}^w\| \neq 0$. Then, from (2.6), we obtain

$$\|y_n - u\| \preceq \frac{1}{c_n - p - \|\widehat{\mathbf{G}}^w\|} \odot \{\Phi(u) \oplus p\|u\| \ominus_{gH} \mathbf{Q}\}.$$

As $(\Phi(u) \oplus p\|u\| \ominus_{gH} \mathbf{Q})$ is bounded below on $\mathscr{N}(u)$, we have $y_n \to u$ as $c_n \to \infty$. Thus, $y_n \in \mathscr{N}(u)$ for large n. Then, it follows from (3.17) that

$$-L\|y_n - u\| \preceq \Phi(y_n) \ominus_{gH} \Phi(u). \tag{3.22}$$

In view of (3.19), we obtain

$$\Phi(y_n) \ominus_{gH} \Phi(u) \preceq \|\widehat{\mathbf{G}}^w\|\|y_n - u\| - c_n\|y_n - u\| = -(c_n - \|\widehat{\mathbf{G}}^w\|)\|y_n - u\|.$$

Since $c_n \to +\infty$ and $L \geq 0$, we can pick c_n sufficiently large so that $c_n - \|\widehat{\mathbf{G}}^w\| \geq L$. So,

$$\Phi(y_n) \ominus_{gH} \Phi(u) \preceq -L\|y_n - u\|.$$

This inequality leads to a contradiction. So, the result follows. □

Theorem 3.5. *Let $\emptyset \neq \mathscr{Y} \subseteq \mathbb{R}^n$. Let $\Psi : \mathscr{Y} \to I(\mathbb{R})$ be gH-Fréchet differentiable at u with gH-Fréchet derivative $\Psi_{\mathscr{F}}(u)$. Then, $\{(\Psi_{\mathscr{F}}(u), c) : c \geq 0\} \subset \partial^w \Psi(u)$.*

Proof. Since Ψ is gH-Fréchet differentiable at u with gH-Fréchet derivative $\Psi_{\mathscr{F}}(u)$, we have

$$\lim_{y\to u} \frac{1}{\|y-u\|} \odot \{\Psi(y) \ominus_{gH} \Psi(u) \ominus_{gH} \Psi_{\mathscr{F}}(u)^\top \odot (y-u)\} = \mathbf{0}$$

$$\Longrightarrow \liminf_{\substack{y\to u\\ y\neq u}} \frac{1}{\|y-u\|} \odot \{\Psi(y) \ominus_{gH} \Psi(u) \ominus_{gH} \Psi_{\mathscr{F}}(u)^\top \odot (y-u)\} = \mathbf{0}.$$

Therefore, by Definition 3.2, $\Psi_{\mathscr{F}}(u) \in \partial^-_{\mathscr{F}}\Psi(u)$, one has

$$\Psi_{\mathscr{F}}(u)^\top \odot (y-u) \preceq \Psi(y) \ominus_{gH} \Psi(u) \; \forall\, y \in \mathscr{Y}$$

$$\Longrightarrow \Psi_{\mathscr{F}}(u)^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \Psi(y) \ominus_{gH} \Psi(u), \text{ for any } c \geq 0.$$

Hence, $(\Psi_{\mathscr{F}}(u), c) \in \partial^w\Psi(u)$. □

Lemma 3.2. *Let $\emptyset \neq \mathscr{Y} \subseteq \mathbb{R}^n$. Let $\Phi : \mathscr{Y} \to I(\mathbb{R})$ be gH-Fréchet differentiable at u with gH-Fréchet derivative $\Phi_{\mathscr{F}}(u)$. Then, $-1 \odot \Phi_{\mathscr{F}}(u) \in \partial^-_{\mathscr{F}}(-1 \odot \Phi)(u)$.*

Proof. Since Φ is gH-Fréchet differentiable at u with gH-Fréchet derivative $\Phi_{\mathscr{F}}(u)$, one sees that

$$\lim_{y\to u} \frac{1}{\|y-u\|} \odot \{\Phi(y) \ominus_{gH} \Phi(u) \ominus_{gH} \Phi_{\mathscr{F}}(u)^\top \odot (y-u)\} = \mathbf{0}.$$

By applying Lemma 2.5, we have

$$\lim_{\substack{y\to u\\ y\neq u}} \frac{1}{\|y-u\|} \odot \left\{\mathbf{0} \ominus_{gH} \{(-1\odot\Phi)(y) \ominus_{gH} (-1\odot\Phi)(u) \ominus_{gH} (-1\odot\Phi_{\mathscr{F}}(u)^\top) \odot (y-u)\}\right\} = \mathbf{0}$$

$$\Longrightarrow \lim_{\substack{y\to u\\ y\neq u}} \frac{1}{\|y-u\|} \odot \left\{(-1\odot\Phi)(y) \ominus_{gH} (-1\odot\Phi)(u) \ominus_{gH} (-1\odot\Phi_{\mathscr{F}}(u))^\top \odot (y-u)\right\} = \mathbf{0}$$

$$\Longrightarrow \liminf_{\substack{y\to u\\ y\neq u}} \frac{1}{\|y-u\|} \odot \{(-1\odot\Phi)(y) \ominus_{gH} (-1\odot\Phi)(u) \ominus_{gH} (-1\odot\Phi_{\mathscr{F}}(u))^\top \odot (y-u)\} = \mathbf{0}.$$

Hence, $-1 \odot \Phi_{\mathscr{F}}(u) \in \partial^-_{\mathscr{F}}(-1\odot\Phi)(u)$. □

Next, we focus on investigating the sum rule of two functions in terms of gH-weak subdifferential. For two real-valued functions f_1 and f_2, the sum rule [15] for their weak subdifferential is $\partial^w(f_1 + f_2)(x) = \partial^w f_1(x) + \partial^w f_2(x)$. However, this sum rule does not hold for interval-valued functions. In the following, we provide such an example.

Consider the interval-valued functions $\Phi_1 : [-1,1] \to I(\mathbb{R})$ and $\Phi_2 : [-1,1] \to I(\mathbb{R})$, defined by

$$\Phi_1(y) = \begin{cases} \left[-y, \frac{1}{2}y\right], & \text{if } y \in [0,1] \\ \left[-\frac{1}{2}y, -y\right], & \text{if } y \in [-1,0] \end{cases} \quad \text{and} \quad \Phi_2(y) = [y^2, -y+3],$$

respectively. For these two functions, the gH-weak subdifferential at $u = 0$ are given by

$$\partial^w\Phi_1(0) = \{(\mathbf{G}^w_1, c_1) \in I(\mathbb{R}) \times \mathbb{R}_+ : \left[-1, -\tfrac{1}{2}\right] \preceq \mathbf{G}^w_1 \oplus c_1,\ \mathbf{G}^w_1 \ominus_{gH} c_1 \preceq \left[-1, \tfrac{1}{2}\right]\}$$

and

$$\partial^w\Phi_2(0) = \{(\mathbf{G}^w_2, c_2) \in I(\mathbb{R}) \times \mathbb{R}_+ : [-1,0] \preceq \mathbf{G}^w_2 \oplus c_2,\ \mathbf{G}^w_2 \ominus_{gH} c_2 \preceq [-1,0]\}.$$

Thus, we have

$$\partial^w\Phi_1(0) \oplus \partial^w\Phi_2(0)$$
$$= \{(\mathbf{H}^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : \left[-2, -\tfrac{1}{2}\right] \preceq \mathbf{H}^w \oplus c,\ \mathbf{H}^w \ominus_{gH} c \preceq \left[-2, \tfrac{1}{2}\right],\ \forall\, y \in [-1,1]\}. \tag{3.23}$$

Now, let $(\mathbf{H}^w, c) \in \partial^w(\Phi_1 \oplus \Phi_2\}(0)$, where

$$(\Phi_1 \oplus \Phi_2)(y) = \begin{cases} \left[y^2 - y, -\frac{1}{2}y + 3\right] & \text{if } y \in [0,1] \\ \left[y^2 - \frac{1}{2}y, -2y + 3\right] & \text{if } y \in [-1,0]. \end{cases}$$

There are the following two cases corresponding to $y \in [0,1]$ and $y \in [-1,0]$.

(i) As $y \geq 0$, we have

$$\begin{aligned} & \mathbf{H}^w \odot y \ominus_{gH} c \odot y \preceq (\Phi_1 \oplus \Phi_2)(y) \ominus_{gH} (\Phi_1 \oplus \Phi_2)(0) \\ \Longrightarrow\ & [\underline{h^w} - c, \overline{h^w} - c] \odot y \preceq \left[y^2 - y, -\tfrac{1}{2}y\right] \\ \Longrightarrow\ & \underline{h^w} - c \leq -1 \text{ and } \overline{h^w} - c \leq -\tfrac{1}{2}. \end{aligned}$$

(ii) As $-1 \leq y \leq 0$, we have

$$\begin{aligned} & [(\overline{h^w} + c)y, (\underline{h^w} + c)y] \preceq \left[y^2 - \tfrac{1}{2}y, -2y + 3\right] \ominus_{gH} [0,3] \\ \Longrightarrow\ & [(\overline{h^w} + c)y, (\underline{h^w} + c)y] \preceq \left[y^2 - \tfrac{1}{2}y, -2y\right] \\ \Longrightarrow\ & -2 - c \leq \underline{h^w} \text{ and } -\tfrac{1}{2} - c \leq \overline{h^w}. \end{aligned}$$

Therefore, from Case (i) and Case (ii), we have

$$\begin{aligned} & \partial^w(\Phi_1 \oplus \Phi_2)(0) \\ = & \left\{(\mathbf{H}^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : \left[-2, -\tfrac{1}{2}\right] \preceq (\mathbf{H}^w \oplus c), (\mathbf{H}^w \ominus_{gH} c) \preceq \left[-1, -\tfrac{1}{2}\right]\right\}. \end{aligned} \tag{3.24}$$

Thus, (3.23) and (3.24) are not equal.

In the following theorem, we show that under some restriction on Φ_1 and Φ_2 one-sided inclusion for the sum rule holds.

Theorem 3.6. *Let $\emptyset \neq \mathscr{Y} \subseteq \mathbb{R}^n$. Let $\Phi_1 : \mathscr{Y} \to I(\mathbb{R})$ be gH-weak subdifferential at u and $\Phi_2 : \mathscr{Y} \to \mathbb{R}$ be gH-Fréchet differentiable at u. Then, $\partial^w(\Phi_1 \oplus \Phi_2)(u) \subset \partial^w\Phi_1(u) \oplus \partial^w\Phi_2(u)$, provided that $w(\widehat{\boldsymbol{G}_1^w}) \leq w(\widehat{\boldsymbol{G}_2^w})$ for all $\widehat{\boldsymbol{G}_1^w} \in \partial\Phi_2(y)$ and $\widehat{\boldsymbol{G}_2^w} \in \partial(\Phi_1 \oplus \Phi_2)(y)$, where $w(\boldsymbol{A})$ is the width of the interval $\boldsymbol{A} \in I(\mathbb{R})$.*

Proof. If $(\widehat{\mathbf{G}^w}, c) \in \partial^w(\Phi_1 \oplus \Phi_2)(u)$, then

$$\widehat{\mathbf{G}^w}^\top \odot (y - u) \ominus_{gH} c\|y - u\| \preceq (\Phi_1 \oplus \Phi_2)(y) \ominus_{gH} (\Phi_1 \oplus \Phi_2)(u). \tag{3.25}$$

We know that $\Phi_2 : \mathscr{Y} \to I(\mathbb{R})$ is gH-Fréchet differentiable at u with the gH-Fréchet derivative $\Phi_{2\mathscr{F}}(u)$. Hence, $\Phi_{2\mathscr{F}}(u) \in \partial_{\mathscr{F}}^- \Phi_2(u)$ implies $-1 \odot \Phi_{2\mathscr{F}}(u) \in \partial_{\mathscr{F}}^-(-1 \odot \Phi_2)(u)$. We can then write

$$\begin{aligned} & -1 \odot \Phi_{2\mathscr{F}}(u) \odot (y - u) \preceq (-1 \odot \Phi_2)(u) \ominus_{gH} (-1 \odot \Phi_2)(u) \\ \Longrightarrow\ & -1 \odot \Phi_{2\mathscr{F}}(u) \odot (y - u) \preceq -1 \odot (\Phi_2(u) \ominus_{gH} \Phi_2(u)) \\ & \text{by properties of } gH\text{-difference (iv) of [28].} \end{aligned} \tag{3.26}$$

In view of Lemma 2.4, (3.25) becomes

$$\widehat{\mathbf{G}^w}^\top \odot (y - u) \ominus_{gH} c\|y - u\| \preceq (\Phi_1(y) \ominus_{gH} \Phi_1(u)) \oplus (\Phi_2(y) \ominus_{gH} \Phi_2(u)).$$

Using (v) of Lemma 2.6, this inequality reduces to

$$\widehat{\mathbf{G}^w}^\top \odot (y - u) \ominus_{gH} (\Phi_2(y) \ominus_{gH} \Phi_2(u)) \ominus_{gH} c\|y - u\| \preceq \Phi_1(y) \ominus_{gH} \Phi_1(u).$$

Now, from the inequality (3.26), we see that

$$\widehat{\mathbf{G}^w}^\top \odot (y - u) \ominus_{gH} \Phi_{2\mathscr{F}}(u) \odot (y - u) \ominus_{gH} c\|y - u\| \preceq \Phi_1(y) \ominus_{gH} \Phi_1(u).$$

Thus,

$$(\widehat{\mathbf{G}^w} \ominus_{gH} \Phi_{2\mathscr{F}}(u))^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \Phi_1(y) \ominus_{gH} \Phi_1(u).$$

Then, $(\widehat{\mathbf{G}^w} \ominus_{gH} \Phi_{2\mathscr{F}}(u), c) \in \partial^w \Phi_1(u)$ and $(\Phi_{2\mathscr{F}}(u), 0) \in \partial^w \Phi_2(u)$. Therefore, $(\widehat{\mathbf{G}^w}, c) \in \partial^w \Phi_1(u) \oplus \partial^w \Phi_2(u)$. Hence, the result follows. □

Theorem 3.7. *Let $\mathscr{Y}$ be a nonempty set of $\mathbb{R}^n$. Let $\Phi_1 : \mathscr{Y} \to I(\mathbb{R})$ be gH-Fréchet differentiable at u. Let $\Phi_2 : \mathscr{Y} \to I(\mathbb{R})$ be an IVF. If u is a weak efficient point of $\Phi_1 \oplus \Phi_2$, then $(-1 \odot \Phi_{1\mathscr{F}}(u), 0) \in \partial^w \Phi_2(u)$.*

Proof. Since u is a weak efficient point of $\Phi_1 \oplus \Phi_2$, for any $y \in \mathscr{Y}$,

$$\begin{aligned}
&(\Phi_1 \oplus \Phi_2)(u) \preceq (\Phi_1 \oplus \Phi_2)(y) \\
\Longrightarrow\ & \Phi_1(u) \oplus \Phi_2(u) \preceq \Phi_1(y) \oplus \Phi_2(y) \\
\Longrightarrow\ & \Phi_1(u) \ominus_{gH} \Phi_1(y) \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u), \text{ using Lemma 2 of [1]} \\
\Longrightarrow\ & (-1) \odot \{\Phi_1(y) \ominus_{gH} \Phi_1(u)\} \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u), \text{ by } \ominus_{gH} \text{ property in (iv) of [28]} \\
\Longrightarrow\ & (-1 \odot \Phi_1)(y) \ominus_{gH} (-1 \odot \Phi_1)(u) \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u), \\
& \text{by } \ominus_{gH} \text{ property in (iv) of [28].}
\end{aligned} \tag{3.27}$$

By Lemma 3.2, we also obtain that

$$(-1) \odot \Phi_{1\mathscr{F}}(u) \odot (y-u) \preceq (-1 \odot \Phi_1)(y) \ominus_{gH} (-1 \odot \Phi_1)(u)\ \forall\, y \in \mathscr{Y}. \tag{3.28}$$

We see from (3.27) and (3.28) that

$$(-1) \odot \Phi_{1\mathscr{F}}(u) \odot (y-u) \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u) \text{ by lemma 1 of [1]},$$

which shows that $((-1) \odot \Phi_{1\mathscr{F}}(u), 0) \in \partial^w \Phi_2(u)$. □

Theorem 3.8. *Let $\emptyset \neq \mathscr{Y} \subseteq \mathbb{R}^n$. Let Ψ be gH-Fréchet differentiable at u with the gH-Fréchet derivative $\Psi_{\mathscr{F}}(u)$. Then, Ψ has weak efficient solution at u if and only if, for any $y \in \mathscr{Y}$, $\Psi_{\mathscr{F}}(u)^\top \odot (y-u) = \boldsymbol{0}$.*

Proof. If Ψ has a weak efficient point at u, then

$$\begin{aligned}
&\Psi(u) \preceq \Psi(y) \\
\text{or, } &\mathbf{0} \preceq \Psi(y) \ominus_{gH} \Psi(u), \text{ by Lemma 2.1 of [13].}
\end{aligned}$$

By gH-Fréchet differentiability of Ψ at u, we have

$$\lim_{\|h\| \to 0} \frac{\|(\Psi(u+h) \ominus_{gH} \Psi(u)) \ominus_{gH} \Psi_{\mathscr{F}}(u)^\top \odot h\|_{I(\mathbb{R})}}{\|h\|} = 0.$$

If we take $h = \lambda(y-u)$, then

$$\lim_{\lambda \to 0} \frac{\|(\Psi(u+\lambda(y-u)) \ominus_{gH} \Psi(u)) \ominus_{gH} \Psi_{\mathscr{F}}(u)^\top \odot \{\lambda(y-u)\}\|_{I(\mathbb{R})}}{\|\lambda(y-u)\|} = 0. \tag{3.29}$$

Since u is a weak efficient point of Ψ, we have from (3.29) that

$$\begin{aligned}
& \lim_{\lambda \to 0} \frac{\|\mathbf{0} \ominus_{gH} \lambda \odot \{\Psi_{\mathscr{F}}(u)^\top \odot (y-u)\}\|_{I(\mathbb{R})}}{\|\lambda(y-u)\|} \leq 0 \text{ by (i) of Lemma 2.6} \\
\Longrightarrow\ & \lim_{\lambda \to 0} \frac{\|\lambda \odot \{\Psi_{\mathscr{F}}(u)^\top \odot (y-u)\}\|_{I(\mathbb{R})}}{\|\lambda(y-u)\|} \leq 0 \\
\Longrightarrow\ & \lim_{\lambda \to 0} \frac{\lambda \|\Psi_{\mathscr{F}}(u)^\top \odot (y-u)\|_{I(\mathbb{R})}}{\lambda \|(y-u)\|} \leq 0.
\end{aligned}$$

Since the norm gives a non-negative value, we see that

$$\frac{1}{\|y-u\|}\odot\{\Psi_{\mathscr{F}}(u)^{\top}\odot(y-u)\}=\mathbf{0}.$$

Thus, we obtain

$$\Psi_{\mathscr{F}}(u)^{\top}\odot(y-u)=\mathbf{0} \text{ for any } y\in\mathscr{Y}.$$

To show the reverse part, we suppose that $\Psi_{\mathscr{F}}(u)^{\top}\odot(y-u)=\mathbf{0}$ for all y. Then, we have $\Psi_{\mathscr{F}}(u)\in\partial^{-}_{\mathscr{F}}\Psi(u)$ and this clearly yields

$$\mathbf{0}=\Psi_{\mathscr{F}}(u)^{\top}\odot(y-u)\preceq\Psi(y)\ominus_{gH}\Psi(u)$$
$$\Longrightarrow\Psi(u)\preceq\Psi(y) \text{ by (ii) of Lemma 2.1 in [13]},$$

which means that u is weak efficient point of Ψ. □

Theorem 3.9. *Let $\emptyset\neq\mathscr{Y}\subseteq\mathbb{R}^n$. If Ψ is gH-Fréchet differentiable at u, then Ψ is gH-weak subdifferentiable at u if and only if $\Psi_{\mathscr{F}}(u)$ is gH-weak subdifferentiable at $0\in\mathscr{Y}$, and $\partial^{w}(\Psi(u))=\partial^{w}(\Psi_{\mathscr{F}}(u)(0))$.*

Proof. By the gH-Fréchet differentiability of Ψ at u, we have

$$\lim_{\|h\|\to 0}\frac{1}{\|h\|}\odot\{(\Psi(u+h)\ominus_{gH}\Psi(u))\ominus_{gH}\Psi_{\mathscr{F}}(u)^{\top}\odot h\}=\mathbf{0}.$$

Inserting $h=\lambda\odot(y-u)$, by gH-weak subdifferentiability of Ψ at u, we see that there exists $(\widehat{\mathbf{G}^{w}},c)\in\partial^{w}\Psi(u)$ such that, for any $y\in\mathscr{Y}$,

$$\widehat{\mathbf{G}^{w}}^{\top}\odot(y-u)\ominus_{gH}c\|y-u\|\preceq\Psi(y)\ominus_{gH}\Psi(u).$$

Hence,

$$\lim_{\lambda\to 0}\frac{1}{\|\lambda(y-u)\|}\odot\{(\Psi(u+\lambda(y-u))\ominus_{gH}\Psi(u))\ominus_{gH}\Psi_{\mathscr{F}}(u)^{\top}\odot\lambda(y-u)\}=\mathbf{0}.$$

In view of the gH-weak subdifferentiability of Ψ at u, we see, for any $y\in\mathscr{Y}$, that

$$\lim_{\lambda\to 0}\frac{1}{\|\lambda(y-u)\|}\odot\left\{(\widehat{\mathbf{G}^{w}}^{\top}\odot\lambda(y-u)\ominus_{gH}\lambda c\|y-u\|)\ominus_{gH}\Psi_{\mathscr{F}}(u)^{\top}\odot\lambda(y-u)\right\}\preceq\mathbf{0},$$
(by (ii) of Lemma 2.6)
$$\Longrightarrow\frac{1}{\|(y-u)\|}\odot\{(\widehat{\mathbf{G}^{w}}^{\top}\odot(y-u)\ominus_{gH}c\|y-u\|)\ominus_{gH}\Psi_{\mathscr{F}}(u)^{\top}\odot(y-u)\}\preceq\mathbf{0}.$$

Therefore, $\widehat{\mathbf{G}^{w}}^{\top}\odot(y-u)\ominus_{gH}c\|y-u\|\ominus_{gH}\Psi_{\mathscr{F}}(u)^{\top}\odot(y-u)\preceq\mathbf{0}$ for all $y\in\mathscr{Y}$. Letting $z=y-u$, we obtain

$$\widehat{\mathbf{G}^{w}}^{\top}\odot z\ominus_{gH}c\|z\|\preceq\Psi_{\mathscr{F}}(u)^{\top}\odot z\;\forall\,z\in\mathscr{Y}. \tag{3.30}$$

Note that the gH-Fréchet derivative $\Psi_{\mathscr{F}}(u)$ is also gH-Gâteaux derivative (see Theorem 5.2 of [13]). Hence, it is a linear IVF as in Definition 4.1 of [13]. By this fact, we have $\Psi_{\mathscr{F}}(u)^{\top}\odot(0)=\mathbf{0}$. Then, inequality (3.30) implies that $(\widehat{\mathbf{G}^{w}},c)\in\partial^{w}(\Psi_{\mathscr{F}}(u)(0))$.

Conversely, let $(\widehat{\mathbf{G}^{w}},c)\in\partial^{w}(\Psi_{\mathscr{F}}(u)(0))$. Then, we can write

$$\widehat{\mathbf{G}^{w}}^{\top}\odot y\ominus_{gH}c\|y\|\preceq\Psi_{\mathscr{F}}(u)^{\top}\odot y\;\forall\,y\in\mathscr{Y}$$
$$\Longrightarrow\widehat{\mathbf{G}^{w}}^{\top}\odot(y-u)\ominus_{gH}c\|y-u\|\preceq\Psi_{\mathscr{F}}(u)^{\top}\odot(y-u)\;\forall\,y\in\mathscr{Y}.$$

Since Ψ has gH-Fréchet derivative $\Psi_{\mathscr{F}}(u)$ and it is also a gH-subgradient, it follows that

$$\Psi_{\mathscr{F}}(y)^{\top} \odot (y-u) \preceq \Psi(y) \ominus_{gH} \Psi(u) \; \forall \, y \in \mathscr{Y}.$$

Thus $\widehat{\mathbf{G}^w}^{\top} \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \Psi(y) \ominus_{gH} \Psi(u)$. Hence the proof is complete. □

Theorem 3.10. *Let $\emptyset \neq \mathscr{Y} \subseteq \mathbb{R}^n$. Let Φ be gH-Fréchet differentiable at u. If u is a weak efficient point of Φ, then*

$$sup \left\{ \widehat{\mathbf{G}^w}^{\top} \odot (y-u) \ominus_{gH} c\|y-u\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w \Phi(u) \right\} = \boldsymbol{0}.$$

Proof. First, we show that

$$\Phi_{\mathscr{F}}(u)^{\top} \odot (y-u) = \sup \left\{ \widehat{\mathbf{G}^w}^{\top} \odot (y-u) \ominus_{gH} c\|y-u\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w \Phi(u) \right\}$$

by which the desired equality can be easily proved. Using the gH-Fréchet differentiability of Φ and taking the supremum on the inequality (3.30), we obtain

$$\begin{aligned} \sup_{(\widehat{\mathbf{G}^w}, c) \in \partial^w \Phi(u)} \{\widehat{\mathbf{G}^w}^{\top} \odot (y-u) \ominus_{gH} c\|y-u\|\} &\preceq \sup_{(\widehat{\mathbf{G}^w}, c) \in \partial^w \Phi(u)} \{\Phi_{\mathscr{F}}(u)^{\top} \odot (y-u)\} \\ &= \Phi_{\mathscr{F}}(u)^{\top} \odot (y-u). \end{aligned}$$

Since $(\Phi_{\mathscr{F}}(u), 0) \in \partial^w \Phi(u)$, one has

$$\Phi_{\mathscr{F}}(u) \odot (y-u) \in \left\{ \widehat{\mathbf{G}^w}^{\top} \odot (y-u) \ominus_{gH} c\|y-u\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w \Phi(u) \right\}$$

and hence the result follows. □

4. Optimality for the Difference of Two IVFs

In this section, we consider the constrained IOP as below:

$$\min_{y \in \mathscr{Y}} \{\Phi_2(y) \ominus_{gH} \Phi_1(y)\}, \tag{4.1}$$

where $\emptyset \neq \mathscr{Y} \subseteq \mathbb{R}^n$ and $\Phi_1, \Phi_2 : \mathscr{Y} \to I(\mathbb{R})$ are two IVFs. We are going to study weak efficiency conditions for the IOP (4.1) under some additional assumptions.

Theorem 4.1. *Let $\emptyset \neq \mathscr{Y} \subseteq \mathbb{R}^n$. Let $\Phi_1, \Phi_2 : \mathscr{Y} \to I(\mathbb{R})$ be gH-weak subdifferentiable at u, which is a weak-efficient point of $\Phi_2 \ominus_{gH} \Phi_1$. If $\Phi_1(u) = \Phi_2(u)$, then $\partial^w \Phi_1(u) \subset \partial^w \Phi_2(u)$.*

Proof. The gH-weak subdifferentiability of Φ_1 at u implies that $\partial^w \Phi_1(u)$ is nonempty. Hence, there exists $(\widehat{\mathbf{U}^w}, c) \in I(\mathbb{R}) \times \mathbb{R}_+$ such that

$$\widehat{\mathbf{U}^w}^{\top} \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \Phi_1(y) \ominus_{gH} \Phi_1(u) \text{ for all } y \in \mathscr{Y}. \tag{4.2}$$

Since $\Phi_2 \ominus_{gH} \Phi_1$ gets the weak efficiency value $\mathbf{0}$ at u for any $y \in \mathscr{Y}$, we have

$$\begin{aligned} &\mathbf{0} \preceq (\Phi_2 \ominus_{gH} \Phi_1)(y) \\ \Longrightarrow\ &\mathbf{0} \preceq \Phi_2(y) \ominus_{gH} \Phi_1(y) \\ \Longrightarrow\ &\Phi_1(y) \preceq \Phi_2(y) \text{ by Lemma 2.1(ii) of [13]} \\ \Longrightarrow\ &\Phi_1(y) \ominus_{gH} \Phi_1(u) \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u) \text{ by Note 2 of [1].} \end{aligned} \tag{4.3}$$

Consequently, inequality (4.3) implies that

$$\widehat{\mathbf{U}^w}^{\top} \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u).$$

This means $(\widehat{\mathbf{U}^w}, c) \in \partial^w \Phi_2(u)$. Hence, the result follows. □

Note 4.1. *If we take an efficient solution of* $\Phi_2 \ominus_{gH} \Phi_1$ *instead of a weak efficient solution, the additional condition* $\Phi_1(u) = \Phi_2(u)$ *becomes essential for Theorem 4.1 to hold. For instance, let two IVFs* $\Phi_1 : \left[-\frac{1}{2}, \frac{1}{2}\right] \to I(\mathbb{R})$ *and* $\Phi_2 : \left[-\frac{1}{2}, \frac{1}{2}\right] \to I(\mathbb{R})$ *be defined as*

$$\Phi_1(y) = [2|y|, |y|+1] \text{ and } \Phi_2(y) = [|y|, 2y^2 + |y|],$$

respectively. Now, according to Theorem 4.1, $(\Phi_2 \ominus_{gH} \Phi_1)(y) = [2y^2 - 1, -|y|]$*, and* 0 *is an efficient point of* $(\Phi_2 \ominus_{gH} \Phi_1)$ *because* $(\Phi_2 \ominus_{gH} \Phi_1)(y)$ *and* $(\Phi_2 \ominus_{gH} \Phi_1)(0)$ *are not comparable for all* $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$*. Note that*

$$\partial^w \Phi_1(0) = \{(\boldsymbol{K}_1^w, c_1) : [-2,-1] \preceq (\boldsymbol{K}_1^w \oplus c_1), (\boldsymbol{K}_1^w \ominus_{gH} c_1) \preceq [1,2]\}$$
$$\text{and } \partial^w \Phi_2(0) = \{(\boldsymbol{K}_2^w, c_2) : [-1,-1] \preceq (\boldsymbol{K}_2^w \oplus c_2), (\boldsymbol{K}_2^w \ominus_{gH} c_2) \preceq [1,1]\}.$$

Hence, $\partial^w \Phi_1(0) \not\subset \partial^w \Phi_2(0)$*. So,* $\Phi_1(u) = \Phi_2(u)$ *is an essential condition.*

As the restriction $\Phi_1(u) = \Phi_2(u)$ is a bit restrictive, in the next result, we give more flexible condition for which the inclusion in Theorem 4.1 holds.

Theorem 4.2. *Let* $\emptyset \neq \mathscr{Y} \subseteq \mathbb{R}^n$*. Let* Φ_1, Φ_2 *have gH-weak subdifferential at* $u \in \mathscr{Y}$*, and* $\Phi_2 \ominus_{gH} \Phi_1$ *attains weak efficient solution at u. Then,*

$$\partial^w \Phi_1(u) \subset \partial^w \Phi_2(u), \tag{4.4}$$

provided that $w(\Phi_1(y)) \geq w(\Phi_2(y))$ *for all* $y \in \mathscr{Y}$ *or* $w(\Phi_1(y)) \leq w(\Phi_2(y))$ *for all* $y \in \mathscr{Y}$*, where* $w(\boldsymbol{A})$ *is the width of the interval* $\boldsymbol{A} \in I(\mathbb{R})$*.*

Proof. The gH-weak subdifferentiability of Φ_1 at u implies that $\partial^w \Phi_1(u)$ is nonempty. Hence, there exists $(\widehat{\mathbf{U}^w}, c) \in I(\mathbb{R}) \times \mathbb{R}_+$ such that

$$\widehat{\mathbf{U}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \Phi_1(y) \ominus_{gH} \Phi_1(u) \text{ for all } y \in \mathscr{Y}. \tag{4.5}$$

Since u is a weak efficient point of $(\Phi_2 \ominus_{gH} \Phi_1)$, one has

$$(\Phi_2 \ominus_{gH} \Phi_1)(u) \preceq (\Phi_2 \ominus_{gH} \Phi_1)(y) \ \forall\, y \in \mathscr{Y}. \tag{4.6}$$

- **Case** 1. If $w(\Phi_1(y)) \geq w(\Phi_2(y))$, then, for all $y \in \mathscr{Y}$, we have from inequality (4.6) that

$$[\overline{\phi}_2(u) - \overline{\phi}_1(u), \underline{\phi}_2(u) - \underline{\phi}_1(u)] \preceq [\overline{\phi}_2(y) - \overline{\phi}_1(y), \underline{\phi}_2(y) - \underline{\phi}_1(y)]$$
$$\Longrightarrow \quad \overline{\phi}_1(y) - \overline{\phi}_1(u) \leq \overline{\phi}_2(y) - \overline{\phi}_2(u), \ \& \ \underline{\phi}_1(u) - \underline{\phi}_1(u) \leq \underline{\phi}_2(y) - \underline{\phi}_2(u) \tag{4.7}$$

  Now there arise two subcases.
  - **Subcase** 1. If $\underline{\phi}_1(y) - \underline{\phi}_1(u) \leq \overline{\phi}_1(y) - \overline{\phi}_1(u)$,
    $\underline{\phi}_1(y) - \underline{\phi}_1(u) \leq \min\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}$ and
    $\overline{\phi}_1(y) - \overline{\phi}_1(u) \leq \max\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}$.
    Clearly, we have $[\underline{\phi}_1(y) - \underline{\phi}_1(u), \overline{\phi}_1(y) - \overline{\phi}_1(u)] \preceq [\min\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}, \max\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}]$.
  - **Subcase** 2. If $\overline{\phi}_1(y) - \overline{\phi}_1(u) \leq \underline{\phi}_1(y) - \underline{\phi}_1(u)$,
    $\overline{\phi}_1(y) - \overline{\phi}_1(u) \leq \min\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}$ and
    $\underline{\phi}_1(y) - \underline{\phi}_1(u) \leq \max\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}$.
    Clearly we have $[\overline{\phi}_1(y) - \overline{\phi}_1(u), \underline{\phi}_1(y) - \underline{\phi}_1(u)] \preceq [\min\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}, \max\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}]$.

Combining Subcase 1 and Subcase 2, we have

$$\Phi_1(y) \ominus_{gH} \Phi_1(u) \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u). \tag{4.8}$$

• **Case** 2. If $w(\Phi_2(u)) \geq w(\Phi_1(u))$, then from the inequality (4.6), for all $y \in \mathscr{Y}$, we have

$$\begin{aligned}
&[\underline{\phi}_2(u) - \underline{\phi}_1(u), \overline{\phi}_2(u) - \overline{\phi}_1(u)] \preceq [\underline{\phi}_2(y) - \underline{\phi}_1(y), \overline{\phi}_2(y) - \overline{\phi}_1(y)] \\
\Longrightarrow \quad & \underline{\phi}_1(y) - \underline{\phi}_1(u) \leq \underline{\phi}_2(y) - \underline{\phi}_2(u) \ \&\ \overline{\phi}_1(y) - \overline{\phi}_1(u) \leq \overline{\phi}_2(y) - \overline{\phi}_2(u).
\end{aligned} \tag{4.9}$$

By a similar manner as in Case 1, we have $\Phi_1(y) \ominus_{gH} \Phi_1(u) \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u)$. Hence, in all cases, we have $\Phi_1(y) \ominus_{gH} \Phi_1(u) \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u)$. In view of (4.5), we have

$$\widehat{\mathbf{U}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \Phi_2(y) \ominus_{gH} \Phi_2(u) \text{ for all } y \in \mathscr{Y}, \text{ by Lemma 2.3 (ii) of [1],}$$

which implies $(\widehat{\mathbf{U}^w}, c) \in \partial^w \Phi_2(u)$. Hence, the result follows. □

Note 4.2. *If we take an efficient solution of $\Phi_2 \ominus_{gH} \Phi_1$ instead of a weak efficient solution, the additional condition $w(\Phi_1(y)) \geq w(\Phi_2(y))$ or $w(\Phi_1(y)) \leq w(\Phi_2(y))$ for all y becomes essential for Theorem 4.2 to hold. For instance, consider the IVFs $\Phi_1 : [-1,1] \to I(\mathbb{R})$ and $\Phi_2 : [-1,1] \to I(\mathbb{R})$ which are defined by*

$$\Phi_1(y) = \begin{cases} [y^3, y], & \text{if } 0 \leq y \leq 1 \\ [4y, y], & \text{if } -1 \leq y < 0 \end{cases} \text{ and } \Phi_2(y) = \begin{cases} [y^3, 5y], & \text{if } 0 \leq y \leq 1 \\ [3y, 2y], & \text{if } -1 \leq y < 0, \end{cases}$$

respectively. Now, according to Theorem 4.2,

$$(\Phi_2 \ominus_{gH} \Phi_1)(y) = \begin{cases} [0, 4y], & \text{if } 0 \leq y \leq 1 \\ [y, -y], & \text{if } -1 \leq y < 0 \end{cases}$$

obtains an efficient solution at 0 because $(\Phi_2 \ominus_{gH} \Phi_1)(0) \preceq (\Phi_2 \ominus_{gH} \Phi_1)(y)$ for all $y \in [0,1]$ and $(\Phi_2 \ominus_{gH} \Phi_1)(0)$ is not comparable with the values $(\Phi_2 \ominus_{gH} \Phi_1)(y)$ for all $y \in [-1,0]$. It is not difficult to check that

$$\begin{aligned}
\partial^w \Phi_1(0) &= \{(\boldsymbol{K}_1^w, c_1) : [1,4] \preceq (\boldsymbol{K}_1^w \oplus c_1), \boldsymbol{K}_1^w \ominus_{gH} c_1 \preceq [0,1]\} \\
\text{and } \partial^w \Phi_2(0) &= \{(\boldsymbol{K}_2^w, c_2) : [2,3] \preceq \boldsymbol{K}_2^w \oplus c_2, \boldsymbol{K}_2^w \ominus_{gH} c_2 \preceq [0,5]\}.
\end{aligned}$$

Here, we see that $\partial^w \Phi_1(0)$ and $\partial^w \Phi_2(0)$ are not comparable and at the same time, and we notice that $w(\Phi_2(y)) \geq w(\Phi_1(y))$ on $[0,1]$ and $w(\Phi_1(y)) \geq w(\Phi_2(y))$ on $[-1,0]$.

Remark 4.1. *In Theorem 4.2, inclusion (4.4) is a necessary but not sufficient condition for weak efficient point of $\Phi_2 \ominus_{gH} \Phi_1$. For instance, consider the IVFs $\Phi_1 : [-1,1] \to I(\mathbb{R})$ and $\Phi_2 : [-1,1] \to I(\mathbb{R})$ that are defined by*

$$\Phi_1(y) = \begin{cases} [y^3, y], & \text{if } 0 \leq y \leq 1 \\ [3y, 1.5y], & \text{if } -1 \leq y < 0 \end{cases} \text{ and } \Phi_2(y) = \begin{cases} [y^3 + y^2, 2y^2 + y], & \text{if } 0 \leq y \leq 1 \\ [3y, 2y], & \text{if } -1 \leq y < 0. \end{cases}$$

We notice that $w(\Phi_2(y)) \geq w(\Phi_1(y))$ on $[0,1]$ and $w(\Phi_2(y)) \leq w(\Phi_1(y))$ on $[-1,0]$. Note that

$$\begin{aligned}
\partial^w \Phi_1(0) &= \{(\boldsymbol{K}_1^w, c_1) : [1.5,3] \preceq \boldsymbol{K}_1^w \oplus c_1, \boldsymbol{K}_1^w \ominus_{gH} c_1 \preceq [0,1]\} \\
\text{and } \partial^w \Phi_2(0) &= \{(\boldsymbol{K}_2^w, c_2) : [2,3] \preceq \boldsymbol{K}_2^w \oplus c_2, \boldsymbol{K}_2^w \ominus_{gH} c_2 \preceq [0,1]\}.
\end{aligned}$$

Hence, $\partial^w \Phi_1(0) \subset \partial^w \Phi_2(0)$ but 0 is not a weak efficient point of $\Phi_2 \ominus_{gH} \Phi_1$ on $[-1,1]$.

Next, we study a relation between the augmented normal cone and gH-weak subdifferential. So, let us define the augmented normal cone to $\mathscr{Y}$ as below.

Definition 4.1. (Augmented normal cone). *An augmented normal cone to* $\mathscr{Y}$ *at* u *is*

$$\mathscr{N}_{\mathscr{Y}}^{c}(u)=\left\{(\widehat{\boldsymbol{G}},c)\in I(\mathbb{R})^{n}\times\mathbb{R}_{+}:\widehat{\boldsymbol{G}}^{\top}\odot(y-u)\ominus_{gH}c\|y-u\|\preceq\boldsymbol{0}\ \forall\, y\in\mathscr{Y}\right\}.$$

Theorem 4.3. (Optimality condition via augmented normal cone). *An IVF* $\Psi:\mathscr{Y}\to I(\mathbb{R})$ *attains weak efficient solution at* u *if and only if* $(\boldsymbol{0},0)\in\partial^{w}\Psi(u)\oplus\mathscr{N}_{\mathscr{Y}}^{c}(u)$, *where* $(\boldsymbol{0},0)$ *denotes the zero of* $I(\mathbb{R})\times\mathbb{R}_{+}$.

Proof. Since u is a weak efficient point of Ψ on $\mathscr{Y}$,

$$\begin{aligned}&\Psi(u)\preceq\Psi(y)\ \forall\, y\in\mathscr{Y}\\ \Longrightarrow\ &\boldsymbol{0}\preceq\Psi(y)\ominus_{gH}\Psi(u)\ \forall\, y\in\mathscr{Y}\ \text{by Lemma 2.1(ii) of [13]}\\ \Longrightarrow\ &(\boldsymbol{0},0)\in\partial^{w}\Psi(u).\end{aligned}$$

Let $\delta_{\mathscr{Y}}:\mathscr{Y}\to I(\mathbb{R})$ be an indicator function, defined by $\delta_{\mathscr{Y}}(y)=\begin{cases}\boldsymbol{0}, & \text{for } y\in\mathscr{Y}\\ \infty, & \text{for } y\notin\mathscr{Y}\end{cases}$. Since

$$(\Psi\oplus\delta_{\mathscr{Y}})(y)=\begin{cases}\Psi(y) & \text{if } y\in\mathscr{Y}\\ \infty & \text{if } y\notin\mathscr{Y},\end{cases}$$

$(\boldsymbol{0},0)\in\partial^{w}\Psi(u)=\partial^{w}(\Psi\oplus\delta_{\mathscr{Y}})(u)$. It needs to show that $\partial^{w}(\Psi\oplus\delta_{\mathscr{Y}})(u)\subset\partial^{w}\Psi(u)\oplus\mathscr{N}_{\mathscr{Y}}^{c}(u)$. To prove this, let $\widehat{\mathbf{G}^{w}}\in\partial^{w}(\Psi_{1}\oplus\delta_{\mathscr{Y}})(u)$. Then,

$$\begin{aligned}&\widehat{\mathbf{G}^{w}}^{\top}\odot(y-u)\ominus_{gH}c\|y-u\|\preceq(\Psi\oplus\delta_{\mathscr{Y}})(y)\ominus_{gH}(\Psi\oplus\delta_{\mathscr{Y}})(u)\\ \Longrightarrow\ &\widehat{\mathbf{G}^{w}}^{\top}\odot(y-u)\ominus_{gH}c\|y-u\|\preceq(\Psi(y)\oplus\delta_{\mathscr{Y}}(y))\ominus_{gH}(\Psi(u)\oplus\delta_{\mathscr{Y}}(u))\\ \Longrightarrow\ &\widehat{\mathbf{G}^{w}}^{\top}\odot(y-u)\ominus_{gH}c\|y-u\|\preceq\Psi(y)\ominus_{gH}\Psi(u),\end{aligned}$$

which implies $\widehat{\mathbf{G}^{w}}\in\partial^{w}\Psi(u)\subset\partial^{w}\Psi(u)\oplus\partial^{w}\delta_{\mathscr{Y}}(u)$, where $\{(\boldsymbol{0},0)\}\subset\partial^{w}\delta_{\mathscr{Y}}(u)$. Hence, $\widehat{\mathbf{G}^{w}}\in\partial^{w}\Psi(u)\oplus\partial^{w}\delta_{\mathscr{Y}}(u)=\partial^{w}\Psi(u)\oplus\mathscr{N}_{\mathscr{Y}}^{c}(u)$.

To show the converse part, let $(\boldsymbol{0},0)\in\partial^{w}\Psi(u)\oplus\mathscr{N}_{\mathscr{Y}}^{c}(u)=\partial^{w}(\Psi(u)\oplus\delta_{\mathscr{Y}}(u))$. Now, for any $y\in\mathscr{Y}$, we have

$$\begin{aligned}&\boldsymbol{0}\odot(y-u)\ominus_{gH}0\|y-u\|\preceq(\Psi(y)\oplus\delta_{\mathscr{Y}}(y))\ominus_{gH}(\Psi(u)\oplus\delta_{\mathscr{Y}}(u))\\ &\text{or, }\boldsymbol{0}\preceq\Psi(y)\ominus_{gH}\Psi(u)\\ &\text{or, }\Psi(u)\preceq\Psi(y)\ \text{by Lemma 2.1(ii) of [13]}.\end{aligned}$$

So, u is a weak efficient solution of Ψ. □

5. gH-Directional Derivative and gH-Weak Subdifferential for IVF

In the section, we investigate a relation between gH-Directional derivative and gH-weak subdifferential for IVFs based on supremum relation, which facilitates the analysis on the existence of efficient solution for nonconvex IVFs. With the help of the proposed relation, we introduce $\mathscr{W}$-gH-weak subgradient method to obtain weak efficient solution of an unconstrained IOP in the coming section.

Lemma 5.1. *Let* $\mathscr{Y}\subseteq\mathbb{R}^{n}$ *be starshaped at* $u\in\mathscr{Y}$. *Let, at* u, *the IVF* $\Phi:\mathscr{Y}\to I(\mathbb{R})$ *has gH-Directional derivative in every direction* $y-u$ *for any* $y\in\mathscr{Y}$, *and*

$$\Phi_{\mathscr{D}}(u;y-u)\preceq\Phi(y)\ominus_{gH}\Phi(u)\ \forall\, y\in\mathscr{Y}. \tag{5.1}$$

Then, u is a weak efficient point of Φ over $\mathscr{Y}$ if and only if

$$\mathbf{0} \preceq \Phi_{\mathscr{D}}(u; y-u) \;\forall\, y \in \mathscr{Y}. \tag{5.2}$$

Proof. Let us assume that condition (5.2) is satisfied. Thus, by using (5.1), we have $\mathbf{0} \preceq \Phi(y) \ominus_{gH} \Phi(u)$ for all $y \in \mathscr{Y}$, which implies that u is a weak efficient point of Φ over $\mathscr{Y}$. It is given that for all $y \in \mathscr{Y}$, $\Phi_{\mathscr{D}}(u; y-u)$ exists. Then,

$$\Phi_{\mathscr{D}}(u; y-u) = \lim_{\beta \to 0} \frac{1}{\beta} \odot [\Phi(u + \beta(y-u) \ominus_{gH} \Phi(u))]. \tag{5.3}$$

As u is a weak efficient point of Φ on $\mathscr{Y}$, we have $\mathbf{0} \preceq \Phi_{\mathscr{D}}(u; y-u)$. □

Theorem 5.1. *Let all the suppositions of Lemma 5.1 be satisfied. In addition, let at u, the gH-Directional derivative $\Phi_{\mathscr{D}}(u,\cdot)$ be gH-lower semicontinuous on $\mathscr{K} = cone(\mathscr{Y} - u)$ and*

$$-\infty \prec \inf\{\Phi_{\mathscr{D}}(u;h) : h \in \mathscr{K} \cap \mathscr{U}\}, \tag{5.4}$$

where $\mathscr{U} = \{v \in \mathbb{R}^n : \|v\| = 1\}$. Then, Φ is gH-weak subdifferentiable at u on $\mathscr{Y}$, that is $\partial^w_{\mathscr{Y}}\Phi(u)$ is nonempty and

$$\Phi_{\mathscr{D}}(u;h) = \sup\{\widehat{\boldsymbol{G}^w}^{\top} \odot h \ominus_{gH} c\|h\| : (\widehat{\boldsymbol{G}^w}, c) \in \partial^w_{\mathscr{Y}}\Phi(u),\; c > 0\}, \forall\, h \in \mathscr{K}. \tag{5.5}$$

Proof. For convenience, we suppose $\Psi(h) = \Phi_{\mathscr{D}}(u;h) \;\forall\, h \in \mathscr{K}$. Clearly, for $\alpha \geq 0$,

$$\begin{aligned}\Psi(\alpha h) = \Phi_{\mathscr{D}}(u;\alpha h) &= \lim_{\beta \to 0} \frac{1}{\beta} \odot [\Phi(u + (\beta\alpha)h) \ominus_{gH} \Phi(u)] \\ &= \alpha \odot \lim_{\beta \to 0} \frac{1}{\beta\alpha} \odot [\Phi(u + (\beta\alpha)h) \ominus_{gH} \Phi(u)] = \alpha \odot \Phi_{\mathscr{D}}(u;h) = \alpha \odot \Psi(h).\end{aligned}$$

So, Ψ is a nonnegative homogeneous IVF and $\Psi(0) = \mathbf{0}$. By the hypothesis, Ψ is bounded below on $\mathscr{K} \cap \mathscr{U}$. Due to this fact, for any given $\widehat{\mathbf{G}^w} \in I(\mathbb{R})^n$, the relation

$$\widehat{\mathbf{G}^w}^{\top} \odot h \ominus_{gH} c\|h\| \preceq \Psi(h) \ominus_{gH} \Psi(0) \;\forall\, h \in \mathscr{K} \cap \mathscr{U} \tag{5.6}$$

holds for sufficiently large c. Inequality (5.6) shows that $(\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathscr{Y}-u}\Psi(0)$, which means Ψ is gH-weak subdifferentiable on $\mathscr{Y} - u$ at 0. So, $\partial^w_{\mathscr{Y}-u}\Psi(0)$ is nonempty. Now it remains to show that

$$\partial^w_{\mathscr{Y}}\Phi(\bar{y}) = \partial^w_{\mathscr{Y}-u}\Psi(0). \tag{5.7}$$

Let $(\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathscr{Y}-u}\Psi(0)$. Thus, from (5.1) and (5.6), it implies that (3.1) is fulfilled, i.e., $(\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathscr{Y}}\Phi(u)$. To prove the reverse inclusion of equality (5.7), let us take $(\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathscr{Y}}\Phi(u)$. Then, for any fixed $y \in \mathscr{Y}$, we have

$$\begin{aligned}\Psi(y-u) &= \Phi_{\mathscr{D}}(u; y-u) \\ &= \lim_{\beta \to 0+} \frac{1}{\beta} \odot [\Phi(u + \beta(y-u)) \ominus_{gH} \Phi(u)] \\ &= \lim_{\beta \to 0+} \frac{1}{\beta} \odot \Big[\min\Big\{\underline{\phi}(u+\beta(y-u)) - \underline{\phi}(u), \overline{\phi}(u+\beta(y-u)) - \overline{\phi}(u)\Big\}, \\ &\qquad \max\Big\{\underline{\phi}(u+\beta(y-u)) - \underline{\phi}(u), \overline{\phi}(u+\beta(y-u)) - \overline{\phi}(u)\Big\}\Big].\end{aligned} \tag{5.8}$$

Let the first m number of components of $(y-u)$ be nonnegative and the rest $n-m$ number of components of $(y-u)$ be negative. Then, by the definition of weak subgradient on $\underline{\phi}$ and $\overline{\phi}$, we have $c_1 > 0$

and $c_2 > 0$ such that

$$\sum_{i=1}^{m} \beta(y_i - u_i)\underline{g}_i + \sum_{j=m+1}^{n} \beta(y_j - u_j)\underline{g}_j - \lambda c_1\|y-u\| \leq \underline{\phi}(u+\beta(y-u)) - \underline{\phi}(u) \tag{5.9}$$

$$\text{and } \sum_{i=1}^{m} \beta(y_i - u_i)\overline{g}_i + \sum_{j=m+1}^{n} \beta(y_j - u_j)\overline{g}_j - \lambda c_2\|y-u\| \leq \overline{\phi}(u+\beta(y-u)) - \overline{\phi}(u). \tag{5.10}$$

With the help of (5.9) and (5.10), (5.8) breaks into two cases.

- **Case** 1.

$$\begin{aligned}
&\lim_{\beta\to 0+} \frac{1}{\beta} \odot [\underline{\phi}(u+\beta(y-u)) - \underline{\phi}(u), \overline{\phi}(u+\beta(y-u)) - \overline{\phi}(u)] = \Psi(y-u) \\
\Longrightarrow\ &\lim_{\beta\to 0+} \frac{1}{\beta} \odot \left[\sum_{i=1}^{m} \beta(y_i-u_i)\underline{g}_i + \sum_{j=m+1}^{n} \beta(y_j-u_j)\underline{g}_j - \beta c_1\|y-u\|,\right. \\
&\left.\sum_{i=1}^{m} \beta(y_i-u_i)\overline{g}_i + \sum_{j=m+1}^{n} \beta(y_j-u_j)\overline{g}_j - \beta c_2\|y-u\|\right] \preceq \Psi(y-u) \\
\Longrightarrow\ &\bigoplus_{i=1}^{m}[\underline{g}_i, \overline{g}_i] \odot (y_i-u_i) \bigoplus_{j=m+1}^{n} [\overline{g}_j, \underline{g}_j] \odot (y_j-u_j) \ominus_{gH} \max\{c_1,c_2\}\|y-u\| \\
&\preceq \Psi(y-u).
\end{aligned} \tag{5.11}$$

- **Case** 2.

$$\begin{aligned}
&\lim_{\beta\to 0^+} \frac{1}{\beta} \odot [\underline{\phi}(u+\beta(y-u)) - \underline{\phi}(u), \overline{\phi}(u+\beta(y-u)) - \overline{\phi}(u)] = \Psi(y-u) \\
\Longrightarrow\ &\bigoplus_{i=1}^{m}[\underline{g}_i, \overline{g}_i] \odot (y_i-u_i) \bigoplus_{j=m+1}^{n} [\overline{g}_j, \underline{g}_j] \odot (y_j-u_j) \ominus_{gH} \max\{c_1,c_2\}\|y-u\| \\
&\preceq \Psi(y-u).
\end{aligned} \tag{5.12}$$

Combining (5.11) and (5.12), we obtain

$$\bigoplus_{i=1}^{m}(y_i-u_i)^{\top} \odot \mathbf{G}_i^w \bigoplus_{j=m+1}^{n} (y_j-u_j)^{\top} \odot \mathbf{G}_j^w \ominus_{gH} c\|y-u\| \preceq \Psi(y-u), \text{ where } c = \max\{c_1,c_2\},$$

which implies $\widehat{\mathbf{G}^w}^{\top} \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \Psi(y-u)$. This leads to (5.6); that is, $(\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathscr{Y}-u}\Psi(0)$.

Now we prove that

$$\Psi(h) = \sup\{\widehat{\mathbf{G}^w}^{\top} \odot h \ominus_{gH} c\|h\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathscr{Y}-u}\Psi(0),\ c \geq 0\}\ \forall\, h \in \mathscr{K}. \tag{5.13}$$

Supposing $h = 0$, equality in (5.13) is obvious. Hence we take care of the case $h \neq 0$. Let $h \in \mathscr{K}$ be a point on the boundary of the unit sphere, i.e., $\|h\| = 1$; that is, $h \in \mathscr{K} \cap \mathscr{U}$. Let $\varepsilon \geq 0$ be arbitrary. It suffices now to prove that

$$(\Psi(h) \ominus_{gH} \varepsilon \oplus c) \odot h^T z \ominus_{gH} c\|z\| \preceq \Psi(z)\ \forall\, z \in \mathscr{K} \cap \mathscr{U} \tag{5.14}$$

is valid for sufficiently large numbers c. Now, we proceed by the contrary that there exist two sequences $\{c_n\}$ and $\{z_n\}$ with $c_n \to \infty$ and $z_n \in \mathscr{K} \cap \mathscr{U}$ such that

$$\begin{aligned}
\Psi(z_n) &\preceq (\Psi(h) \ominus_{gH} \varepsilon \oplus c_n) \odot h^T z_n \ominus_{gH} c_n\|z_n\| \text{ for all } n = 1,2,\ldots \\
&= (\Psi(h) \ominus_{gH} \varepsilon) \odot h^T z_n \oplus c_n \odot (h^T z_n - 1) \text{ for all } n = 1,2,\ldots.
\end{aligned} \tag{5.15}$$

Since $\mathscr{K} \cap \mathscr{U}$ is closed and bounded, $\{z_n\}$ has a convergent subsequence. Without loss of generality, we presume that z_n converges to $z \in \mathscr{K} \cap \mathscr{U}$. Let $z \neq h$ and $\|h\| = 1$. Then $h^\top z \leq h^\top h = \|h\|^2 = 1$ follows. Thus, letting c_n approaches to ∞ in (5.15), we have $\Psi(z) = -\infty$, which contradicts (5.4). Thus, $z = h$ which ensures that $\|h\|^2 = 1$. By gH-lower semicontinuity of gH-Directional derivative $\Phi_{\mathscr{D}}(u;h)$, we have

$$\Psi(h) \preceq \liminf_{n\to\infty} \Psi(z_n) \preceq (\Psi(h) \ominus_{gH} \varepsilon)\|h\|^2 = \Psi(h) \ominus_{gH} \varepsilon, \tag{5.16}$$

which leads to a contradiction. Note that the inequality (5.14) is true for some $c \geq 0$. Denote $\widehat{\mathbf{G}^w} = (\Psi(h) \ominus_{gH} \varepsilon \oplus c) \odot h^\top$. The inequality (5.14) then gives $(\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathscr{Y}-u}\Psi(0)$. It is obvious that

$$(\Psi(h) \ominus_{gH} \varepsilon \oplus c) \odot h^\top h \ominus_{gH} c\|h\| \preceq \sup\{\widehat{\mathbf{G}^w}^\top \odot h \ominus_{gH} c\|h\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathscr{Y}-u}\Psi(0),\ c \geq 0\}.$$

As for $\|h\| = 1$, we have

$$\Psi(h) \ominus_{gH} \varepsilon \preceq \sup\{\widehat{\mathbf{G}^w}^\top \odot h \ominus_{gH} c\|h\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathscr{Y}-u}\Psi(0),\ c \geq 0\}.$$

Since this inequality holds for every $\varepsilon > 0$, we deduce that

$$\Psi(h) \preceq \sup\{\widehat{\mathbf{G}^w}^\top \odot h \ominus_{gH} c\|h\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathscr{Y}-u}\Psi(0),\ c \geq 0\}. \tag{5.17}$$

In the other words, $\widehat{\mathbf{G}^w}^\top \odot h \ominus_{gH} c\|h\| \preceq \Psi(h)$ for all $(\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathscr{Y}-u}\Psi(0)$, which yields

$$\Psi(h) = \sup\{\widehat{\mathbf{G}^w}^\top \odot h \ominus_{gH} c\|h\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathscr{Y}-u}\Psi(0),\ c \geq 0\}. \tag{5.18}$$

Thus, (5.13) is true. Then, (5.6) is followed by (5.7) and (5.14), which completes the proof of the theorem. □

6. $\mathscr{W}$-gH-Weak Subgradient Method

In this section, we illustrate a $\mathscr{W}$-gH-weak subgradient method to obtain a weak efficient solution of the following unconstrained IOP:

$$\min_{y\in\mathbb{R}^n} \Phi(y), \tag{6.1}$$

where $\Phi : \mathbb{R}^n \to I(\mathbb{R})$ is a nonsmooth nonconvex gH-Lipschitz continuous IVF. In order to derive the method, we define the weak efficient direction of an IVF.

Definition 6.1. (Weak efficient-direction). *A direction $d \in \mathbb{R}^n$ is said to be a weak efficient-direction of an IVF $\Phi : \mathbb{R}^n \to I(\mathbb{R})$ at $u \in \mathbb{R}^n$ if there exists a $\delta > 0$ such that*

(i) $\Phi(u + \lambda d) \preceq \Phi(u)$ for all $\lambda \in (0, \delta)$, and

(ii) there also exists a point $y' = u + \alpha d$ with $\alpha \in (0, \delta)$ and a positive real number $\delta' \leq \alpha$ such that $\Phi(y') \preceq \Phi(y' + \lambda d)$ for all $\lambda \in (-\delta', \delta')$. The point y' is known as a weak efficient solution of Φ in the direction d.

In the proposed method, similar to the existing result for gH-differentiable IVF [14, Theorem 5.4], we use the weak efficient direction $-\mathscr{W}(\widehat{\mathbf{G}^w})$, where $(\widehat{\mathbf{G}^w}, c) \in \partial^w\Phi(u)$ at any point $u \in \mathbb{R}^n$ and the mapping $\mathscr{W} : I(\mathbb{R})^n \to \mathbb{R}^n$ is defined by

$$\mathscr{W}(\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n) = (w_1\underline{b}_1 + w_2\overline{b}_1, w_1\underline{b}_2 + w_2\overline{b}_2, \ldots, w_1\underline{b}_n + w_2\overline{b}_n)^\top$$

for two given numbers $w_1, w_2 \in [0,1]$ with $w_1 + w_2 = 1$ and $\mathbf{B}_j = [\underline{b}_j, \overline{b}_j] \in I(\mathbb{R})$. To identify the weak efficient solution, we employ the $\mathscr{W}$-map. Applying $\mathscr{W}$-map, the weak efficient solution of IOP (6.1) can be found by solving the following problem:

$$\min_{y \in \mathbb{R}^n} w_1 \underline{\phi}(y) + w_2 \overline{\phi}(y). \tag{6.2}$$

The reason is as follows:

Clearly, $(w_1 \underline{g}^w + w_2 \overline{g}^w, c) \in \partial^w (w_1 \underline{\phi}(y) + w_2 \overline{\phi}(y))$ for any $y \in \mathbb{R}^n$, where $(\underline{g}^w, c) \in \partial^w \underline{\phi}(y)$ and $(\overline{g}^w, c) \in \partial^w \overline{\phi}(y)$. It can be noted that $w_1 \underline{g}^w + w_2 \overline{g}^w \in [\underline{g}^w, \overline{g}^w]$, which implies $(w_1 \underline{g}^w + w_2 \overline{g}^w, c) \in \partial^w \Phi(y)$. To be certain, we will prove a theorem to show that $(w_1 \underline{g}^w + w_2 \overline{g}^w, c) \in \partial^w \Phi(y)$ is correct.

Since Φ is gH-Lipschitz continuous IVF with Lipschitz constant L, Φ is gH-lower Lipschitz at any $u \in \mathbb{R}^n$ as well. Then, the gH-weak subdifferential set of Φ, $\partial^w \Phi(u)$ is nonempty. Along with this, we also assume that L be a positive real number with

$$\partial^w_L \Phi(u) = \{(\widehat{\mathbf{G}^w}, c) \in \partial^w \Phi(u) : c \leq L, j \in \mathbb{N}\} \neq \emptyset,$$

is clearly found to be compact set and $\|\widehat{\mathbf{G}^w}\| \leq l + L$ for every $(\widehat{\mathbf{G}^w}, c) \in \partial^w_L \Phi(u)$. This compactness of $\partial^w_L \Phi(u)$ will be used to produce an algorithm for the computation of weak efficient direction using the computation of gH-weak subgradients at any given point. To compute an approximate gH-weak subgradients, we will use the relation between gH-Direction derivative and gH-weak subdifferential (see Theorem 5.1) and consider all assumptions that given in Lemma 5.1 and Theorem 5.1.

To describe the algorithm for computing gH-weak subgradient, we first consider the following set and sequence for using the relation (5.5):

$Q = \{\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n) \in \mathbb{R}^n : |\vartheta_j| = 1, j = 1, 2, \ldots, n\}$. For $\vartheta \in Q$, consider the sequence of n vectors $\vartheta^j = \vartheta^j(\mu), j = 1, 2, \ldots, n$ with $\mu \in (0,1]$, where $\vartheta^j = (\mu \vartheta_1, \mu^2 \vartheta_2, \ldots, \mu^j \vartheta_j, 0, \ldots, 0)$. From the compactness of gH-weak subdifferential set $\partial^w_L \Phi(u)$ and the relation (5.5), there exists a gH-weak subgradients $(\widehat{\bar{\mathbf{G}}^w}, \bar{c})$ such that

$$\Phi_{\mathscr{D}}(u; \vartheta^j(\mu)) = \widehat{\bar{\mathbf{G}}^w}^\top \odot \vartheta^j(\mu) \ominus_{gH} \bar{c} \|\vartheta^j(\mu)\|.$$

Then, the set $\mathscr{G}_c = \{\widehat{\mathbf{G}^w} \in I(\mathbb{R})^n : (\widehat{\mathbf{G}^w}, \bar{c}) \in \partial^w_L \Phi(u)\}$ is nonempty. Suppose that there is a set $\mathscr{A} \subset \mathscr{G}_c$ such that

$$\Phi_{\mathscr{D}}(u; \vartheta^j(\mu)) = \sup\{\widehat{\bar{\mathbf{G}}^w}^\top \odot \vartheta^j(\mu) \ominus_{gH} \bar{c} \|\vartheta^j(\mu)\| : \widehat{\mathbf{G}^w} \in \mathscr{G}_c\}.$$

Next, we reconstruct a few following auxiliary sets similar to existing construction for weak subgradients (see [9] Remark 3.1): For any $\vartheta \in Q$ and $\mu > 0$, $\mathscr{R}_0(\vartheta) = \mathscr{A}$,

$$\begin{aligned}\mathscr{R}_j(\vartheta) = \{\widehat{\mathbf{M}^w} = (\mathbf{M}^w_1, \mathbf{M}^w_2, \ldots, \mathbf{M}^w_n) \in \mathscr{A} : \vartheta_j \odot \mathbf{M}^w_j = \sup\{\vartheta_j \odot \mathbf{G}^w_j : \widehat{\mathbf{G}^w} \\ = (\mathbf{G}^w_1, \mathbf{G}^w_2, \ldots, \mathbf{G}^w_n) \in \mathscr{R}_{j-1},\end{aligned}$$

and

$$\begin{aligned}\mathscr{R}(u, \vartheta^j(\mu)) = \{\widehat{\mathbf{M}^w} \in \mathscr{A} : \vartheta^j(\mu) \odot \widehat{\mathbf{M}} \\ = \sup\{\vartheta^j(\mu) \odot \widehat{\mathbf{G}^w} : \widehat{\mathbf{G}^w} \in \mathscr{A}\} \text{for all } j = 1, 2, \ldots, n.\end{aligned}$$

By using this construction, we have that, for every $\vartheta^j(\mu), j = 1, 2, \ldots, n$, there is an element $\widehat{\bar{\mathbf{G}}^w} \in \mathscr{R}(u, \vartheta^j(\mu))$ such that

$$\Phi_{\mathscr{D}}(u; \vartheta^j(\mu)) = \widehat{\bar{\mathbf{G}}^w}^\top \odot \vartheta^j(\mu) \ominus_{gH} \bar{c} \|\vartheta^j(\mu)\|. \tag{6.3}$$

In the sequel, like to the existing definition in p. 1527 of [9], we are ready to define a vector $\widehat{\mathbf{G}^w}(\vartheta, \mu, \lambda) \in I(\mathbb{R})^n$ and a set $\mathscr{U}(\vartheta, \mu)$ as follows: For any given $\vartheta \in Q$, $\lambda > 0$ and $\mu > 0$

, consider the following points: $y_0 = u$, $y_j = y_0 + \lambda\vartheta^j(\mu)$, $j = 1,2,\ldots,n$. Then, clearly $y_j = y_{j-1} + (0,\ldots,0,\lambda\mu^j\vartheta_j,0,\ldots)$ for every $j = 1,2,\ldots,n$. Let $\widehat{\mathbf{G}^w} = \widehat{\mathbf{G}^w}(\vartheta,\mu,\lambda) \in I(\mathbb{R})^n$ be a vector with n coordinates:

$$\widehat{\mathbf{G}_j^w}(\vartheta,\mu,\lambda) = \frac{1}{\lambda\mu^j\vartheta_j} \odot \{\Phi(y_j) \ominus_{gH} \Phi(y_{j-1})\} + \frac{\bar{c}}{\vartheta_j},\ j = 1,2,\ldots,n.$$

For any fixed $\vartheta \in Q$ and $\mu > 0$, we define the set:

$$\mathscr{U}(\vartheta,\mu) = \{(\widehat{\mathbf{M}^w},\bar{c}) \in I(\mathbb{R})^n \times \mathbb{R}_+ : \exists(\lambda_k \to +0, k \to +\infty),\ \widehat{\mathbf{M}^w} = \lim_{k\to\infty} \widehat{\mathbf{G}^w}(\vartheta,\mu,\lambda_k)\}.$$

We claim that $(\widehat{\mathbf{G}_j^w},\bar{c})$ is an approximate gH-weak subgradient of Φ at u, which need to satisfy the relation (6.3). To show $(\widehat{\mathbf{G}_j^w},\bar{c})$ certainly satisfies the relation (6.3), it is sufficient to prove Theorem 6.1. This theorem will also show that $(\mathscr{W}(\widehat{\mathbf{G}_j^w}),\bar{c})$ is also an approximate gH-weak subgradient of Φ at u. So, it indicates that $-\mathscr{W}(\widehat{\mathbf{G}_j^w})$ is an appropriate choice for weak efficient direction in the proposed $\mathscr{W}$-gH-weak subgradient method. Therefore, this method easily reduces to the conventional weak subgradient method of optimization problems in [9].

For establishing Theorem 6.1, we need the following two lemmas.

Lemma 6.1. *For any $\vartheta \in Q$, $\mathscr{R}_n(\vartheta)$ is singleton set.*

Proof. The proof is analogous to the proof of Proposition 3.1 for real-valued functions of real variables (see p. 1525 of [9]). □

Lemma 6.2. *There exist $\mu_0 > 0$ and $\widehat{\mathbf{M}^w} \in \mathscr{R}_j(\vartheta)$ such that*

$$\Phi_{\mathscr{D}}(u,\vartheta^j(\mu)) = \Phi_{\mathscr{D}}(u,\vartheta^{j-1}(\mu)) \oplus \mu^j\vartheta_j \odot \widehat{\mathbf{M}^w} \ominus_{gH} \bar{c}\mu^j$$

for all $\mu \in (0,\mu_0]$ and for every $j = 1,2,\ldots,n$.

Proof. The proof is analogous to the proof of Corollary 3.4 for real-valued functions of real-variables (see p. 1527 of [9]). □

In order to show that $(\widehat{\mathbf{G}_j^w},\bar{c})$ is an approximate gH-weak subgradient of Φ at u, we establish a relationship between the sets $\mathscr{U}(\vartheta,\mu)$ and $\partial_L^w\Phi(u)$ via the following theorem.

Theorem 6.1. *There exists $\mu_0 > 0$ such that $\mathscr{U}(\vartheta,\mu) \subseteq \partial_L^w\Phi(u)$ for all $\mu \in (0,\mu_0]$.*

Proof. Let $\widehat{\mathbf{G}_j^w}(\vartheta,\mu,\lambda) = [\underline{g}_j^w(\vartheta,\mu,\lambda),\overline{g}_j^w(\vartheta,\mu,\lambda)] = \frac{1}{\lambda\mu^j\vartheta_j} \odot \{\Phi(y_j) \ominus_{gH} \Phi(y_{j-1})\} \oplus \frac{\bar{c}}{\vartheta_j}$. It implies that

$$\widehat{\mathbf{G}_j^w}(\vartheta,\mu,\lambda) \subseteq \frac{1}{\lambda\mu^j\vartheta_j} \odot \{\{\Phi(y_j) \ominus_{gH} \Phi(u)\} \ominus_{gH} \{\Phi(y_{j-1}) \ominus_{gH} \Phi(u)\}\}.$$

Since $\Phi_{\mathscr{D}}(u,\vartheta^j(\mu)) = \lim_{\lambda\to+0} \frac{1}{\lambda} \odot \{\Phi(y_j) \ominus_{gH} \Phi(u)\}$, we have

$$\begin{aligned}&\widehat{\mathbf{G}_j^w}(\vartheta,\mu,\lambda)\\ &\subseteq \frac{1}{\lambda\mu^j\vartheta_j} \odot \{\lambda \odot \Phi_{\mathscr{D}}(u,\vartheta^j(\mu)) \ominus_{gH} \lambda \odot \Phi_{\mathscr{D}}(u,\vartheta^{j-1}(\mu)) \oplus o(\lambda,\vartheta^j(\mu)) - o(\lambda,\vartheta^{j-1}(\mu))\} + \frac{\bar{c}}{\vartheta_j},\end{aligned}$$

where $\lambda^{-1}o(\lambda,\vartheta^i) \to 0, \lambda \to +0, i = j-1, j$. Due to nonemptiness of $\mathscr{R}_j(\vartheta)$ for all $j = 1,2,\ldots,n$, we let $\mathbf{M} = (\mathbf{M}_1,\mathbf{M}_2,\ldots,\mathbf{M}_n) = ([\underline{m}_1,\overline{m}_1],[\underline{m}_2,\overline{m}_2],\ldots,[\underline{m}_n,\overline{m}_n]) \in \mathscr{R}_n(\vartheta)$. By Lemma 6.1, $\mathbf{M}$ is unique element of $\mathscr{R}_n(\vartheta)$. From the definition $\mathscr{R}_j(\vartheta)$ for all $j = 1,2,\ldots,n$, it is clear that $\mathscr{R}_n(\vartheta) \subseteq$

$\mathscr{R}_j(\vartheta)$ for all $j = 1, 2, \ldots, n$. Then, from this inclusion and Lemma 6.2, we have that there exist $\mu_0 > 0$ such that

$$\begin{aligned}\widehat{\mathbf{G}_j^w}(\vartheta,\mu,\lambda) &\subseteq \frac{1}{\lambda\mu^j\vartheta_j}\odot\{\lambda\odot(\mu^j\vartheta_j\odot\widehat{\mathbf{M}_j^w}\ominus_{gH}\bar{c}\mu^j)+o(\lambda,\vartheta^j(\mu))-o(\lambda,\vartheta^{j-1}(\mu))\}+\frac{\bar{c}}{\vartheta_j}\\ &=\widehat{\mathbf{M}_j^w}\ominus_{gH}\frac{\bar{c}}{\vartheta_j}\oplus\frac{o(\lambda,\vartheta^j(\mu))-o(\lambda,\vartheta^{j-1}(\mu))}{\lambda\mu^j\vartheta^j}\oplus\frac{\bar{c}}{\vartheta_j}\\ &=\widehat{\mathbf{M}_j^w}\oplus\frac{o(\lambda,\vartheta^j(\mu))-o(\lambda,\vartheta^{j-1}(\mu))}{\lambda\mu^j\vartheta^j}\end{aligned}$$

for all $\mu \in (0, \mu_0]$. Then, for any $\mu \in (0, \mu_0]$, we have

$$\lim_{\lambda\to+0}\widehat{\mathbf{G}_j^w}(\vartheta,\mu,\lambda)\ominus_{gH}\widehat{\mathbf{M}_j^w}\subseteq\{\mathbf{0}\}\implies\lim_{\lambda\to+0}\widehat{\mathbf{G}_j^w}(\vartheta,\mu,\lambda)=\widehat{\mathbf{M}_j^w}.$$

Consequently, $\lim_{\lambda\to+0}\widehat{\mathbf{G}^w}(\vartheta,\mu,\lambda)=\widehat{\mathbf{M}^w}\in\mathscr{G}_c$. On the other hand,

$$\begin{aligned}\mathscr{W}(\widehat{\mathbf{G}_j^w}(\vartheta,\mu,\lambda)) &= w_1\underline{g}_j^w(\vartheta,\mu,\lambda)+w_2\overline{g}_j^w(\vartheta,\mu,\lambda)\\ &=\frac{(w_1\underline{\phi}(y^j)+w_2\overline{\phi}(y^j))-(w_1\underline{\phi}(y^{j-1})+w_2\overline{\phi}(y^{j-1}))}{\lambda\mu^j\vartheta_j}+\frac{(w_1+w_2)\bar{c}}{\vartheta_j}\\ &=w_1\left\{\frac{(\underline{\phi}(y^j)-\underline{\phi}(y^{j-1}))}{\lambda\mu^j\vartheta_j}+\frac{\bar{c}}{\vartheta_j}\right\}+w_2\left\{\frac{(\overline{\phi}(y^j)-\overline{\phi}(y^{j-1}))}{\lambda\mu^j\vartheta_j}+\frac{\bar{c}}{\vartheta_j}\right\}\\ &=\frac{w_1\lambda\{\underline{\phi}_{\mathscr{D}}(\lambda,\vartheta^j(\mu))-\underline{\phi}_{\mathscr{D}}(\lambda,\vartheta^{j-1}(\mu))\}+o(\lambda,\vartheta^j(\mu))-o(\lambda,\vartheta^{j-1}(\mu))}{\lambda\mu^j\vartheta_j}+\frac{\bar{c}}{\vartheta_j}\\ &\quad+\frac{w_2\lambda\{\overline{\phi}_{\mathscr{D}}(\lambda,\vartheta^j(\mu))-\overline{\phi}_{\mathscr{D}}(\lambda,\vartheta^{j-1}(\mu))\}+o(\lambda,\vartheta^j(\mu))-o(\lambda,\vartheta^{j-1}(\mu))}{\lambda\mu^j\vartheta_j}+\frac{\bar{c}}{\vartheta_j}\\ &=\frac{w_1\lambda\{\underline{m}_j\mu^j\vartheta_j-\bar{c}\mu^j\}+o(\lambda,\vartheta^j(\mu))-o(\lambda,\vartheta^{j-1}(\mu))}{\lambda\mu^j\vartheta_j}+\frac{\bar{c}}{\vartheta_j}\\ &\quad+\frac{w_2\lambda\{\overline{m}_j\mu^j\vartheta_j-\bar{c}\mu^j\}+o(\lambda,\vartheta^j(\mu))-o(\lambda,\vartheta^{j-1}(\mu))}{\lambda\mu^j\vartheta_j}+\frac{\bar{c}}{\vartheta_j}\\ &=\frac{\lambda\{(w_1\underline{m}_j+w_2\overline{m}_j)\mu^j\vartheta_j-\bar{c}\mu^j\}+o(\lambda,\vartheta^j(\mu))-o(\lambda,\vartheta^{j-1}(\mu))}{\lambda\mu^j\vartheta_j}+\frac{\bar{c}}{\vartheta_j}.\end{aligned}$$

Similarly, $\lim_{\lambda\to+0} w_1\underline{g}_j^w(\vartheta,\mu,\lambda)+w_2\overline{g}_j^w(\vartheta,\mu,\lambda)=(w_1\underline{m}_j+w_2\overline{m}_j)$. Since $w_1\underline{g}_j^w(\vartheta,\mu,\lambda)+w_2\overline{g}_j^w(\vartheta,\mu,\lambda)\in\widehat{\mathbf{G}_j^w}(\vartheta,\mu,\lambda)$, is closed and bounded interval, then each point of $\widehat{\mathbf{G}_j^w}(\vartheta,\mu,\lambda)$ is a limit point of $\widehat{\mathbf{G}_j^w}(\vartheta,\mu,\lambda)$ and $\lim_{\lambda\to+0} w_1\underline{g}_j^w(\vartheta,\mu,\lambda)+w_2\overline{g}_j^w(\vartheta,\mu,\lambda)=(w_1\underline{m}_j+w_2\overline{m}_j)\in\widehat{\mathbf{M}_j^w}$. Therefore, $\lim_{\lambda\to+0} w_1\underline{g}^w(\vartheta,\mu,\lambda)+w_2\overline{g}^w(\vartheta,\mu,\lambda)=(w_1\underline{m}+w_2\overline{m})\in\widehat{\mathbf{M}^w}\in\mathscr{G}_c$. □

In the Algorithm 1 below, we describe a step-by-step procedure for computing gH-Weak subgradient $(\widehat{\mathbf{G}^w}, c)$ approximately of the given IVF Φ at the point $u \in \mathbb{R}^n$ based on the above assumptions, lemmas and theorems.

From Algorithm 1, we obtain gH-weak subgradients of the objective IVF Φ at every iteration. Algorithm 2 is initialized by choosing a point. We take the function value at this initial point and name this value UB. Algorithm 2 uses one of gH-weak subgradients obtained from Algorithm 1 for computing weak efficient direction at every iterative step and attempts to find a weak efficient solution by sequentially moving along the weak efficient direction for the diminishing stepsize. This

Algorithm 1 Approximate estimating of the gH-weak subgradient $(\widehat{\mathbf{G}^w}, c) \in \partial_L^w \Phi(u)$.

Let $\vartheta \in Q = \{\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n) \in \mathbb{R}^n : |\vartheta_j| = 1, j = 1, 2, \ldots, n\}$ and $\lambda > 0, \mu \in (0,1], u \in \mathbb{R}^n$.
Set $\vartheta^j(\mu) = (\vartheta_1 \mu, \vartheta_2 \mu^2, \ldots, \vartheta_j \mu^j, 0, \ldots, 0), j = 1, 2, \ldots, n$.
Let $y^0 = u$.
Select a number $c > 0$.
$j \leftarrow 1$.
while $j \leq n$ **do**
  $y_j = y_0 + \lambda \vartheta^j(\mu)$
  $\widehat{\mathbf{G}_j^w} = \frac{1}{\lambda \mu^j \vartheta_j} \odot \{\Phi(y_j) \ominus_{gH} \Phi(y_{j-1})\} + \frac{c}{\vartheta_j}$.
  $j = j + 1$
end while

algorithm will not stop until the function value at any point of the sequence $\{y_k\}_{k=1}^{\infty}$ is not less than UB. In the below, we present a step-by-step procedure via Algorithm 2 for finding weak efficient points for a given IOP (6.1) with the help of the above process.

Algorithm 2 $\mathscr{W}$-gH-weak subgradient method

Require: Given an initial solution $y_0 \in \mathbb{R}^n$, $w_1, w_2 \in [0,1]$ such that $w_1 + w_2 = 1$, let the current upper bound be UB $= \Phi(y_0)$, and weak efficient solution be $y_{eff} = y_0$.
Define the initial iteration and let $k \leftarrow 1$.
while $k \leq n$ **do**
  From Algorithm 1, choose a $(\widehat{\mathbf{G}_k^w}, c) \in \partial_L^w \Phi(y_k)$ such that $\mathscr{W}(\widehat{\mathbf{G}_k^w}) \neq 0$ and an α_k such that

$$\alpha_k > 0, \lim_{k \to \infty} \alpha_k = 0 \text{ and } \sum_{k=1}^{\infty} \alpha_k = \infty.$$

  Calculate

$$y_{k+1} = y_k - \alpha_k \mathscr{W}(\widehat{\mathbf{G}_k^w}).$$

  if $\Phi(y_{k+1}) \prec$ UB **then**
  UB $= \Phi(y_{k+1})$
  $y_{eff} = y_{k+1}$.
  end if
  Set $k = k + 1$
end while
return : the weak efficient solution

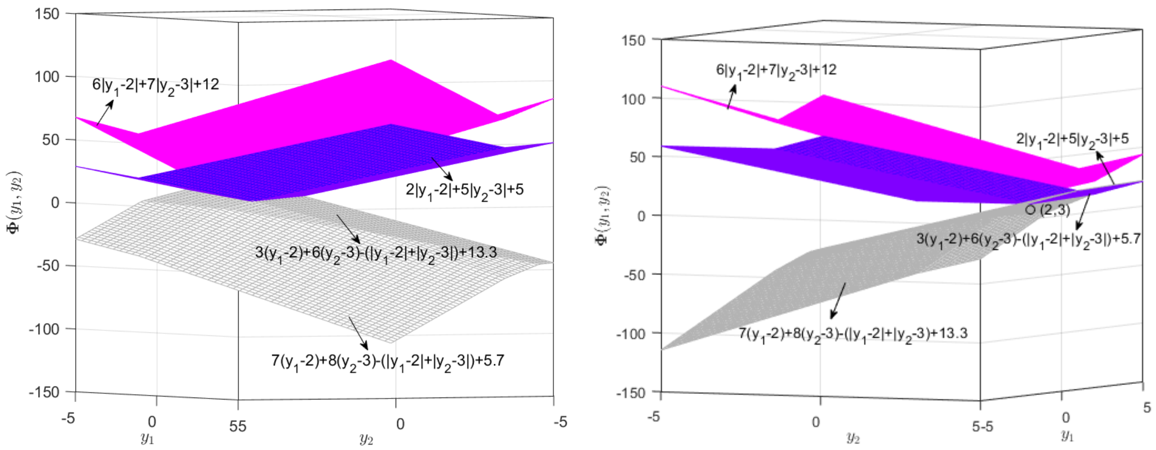
In the numerical example below, we apply the proposed Algorithm 1 to calculate a gH-weak subgradient of the objective function to the IOP.

Example 6.1. *Consider the following IOP:*

$$\min_{(y_1, y_2) \in \mathbb{R}^2} \Phi(y_1, y_2) = [2,6] \odot |y_1 - 2| \oplus [5,7] \odot |y_2 - 3| \oplus [5,12]. \tag{6.4}$$

We solve this IOP (6.4) by the method of gH-weak subgradient method. For this, we first start Algorithm 1 with initial point $u = [2.1, 3.1]$ *and perform two iterations with the parameters* $\vartheta = (1,1), \lambda = 0.1, \mu = 0.5, c = 1$. *Thereafter, we obtain pair of two gH-weak subgradients* $(\widehat{\boldsymbol{G}^w}, c) = ((\boldsymbol{G}_1^w, \boldsymbol{G}_2^w), c) = (([3,7], [6,8]), 1) \in \partial^w \Phi((2.1, 3.1))$ *in two successive iterations.*

Geometrically, $(\widehat{\boldsymbol{G}^w}, c)$ *represents that there exists a concave and gH-continuous IVF* $\boldsymbol{H}(y_1, y_2) = [3,7] \odot (y_1 - 2) \oplus [6,8] \oplus (y_2 - 3) - (|y_1 - 2| + |y_2 - 3|) \oplus \Phi(2.1, 3.1) = [3,7] \odot (y_1 - 2) \oplus [6,8] \odot$



(a) The IVFs Φ and $\mathbf{H}$

(b) Weak efficient point of the IOP (6.4) in Example 6.1

FIGURE 2. Visualization of the IVF Φ with its supporting below conic surface $\mathbf{H}$ in Example 6.1

$(y_2-3)-(|y_1-2|+|y_2-3|)\oplus[5.7,13.3]$, *is a conic surface that coincides with some section of* $\underline{\phi}(y_1,y_2)=2|y_1-2|+5|y_2-3|+5$ *and also intersects* $\underline{\phi}$ *at least the point* $(2,3)$ *from bottom.*

Taking a gH-weak subgradient $\widehat{G_1^w}$ *in first iteration of Algorithm 1, we start Algorithm 2 with diminishing step length* $\alpha_k=\frac{1}{k}$ *at k-th iteration, we compute an unique weak efficient point* $(2,3)$ *(shown in Figure 2) of IOP (6.4) after four iterations for seven different combinations of* w_1 *and* w_2 *with different initial points, depicted in Table 1.*

TABLE 1. Result of Algorithm 2 to find efficient solutions of IOP (6.4)

w_1	w_2	Initial point	Weak efficient solution
0.1	0.9	$(3.95,4.95)$	$(2,3)$
0.3	0.7	$(3.85,4.85)$	$(2,3)$
0.4	0.6	$(3.80,4.80)$	$(2,3)$
0.5	0.5	$(3.75,4.75)$	$(2,3)$
0.6	0.4	$(3.70,4.70)$	$(2,3)$
0.9	0.1	$(3.55,4.55)$	$(2,3)$
0.7	0.3	$(3.65,4.65)$	$(2,3)$

6.1. Convergence analysis of $\mathscr{W}$-gH-weak subgradient algorithm.

$\mathscr{W}$-gH-weak subgradient algorithm generates the sequence of points $\{y_k\}_{k=1}^{\infty}\subseteq\mathbb{R}^n$, given by

$$y_{k+1}=y_k-\mu_k\mathscr{W}(\widehat{\mathbf{G}_k^w}),\ \text{where}\ (\widehat{\mathbf{G}_k^w},c_k)\in\partial^w\Phi(y_k).$$

Towards the convergence of $\mathscr{W}$-gH-weak subgradient method, we need the following lemma.

Lemma 6.3. *Let* $\{y_k\}$ *be the sequence generated by* $\mathscr{W}$-gH-*weak subgradient method. Then, for all* $k\geq 0$, *we have*

$$\|y_{k+1}-y^*\|^2\leq\|y_k-y^*\|^2-2\mu_k\{\mathscr{W}(\Phi(y_k))-\mathscr{W}(\Phi(y^*))-c_k\|y^*-y_k\|\}+\mu_k^2\|\mathscr{W}(\widehat{\mathbf{G}_k^w})\|^2.$$

Proof. From Definition 3.1, we have for every $(\widehat{\mathbf{G}_k^w},c_k)$ that

$$\widehat{\mathbf{G}_k^w}^\top\odot(y^*-y_k)\ominus_{gH}c_k\|y^*-y_k\|\preceq\Phi(y^*)\ominus_{gH}\Phi(y_k)$$

$$\Longrightarrow \{\Phi(y_k) \ominus_{gH} \Phi(y^*)\} \ominus_{gH} c_k\|y^* - y_k\| \preceq \widehat{\mathbf{G}^w}^\top \odot (y_k - y^*)$$
$$\Longrightarrow \Phi(y_k) \ominus_{gH} \Phi(y^*) \preceq \widehat{\mathbf{G}^w}^\top \odot (y_k - y^*) \oplus c_k\|y^* - y_k\|$$
$$\Longrightarrow \mathscr{W}(\{\Phi(y_k) \ominus_{gH} \Phi(y^*)\}) \preceq \mathscr{W}(\widehat{\mathbf{G}^w}^\top \odot (y_k - y^*) \oplus c_k\|y^* - y_k\|). \tag{6.5}$$

We note that

$$\Phi(y_k) \ominus_{gH} \Phi(y^*) = [\min\{\underline{\phi}(y_k) - \underline{\phi}(y^*), \overline{\phi}(y_k) - \overline{\phi}(y^*)\}, \max\{\underline{\phi}(y_k) - \underline{\phi}(y^*), \overline{\phi}(y_k) - \overline{\phi}(y^*)\}].$$

We now consider the following two cases.

- **Case** 1. If $\underline{\phi}(y_k) - \underline{\phi}(y^*) < \overline{\phi}(y_k) - \overline{\phi}(y^*)$, then

$$\begin{aligned}\mathscr{W}(\Phi(y_k) \ominus_{gH} \Phi(y^*)) &= w_1(\underline{\phi}(y_k) - \underline{\phi}(y^*)) + w_2(\overline{\phi}(y_k) - \overline{\phi}(y^*))\\ &= (w_1\underline{\phi}(y_k) + w_2\overline{\phi}(y_k)) - (w_1\underline{\phi}(y^*) + w_2\overline{\phi}(y^*))\\ &= \mathscr{W}(\Phi(y_k)) \ominus_{gH} \mathscr{W}(\Phi(y^*)).\end{aligned}$$

- **Case** 2. If $\overline{\phi}(y_k) - \overline{\phi}(y^*) < \underline{\phi}(y_k) - \underline{\phi}(y^*)$, then

$$\begin{aligned}\mathscr{W}(\Phi(y_k) \ominus_{gH} \Phi(y^*)) &= w_1(\overline{\phi}(y_k) - \overline{\phi}(y^*)) + w_2(\underline{\phi}(y_k) - \underline{\phi}(y^*))\\ &= (w_1\overline{\phi}(y_k) + w_2\underline{\phi}(y_k)) - (w_1\overline{\phi}(y^*) + w_2\underline{\phi}(y^*))\\ &= \mathscr{W}(\Phi(y_k)) \ominus_{gH} \mathscr{W}(\Phi(y^*)).\end{aligned}$$

Accumulating the above two cases, we have from (6.5) that

$$\mathscr{W}(\Phi(y_k)) \ominus_{gH} \mathscr{W}(\Phi(y^*)) \preceq \mathscr{W}(\widehat{\mathbf{G}^w}^\top \odot (y_k - y^*) \oplus c_k\|y^* - y_k\|)$$
$$\Longrightarrow \mathscr{W}(\Phi(y_k)) \ominus_{gH} \mathscr{W}(\Phi(y^*)) \preceq \mathscr{W}(\widehat{\mathbf{G}^w})^\top (y_k - y^*) \oplus c_k\|y^* - y_k\|. \tag{6.6}$$

Using (6.6), we obtain

$$\begin{aligned}&\|y_{k+1} - y^*\|^2 = \|y_k - \mu_k\mathscr{W}(\widehat{\mathbf{G}^w}) - y^*\|^2\\ &= \|y_k - y^*\|^2 - 2\mu_k\mathscr{W}(\widehat{\mathbf{G}^w})^\top (y_k - y^*) + \mu_k^2\|\mathscr{W}(\widehat{\mathbf{G}^w})\|^2\\ &\leq \|y_k - y^*\|^2 - 2\mu_k\{\mathscr{W}(\Phi(y_k)) - \mathscr{W}(\Phi(y^*)\} - c_k\|y^* - y_k\|\} + \mu_k^2\|\mathscr{W}(\widehat{\mathbf{G}_k^w})\|^2.\end{aligned}$$

□

Theorem 6.2. (Convergence analysis of $\mathscr{W}$-gH-weak subgradient method for the constant stepsize). *For the sequence $\{y_k\}$ generated by $\mathscr{W}$-gH-weak subgradient method with constant stepsize μ, we have*

(i) if $\Phi(y^*) = -\infty$, then

$$\liminf_{k\to\infty} \Phi(y_k) = -\infty, \text{ and} \tag{6.7}$$

(ii) if $-\infty \prec \Phi(y^*)$, then

$$\liminf_{k\to\infty} \Phi(y_k) \preceq \Phi(y^*) \oplus \mu\frac{(l+L)^2}{2} \oplus \liminf_{k\to\infty} c_k d_{\mathscr{Y}}, \tag{6.8}$$

where $d_{\mathscr{Y}}$ is the diameter of $\mathscr{Y}$, denoted by $d_{\mathscr{Y}} = diam(\mathscr{Y}) = \max_{y_1,y_2\in\mathscr{Y}} \|y_1 - y_2\|$.

Proof. The statements (6.7) and (6.8) can be proven simultaneously. If possible, let there exist an $\varepsilon > 0$ such that

$$\Phi(y^*) \oplus \mu \frac{(l+L)^2}{2} \oplus \liminf_{k\to\infty} c_k d_{\mathscr{Y}} \oplus \varepsilon \prec \liminf_{k\to\infty} \Phi(y_k),$$

and let k_0 be sufficiently large such that for all $k \geq k_0$. It follows that

$$\begin{aligned} &\mu \frac{(l+L)^2}{2} \oplus \varepsilon \prec (\Phi(y_k) \ominus_{gH} \Phi(y^*)) \ominus_{gH} c_k d_{\mathscr{Y}} \\ \implies &\mu \frac{(l+L)^2}{2} \oplus \varepsilon \prec \mathscr{W}((\Phi(y_k) \ominus_{gH} \Phi(y^*)) \ominus_{gH} c_k d_{\mathscr{Y}}) \\ \implies &\mu \frac{(l+L)^2}{2} \oplus \varepsilon \prec \mathscr{W}(\Phi(y_k)) \ominus_{gH} \mathscr{W}(\Phi(y^*)) \ominus_{gH} c_k d_{\mathscr{Y}} \text{ by Lemma 6.3.} \end{aligned}$$

Since $\|y_k - y^*\| \leq d_{\mathscr{Y}}$, we have, from Lemma 6.3, that

$$\begin{aligned} &\|y_{k+1} - y^*\|^2 \\ \leq\ &\|y_k - y^*\|^2 - 2\mu\{\mathscr{W}(\Phi(y_k)) - \mathscr{W}(\Phi(y^*)) - c_k\|y^* - y_k\|\} + \mu^2\|\mathscr{W}(\widehat{\mathbf{G}_k^w})\|^2 \\ \leq\ &\|y_k - y^*\|^2 - 2\mu\{\mathscr{W}(\Phi(y_k)) - \mathscr{W}(\Phi(y^*)) - c_k d_{\mathscr{Y}}\|\} + \mu^2\|\mathscr{W}(\widehat{\mathbf{G}_k^w})\|^2 \\ \leq\ &\|y_k - y^*\|^2 - 2\mu\left[\mu\frac{(l+L)^2}{2} \oplus \varepsilon\right] + \mu^2\|\widehat{\mathbf{G}_k^w}\|^2 \\ \leq\ &\|y_k - y^*\|^2 - \mu^2(l+L)^2 - 2\mu\varepsilon + \mu^2(l+L)^2 \\ =\ &\|y_k - y^*\| - 2\mu\varepsilon \\ \leq\ &\|y_{k-1} - y^*\| - 4\mu\varepsilon \\ \leq\ &\cdots \leq \|y_{k_0} - y^*\|^2 - 2(k+1-k_0)\mu\varepsilon, \end{aligned}$$

which may not hold for k large enough, so it is a contradiction. □

Theorem 6.3. (Convergence analysis of $\mathscr{W}$-gH-weak subgradient method for the diminishing stepsize). *Let the stepsize μ_k be such that $\lim\limits_{k\to\infty} \mu_k = 0$ and $\sum_{k=0}^{\infty} \mu_k = \infty$. Then, for sequence $\{y_k\}$ generated by the $\mathscr{W}$-gH-weak subgradient method with the diminishing stepsize μ_k,*

$$\liminf_{k\to\infty} \Phi(y_k) \preceq \Phi(y^*) \oplus \liminf_{k\to\infty} c_k d_{\mathscr{Y}}.$$

Proof. On contrary, if possible let there exist an $\varepsilon > 0$ such that

$$\Phi(y^*) \oplus \liminf_{k\to\infty} c_k d_{\mathscr{Y}} \oplus \varepsilon \prec \liminf_{k\to\infty} \Phi(y_k).$$

Letting k_0 be sufficiently large so that for all $k \geq k_0$, we have $\varepsilon \prec (\Phi(y_k) \ominus_{gH} \Phi(y^*)) \ominus_{gH} c_k d_{\mathscr{Y}}$. By using Lemma 6.3 and following similar steps used in the proof of Theorem 6.2, we obtain

$$\|y_{k+1} - y^*\|^2 \leq \|y_k - y^*\|^2 - 2\mu_k\varepsilon + \mu_k(\mu_k\|\mathscr{W}(\widehat{\mathbf{G}_k^w})\|^2).$$

Since $\mu_k \to 0$ and $\{\widehat{\mathbf{G}_k^w}\}$ is bounded, then k_0 is large enough so that $\mu_k\|\widehat{\mathbf{G}_k^w}\|^2 < \varepsilon$ for all $k \geq k_0$. Consequently, $\mu_k\|\mathscr{W}(\widehat{\mathbf{G}_k^w})\|^2 < \varepsilon$ for all $k \geq k_0$. This implies that

$$\begin{aligned} \|y_{k+1} - y^*\|^2 &\leq \|y_k - y^*\|^2 - 2\mu_k\varepsilon + \mu_k\varepsilon \\ &= \|y_k - y^*\|^2 - \mu_k\varepsilon \leq \|y_{k-1} - y^*\|^2 - (\mu_{k-1} + \mu_k)\varepsilon \\ &\leq \cdots \leq \|y_{k_0} - y^*\|^2 - \varepsilon \sum_{j=k_0}^{k} \mu_j. \end{aligned}$$

Since $\sum_{k=0}^{\infty} \mu_k = \infty$, this relation may not hold for k sufficient large, so it leads to a contradiction. □

7. Conclusion

In this paper, the concepts of gH-weak subdifferentials and gH-weak subgradients (Definition 3.1) for IVFs with illustrative examples were provided. The gH-weak subdifferential set of an IVF was shown to be convex (Theorem 3.1) and closed (Theorem 3.2). We further introduced a necessary and sufficient condition (Theorem 3.4) for the set of gH-weak subgradients to be non-empty. We derived the necessary optimality condition (Theorem 3.10) involving gH-Fréchet differential and gH-weak subdifferential for IVFs. We derived a necessary optimality criterion for the difference of two IVFs (Theorem 4.1 and Theorem 4.3). We provided a necessary and sufficient condition for a weak efficient solution in terms of two notions of augmented normal cone and gH-weak subdifferential. Towards the end of the paper, we proposed the $\mathscr{W}$-gH-weak subgradient method and its algorithmic implementations (Algorithm 1 and Algorithm 2) to obtain efficient solutions of an unconstrained IOP with the nonconvex and nonsmooth objective IVF. The convergence of the proposed method using the constant and diminishing stepsize was explained (Theorem 6.2 and Theorem 6.3).

Continuing the present study, in the forthcoming work, we attempt to solve the following three problems.

- Introducing a gH-weak subgradient algorithm with the dynamic stepsize, which characterizes efficient solutions for nonsmooth nonconvex interval optimization problems.
- In the future, we take up the practical optimization problems to be solved by gH-weak subgradient algorithm.
- Analogous to the notion of weak-stability for conventional optimization problems [26], in the future, one may attempt to extend the notion for the following IOP (P):

$$\begin{aligned} &\min \Phi(y) \\ &\text{subject to } g_j(y) \leq 0, \ j = 1, 2, \ldots, p \\ &y \in \mathscr{Y}, \end{aligned}$$

where $\Phi : \mathscr{Y} \to I(\mathbb{R}) \cup \{-\infty, +\infty\}$ is an IVF and $g_j : \mathscr{Y} \to \mathbb{R}$ is a real-valued constraint, $j = 1, 2, \ldots, p$, and the feasible set C is

$$C = \{y \in \mathbb{R}^n : y \in \mathscr{Y}, \ g_j(y) \leq 0, j = 1, 2, \ldots, p\}.$$

To establish an interrelation between strong duality and weak stability for (P), one may define the augmented Lagrange interval-valued function for (P) as follows. Let J be an arbitrary index set, for which we define

$$\begin{aligned} \mathbb{R}_{\lambda}^{(J)} &:= \{e \in \mathbb{R}^{(J)} : |e_j| \leq 1, j \in J(\lambda)\} \\ \text{and } \Lambda &:= \left\{(\lambda, k) \in \mathbb{R}^{(J)} : \exists\, e \in \mathbb{R}_{\lambda}^{(J)}, ke - \lambda \in \mathbb{R}_{+}^{(J)}\right\}, \end{aligned}$$

where

$$\begin{aligned} \mathbb{R}^{(J)} &:= \{\lambda = (\lambda_j)_{j \in J} : \lambda_j = 0 \text{ for all } j \in J \text{ but only finitely many } \lambda_j \neq 0\}, \\ J(\lambda) &:= \{j \in J : \lambda_j \neq 0\}, \text{ is a finite subset of } J \\ \text{and } \mathbb{R}_{+}^{J} &:= \{\lambda = (\lambda_j)_{j \in J} \in \mathbb{R}^{(J)} : \lambda_j \geq 0, j \in J\}. \end{aligned}$$

For each $j \in J$, the augmented Lagrange interval-valued function for (P) can be defined by

$$\mathbf{L}(y, \Lambda, k) = \Phi(y) \ominus_{gH} \langle \lambda, (g_j(y))_j \rangle \oplus \beta((g_j(y)), \lambda, k),$$

where $\beta(u,\lambda,k):\mathbb{R}^J\times\mathbb{R}^{(J)}\times\mathbb{R}_+\to\mathbb{R}$ is such that

$$\beta(y,\lambda,k)=\begin{cases}\sup_{e\in\mathbb{R}^{(J)}_\lambda}\left\{\langle ke,u\rangle: ke-\lambda\in\mathbb{R}^{(J)}_+\right\} & \text{if } J(\lambda)\neq\emptyset,\\ 0 & \text{if } J(\lambda)=\emptyset.\end{cases}$$

The dual of (P) can be found as

$$\begin{aligned}\max\ \inf\ &\mathbf{L}(x,\lambda,k)\\ \text{subject to } &(\lambda,k)\in\Lambda.\end{aligned}$$

We will make an effort to reduce the duality gap by the weak-stability property of the following perturbation function $\Psi:\mathscr{Y}\times\mathbb{R}^n\to I(\mathbb{R})\cup\{+\infty\}$ associated to the IOP (P):

$$\Psi(y,u)=\begin{cases}\Phi(y) & \text{if } y\in\mathscr{Y}\subset\mathbb{R}^n \text{ and } g_j(y)\leq u_j,\ \forall\ j=1,2,3,\ldots,p\\ +\infty & \text{otherwise,}\end{cases}$$

where $u=(u_1,u_2,\ldots,u_n)$ is called the perturbation vector.

- One may also try to apply the gH-weak subdifferential in the context of zero duality gap in IOPs and interval-valued differential equations. The method for eliminating the duality gap will be immediately applicable in the following areas:
  - two-person zero-sum game [22],
  - optimal solutions of control problems with first order differential equations [23],
  - Hamilton-Jacobi field theory [23],
  - difference of convex programming [10].
- The newly defined augmented normal cone and gH-weak subdifferential together lead to the thought of introducing supporting cones for a set of intervals in the future. This new concept may be used later to describe the conic gap, which may be a crucial property to capturing the geometry of a nonconvex set of intervals.

Appendix A. Proof of Lemma 2.3

Proof. Let $\mathbf{W}=[\underline{w},\overline{w}]$, $\mathbf{Y}=[\underline{y},\overline{y}]$ and $\mathbf{Z}=[\underline{z},\overline{z}]$. From the gH-difference, we have the following four possible cases:

(i) Give $\varepsilon\preceq(\mathbf{W}\ominus_{gH}\mathbf{Y})\ominus_{gH}\mathbf{Z}=[\underline{w}-\underline{y}-\underline{z},\overline{w}-\overline{y}-\overline{z}]$. Since $\underline{w}-\underline{y}\geq\underline{z}+\varepsilon$ and $\overline{w}-\overline{y}\geq\overline{z}+\varepsilon$, we have $\underline{z}+\varepsilon\leq\overline{z}+\varepsilon\leq\overline{w}-\overline{y}$. This implies $\underline{z}+\varepsilon\leq\min\{\underline{w}-\underline{y},\overline{w}-\overline{y}\}$. Also, $\overline{z}+\varepsilon\leq\overline{w}-\overline{y}\leq\max\{\underline{w}-\underline{y},\overline{w}-\overline{y}\}$. Clearly we have $[\underline{z}+\varepsilon,\overline{z}+\varepsilon]\preceq[\min\{\underline{w}-\underline{y},\overline{w}-\overline{y}\},\max\{\underline{w}-\underline{y},\overline{w}-\overline{y}\}]$ and hence $\mathbf{Z}\oplus\varepsilon\preceq\mathbf{W}\ominus_{gH}\mathbf{Y}$.

(ii) $(\mathbf{W}\ominus_{gH}\mathbf{Y})\ominus_{gH}\mathbf{Z}=[\overline{w}-\overline{y}-\overline{z},\underline{w}-\underline{y}-\underline{z}]$. Thus, the proof is straightforward and identical to **Case** (i).

(iii) $(\mathbf{W}\ominus_{gH}\mathbf{Y})\ominus_{gH}\mathbf{Z}=[\overline{w}-\overline{y}-\underline{z},\underline{w}-\underline{y}-\overline{z}]$. Since $\overline{w}-\overline{y}\geq\underline{z}+\varepsilon,\underline{w}-\underline{y}\geq\overline{z}+\varepsilon$, we have $\underline{z}+\varepsilon\leq\overline{z}+\varepsilon\leq\underline{w}-\underline{y}$. This implies $\underline{z}+\varepsilon\leq\min\{\underline{w}-\underline{y},\overline{w}-\overline{y}\}$. Also, $\overline{z}+\varepsilon\leq\underline{w}-\underline{y}\leq\max\{\underline{w}-\underline{y},\overline{w}-\overline{y}\}$. Clearly we have $[\underline{z}+\varepsilon,\overline{z}+\varepsilon]\preceq[\min\{\underline{w}-\underline{y},\overline{w}-\overline{y}\},\max\{\underline{w}-\underline{y},\overline{w}-\overline{y}\}]$ and hence $\mathbf{Z}\oplus\varepsilon\preceq\mathbf{W}\ominus_{gH}\mathbf{Y}$.

(iv) $(\mathbf{W}\ominus_{gH}\mathbf{Y})\ominus_{gH}\mathbf{Z}=[\underline{w}-\underline{y}-\overline{z},\overline{w}-\overline{y}-\underline{z}]$. Thus, the proof is identical to **Case** (iii).

□

Appendix B. Proof of Lemma 2.4

Proof. Let $\mathbf{X}=[\underline{x},\overline{x}],\mathbf{Y}=[\underline{y},\overline{y}]$, $\mathbf{Z}=[\underline{z},\overline{z}]$ and $\mathbf{W}=[\underline{w},\overline{w}]$. Then,

$$(\mathbf{X}\oplus\mathbf{Y})\ominus_{gH}(\mathbf{Z}\oplus\mathbf{W})$$

$$= [\min\{\underline{x}+\underline{y}-\underline{z}-\underline{w}, \overline{x}+\overline{y}-\overline{z}-\overline{w}\}, \max\{\underline{x}+\underline{y}-\underline{z}-\underline{w}, \overline{x}+\overline{y}-\overline{z}-\overline{w}\}]$$
$$= [\min\{\underline{x}-\underline{z}+\underline{y}-\underline{w}, \overline{x}-\overline{z}+\overline{y}-\overline{w}\}, \max\{\underline{x}-\underline{z}+\underline{y}-\underline{w}, \overline{x}-\overline{z}+\overline{y}-\overline{w}\}]. \quad \text{(B.1)}$$

We have

$$\min\{\underline{x}-\underline{z}+\underline{y}-\underline{w}, \overline{x}-\overline{z}+\overline{y}-\overline{w}\} \geq \min\{\underline{x}-\underline{z}, \overline{x}-\overline{z}\} + \min\{\underline{y}-\underline{w}, \overline{y}-\overline{w}\} \quad \text{(B.2)}$$
$$\text{and } \max\{\underline{x}-\underline{z}+\underline{y}-\underline{w}, \overline{x}-\overline{z}+\overline{y}-\overline{w}\} \leq \max\{\underline{x}-\underline{z}, \overline{x}-\overline{z}\} + \max\{\underline{y}-\underline{w}, \overline{y}-\overline{w}\}. \quad \text{(B.3)}$$

By (B.2) and (B.3), from (B.1), we write

$$\begin{aligned}
&(\mathbf{X} \oplus \mathbf{Y}) \ominus_{gH} (\mathbf{Z} \oplus \mathbf{W}) \\
&= [\min\{\underline{x}-\underline{z}+\underline{y}-\underline{w}, \overline{x}-\overline{z}+\overline{y}-\overline{w}\}, \max\{\underline{x}-\underline{z}+\underline{y}-\underline{w}, \overline{x}-\overline{z}+\overline{y}-\overline{w}\}] \\
&\subseteq [\min\{\underline{x}-\underline{z}, \overline{x}-\overline{z}\} + \min\{\underline{y}-\underline{w}, \overline{y}-\overline{w}\}, \max\{\underline{x}-\underline{z}, \overline{x}-\overline{z}\} + \max\{\underline{y}-\underline{w}, \overline{y}-\overline{w}\}] \\
&= [\min\{\underline{x}-\underline{z}, \overline{x}-\overline{z}\}, \max\{\underline{x}-\underline{z}, \overline{x}-\overline{z}\}] + [\min\{\underline{y}-\underline{w}, \overline{y}-\overline{w}\}, \max\{\underline{y}-\underline{w}, \overline{y}-\overline{w}\}] \\
&= (\mathbf{X} \ominus_{gH} \mathbf{Z}) \oplus (\mathbf{Y} \ominus_{gH} \mathbf{W}).
\end{aligned}$$

□

APPENDIX C. PROOF OF LEMMA 2.5

Proof. Let $\mathbf{W} = [\underline{w}, \overline{w}]$, $\mathbf{Y} = [\underline{y}, \overline{y}]$, and $\mathbf{Z} = [\underline{z}, \overline{z}]$. Then,

$$-1 \odot \mathbf{W} = [-\overline{w}, -\underline{w}], -1 \odot \mathbf{Y} = [-\overline{y}, -\underline{y}], -1 \odot \mathbf{Z} = [-\overline{z}, -\underline{z}].$$

From Definition of gH-difference of two intervals, we have: either

$$-1 \odot \mathbf{W} \ominus_{gH} -1 \odot \mathbf{Y} = [\overline{y}-\overline{w}, \underline{y}-\underline{w}]$$

or

$$-1 \odot \mathbf{W} \ominus_{gH} -1 \odot \mathbf{Y} = [\underline{y}-\underline{w}, \overline{y}-\overline{w}].$$

Then, one of the following holds true

(a) $((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z}) = [\overline{y}-\overline{w}+\overline{z}, \underline{y}-\underline{w}+\underline{z}]$,
(b) $((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z}) = [\underline{y}-\underline{w}+\underline{z}, \overline{y}-\overline{w}+\overline{z}]$,
(c) $((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z}) = [\underline{y}-\underline{w}+\overline{z}, \overline{y}-\overline{w}+\underline{z}]$,
(d) $((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z}) = [\overline{y}-\overline{w}+\underline{z}, \underline{y}-\underline{w}+\overline{z}]$.

From this, we have

(a) $\mathbf{0} \ominus_{gH} \{((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z})\} = [\underline{w}-\underline{y}-\underline{z}, \overline{w}-\overline{y}-\overline{z}]$,
(b) $\mathbf{0} \ominus_{gH} \{((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z})\} = [\overline{w}-\overline{y}-\overline{z}, \underline{w}-\underline{y}-\underline{z}]$,
(c) $\mathbf{0} \ominus_{gH} \{((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z})\} = [\overline{w}-\overline{y}-\underline{z}, \underline{w}-\underline{y}-\overline{z}]$,
(d) $\mathbf{0} \ominus_{gH} \{((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z})\} = [\underline{w}-\underline{y}-\overline{z}, \overline{w}-\overline{y}-\underline{z}]$.

On the other hand, we have

(a) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{w}-\underline{y}-\underline{z}, \overline{w}-\overline{y}-\overline{z}]$,
(b) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\overline{w}-\overline{y}-\overline{z}, \underline{w}-\underline{y}-\underline{z}]$,
(c) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\overline{w}-\overline{y}-\underline{z}, \underline{w}-\underline{y}-\overline{z}]$,
(d) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{w}-\underline{y}-\overline{z}, \overline{w}-\overline{y}-\underline{z}]$.

Hence, the desired result follows. □

APPENDIX D. PROOF OF LEMMA 2.6

Proof. Let $\mathbf{X} = [\underline{x}, \overline{x}], \mathbf{Y} = [\underline{y}, \overline{y}]$ and $\mathbf{Z} = [\underline{z}, \overline{z}]$.

(i) Let us consider the following four representations:

(a) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{x} - \underline{y} - \underline{z}, \overline{x} - \overline{y} - \overline{z}]$,

(b) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{x} - \underline{y} - \overline{z}, \overline{x} - \overline{y} - \underline{z}]$,

(c) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\overline{x} - \overline{y} - \underline{z}, \underline{x} - \underline{y} - \overline{z}]$,

(d) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\overline{x} - \overline{y} - \overline{z}, \underline{x} - \underline{y} - \underline{z}]$.

• **Case** 1. Give that $\mathbf{0} \preceq \mathbf{X} \ominus_{gH} \mathbf{Y}$. Then we have

$$\begin{aligned} & 0 \le \underline{x} - \underline{y} \text{ and } 0 \le \overline{x} - \overline{y} \\ \implies & 0 - \underline{z} \le \underline{x} - \underline{y} - \underline{z} \text{ and } 0 - \overline{z} \le \overline{x} - \overline{y} - \overline{z} \\ \implies & [0 - \underline{z}, 0 - \overline{z}] \preceq [\underline{x} - \underline{y} - \underline{z}, \overline{x} - \overline{y} - \overline{z}]. \end{aligned} \tag{D.1}$$

So, from (D.1), we have $\mathbf{0} \ominus_{gH} \mathbf{Z} \preceq (\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z}$.

• **Case** 2. Similarly, we can arrive at this conclusion (D.1). So, from (D.1), we have $\mathbf{0} \ominus_{gH} \mathbf{Z} \preceq (\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z}$.

• **Case** 3. This case can be proved by using the same steps as **Case 1**.

• **Case** 4. This case can be proved by using the same steps as **Case 2**.

(ii) Let $\mathbf{W} = [\underline{w}, \overline{w}]$. By the definition of gH-difference, there may be the following four cases.

(a) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W} = [\underline{x} - \underline{y} - \underline{w}, \overline{x} - \overline{y} - \overline{w}]$,

(b) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W} = [\underline{x} - \underline{y} - \overline{w}, \overline{x} - \overline{y} - \underline{w}]$,

(c) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W} = [\overline{x} - \overline{y} - \underline{w}, \underline{x} - \underline{y} - \overline{w}]$,

(d) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W} = [\overline{x} - \overline{y} - \overline{w}, \underline{x} - \underline{y} - \underline{w}]$.

The following two cases are needed to consider for the representation of these above four cases.

• **Case** 1. Since $\mathbf{Z} \preceq \mathbf{X} \ominus_{gH} \mathbf{Y}$, we have

$$\begin{aligned} & \underline{z} \le \underline{x} - \underline{y} \text{ and } \overline{z} \le \overline{x} - \overline{y} \\ \implies & \underline{z} - \underline{w} \le \underline{x} - \underline{y} - \underline{w} \text{ and } \overline{z} - \overline{w} \le \overline{x} - \overline{y} - \overline{w} \\ \implies & \text{either } [\underline{z} - \underline{w}, \overline{z} - \overline{w}] \preceq [\underline{x} - \underline{y} - \underline{w}, \overline{x} - \overline{y} - \overline{w}] \qquad \text{(D.2)} \\ & \text{or } [\overline{z} - \overline{w}, \underline{z} - \underline{w}] \preceq [\overline{x} - \overline{y} - \overline{w}, \underline{z} - \underline{y} - \underline{w}]. \qquad \text{(D.3)} \end{aligned}$$

From (D.2) and (D.3), we have $\mathbf{Z} \ominus_{gH} \mathbf{W} \preceq (\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W}$.

• **Case** 2. Similarly, at the last step, we have

$$\begin{aligned} & \text{either } [\underline{z} - \underline{w}, \overline{z} - \overline{w}] \preceq [\overline{x} - \overline{y} - \underline{w}, \underline{x} - \underline{y} - \overline{w}] \qquad \text{(D.4)} \\ & \text{or } [\overline{z} - \overline{w}, \underline{z} - \underline{w}] \preceq [\underline{x} - \underline{y} - \overline{w}, \overline{x} - \overline{y} - \underline{w}]. \qquad \text{(D.5)} \end{aligned}$$

From (D.4) and (D.5), we have $\mathbf{Z} \ominus_{gH} \mathbf{W} \preceq (\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W}$.

(iii) Give $\mathbf{X} \ominus_{gH} \mathbf{Y} \preceq [L, L]$. From the formula of gH-difference of intervals,

$$\begin{aligned} & \underline{x} - \underline{y} \le L \text{ and } \overline{x} - \overline{y} \le L \\ \implies & -L \le \underline{y} - \underline{x}, -L \le \overline{y} - \overline{x} \\ \implies & \text{either } [-L, -L] \preceq [\underline{y} - \underline{x}, \overline{y} - \overline{x}] \text{ or } [-L, -L] \preceq [\overline{y} - \overline{x}, \underline{y} - \underline{x}]. \end{aligned}$$

Hence, $[-L, -L] \preceq \mathbf{Y} \ominus_{gH} \mathbf{X}$.

(iv) Give $[-\gamma, -\gamma] \preceq \mathbf{X} \ominus_{gH} \mathbf{Y}$. From the formula of gH-difference of intervals,

$$\begin{aligned} & -\gamma \le \underline{x} - \underline{y} \text{ and } -\gamma \le \overline{x} - \overline{y} \\ \implies & \underline{y} - \gamma \le \underline{x} \text{ and } \overline{y} - \gamma \le \overline{x} \end{aligned}$$

$$\implies [\underline{y}-\gamma, \overline{y}-\gamma] \preceq [\underline{x}, \overline{x}].$$

Hence, $\mathbf{Y} \ominus_{gH} [\gamma, \gamma] \preceq \mathbf{X}$.

(v) Give $\mathbf{Z} \preceq \mathbf{X} \oplus \mathbf{Y}$. Then,

$$\begin{aligned} \underline{z}, \overline{z} \preceq [\underline{x}, \overline{x}] \oplus [\underline{y}, \overline{y}] &\implies \underline{z} \leq \underline{x} + \underline{y}, \overline{z} \leq \overline{x} + \overline{y} \\ &\implies \underline{z} - \underline{y} \leq \underline{x}, \overline{z} - \overline{y} \leq \overline{x} \\ &\implies [\underline{z} - \underline{y}, \overline{z} - \overline{y}] \preceq [\underline{x}, \overline{x}]. \end{aligned}$$

Hence, $\mathbf{Z} \ominus_{gH} \mathbf{Y} \preceq \mathbf{X}$.

□

APPENDIX E. PROOF OF LEMMA 3.1

Proof. Let $y^\top \odot \widehat{\mathbf{C}} = \mathbf{D}$ and $\mathbf{D} = [\underline{d}, \overline{d}]$. Note that

$$\|\mathbf{D}\|_{I(\mathbb{R})} = \max\{|\underline{d}|, |\overline{d}|\}. \tag{E.1}$$

On the other hand,

$$\begin{aligned} \|\mathbf{D}\|_{I(\mathbb{R})} &= \|y_1 \odot \mathbf{C}_1 \oplus y_2 \odot \mathbf{C}_2 \oplus \cdots \oplus y_n \odot \mathbf{C}_n\|_{I(\mathbb{R})} \\ &\leq \|y_1 \odot \mathbf{C}_1\|_{I(\mathbb{R})} + \|y_2 \odot \mathbf{C}_2\|_{I(\mathbb{R})} + \cdots + y_n \odot \mathbf{C}_n\|_{I(\mathbb{R})} \\ &= |y_1|\|\mathbf{C}_1\|_{I(\mathbb{R})} \oplus |y_2|\|\mathbf{C}_2\|_{I(\mathbb{R})} + \cdots + |y_n|\|\mathbf{C}_n\|_{I(\mathbb{R})} \\ &\leq \|y\| \sum_{i=1}^{n} \|\mathbf{C}_i\|_{I(\mathbb{R})} \\ &= \|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n}. \end{aligned} \tag{E.2}$$

Then, taking into account (E.1) and (E.2), we obtain

$$\begin{aligned} & |\underline{d}| \leq \|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \text{ and } |\overline{d}| \leq \|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \\ \implies & -\|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \leq \underline{d} \text{ and } -\|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \leq \overline{d} \\ \implies & -\|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \leq |\underline{d}| \text{ and } -\|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \leq |\overline{d}| \\ \implies & -\|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \leq \max\{|\underline{d}|, |\overline{d}|\} \\ \implies & -\|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \leq \|\mathbf{D}\|_{I(\mathbb{R})}. \end{aligned}$$

Thus, we arrived at the desired result. □

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