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THE LEVEL-SET SUBDIFFERENTIAL ERROR BOUND VIA MOREAU ENVELOPES

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Abstract. The level-set subdifferential error bound (LSEB) is weaker than the Kurdyka-Łojasiewicz (KL) property and can replace it to establish linear convergence for various first-order algorithms. In this paper, we mainly study the behaviour of the level-set subdifferential error bound via Moreau envelopes under suitable assumptions. We provide an example that the Moreau envelope does not have the KL property but has the LSEB when the original function does not satisfy the KL property but only the LSEB.

Keywords. The Kurdyka-Łojasiewicz property; Level-set subdifferential error bound; Local Hölder error bound; Moreau envelope.

1. Introduction

The theory of error bounds has long been known to be important in variational analysis and optimization theory [1, 2], and is central to subdifferential calculus, exact penalty functions, stability and sensitivity analysis, optimality conditions, and convergence analysis and convergence rate analysis of various iterative methods [3–12] and the references therein. There are many types of error bounds that have been widely studied in recent years, such as the Kurdyka-Łojasiewicz (KL) property (see [6, 9, 12–15]), the level-set subdifferential error bound (LSEB) (see [8, 14, 16]), and the local Hölder error bound (LHEB) (see [8, 14, 17–20]). Under the proper and lower semicontinuous function assumption, the LSEB is weaker than the KL property and stronger than the LHEB, which can be found in [8, 13, 17, 21, 22] and the references therein. Under the convexity assumption, the authors in [13] obtained the equivalence between the KL property and the LHEB, and the authors in [8] showed the equivalence between the LSEB with exponent 1 and the LHEB with exponent 2. Kruger et al. [17] proved that the existence of the KL property implies the existence of the LSEB and the existence of the LSEB implies the existence of the LHEB in Asplund spaces or the function is convex. Recently, under a weak convexity assumption, Zhu et al. established the equivalence of the KL property and the LSEB in [16].

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Bai et al. established the equivalence among the KL property, the LSEB, and the LHEB for weakly convex functions with exponent $\alpha \in [0,1]$ and approximately convex functions in [14].

The KL property has been used by many researchers to analyze the local convergence rate of various first-order methods for a wide variety of problems; see [4, 9, 12, 23–26]. And the KL exponent is closely related to the convergence rate. Recently, by developing several important calculus rules for the KL exponent, Li and Pong [9] estimated the KL exponents for the functions appearing in structured optimization problems. This research line was further explored by Yu et al. [12]. They proved that the KL exponent can be preserved under the inf-projection operation, which is a significant generalization of the operation of taking the minimum of finitely many functions. Additionally, they demonstrated that the KL exponent can be preserved under the Bregman envelope, which is a generalised form of the Moreau envelope. Moreau envelope, which was first mentioned in [27], plays an important role in the structure of the algorithm. Since then, it has been widely studied by many scholars; see, e.g. [8, 9, 12, 28–32] and the references therein. However, there is no investigation on whether the LSEB and the LHEB properties of the original function are maintained via the Moreau envelope.

In this paper, motivated by [9, 12, 14, 16], we first establish the equivalence among the KL property, the LSEB, and the LHEB for a prox-regular function, which is a generalisation of the corresponding conclusion in [14]. Then we show that the LSEB (u-LSEB) and the LHEB (u-HEB) exponents and constants are expressed via the Moreau envelopes of a function, which is restrained by that of the function itself. It differs from the corresponding result in [12]. We also give some examples to illustrate our results.

The rest of this paper is structured as follows. Section 2 provides notations and preliminaries, and gives some results on the uniform error bounds. Section 3, which is also the last section, presents the different behavior under the Moreau envelopes of a proper and closed function. Two examples are also presented in this section to illustrate our main results.

2. Preliminaries

Throughout this paper, we use $\mathbb{R}^n$ to denote the n-dimensional Euclidean space with scalar product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$. Let $B(c,r)$ denote the open ball with centered at c and radius r. For an nonempty set $A \subseteq \mathbb{R}^n$, we denote the distance from a point $x \in \mathbb{R}^n$ to A by

$$d(x,A) = \inf_{y \in A} \|x - y\|.$$

The set of points in A that achieve this infimum is called the projection of x onto A and denoted by $\mathrm{Proj}_A(x)$. For an empty set A, we define $d(x,A) = +\infty$.

An extended real valued function $f : \mathbb{R}^n \to \bar{\mathbb{R}} := (-\infty, \infty]$ is called proper if its domain is nonempty, $\mathrm{dom}\, f := \{x : f(x) < \infty\} \neq \emptyset$. The function f is called closed if the inequality $\liminf_{x \to \bar{x}} f(x) \geq f(\bar{x})$ holds for any point $\bar{x} \in \mathbb{R}^n$. The level set of f

$$\mathrm{lev}_{\leq \alpha} f := \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$$

is closed for a closed function f and $\alpha \in \mathbb{R}$. A proper and closed function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to satisfy a Lipschitz condition of rank K on a given set S provided that f is finite on S and satisfies

$$|f(x) - f(y)| \leq K\|x - y\| \;\forall x, y \in S.$$

The regular subdifferential and limiting subdifferential of f at $\bar{x} \in \operatorname{dom} f$ are defined in [2, Definition 8.3], respectively, by

$$\hat{\partial} f(\bar{x}) := \left\{ v \in \mathbb{R}^n : \liminf_{x \to \bar{x}, x \neq \bar{x}} \frac{f(x) - f(\bar{x}) - \langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}$$

and

$$\partial f(\bar{x}) := \left\{ v \in \mathbb{R}^n : \exists x_k \xrightarrow{f} \bar{x}, v_k \to v \text{ with } v_k \in \hat{\partial} f(x_k) \text{ for each } k \right\},$$

where $x_k \xrightarrow{f} \bar{x}$ means both $x_k \to \bar{x}$ and $f(x_k) \to f(\bar{x})$. By the definitions of subdifferentials, the inclusion relationship $\hat{\partial} f(x) \subseteq \partial f(x)$ always holds. By convention, we set $\partial f(x) = \emptyset$ for $x \notin \operatorname{dom} f$, and write $\operatorname{dom} \partial f := \{x \in \mathbb{R}^n : \partial f(x) \neq \emptyset\}$.

A proper and closed function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be prox-regular at $\bar{x}$ for $\bar{v}$ with a constant ρ if $\bar{x} \in \operatorname{dom} f$ with $\bar{v} \in \partial f(\bar{x})$, and there exist $\varepsilon > 0$ and $\rho \geq 0$ such that

$$f(y) \geq f(x) + \langle v, y - x \rangle - \frac{\rho}{2} \|y - x\|^2 \ \forall y \in B(\bar{x}, \varepsilon)$$

when $v \in \partial f(x) \cap B(\bar{v}, \varepsilon)$ and $x \in B(\bar{x}, \varepsilon)$ with $f(x) < f(\bar{x}) + \varepsilon$.

For simplicity, for any $\bar{x} \in \operatorname{dom} f$ and $\nu, \eta > 0$,

$$[f(\bar{x}) < f < f(\bar{x}) + \nu] := \left\{ x \in \mathbb{R}^n : f(\bar{x}) < f(x) < f(\bar{x}) + \nu \right\}$$

and

$$\mathfrak{B}(\bar{x}, \eta, \nu) := B(\bar{x}, \eta) \cap [f(\bar{x}) < f < f(\bar{x}) + \nu].$$

Recall that a proper and closed function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ satisfies the Kurdyka-Łojasiewicz (KL) property or Łojasiewicz inequality at $\bar{x} \in \operatorname{dom} \partial f$ with an exponent $\gamma \in [0, 1)$ and a constant $\mu > 0$ if there exist $\eta > 0$ and $\nu > 0$ such that

$$\big(f(x) - f(\bar{x})\big)^{\gamma} \leq \mu d\big(0, \partial f(x)\big) \ \forall x \in \mathfrak{B}(\bar{x}, \eta, \nu).$$

The function f satisfies the level-set subdifferential error bound (LSEB) at $\bar{x} \in \operatorname{dom} \partial f$ with an exponent $\gamma \geq 0$ and a constant $\mu > 0$ if there exist $\eta > 0$ and $\nu > 0$ such that

$$d^{\gamma}(x, \operatorname{lev}_{\leq f(\bar{x})} f) \leq \mu d\big(0, \partial f(x)\big) \ \forall x \in \mathfrak{B}(\bar{x}, \eta, \nu).$$

And the function f satisfies the local Hölder error bound (LHEB) at $\bar{x} \in \operatorname{dom} f$ with an exponent $\gamma > 0$ and a constant $\mu > 0$ if there exists $\eta > 0$ such that

$$d^{\gamma}(x, \operatorname{lev}_{\leq f(\bar{x})} f) \leq \mu \big(f(x) - f(\bar{x})\big)$$

for any $x \in B(\bar{x}, \eta)$ with $f(\bar{x}) < f(x)$.

Now, we give the following definition of the uniform Kurdyka-Łojasiewicz property, which is inspired by [12, Lemma 2.2].

Definition 2.1. Let $\Omega \subseteq \operatorname{dom} \partial f$ be a nonempty set. A proper and closed function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ satisfies the uniform Kurdyka-Łojasiewicz (u-KL) property on Ω with an exponent $0 \leq \gamma < 1$ and a constant $\mu > 0$ if there exist $\varepsilon > 0$ and $\nu > 0$ such that

$$\big(f(x) - f(\bar{x})\big)^{\gamma} \leq \mu d\big(0, \partial f(x)\big)$$

for any $\bar{x} \in \Omega$ and any x with $d(x, \Omega) < \varepsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \nu$.

Motivated by Definition 2.1, we introduce the following definitions.

Definition 2.2. Let $\Omega \subseteq \operatorname{dom}\partial f$ be a nonempty set. The proper and closed function f satisfies the uniform level-set subdifferential error bound (u-LSEB) on Ω with an exponent $\gamma \geq 0$ and a constant $\mu > 0$ if there exist $\varepsilon > 0$ and $\nu > 0$ such that

$$d^{\gamma}(x, \operatorname{lev}_{\leq f(\bar{x})} f) \leq \mu d\big(0, \partial f(x)\big)$$

for any $\bar{x} \in \Omega$ and any x with $d(x,\Omega) < \varepsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \nu$.

Definition 2.3. Let $\Omega \subseteq \operatorname{dom} f$ be a nonempty set. The proper and closed function f satisfies the uniform Hölder error bound (u-HEB) on Ω with an exponent $\gamma > 0$ and a constant $\mu > 0$ if there exists $\varepsilon > 0$ such that

$$d^{\gamma}(x, \operatorname{lev}_{\leq f(\bar{x})} f) \leq \mu\big(f(x) - f(\bar{x})\big)$$

for any $\bar{x} \in \Omega$ and any x with $d(x,\Omega) < \varepsilon$ and $f(\bar{x}) < f(x)$.

Remark 2.1. Note that f satisfying the uniform error bounds (u-KL property, u-LSEB, and u-HEB) on Ω is stronger than f satisfying the error bounds (KL property, LSEB, and LHEB) at each point in Ω, since the uniform error bounds require that the error bounds exponents and constants are the same at each point in Ω, but the latter does not. If Ω is a compact set and f takes a constant value on Ω, they are equivalent. See the conclusions below.

Lemma 2.1. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper and closed function and $\Omega \subseteq \operatorname{dom}\partial f$ be a nonempty and compact set. If f takes a constant value on Ω and satisfies the LSEB at each point of Ω. Then f satisfies the u-LSEB on Ω with an exponent $\gamma \geq 0$ and a constant $\mu > 0$.*

Proof. For any $\bar{x} \in \Omega$. Since f takes a constant value on Ω and satisfies the LSEB at each point of Ω, then, for any $z \in \Omega$, there exist $\gamma_z \geq 0$ and $\mu_z, \varepsilon_z, \nu_z > 0$ such that

$$d^{\gamma_z}(x, \operatorname{lev}_{\leq f(\bar{x})} f) \leq \mu_z d\big(0, \partial f(x)\big) \ \forall x \in \mathfrak{B}(z, \varepsilon_z, \nu_z).$$

Hence, $\{B(z_i, \frac{\varepsilon_{z_i}}{2}) : z_i \in \Omega\}$ is an open cover of Ω. It follows from the compactness of Ω that there exist $z_i \in \Omega$, $i = 1, ..., p$ such that

$$\Omega \subseteq \bigcup_{i=1}^{p} B(z_i, \frac{\varepsilon_{z_i}}{2}). \tag{2.1}$$

Set $\nu := \min\{\nu_{z_i} : i = 1, ..., p\}$, $\varepsilon := \min\{\varepsilon_{z_i} : i = 1, ..., p\}$ and $U_\varepsilon := \{x \in \mathbb{R}^n : d\big(x,\Omega\big) < \frac{\varepsilon}{2}\}$. For any $x \in U_\varepsilon$, it follows from (2.1) that there exists an index $j \in \{1, ...p\}$ such that $x_j \in B(z_j, \frac{\varepsilon_{z_j}}{2})$, where x_j is one of the projection of x onto Ω. Then we have

$$\|x - z_j\| \leq \|x - x_j\| + \|x_j - z_j\| < \frac{\varepsilon}{2} + \frac{\varepsilon_{z_j}}{2} \leq \varepsilon_{z_j},$$

which means that $U_\varepsilon \subseteq \bigcup_{i=1}^{p} B(z_i, \varepsilon_{z_i})$ and

$$U_\varepsilon \cap [f(\bar{x}) < f < f(\bar{x}) + \nu] \subseteq \bigcup_{i=1}^{p} \left\{ B(z_i, \varepsilon_{z_i}) \cap [f(\bar{x}) < f < f(\bar{x}) + \nu_{z_i}] \right\} = \bigcup_{i=1}^{p} \mathfrak{B}(z_i, \varepsilon_{z_i}, \nu_{z_i}).$$

Then, for any $x \in U_\varepsilon \cap [f(\bar{x}) < f < f(\bar{x}) + \nu]$, one has

$$d^{\gamma}(x, \operatorname{lev}_{\leq f(\bar{x})} f) \leq \mu d\big(0, \partial f(x)\big),$$

where $\gamma := \max\{\gamma_{z_i} : i = 1, ..., p\}$ and $\mu := \max\{\mu_{z_i} : i = 1, ..., p\}$. The proof is completed. □

The following proof is similar to the previous lemma, so we omit it here.

Lemma 2.2. *Let $f:\mathbb{R}^n\to\bar{\mathbb{R}}$ be a proper and closed function, and let $\Omega\subseteq \mathrm{dom} f$ be a nonempty and compact set. If f takes a constant value on Ω and satisfies the LHEB at each point of Ω, then f satisfies the u-HEB on Ω with an exponent $\gamma>0$ and a constant $\mu>0$.*

Next, we recall the following lemma which follows from the Theorem 2.1 in [14].

Lemma 2.3. *Let f be a proper and closed function and $\bar{x}\in\mathbb{R}^n$. Assume that f satisfies the KL property at $\bar{x}$ with an exponent $\gamma_1\in[0,1)$ and a constant $\mu_1>0$. Then, f satisfies LSEB at $\bar{x}$ with an exponent $\gamma_2=\frac{\gamma_1}{1-\gamma_1}$ and a constant $\mu_2=(1-\gamma_1)^{\frac{-\gamma_1}{1-\gamma_1}}\mu_1^{\frac{1}{1-\gamma_1}}$. Furthermore, f satisfies LHEB at $\bar{x}$ with an exponent $\gamma_3=\gamma_2+1$ and a constant $\mu_3=\frac{(\gamma_2+1)^{\gamma_2+1}}{\gamma_2^{\gamma_2}}\mu_2$.*

Motivated by [14, Theorem 2.1], we give the following result.

Proposition 2.1. *Let $f:\mathbb{R}^n\to\bar{\mathbb{R}}$ be a proper and closed function and $\bar{x}\in(\partial f)^{-1}(0)$. Assume that f is prox-regular at $\bar{x}$ for $\bar{v}=0$ with a constant $\rho\geq 0$. Consider the following conditions:*

(i) f satisfies the KL property at $\bar{x}$ with an exponent $1>\gamma_1\geq 0$ and a constant $\mu_1>0$;
(ii) f satisfies the LSEB at $\bar{x}$ with an exponent $\gamma_2\geq 0$ and a constant $\mu_2>0$;
(iii) f satisfies the LHEB at $\bar{x}$ with an exponent $\gamma_3>0$ and a constant $\mu_3>0$.

Then, we have the following results:

(a) If $1\leq\gamma_3<2$, then (iii) $\Rightarrow$ (ii) with the exponent $\gamma_2=\gamma_3-1$ and the constant $\mu_2>\mu_3$; If $\gamma_3=2$ and $\mu_3<\frac{2}{\rho}$, then (iii) $\Rightarrow$ (ii) with the exponent $\gamma_2=1$ and the constant $\mu_2=\frac{2\mu_3}{2-\rho\mu_3}$;
(b) (ii) $\Rightarrow$ (i) with the exponent $\gamma_1=\max\{\frac{\gamma_2}{1+\gamma_2},\frac{\gamma_2}{2}\}$ and the constant

$$\mu_1=(\mu_2^{\frac{1}{\gamma_2}}+\frac{\rho}{2}\mu_2^{\frac{2}{\gamma_2}})^{\max\{\frac{\gamma_2}{1+\gamma_2},\frac{\gamma_2}{2}\}}.$$

Proof. (a) The proof is similar to the Theorem 2.1 in [14] and we omit it.

(b) Assume that f does not have the KL property at $\bar{x}$ with the exponent $\gamma_1=\max\{\frac{\gamma_2}{1+\gamma_2},\frac{\gamma_2}{2}\}$ and the constant $\mu_1=(\mu_2^{\frac{1}{\gamma_2}}+\frac{\rho}{2}\mu_2^{\frac{2}{\gamma_2}})^{\max\{\frac{\gamma_2}{1+\gamma_2},\frac{\gamma_2}{2}\}}$. Then there exist a sequence $\{x_k\}$ with $x_k\to\bar{x}$ and $f(x_k)\downarrow f(\bar{x})$ such that

$$\big(f(x_k)-f(\bar{x})\big)^{\gamma_1}>\mu_1 d\big(0,\partial f(x_k)\big)\ \forall k\in\mathbb{N}. \tag{2.2}$$

Since $\partial f(x_k)$ is closed, we can choose that $v_k\in\partial f(x_k)$ with $d\big(0,\partial f(x_k)\big)=\|v_k\|\to 0$. In view of (2.2), one has

$$\|v_k\|^{\max\{\frac{\gamma_2}{1+\gamma_2},\frac{\gamma_2}{2}\}^{-1}}<\mu_1^{-\max\{\frac{\gamma_2}{1+\gamma_2},\frac{\gamma_2}{2}\}^{-1}}\big(f(x_k)-f(\bar{x})\big). \tag{2.3}$$

Since f is prox-regular at $\bar{x}$ for $\bar{v}=0$ with a constant $\rho\geq 0$, one sees that there exists $\varepsilon>0$ such that

$$f(y)\geq f(x)+\langle v,y-x\rangle-\frac{\rho}{2}\|y-x\|^2\ \forall y\in B(\bar{x},\varepsilon)$$

when $v\in\partial f(x)\cap B(0,\varepsilon)$ and $x\in B(\bar{x},\varepsilon)$ with $f(x)<f(\bar{x})+\varepsilon$. Let $y_k\in \mathrm{Proj}_{\mathrm{lev}_{\leq f(\bar{x})}f}(x_k)$. Then $y_k\to\bar{x}$ and $\|y_k-x_k\|\to 0$ as $k\to\infty$ due to $\|y_k-\bar{x}\|\leq\|y_k-x_k\|+\|x_k-\bar{x}\|\leq 2\|x_k-\bar{x}\|$ and

$x_k \to \bar{x}$. Then there exists k large enough such that $x_k, y_k \in B(\bar{x}, \varepsilon)$, $f(x_k) < f(\bar{x}) + \varepsilon$, $v_k \in \partial f(x_k)$ with $d\big(0, \partial f(x_k)\big) = \|v_k\| < \varepsilon$, and

$$f(y_k) \geq f(x_k) + \langle v_k, y_k - x_k \rangle - \frac{\rho}{2}\|y_k - x_k\|^2.$$

It follows that

$$\begin{aligned} d\big(0, \partial f(x_k)\big) &\geq \frac{f(x_k) - f(y_k)}{\|y_k - x_k\|} - \frac{\rho}{2}\|y_k - x_k\| \\ &\geq \frac{f(x_k) - f(\bar{x})}{\|y_k - x_k\|} - \frac{\rho}{2}\|y_k - x_k\|. \end{aligned}$$

Since f satisfies the LSEB at $\bar{x}$ with an exponent $\gamma_2 \geq 0$ and a constant $\mu_2 > 0$, we can choose a suitable neighborhood of $\bar{x}$ such that

$$d^{\gamma_2}(x_k, \mathrm{lev}_{\leq f(\bar{x})} f) = \|y_k - x_k\|^{\gamma_2} \leq \mu_2 d\big(0, \partial f(x_k)\big) = \mu_2 \|v_k\|,$$

which means that $\|y_k - x_k\| \leq \mu_2^{\frac{1}{\gamma_2}} \|v_k\|^{\frac{1}{\gamma_2}}$. Combining (2.3), (2.4), $\gamma_1 = \max\{\frac{\gamma_2}{1+\gamma_2}, \frac{\gamma_2}{2}\}$, and $\mu_1 = (\mu_2^{\frac{1}{\gamma_2}} + \frac{\rho}{2}\mu_2^{\frac{2}{\gamma_2}})^{\max\{\frac{\gamma_2}{1+\gamma_2}, \frac{\gamma_2}{2}\}}$, one has

$$\begin{aligned} f(x_k) - f(\bar{x}) &\leq \mu_2^{\frac{1}{\gamma_2}} \|v_k\|^{\frac{1+\gamma_2}{\gamma_2}} + \frac{\rho}{2}\mu_2^{\frac{2}{\gamma_2}} \|v_k\|^{\frac{2}{\gamma_2}} \\ &\leq \big(\mu_2^{\frac{1}{\gamma_2}} + \frac{\rho}{2}\mu_2^{\frac{2}{\gamma_2}}\big) \|v_k\|^{\min\{\frac{1+\gamma_2}{\gamma_2}, \frac{2}{\gamma_2}\}} \\ &= \big(\mu_2^{\frac{1}{\gamma_2}} + \frac{\rho}{2}\mu_2^{\frac{2}{\gamma_2}}\big) \|v_k\|^{\max\{\frac{\gamma_2}{1+\gamma_2}, \frac{\gamma_2}{2}\}^{-1}} \\ &< \big(\mu_2^{\frac{1}{\gamma_2}} + \frac{\rho}{2}\mu_2^{\frac{2}{\gamma_2}}\big) \mu_1^{-\max\{\frac{\gamma_2}{1+\gamma_2}, \frac{\gamma_2}{2}\}^{-1}} \big(f(x_k) - f(\bar{x})\big) \\ &= f(x_k) - f(\bar{x}), \end{aligned}$$

which is a contradiction. The proof is completed. □

Remark 2.2. Since the weakly convex functions are prox-regular functions, Proposition 2.1 generalizes the Theorem 2.1 in [14]. And we improve the proof of Theorem 2.1 (e) and (f) in [14] so that the LSEB exponent is not bounded by the weakly convex exponent.

Now we consider the equivalence among the u-KL property, the u-LSEB and the u-HEB.

Proposition 2.2. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper and closed function, and let $\Omega \subseteq \mathrm{dom}\,\partial f$ be a nonempty and compact set. Suppose that f takes a constant value on Ω. Consider the following conditions:*

(i) f satisfies the u-KL property on Ω with an exponent $0 \leq \gamma_1 < 1$ and a constant $\mu_1 > 0$;
(ii) f satisfies the u-LSEB on Ω with an exponent $\gamma_2 \geq 0$ and a constant $\mu_2 > 0$;
(iii) f satisfies the u-HEB on Ω with an exponent $\gamma_3 > 0$ and a constant $\mu_3 > 0$.

Then, we have the following results:

(a) (i) $\Rightarrow$ (ii) with the exponent $\gamma_2 = \frac{\gamma_1}{1-\gamma_1}$ and the constant $\mu_2 = (1-\gamma_1)^{\frac{-\gamma_1}{1-\gamma_1}} \mu_1^{\frac{1}{1-\gamma_1}}$;
(b) (ii) $\Rightarrow$ (iii) with the exponent $\gamma_3 = \gamma_2 + 1$ and the constant $\mu_3 = \frac{(\gamma_2+1)^{\gamma_2+1}}{\gamma_2^{\gamma_2}}\mu_2$.

Moreover, if f is prox-regular at every point of Ω with a constant $\rho \geq 0$, then we have the following results:

- *(c) If $1 \leq \gamma_3 < 2$, then (iii) $\Rightarrow$ (ii) with the exponent $\gamma_2 = \gamma_3 - 1$ and the constant $\mu_2 > \mu_3$; If $\gamma_3 = 2$ and $\mu_3 < \frac{2}{\rho}$, then (iii) $\Rightarrow$ (ii) with the exponent $\gamma_2 = 1$ and the constant $\mu_2 = \frac{2\mu_3}{2-\rho\mu_3}$;*
- *(d) (ii) $\Rightarrow$ (i) with the exponent $\gamma_1 = \max\{\frac{\gamma_2}{1+\gamma_2}, \frac{\gamma_2}{2}\}$ and the constant*

$$\mu_1 = (\mu_2^{\frac{1}{\gamma_2}} + \frac{\rho}{2}\mu_2^{\frac{2}{\gamma_2}})^{\max\{\frac{\gamma_2}{1+\gamma_2}, \frac{\gamma_2}{2}\}}.$$

Proof. By Lemmas 2.1 and 2.3, (a) is established immediately. (b) holds by the Lemmas 2.2 and 2.3. (c) follows from Lemma 2.1 and Proposition 2.1 (a). (d) holds by Lemma 2.2 and Proposition 2.1 (b). □

3. BEHAVIOR OF ERROR BOUNDS VIA MOREAU ENVELOPES

In this section, we consider the Moreau envelopes and proximal mappings which were defined in [2, Definition 1.22]. For a proper and closed function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and parameter $\lambda > 0$, the Moreau envelope $e_\lambda f$ and proximal mapping $P_\lambda f(x)$ are defined, respectively, by

$$e_\lambda f(x) := \inf_{y\in\mathbb{R}^n} \{f(y) + \frac{1}{2\lambda} \| y - x \|^2 \}$$

and

$$P_\lambda f(x) := \arg\min_{y\in\mathbb{R}^n} \{f(y) + \frac{1}{2\lambda} \| y - x \|^2 \}.$$

A function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is prox-bounded if there exists $\lambda > 0$ such that $e_\lambda f(x) > -\infty$ for some $x \in \mathbb{R}^n$. The supremum of the set of all such λ is the threshold λ_f of prox-boundedness for f; see [2, Definition 1.23].

It immediately follows from [9, Lemma 2.1] and Lemma 2.3 that $e_\lambda f$ satisfies the KL property with an exponent 0, the LSEB with an exponent 0, and the LHEB with an exponent 1 at x with $0 \notin \partial e_\lambda f(x)$. So we only consider the point $\bar{x}$ with $0 \in \partial e_\lambda f(\bar{x})$ in what follows.

A local minimum occurs at $\bar{x} \in \operatorname{dom} f$ if $f(x) \geq f(\bar{x})$ for all $x \in V$, where V is a neighborhood of $\bar{x}$. At first, we consider the behavior of the LSEB via Moreau envelopes.

Theorem 3.1. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper and closed function and prox-bounded with threshold λ_f. Assume that the following conditions hold:*

- *(i) For any $\lambda \in (0, \lambda_f)$, $\bar{x} \in \operatorname{dom} f$ with $0 \in \partial e_\lambda f(\bar{x})$;*
- *(ii) f satisfies the LSEB at $\bar{x}$ with an exponent $\gamma > 0$ and a constant $\mu > 0$.*

Then $e_\lambda f$ satisfies the LSEB at $\bar{x}$ with an exponent $\max\{1,\gamma\}$ and a constant $\lambda\left(1 + (\frac{\mu}{\lambda})^{\gamma^{-1}}\right)^{\max\{1,\gamma\}}$.

Proof. For any $\bar{x} \in \operatorname{dom} f$ with $0 \in \partial e_\lambda f(\bar{x})$, we have $0 \in \lambda^{-1}\big(\bar{x} - P_\lambda f(\bar{x})\big)$ by condition (i) and [2, Example 10.32]. Then $\bar{x} \in P_\lambda f(\bar{x})$ and $f(\bar{x}) = e_\lambda f(\bar{x})$.

Next, since f satisfies the LSEB at $\bar{x}$ with an exponent $\gamma > 0$ and a constant $\mu > 0$, we find that there exist $\eta > 0$ and $\nu > 0$ such that

$$d^\gamma(x, \operatorname{lev}_{\leq f(\bar{x})} f) \leq \mu d\big(0, \partial f(x)\big) \ \forall x \in \mathfrak{B}(\bar{x}, \eta, \nu). \tag{3.1}$$

For any $x \in B(\bar{x}, \tilde{\eta})$ with $e_\lambda f(\bar{x}) < e_\lambda f(x) < e_\lambda f(\bar{x}) + \tilde{\nu}$, where $\tilde{\eta} \in (0, \frac{\eta}{2})$ and $\tilde{\nu} \in (0, \nu)$. Then, there exists a point $y \in P_\lambda f(x)$ such that $d\big(x, P_\lambda f(x)\big) = \| x - y \|$. Obviously, one has

$$f(y) \le e_\lambda f(x) < e_\lambda f(\bar{x}) + \tilde{\nu} < f(\bar{x}) + \nu$$

and $\frac{1}{\lambda}(x - y) \in \partial f(y)$. If $f(y) \le f(\bar{x})$, it means that $y \in \mathrm{lev}_{\le f(\bar{x})} f$ and

$$d(x, \mathrm{lev}_{\le f(\bar{x})} f) \le \| x - y \| . \tag{3.2}$$

If $f(\bar{x}) < f(y) < f(\bar{x}) + \nu$, we have

$$f(y) + \frac{1}{2\lambda} \| y - x \|^2 \le f(\bar{x}) + \frac{1}{2\lambda} \| \bar{x} - x \|^2,$$

due to $y \in P_\lambda f(x)$. It follows from $f(\bar{x}) < f(y)$ and $\lambda > 0$ that we have $\| y - x \| < \| x - \bar{x} \| < \frac{\eta}{2}$. Hence $\| y - \bar{x} \| \le \| y - x \| + \| x - \bar{x} \| < \eta$, which means that $y \in \mathfrak{B}(\bar{x}, \eta, \nu)$ and (3.1) holds. Then, for any such x, we have

$$\begin{aligned} d(x, \mathrm{lev}_{\le f(\bar{x})} f) &\le \| x - y \| + d(y, \mathrm{lev}_{\le f(\bar{x})} f) \\ &\le \| x - y \| + \big(\mu d\big(0, \partial f(y)\big)\big)^{\gamma^{-1}} \\ &\le \| x - y \| + \big(\frac{\mu}{\lambda} \| x - y \| \big)^{\gamma^{-1}}. \end{aligned} \tag{3.3}$$

The last inequality holds with $\frac{1}{\lambda}(x - y) \in \partial f(y)$. Combining (3.2) and (3.3), for any $x \in B(\bar{x}, \tilde{\eta})$ with $e_\lambda f(\bar{x}) < e_\lambda f(x) < e_\lambda f(\bar{x}) + \tilde{\nu}$ and $y \in \mathrm{Proj}_{P_\lambda f(x)}(x)$, one has

$$d(x, \mathrm{lev}_{\le f(\bar{x})} f) \le \| x - y \| + \big(\frac{\mu}{\lambda} \| x - y \| \big)^{\gamma^{-1}}.$$

Shrink η if necessary so that $\| x - y \| < \eta \le 1$. If $0 < \gamma < 1$, then

$$d(x, \mathrm{lev}_{\le f(\bar{x})} f) \le \big(1 + (\frac{\mu}{\lambda})^{\gamma^{-1}}\big) \| x - y \| .$$

Setting $\tilde{\mu} := \lambda \big(1 + (\frac{\mu}{\lambda})^{\gamma^{-1}}\big)$. Then for any $x \in B(\bar{x}, \tilde{\eta})$ with $e_\lambda f(\bar{x}) < e_\lambda f(x) < e_\lambda f(\bar{x}) + \tilde{\nu}$, one has

$$\begin{aligned} \tilde{\mu} d\big(0, \partial e_\lambda f(x)\big) &\ge \tilde{\mu} d\big(0, \lambda^{-1}(x - P_\lambda f(x))\big) = \frac{\tilde{\mu}}{\lambda} \| x - y \| \\ &\ge d(x, \mathrm{lev}_{\le f(\bar{x})} f) \\ &\ge d(x, \mathrm{lev}_{\le e_\lambda f(\bar{x})} e_\lambda f). \end{aligned} \tag{3.4}$$

The first inequality follows from $\partial e_\lambda f(x) \subseteq \lambda^{-1}\big(x - P_\lambda f(x)\big)$; see [2, Example 10.32]; the last inequality holds with the fact $e_\lambda f(z) \le f(z) \le f(\bar{x}) = e_\lambda f(\bar{x})$ for any $z \in \mathrm{lev}_{\le f(\bar{x})} f$ and then $\mathrm{lev}_{f(\bar{x})} f \subseteq \mathrm{lev}_{\le e_\lambda f(\bar{x})} e_\lambda f$.

If $\gamma \ge 1$, one has $d^\gamma(x, \mathrm{lev}_{\le f(\bar{x})} f) \le \big(1 + (\frac{\mu}{\lambda})^{\gamma^{-1}}\big)^\gamma \| x - y \|$ for any $x \in B(\bar{x}, \tilde{\eta})$ with $e_\lambda f(\bar{x}) < e_\lambda f(x) < e_\lambda f(\bar{x}) + \tilde{\nu}$. Taking $\bar{\mu} := \lambda \big(1 + (\frac{\mu}{\lambda})^{\gamma^{-1}}\big)^\gamma$, then for every such x, one has

$$\begin{aligned} \bar{\mu} d\big(0, \partial e_\lambda f(x)\big) &\ge \bar{\mu} d\big(0, \lambda^{-1}(x - P_\lambda f(x))\big) = \frac{\bar{\mu}}{\lambda} \| x - y \| \\ &\ge d^\gamma(x, \mathrm{lev}_{\le f(\bar{x})} f) \\ &\ge d^\gamma(x, \mathrm{lev}_{\le e_\lambda f(\bar{x})} e_\lambda f). \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we have the desired conclusion immediately. □

Remark 3.1. If the condition (ii) in Theorem 3.1 is replaced as follows: f satisfies the LSEB at $\bar{x}$ with an exponent $\gamma = 0$ and a constant $\mu > 0$. It is easy to see that $e_\lambda f$ satisfies the LSEB at $\bar{x}$ with the exponent 1 and the constant $\mu + \lambda$.

Under suitable assumptions, we now consider the behaviour of the LHEB via Moreau envelopes.

Theorem 3.2. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper and closed function and prox-bounded with threshold λ_f. Assume that the following conditions hold:*

(i) for any $\lambda \in (0, \lambda_f)$, $\bar{x} \in \operatorname{dom} f$ with $0 \in \partial e_\lambda f(\bar{x})$;
(ii) f satisfies the LHEB at $\bar{x}$ with an exponent $\gamma > 0$ and a constant $\mu \geq 2\lambda$;
(iii) for $\bar{\varepsilon} > 0$ and any $x \in B(\bar{x}, \bar{\varepsilon})$ with $f(x) > f(\bar{x})$, there exists a point $y \in P_\lambda f(x)$ such that $f(y) \geq f(\bar{x})$.

Then $e_\lambda f$ satisfies the LHEB at $\bar{x}$ with an exponent $\max\{2, \gamma\}$ *and a constant* $2^{\max\{1,\gamma-1\}}\mu$.

Proof. By the condition (i) and the Example 10.32 in [2], for any $\bar{x} \in \operatorname{dom} f$ with $0 \in \partial e_\lambda f(\bar{x})$, we have $\bar{x} \in P_\lambda f(\bar{x})$ and $f(\bar{x}) = e_\lambda f(\bar{x})$. Since f satisfies the LHEB at $\bar{x}$ with an exponent $\gamma > 0$ and a constant $\mu \geq 2\lambda$, we sees that there exists $0 < \eta < 1$ such that

$$d^\gamma(x, \operatorname{lev}_{\leq f(\bar{x})} f) \leq \mu\big(f(x) - f(\bar{x})\big) \tag{3.6}$$

for any $x \in B(\bar{x}, \eta)$ with $f(\bar{x}) \leq f(x)$. Take $\varepsilon := \min\{\frac{\eta}{2}, \bar{\varepsilon}\}$. For any $x \in B(\bar{x}, \varepsilon)$ with $e_\lambda f(\bar{x}) < e_\lambda f(x)$, by $e_\lambda f(x) \leq f(x)$ and condition (iii), we have that there exists a point $y \in P_\lambda f(x)$ such that $f(y) \geq f(\bar{x})$ and

$$f(y) + \frac{1}{2\lambda} \parallel y - x \parallel^2 \leq f(\bar{x}) + \frac{1}{2\lambda} \parallel \bar{x} - x \parallel^2,$$

which means that $\parallel y - x \parallel \leq \parallel x - \bar{x} \parallel < \frac{\eta}{2}$. Then $\parallel y - \bar{x} \parallel \leq \parallel y - x \parallel + \parallel x - \bar{x} \parallel < \eta$. Hence (3.6) holds with y. Then, for any $x \in B(\bar{x}, \varepsilon)$ with $e_\lambda f(\bar{x}) < e_\lambda f(x)$ and $\gamma \geq 2$, we have

$$\begin{aligned}
d^\gamma(x, \operatorname{lev}_{\leq e_\lambda f(\bar{x})} e_\lambda f) \leq& d^\gamma(x, \operatorname{lev}_{\leq f(\bar{x})} f) \leq \big(d(y, \operatorname{lev}_{\leq f(\bar{x})} f) + \|y - x\|\big)^\gamma \\
\leq& 2^{\gamma-1}\big(d^\gamma(y, \operatorname{lev}_{\leq f(\bar{x})} f) + \|y - x\|^\gamma\big) \\
\leq& 2^{\gamma-1}\big(\mu\big(f(y) - f(\bar{x})\big) + \|y - x\|^\gamma\big) \\
=& 2^{\gamma-1}\mu\big(f(y) + \frac{1}{\mu}\|y - x\|^\gamma - f(\bar{x})\big) \\
\leq& 2^{\gamma-1}\mu\big(f(y) + \frac{1}{2\lambda}\|y - x\|^2 - f(\bar{x})\big) \\
=& 2^{\gamma-1}\mu\big(e_\lambda f(x) - e_\lambda f(\bar{x})\big),
\end{aligned} \tag{3.7}$$

where the first inequality holds with the fact $\operatorname{lev}_{f(\bar{x})} f \subseteq \operatorname{lev}_{\leq e_\lambda f(\bar{x})} e_\lambda f$, which follows from $e_\lambda f(x) \leq f(x)$ and $e_\lambda f(\bar{x}) = f(\bar{x})$; the third inequality holds with the fact $(a+b)^\alpha \leq 2^{\alpha-1}(a^\alpha + b^\alpha)$ while $a, b \geq 0$ and $\alpha \geq 1$, since $g : x \mapsto x^\alpha$ is a convex function on $[0, \infty)$ with $\alpha \geq 1$; the fourth inequality holds with (3.6); and the last inequality holds with the condition $\mu \geq 2\lambda$,

$\|y-x\|<1$ and $\gamma \geq 2$. If $0<\gamma<2$, for any $x \in B(\bar{x}, \varepsilon)$ with $e_\lambda f(\bar{x}) < e_\lambda f(x)$, one has

$$\begin{aligned} d^2(x, \operatorname{lev}_{\leq e_\lambda f(\bar{x})} e_\lambda f) \leq& d^2(x, \operatorname{lev}_{\leq f(\bar{x})} f) \leq \left(d(y, \operatorname{lev}_{\leq f(\bar{x})} f) + \|y-x\|\right)^2 \\ \leq& 2\left(d^2(y, \operatorname{lev}_{\leq f(\bar{x})} f) + \|y-x\|^2\right) \\ \leq& 2\left(d^\gamma(y, \operatorname{lev}_{\leq f(\bar{x})} f) + \|y-x\|^2\right) \\ \leq& 2\left(\mu\left(f(y) - f(\bar{x})\right) + \|y-x\|^2\right) \\ \leq& 2\mu\left(e_\lambda f(x) - e_\lambda f(\bar{x})\right), \end{aligned} \tag{3.8}$$

where the fourth inequality holds with $\gamma < 2$. Combining (3.7) and (3.8), we obtain the desired conclusion immediately. □

The following proposition gives sufficient conditions of (iii) in Theorem 3.2.

Proposition 3.1. *Let $f:\mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper and closed function and $\bar{x} \in \operatorname{dom} f$ with $0 \in \partial f(\bar{x})$. Assume that f is prox-bounded with threshold λ_f and $\lambda \in (0, \lambda_f)$. Suppose that either*

(i) $\bar{x}$ is a global optimal point; or

(ii) $\bar{x}$ is a local optimal point and f is prox-regular at $\bar{x}$ for $\bar{v} = 0$.

Then, the condition (iii) in Theorem 3.2 holds.

Proof. (i) holds naturally and we only consider (ii). By [2, Proposition 13.37] we known that $P_\lambda f$ is satisfied a Lipschitz condition of rank K on $\mathbb{R}^n$ and $P_\lambda f(\bar{x}) = \bar{x}$. Then, for $\bar{\varepsilon} > 0$ and any $x \in B(\bar{x}, \bar{\varepsilon})$ with $f(x) > f(\bar{x})$,

$$\|P_\lambda f(x) - P_\lambda f(\bar{x})\| = \|y-x\| \leq K\|x-\bar{x}\| < K\bar{\varepsilon},$$

where $y = P_\lambda f(x)$. Since $\bar{x}$ is a local optimal point, one has that $f(y) \geq f(\bar{x})$. The proof is completed. □

Consider the u-LSEB and the u-HEB, we have the following results.

Corollary 3.1. *Let $f:\mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper and closed function and prox-bounded with threshold λ_f. Assume that the following conditions hold:*

(i) for any $\lambda \in (0, \lambda_f)$, $\bar{x} \in \operatorname{dom} f$ with $0 \in \partial e_\lambda f(\bar{x})$;

(ii) f satisfies the u-LSEB on $P_\lambda f(\bar{x})$ with an exponent $\gamma > 0$ and a constant $\mu > 0$.

Then $e_\lambda f$ satisfies the u-LSEB on $P_\lambda f(\bar{x})$ with an exponent $\max\{1,\gamma\}$ and a constant $\lambda\left(1 + (\frac{\mu}{\lambda})^{\gamma^{-1}}\right)^{\max\{1,\gamma\}}$.

Proof. For any $x' \in P_\lambda f(\bar{x})$, f satisfies the LSEB at x' with an exponent γ and a constant μ by condition (ii) and Definition 2.1. Then, by condition (i) and the Theorem 3.1, $e_\lambda f$ satisfies the LSEB at x' with an exponent $\max\{1,\gamma\}$ and a constant $\lambda\left(1 + (\frac{\mu}{\lambda})^{\gamma^{-1}}\right)^{\max\{1,\gamma\}}$. Since $P_\lambda f(\bar{x})$ is a compact set and f takes a constant value $f(\bar{x})$ on $P_\lambda f(\bar{x})$, we have that $e_\lambda f$ satisfies the u-LSEB on $P_\lambda f(\bar{x})$ with an exponent $\max\{1,\gamma\}$ and a constant $\lambda\left(1 + (\frac{\mu}{\lambda})^{\gamma^{-1}}\right)^{\max\{1,\gamma\}}$ by Lemma 2.1. □

We omit the proof of Corollary 3.2 since the proof is similar to Corollary 3.1.

Corollary 3.2. *Let $f:\mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper and closed function and prox-bounded with threshold λ_f. Assume that the following conditions hold:*

(i) for any $\lambda \in (0, \lambda_f)$*,* $\bar{x} \in \mathrm{dom}\, f$ *with* $0 \in \partial e_\lambda f(\bar{x})$*;*
(ii) f *satisfies the u-HEB on* $P_\lambda f(\bar{x})$ *with an exponent* $\gamma > 0$ *and a constant* $\mu \geq 2\lambda$*;*
(iii) for $\bar{\varepsilon} > 0$ *and any* $x \in \{y : d(y, P_\lambda f(\bar{x})) < \bar{\varepsilon}\}$ *with* $f(x) > f(\bar{x})$*, there exists a point* $y \in P_\lambda f(x)$ *such that* $f(y) \geq f(\bar{x})$*.*

Then $e_\lambda f$ *satisfies the u-HEB on* $P_\lambda f(\bar{x})$ *with an exponent* $\max\{2, \gamma\}$ *and a constant* $2^{\max\{1, \gamma - 1\}}\mu$*.*

From Remark 5.1 (i) in [12], if f is a KL function with exponent $\alpha \in (\frac{1}{2}, 1]$ and $\inf f > -\infty$, then $e_\lambda f$ is a KL function with exponent α. However, the following example shows that $e_\lambda f$ does not necessarily have the KL property but has the LSEB when f does not satisfy the KL property but only the LSEB.

Example 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ x^2 + \frac{1}{n} - \frac{1}{n^2} & \text{if } \frac{1}{n} < x \leq \frac{1}{n-1}, n = 3, 4, \ldots, \\ x^2 + \frac{1}{4} & \text{if } x > \frac{1}{2}. \end{cases}$$

It follows from the Example 3.19 in [17] that f is proper and closed function, and from Example 2.2 in [14] that f satisfies the LSEB at $\bar{x} = 0$ with the exponent 1 and the constant $\frac{1}{2}$. However, f does not have the KL property at 0 with any exponent $\alpha \in [0, 1)$, since, for $x_n = \frac{1}{n-1}$ with n sufficiently large, one has

$$\big(f(x_n) - f(0)\big)^{-\alpha} d\big(0, \partial f(x_n)\big) \leq (\frac{1}{n})^{-\alpha} \frac{2}{n-1} = \frac{2n^\alpha}{n-1} \to 0 \text{ as } n \to \infty.$$

Now, we consider the following function:

$$e_{\frac{1}{2}} f(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ \frac{1}{2}x^2 + \frac{1}{n} - \frac{1}{n^2} & \text{if } \frac{1}{n} < x \leq \frac{1}{n-1}, n = 3, 4, \ldots, \\ \frac{1}{2}x^2 + \frac{1}{4} & \text{if } x > \frac{1}{2}. \end{cases}$$

It is easy to see that $e_{\frac{1}{2}} f$ is a proper and closed function and f satisfies the conditions (i) and (ii) in Theorem 3.1, which means that $e_{\frac{1}{2}} f$ satisfies the LSEB at $\bar{x} = 0$ with the exponent 1 and the constant 1. However, $e_{\frac{1}{2}} f$ does not have KL property at 0 with any exponent $\alpha \in [0, 1)$, since, for $x_n = \frac{1}{n-1}$ with n sufficiently large, one has

$$\begin{aligned} \big(e_{\frac{1}{2}} f(x_n) - e_{\frac{1}{2}} f(0)\big)^{-\alpha} d\big(0, \partial e_{\frac{1}{2}} f(x_n)\big) &= \big(\frac{1}{2(n-1)^2} + \frac{1}{n} - \frac{1}{n^2}\big)^{-\alpha} \frac{1}{n-1} \\ &\leq (\frac{1}{n} - \frac{1}{2n^2})^{-\alpha} \frac{1}{n-1} \\ &\leq (\frac{5}{6n})^{-\alpha} \frac{1}{n-1} = \frac{(1.2n)^\alpha}{n-1} \to 0 \text{ as } n \to \infty. \end{aligned}$$

The following example shows that $e_\lambda f$ does not necessarily have the LSEB but has the LHEB when f does not satisfy the LSEB but only the LHEB.

Example 3.2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} x^2(2 + \cos \frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

It easy to see that closed function f satisfies the LHEB at $\bar{x}=0$ with the exponent 2 and the constant 1. But f does not satisfy the LSEB at 0 with any exponent $\gamma\in[0,+\infty)$, since, for any x sufficiently near 0 with $x\neq 0$, we have

$$\nabla f(x)=4x+2x\cos\frac{1}{x}+\sin\frac{1}{x}.$$

Picking $x_k^1=\frac{1}{2k\pi}$ and $x_k^2=\frac{1}{2k\pi+1.5\pi}$, for sufficiently large k, one has $\nabla f(x_k^1)=\frac{3}{k\pi}>0$ and $\nabla f(x_k^2)=\frac{4}{2k\pi+1.5\pi}-1<0$, which means that there is a sequence $\{x_k\}$ converging to 0 with $\nabla f(x_k)=0$ for all $k\in N$. Now, we consider the following function:

$$e_{\frac{1}{2}}f(x):=\begin{cases}\inf_{y\in\mathbb{R}^n}\left\{y^2(2+\cos\frac{1}{y})+\|y-x\|^2\right\} & \text{if } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$$

Take into account that 0 is a global optimal point and $\mu=2\lambda=1$. From Proposition 3.1 and Theorem 3.2, we have that $e_{\frac{1}{2}}f$ satisfies the LHEB at 0 with the exponent 2 and the constant 2. Noting that, for any x near 0 with $x\neq 0$, $P_{\frac{1}{2}}f(x)\subseteq[\frac{x}{4},\frac{x}{2}]$. So we can assume that $ax\in P_{\frac{1}{2}}f(x)$ with $a\in[\frac{1}{4},\frac{1}{2}]$. For any x sufficiently near 0, if $x\neq 0$, we have

$$\nabla e_{\frac{1}{2}}f(x)=(6a^2-4a+2)x+2a^2x\cos\frac{1}{ax}+a\sin\frac{1}{ax}.$$

Picking $x_k^1=\frac{1}{2ak\pi}$ and $x_k^2=\frac{1}{2ak\pi+1.5a\pi}$, for sufficiently large k, one has $\nabla f(x_k^1)=\frac{8a^2-4a+2}{2ak\pi}>0$ and $\nabla f(x_k^2)=\frac{6a^2-4a+2}{2ak\pi+1.5a\pi}-a<0$, which means that there is a sequence $\{x_k\}$ converging to 0 with $\nabla f(x_k)=0$ for all $k\in N$. Hence $e_{\frac{1}{2}}f$ does not satisfy the LSEB at 0 with any exponent $\gamma\in[0,+\infty)$.

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