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ROBUST VARIATIONAL INEQUALITIES GOVERNED BY CURVILINEAR INTEGRAL FUNCTIONALS

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Abstract. In this paper, we state that generalized convexity, invex sets, and the Fréchet differentiability assumption associated with curvilinear integral type functionals represent some necessary and sufficient mathematical tools for establishing various connections between the solutions of some robust (weak) vector commanded variational inequalities and (weak, proper) robust efficient solutions of the corresponding multi-objective variational control problem. In addition, the physical motivation of the problem under investigation is formulated in the illustrative application given at the end of this paper.

Keywords. Curvilinear integral; Fréchet differentiability; Invex set; Robust efficient solution.

1. Introduction

Over times, in order to investigate the multi-objective (vector) problems in optimization theory, it was needed to introduce the notions of *efficient solutions*, namely: *proper efficient solutions*, introduced by Geoffrion [9], *improper solutions*, studied by Klinger [17], *weak minimum*, analyzed in some multi-ojective optimization problems with constraints (see Kazmi [15]), formulating conditions of efficiency for *(weakly, properly) approximating efficient points* in a generalized optimization problem (see Ghaznavi-ghosoni and Khorram [10]).

The generalization of *convexity* was necessary in order to study concrete problems in applied sciences, or engineering. In this regard, Hanson [12] introduced *invexity* and, over times, many other extensions have been formulated (see Ahmad [1], Antczak [2], Arana-Jiménez et al. [5], and Mishra et al. [22]), namely: univexity, preinvexity, pseudoinvexity, approximate convexity, quasiinvexity, etc. Moreover, these extended concepts have been considered in the multi-dimensional optimization problems (Treanţă [28–30]).

Since variational inequalities are useful to model and investigate concrete problems in natural phenomena, mechanics, engineering, physics, Giannessi [11] analyzed vector variational

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inequalities. Also, since scalar and vector-type variational inequalities can offer the existence of solutions in scalar and multi-objective optimization problems, a lot of papers investigated these connections (see [3, 4, 6, 13, 18, 20, 25]). Kim [16] established some results on multi-objective continuous-time variational program and vector variational inequality. Obviously, (robust) optimal control problems can be seen as continuous-time variational problems. In this direction, Treanţă [26, 27] and Jha et al. [14] have studied efficiency conditions, well-posedness results, saddle-point criterion, and modified objective function technique in commanded variational problems with special objective functionals. Very recently, Zeng et al. [31] investigated existence, convergence and optimal control associated with a class of double phase mixed boundary value problems. For more details concerning multivalued boundary problems and distributed optimal control problems, the reader is welcome to consult Liu et al. [19], Migórski et al. [21], Papageorgiou et al. [23, 24], Colombo and Mingione [7], Farkas and Winkert [8] and the references therein.

In the present paper, we introduce a class of robust (weak) vector variational control inequalities and formulate the corresponding multi-objective variational control problem given by curvilinear integral type functionals. Under generalized convexity and differentiability hypotheses, we establish several connections between the two multi-dimensional commanded variational problems. Compared with the above-mentioned papers in the literature, the main novelty element in our mathematical framework is the presence of uncertain parameters in the vector variational control inequalities under study. The extended concept of invex set represents another important element in proving the main results. Also, the curvilinear integral functionals (mechanical work) are new ingredients in such a mathematical context. Moreover, an illustrative example is presented to justify the outstanding applicability of the paper.

In the following, the paper includes some preliminaries and problem formulation. In Section 3, we state several characterization theorems of the solutions in the robust variational problems under study. Finally, Section 4 states the conclusions of the current paper.

2. Preliminaries

Let A be a compact set in $\mathbb{R}^b$, and $A \ni u = (u^\xi)$, $\xi = \overline{1,b}$, as a multiple variable of evolution. Let $A \supset \mathsf{C} : u = u(\varsigma), \varsigma \in [t_0, t_1]$ be a piece-wise differentiable curve joining the following two multiple variables of evolution $u_1 = (u_1^1, \dots, u_1^b)$, $u_2 = (u_2^1, \dots, u_2^b)$ in A. Consider Φ is the space of piece-wise differentiable *state* functions $x : A \to \mathbb{R}^a$, and Ψ is the space of *control* functions $y : A \to \mathbb{R}^k$ (piece-wise continuous functions). Also, we define on $\Phi \times \Psi$ the scalar product

$$\begin{aligned}\langle (x,y),(\pi,v)\rangle &= \int_{\mathsf{C}} \Big[x(u)\cdot \pi(u) + y(u)\cdot v(u)\Big] du^\xi \\ &= \int_{\mathsf{C}} \Big[\sum_{i=1}^{a} x^i(u)\pi^i(u) + \sum_{j=1}^{k} y^j(u)v^j(u)\Big] du^1 \\ &\quad + \cdots + \Big[\sum_{i=1}^{a} x^i(u)\pi^i(u) + \sum_{j=1}^{k} y^j(u)v^j(u)\Big] du^b, \quad \forall (x,y),(\pi,v) \in \Phi\times\Psi\end{aligned}$$

together with the norm induced by it. Now, let us consider the vector-valued functions $\gamma_\xi = (\gamma_\xi^l) : A \times \mathbb{R}^a \times \mathbb{R}^k \times W \to \mathbb{R}^n$, $\xi = \overline{1,b}$, $l = \overline{1,n}$, of C^1-class, and introduce the vector functional

(curvilinear integrals)

$$\begin{aligned}
&P:\Phi\times\Psi\times W\to\mathbb{R}^n,\\
&P(x,y;\omega)=\int_{\mathsf{C}}\gamma_\xi\left(u,x(u),y(u),\omega\right)du^\xi\\
&=\left(\int_{\mathsf{C}}\gamma_\xi^1\left(u,x(u),y(u),\omega_1\right)du^\xi,\cdots,\int_{\mathsf{C}}\gamma_\xi^n\left(u,x(u),y(u),\omega_n\right)du^\xi\right),
\end{aligned}$$

where $\mathbb{R}^n\supset W=W_1\times W_2\times\cdots\times W_n$ are convex compact sets containing the uncertain parameters $\omega=(\omega_l),\ l=\overline{1,n}$. Further, let $D_\rho,\ \rho\in\{1,\ldots,b\}$, denote the operator associated with the total derivative, and we assume that the 1-form densities $\gamma_\xi=\left(\gamma_\xi^1,\ldots,\gamma_\xi^n\right):A\times\mathbb{R}^a\times\mathbb{R}^k\times W\to\mathbb{R}^n$, $\xi=\overline{1,b}$, satisfy $D_\rho\gamma_\xi^l=D_\xi\gamma_\rho^l,\ \xi,\rho=\overline{1,b},\ \xi\neq\rho,\ l=\overline{1,n}$. Also, we assume the following rules: $\zeta=\eta\Leftrightarrow\zeta^l=\eta^l$, $\zeta\leq\eta\Leftrightarrow\zeta^l\leq\eta^l$, $\zeta<\eta\Leftrightarrow\zeta^l<\eta^l$, $\zeta\preceq\eta\Leftrightarrow\zeta\leq\eta,\ \zeta\neq\eta,\ l=\overline{1,n}$, for any $\zeta=\left(\zeta^1,\cdots,\zeta^n\right)$ and $\eta=\left(\eta^1,\cdots,\eta^n\right)$ in $\mathbb{R}^n$.

Next, we introduce the following *vector commanded variational problem* with partial differential equation constraints and data uncertainty in the objective (cost) functional

$$(VP)\quad\min_{(x,y;\omega)}\left\{P(x,y;\omega)=\int_{\mathsf{C}}\gamma_\xi\left(u,x(u),y(u),\omega\right)du^\xi\right\}\text{ subject to }(x,y)\in Sol,$$

where

$$P(x,y;\omega)=\int_{\mathsf{C}}\gamma_\xi\left(u,x(u),y(u),\omega\right)du^\xi=\left(P^1(x,y;\omega_1),\cdots,P^n(x,y;\omega_n)\right)$$

and $Sol=\{(x,y)\in\Phi\times\Psi\mid x_\rho^i(u):=\frac{\partial x^i}{\partial u^\rho}(u)=H_\rho^i\left(u,x(u),y(u)\right),\ F\left(u,x(u),y(u)\right)\leq0,x|_{u=u_1,u_2}$ $=\text{given},\ \omega\in W\}$. In the definition of Sol, we have considered that the C^1-class functions $H_\rho=(H_\rho^i):A\times\mathbb{R}^a\times\mathbb{R}^k\to\mathbb{R}^a,\ i=\overline{1,a},\ \rho=\overline{1,b}$, define the following partial differential equations of evolution $x_\rho^i(u)=H_\rho^i\left(u,x(u),y(u)\right),\ i=\overline{1,a}$ and $\rho=\overline{1,b}$ and verify the closeness relations $D_\xi H_\rho^i=D_\rho H_\xi^i,\ \rho,\xi=\overline{1,b},\ \rho\neq\xi,\ i=\overline{1,a}$. Also, we assume that $F=(F^r):A\times\mathbb{R}^a\times\mathbb{R}^k\to\mathbb{R}^q,\ r=\overline{1,q}$, are functions of C^1-class.

The associated robust counterpart of the multi-dimensional multi-objective optimization problem (VP) is defined as:

$$(RVP)\quad\min_{(x,y)}\int_{\mathsf{C}}\max_{\omega\in W}\gamma_\xi\left(u,x(u),y(u),\omega\right)du^\xi\ \text{ subject to }\ (x,y)\in Sol,$$

where

$$\begin{aligned}
&\int_{\mathsf{C}}\max_{\omega\in W}\gamma_\xi\left(u,x(u),y(u),\omega\right)du^\xi\\
&=\left(\int_{\mathsf{C}}\max_{\omega_1\in W_1}\gamma_\xi^1\left(u,x(u),y(u),\omega_1\right)du^\xi,\cdots,\int_{\mathsf{C}}\max_{\omega_n\in W_n}\gamma_\xi^n\left(u,x(u),y(u),\omega_n\right)du^\xi\right)\\
&=\left(\max_{\omega_1\in W_1}P^1(x,y;\omega_1),\ldots,\max_{\omega_n\in W_n}P^n(x,y;\omega_n)\right).
\end{aligned}$$

The set of all feasible solutions Sol in (RVP) is called the *robust feasible solution set* to the problem (VP). By the *robust solution* associated to (VP), we mean a robust feasible point (solution) that simultaneously minimizes all objective functions $P^l(x,y;\omega_l),\ l=\overline{1,n}$. In this regard, we consider the following types of solutions for the study of (VP).

Definition 2.1. A point $(x^0,y^0) \in Sol$ is a *robust efficient solution* in (VP) if there exists no other $(x,y) \in Sol$ such that $\max_{\omega \in W} P(x,y;\omega) \preceq \max_{\omega \in W} P(x^0,y^0;\omega)$, or, equivalently, $\max_{\omega_l \in W_l} P^l(x,y;\omega_l) - \max_{\omega_l \in W_l} P^l(x^0,y^0;\omega_l) \leq 0$, $\forall l = \overline{1,n}$, with "$<$" for at least one l, where we used the notation

$$\max_{\omega \in W} P(x,y;\omega) := \int_C \max_{\omega \in W} \gamma_\xi \left(u, x(u), y(u), \omega\right) du^\xi.$$

Definition 2.2. The pair $(x^0,y^0) \in Sol$ is a *proper robust efficient solution* of (VP) if $(x^0,y^0) \in Sol$ is a robust efficient solution in (VP) and, for all $l = \overline{1,n}$, there exists a positive real number M satisfying

$$\max_{\omega_l \in W_l} P^l(x^0,y^0;\omega_l) - \max_{\omega_l \in W_l} P^l(x,y;\omega_l) \leq M \left(\max_{\omega_s \in W_s} P^s(x,y;\omega_s) - \max_{\omega_s \in W_s} P^s(x^0,y^0;\omega_s) \right),$$

for some $s \in \{1,\cdots,n\}$ such that $\max_{\omega_s \in W_s} P^s(x,y;\omega_s) > \max_{\omega_s \in W_s} P^s(x^0,y^0;\omega_s)$, whenever $(x,y) \in Sol$ and $\max_{\omega_l \in W_l} P^l(x,y;\omega_l) < \max_{\omega_l \in W_l} P^l(x^0,y^0;\omega_l)$.

Definition 2.3. A point $(x^0,y^0) \in Sol$ is a *weak robust efficient solution* in (VP) if there exists no other $(x,y) \in Sol$ satisfying $\max_{\omega \in W} P(x,y;\omega) < \max_{\omega \in W} P(x^0,y^0;\omega)$, or $\max_{\omega_l \in W_l} P^l(x,y;\omega_l) - \max_{\omega_l \in W_l} P^l(x^0,y^0;\omega_l) < 0$, $\forall l = \overline{1,n}$.

Next, we define a vector uncertain functional

$$K : \Phi \times \Psi \times W \to \mathbb{R}^n, \quad K(x,y;\overline{\omega}) = \int_C \kappa_\xi \left(u, x(u), x_\rho(u), y(u), \overline{\omega}\right) du^\xi$$

and introduce the concepts of invexity and pseudoinvexity for K.

Definition 2.4. The functional K is *invex at* $\left(x^0,y^0\right) \in \Phi \times \Psi$ *with respect to* φ *and* χ *if there exist*

$$\varphi : A \times \mathbb{R}^a \times \mathbb{R}^k \times \mathbb{R}^a \times \mathbb{R}^k \to \mathbb{R}^a,$$
$$\varphi = \varphi\left(u, x(u), y(u), x^0(u), y^0(u)\right) = \left(\varphi^i\left(u, x(u), y(u), x^0(u), y^0(u)\right)\right), \quad i = \overline{1,a},$$

of C^1*-class with* $\varphi\left(u, x^0(u), y^0(u), x^0(u), y^0(u)\right) = 0$, $\forall u \in A$, $\varphi|_{u=u_1,u_2} = 0$, *and*

$$\chi : A \times \mathbb{R}^a \times \mathbb{R}^k \times \mathbb{R}^a \times \mathbb{R}^k \to \mathbb{R}^k,$$
$$\chi = \chi\left(u, x(u), y(u), x^0(u), y^0(u)\right) = \left(\chi^j\left(u, x(u), y(u), x^0(u), y^0(u)\right)\right), \quad j = \overline{1,k},$$

of C^0*-class with* $\chi\left(u, x^0(u), y^0(u), x^0(u), y^0(u)\right) = 0$, $\forall u \in A$, $\chi|_{u=u_1,u_2} = 0$, *satisfying*

$$K(x,y;\overline{\omega}) - K\left(x^0,y^0;\overline{\omega}\right)$$
$$\geq \int_C \left[\frac{\partial \kappa_\xi}{\partial x}\left(u, x^0(u), x^0_\rho(u), y^0(u), \overline{\omega}\right) \varphi + \frac{\partial \kappa_\xi}{\partial \sigma_\rho}\left(u, x^0(u), x^0_\rho(u), y^0(u), \overline{\omega}\right) D_\rho \varphi \right] du^\xi$$
$$+ \int_C \left[\frac{\partial \kappa_\xi}{\partial y}\left(u, x^0(u), x^0_\rho(u), y^0(u), \overline{\omega}\right) \chi \right] du^\xi,$$

for any $(x,y) \in \Phi \times \Psi$.

Definition 2.5. If we replace $\geq$ with $>$, for $(x,y) \neq \left(x^0,y^0\right)$, we obtain *strictly invexity at* $\left(x^0,y^0\right) \in \Phi \times \Psi$ *with respect to* φ *and* χ of K.

Definition 2.6. The functional K is *pseudoinvex at* $\left(x^0,y^0\right)\in\Phi\times\Psi$ *with respect to* φ *and* χ *if there exist*

$$\varphi: A\times\mathbb{R}^a\times\mathbb{R}^k\times\mathbb{R}^a\times\mathbb{R}^k\to\mathbb{R}^a,$$

$$\varphi=\varphi\left(u,x(u),y(u),x^0(u),y^0(u)\right)=\left(\varphi^i\left(u,x(u),y(u),x^0(u),y^0(u)\right)\right),\quad i=\overline{1,a},$$

of C^1*-class with* $\varphi\left(u,x^0(u),y^0(u),x^0(u),y^0(u)\right)=0,\ \forall u\in A,\ \varphi|_{u=u_1,u_2}=0$*, and*

$$\chi: A\times\mathbb{R}^a\times\mathbb{R}^k\times\mathbb{R}^a\times\mathbb{R}^k\to\mathbb{R}^k,$$

$$\chi=\chi\left(u,x(u),y(u),x^0(u),y^0(u)\right)=\left(\chi^j\left(u,x(u),y(u),x^0(u),y^0(u)\right)\right),\quad j=\overline{1,k},$$

of C^0*-class with* $\chi\left(u,x^0(u),y^0(u),x^0(u),y^0(u)\right)=0,\ \forall u\in A,\ \chi|_{u=u_1,u_2}=0$*, satisfying*

$$K(x,y;\overline{\omega})-K\left(x^0,y^0;\overline{\omega}\right)<0$$

$$\Rightarrow\int_{\mathsf{C}}\left[\frac{\partial\kappa_\xi}{\partial x}\left(u,x^0(u),x^0_\rho(u),y^0(u),\overline{\omega}\right)\varphi+\frac{\partial\kappa_\xi}{\partial\sigma_\rho}\left(u,x^0(u),x^0_\rho(u),y^0(u),\overline{\omega}\right)D_\rho\varphi\right]du^\xi$$

$$+\int_{\mathsf{C}}\left[\frac{\partial\kappa_\xi}{\partial y}\left(u,x^0(u),x^0_\rho(u),y^0(u),\overline{\omega}\right)\chi\right]du^\xi<0,$$

or,

$$\int_{\mathsf{C}}\left[\frac{\partial\kappa_\xi}{\partial x}\left(u,x^0(u),x^0_\rho(u),y^0(u),\overline{\omega}\right)\varphi+\frac{\partial\kappa_\xi}{\partial\sigma_\rho}\left(u,x^0(u),x^0_\rho(u),y^0(u),\overline{\omega}\right)D_\rho\varphi\right]du^\xi$$

$$+\int_{\mathsf{C}}\left[\frac{\partial\kappa_\xi}{\partial y}\left(u,x^0(u),x^0_\rho(u),y^0(u),\overline{\omega}\right)\chi\right]du^\xi\geq 0\Rightarrow K(x,y;\overline{\omega})-K\left(x^0,y^0;\overline{\omega}\right)\geq 0,$$

for any $(x,y)\in\Phi\times\Psi$.

The next definition is very important in our investigation. It represents a key element in proving the main results derived in the current paper.

Definition 2.7. The subset $\emptyset\neq\mathsf{X}\times\mathsf{U}\subset\Phi\times\Psi$ is named *invex with respect to* φ *and* χ if $(x^0,y^0)+\theta\left(\varphi\left(u,x,y,x^0,y^0\right),\chi\left(u,x,y,x^0,y^0\right)\right)\in\mathsf{X}\times\mathsf{U}$, for all $(x,y),\ (x^0,y^0)\in\mathsf{X}\times\mathsf{U}$ and $\theta\in[0,1]$.

Now, for obtaining some existence results for problem (VP), we define the following *robust (weak) vector variational control inequalities*:

I. For $\max\limits_{\omega\in W}P(x,y;\omega)=P(x,y;\overline{\omega})$, find $(x^0,y^0)\in Sol$ such that there exists no $(x,y)\in Sol$ satisfying

$$(VI)\quad\left(\int_{\mathsf{C}}\left[\frac{\partial\gamma^1_\xi}{\partial x}\left(u,x^0(u),y^0(u),\overline{\omega}_1\right)\varphi+\frac{\partial\gamma^1_\xi}{\partial y}\left(u,x^0(u),y^0(u),\overline{\omega}_1\right)\chi\right]du^\xi,\cdots,\right.$$

$$\left.\int_{\mathsf{C}}\left[\frac{\partial\gamma^n_\xi}{\partial x}\left(u,x^0(u),y^0(u),\overline{\omega}_n\right)\varphi\varphi+\frac{\partial\gamma^n_\xi}{\partial y}\left(u,x^0(u),y^0(u),\overline{\omega}_n\right)\chi\right]du^\xi\right)\leq 0;$$

II. For $\max\limits_{\omega\in W}P(x,y;\omega)=P(x,y;\overline{\omega})$, find $(x^0,y^0)\in Sol$ such that there exists no $(x,y)\in Sol$ satisfying

$$(WVI)\quad\left(\int_{\mathsf{C}}\left[\frac{\partial\gamma^1_\xi}{\partial x}\left(u,x^0(u),y^0(u),\overline{\omega}_1\right)\varphi+\frac{\partial\gamma^1_\xi}{\partial y}\left(u,x^0(u),y^0(u),\overline{\omega}_1\right)\chi\right]du^\xi,\cdots,\right.$$

$$\int_{C}\left[\frac{\partial\gamma_{\xi}^{n}}{\partial x}\left(u,x^{0}(u),y^{0}(u),\overline{\omega}_{n}\right)\varphi+\frac{\partial\gamma_{\xi}^{n}}{\partial y}\left(u,x^{0}(u),y^{0}(u),\overline{\omega}_{n}\right)\chi\right]du^{\xi}\Big)\Big)<0.$$

3. Main Results

In the following, we establish some connections between (VP), (VI), and (WVI). In this regard, we start with the next result.

Theorem 3.1. *Consider $Sol\subset\Phi\times\Psi$ an invex set (related to φ and χ). Suppose that $(x^0,y^0)\in Sol$ is a proper robust efficient solution in (VP), and each integral $\int_C\gamma_{\xi}^{l}\left(u,x(u),y(u),\overline{\omega}_l\right)du^{\xi}$, $l=\overline{1,n}$, is Fréchet differentiable at $(x^0,y^0)\in Sol$. Then the pair (x^0,y^0) solves (VI).*

Proof. By reductio ad absurdum, we consider that $(x^0,y^0)\in Sol$ is a proper robust efficient solution to (VP) and it does not satisfy (VI). Then, for all $l=\overline{1,n}$, there exists $(x,y)\in Sol$ satisfying

$$\int_{C}\left[\frac{\partial\gamma_{\xi}^{l}}{\partial x}\left(u,x^{0}(u),y^{0}(u),\overline{\omega}_{l}\right)\varphi+\frac{\partial\gamma_{\xi}^{l}}{\partial y}\left(u,x^{0}(u),y^{0}(u),\overline{\omega}_{l}\right)\chi\right]du^{\xi}<0 \tag{3.1}$$

and, for $s\neq l$,

$$\int_{C}\left[\frac{\partial\gamma_{\xi}^{s}}{\partial x}\left(u,x^{0}(u),y^{0}(u),\overline{\omega}_{s}\right)\varphi+\frac{\partial\gamma_{\xi}^{s}}{\partial y}\left(u,x^{0}(u),y^{0}(u),\overline{\omega}_{s}\right)\chi\right]du^{\xi}\leq 0. \tag{3.2}$$

Since $Sol\subset\Phi\times\Psi$ is an invex set (by hypothesis), we consider

$$(z,w)=(x^0,y^0)+\theta_a\left(\varphi\left(u,x,y,x^0,y^0\right),\chi\left(u,x,y,x^0,y^0\right)\right)\in Sol,\ \forall a,$$

for some sequence $\{\theta_a\}$ of positive real numbers, satisfying $\theta_a\to 0$ as $a\to\infty$. Further, since each curvilinear integral $\int_C\gamma_{\xi}^{l}\left(u,x(u),y(u),\overline{\omega}_l\right)du^{\xi}$, $l=\overline{1,n}$, is Fréchet differentiable at $(x^0,y^0)\in Sol$, we have

$$\begin{aligned}&P^{l}(z,w;\overline{\omega}_l)-P^{l}(x^0,y^0;\overline{\omega}_l)\\&=\int_{C}\theta_a\left[\frac{\partial\gamma_{\xi}^{l}}{\partial x}\left(u,x^{0}(u),y^{0}(u),\overline{\omega}_{l}\right)\varphi+\frac{\partial\gamma_{\xi}^{l}}{\partial y}\left(u,x^{0}(u),y^{0}(u),\overline{\omega}_{l}\right)\chi\right]du^{\xi}\\&\quad+\parallel\theta_a\left(\varphi\left(u,x,y,x^0,y^0\right),\chi\left(u,x,y,x^0,y^0\right)\right)\parallel\cdot G^{l}(z,w),\end{aligned} \tag{3.3}$$

where $G^l:V_{(x^0,y^0)}\to\mathbb{R}$ is a continuous function defined on $V_{(x^0,y^0)}$ (a neighborhood of (x^0,y^0)), with $\lim_{a\to\infty}G^l(z,w)=0$. Now, by dividing (3.3) with θ_a and considering the limit, it results

$$\begin{aligned}&\lim_{a\to\infty}\frac{1}{\theta_a}\left[P^{l}(z,w;\overline{\omega}_l)-P^{l}(x^0,y^0;\overline{\omega}_l)\right]\\&=\int_{C}\left[\frac{\partial\gamma_{\xi}^{l}}{\partial x}\left(u,x^{0}(u),y^{0}(u),\overline{\omega}_{l}\right)\varphi+\frac{\partial\gamma_{\xi}^{l}}{\partial y}\left(u,x^{0}(u),y^{0}(u),\overline{\omega}_{l}\right)\chi\right]du^{\xi}.\end{aligned} \tag{3.4}$$

By (3.1) and (3.4), we obtain $P^{l}(z,w;\overline{\omega}_l)-P^{l}(x^0,y^0;\overline{\omega}_l)<0$, for some $a\geq N$ (see N as a natural number). Let $(x^0,y^0)\in Sol$ be a proper robust efficient solution in (VP). Now, we consider the following nonempty set $\mathscr{B}=\left\{s\in\{1,\cdots,n\}\mid P^{s}(x^0,y^0;\overline{\omega}_s)-P^{s}(z,w;\overline{\omega}_s)\leq 0,\ \forall a\geq N\right\}$.

For $s \in \mathscr{B}$,taking into account the Fréchet differentiability of $\int_C \gamma_\xi^s(u,x(u),y(u),\overline{\omega}_s)\,du^\xi$ at $(x^0,y^0) \in Sol$, we obtain

$$\begin{aligned} &P^s(z,w;\overline{\omega}_s) - P^s(x^0,y^0;\overline{\omega}_s) \\ &= \int_C \theta_a \left[\frac{\partial \gamma_\xi^s}{\partial x}\left(u,x^0(u),y^0(u),\overline{\omega}_s\right)\varphi + \frac{\partial \gamma_\xi^s}{\partial y}\left(u,x^0(u),y^0(u),\overline{\omega}_s\right)\chi \right] du^\xi \\ &\quad + \parallel \theta_a\left(\varphi\left(u,x,y,x^0,y^0\right),\chi\left(u,x,y,x^0,y^0\right)\right) \parallel \cdot G^s(z,w), \end{aligned} \tag{3.5}$$

where $G^s : V_{(x^0,y^0)} \to \mathbb{R}$ is a continuous function on $V_{(x^0,y^0)}$, with $\lim_{a\to\infty} G^s(z,w) = 0$. Also, by dividing (3.5) with θ_a, and computing the limit, we conclude that

$$\begin{aligned} &\lim_{a\to\infty} \frac{1}{\theta_a}\left[P^s(z,w;\overline{\omega}_s) - P^s(x^0,y^0;\overline{\omega}_s)\right] \\ &= \int_C \left[\frac{\partial \gamma_\xi^s}{\partial x}\left(u,x^0(u),y^0(u),\overline{\omega}_s\right)\varphi\varphi + \frac{\partial \gamma_\xi^s}{\partial y}\left(u,x^0(u),y^0(u),\overline{\omega}_s\right)\chi \right] du^\xi. \end{aligned}$$

For $a \geq N$, by considering the set $\mathscr{B}$, we obtain

$$\int_C \left[\frac{\partial \gamma_\xi^s}{\partial x}\left(u,x^0(u),y^0(u),\overline{\omega}_s\right)\varphi + \frac{\partial \gamma_\xi^s}{\partial y}\left(u,x^0(u),y^0(u),\overline{\omega}_s\right)\chi \right] du^\xi \geq 0. \tag{3.6}$$

Further, it follows from (3.2) and (3.6) that

$$\int_C \left[\frac{\partial \gamma_\xi^s}{\partial x}\left(u,x^0(u),y^0(u),\overline{\omega}_s\right)\varphi + \frac{\partial \gamma_\xi^s}{\partial y}\left(u,x^0(u),y^0(u),\overline{\omega}_s\right)\chi \right] du^\xi = 0$$

for some $a \geq N$, and $s \neq l,\ s \in \mathscr{B}$.

Now, for $s \neq l,\ s \in \mathscr{B}$, we find that $\frac{\frac{1}{\theta_a}\left[P^l(x^0,y^0;\overline{\omega}_l) - P^l(z,w;\overline{\omega}_l)\right]}{\frac{1}{\theta_a}\left[P^s(z,w;\overline{\omega}_s) - P^s(x^0,y^0;\overline{\omega}_s)\right]} \to \infty$ as $a \to \infty$, which contradicts the proper efficiency of (x^0,y^0) in (VP). The proof is now complete. □

From the vector variational commanded inequality (VI), a charaterization result of robust efficient solutions in (VP) can be formulated below.

Theorem 3.2. *Let $(x^0,y^0) \in Sol$ be a solution to (VI), and let each curvilinear integral $\int_C \gamma_\xi^l(u,x(u),y(u),\overline{\omega}_l)du^\xi$, $l = \overline{1,n}$, be Fréchet differentiable and invex at $(x^0,y^0) \in Sol$ with respect to φ and χ. Then the pair (x^0,y^0) is a robust efficient solution in (VP).*

Proof. By means of contradiction, we consider that $(x^0,y^0) \in Sol$ is a solution to (VI), but it is not a robust efficient solution in (VP). Thus, for all $l = \overline{1,n}$, there exists $(x,y) \in Sol$ satisfying

$$P^l(x,y;\overline{\omega}_l) - P^l(x^0,y^0;\overline{\omega}_l) \leq 0, \tag{3.7}$$

with $<$ for at least one l. Since the curvilinear integrals $\int_C \gamma_\xi^l(u,x(u),y(u),\overline{\omega}_l)\,du^\xi$, $l = \overline{1,n}$, are Fréchet differentiable and invex at $(x^0,y^0) \in Sol$ with respect to φ and χ, we have

$$\begin{aligned} &P^l(x,y;\overline{\omega}_l) - P^l(x^0,y^0;\overline{\omega}_l) \\ &\geq \int_C \left[\frac{\partial \gamma_\xi^l}{\partial x}\left(u,x^0(u),y^0(u),\overline{\omega}_l\right)\varphi + \frac{\partial \gamma_\xi^l}{\partial y}\left(u,x^0(u),y^0(u),\overline{\omega}_l\right)\chi \right] du^\xi, \end{aligned} \tag{3.8}$$

for any $(x,y) \in Sol$ and $l = \overline{1,n}$. Combining (3.7) and (3.8), for all $l = \overline{1,n}$, we find that there exists $(x,y) \in Sol$ satisfying

$$\int_C \left[\frac{\partial \gamma_\xi^l}{\partial x} \left(u, x^0(u), y^0(u), \overline{\omega}_l\right) \varphi + \frac{\partial \gamma_\xi^l}{\partial y} \left(u, x^0(u), y^0(u), \overline{\omega}_l\right) \chi \right] du^\xi \leq 0,$$

with $<$ for at least one l, which contradicts that $(x^0,y^0) \in Sol$ is solution in (VI). □

Further, a sufficient condition for a pair $(x^0,y^0) \in Sol$ to be a solution of (WVI) is stated in the following result.

Theorem 3.3. *Consider $Sol \subset \Phi \times \Psi$ an invex set. Suppose that $(x^0,y^0) \in Sol$ is a weak robust efficient solution of (VP), and each curvilinear integral $\int_C \gamma_\xi^l (u, x(u), y(u), \overline{\omega}_l)\, du^\xi$, $l = \overline{1,n}$, is Fréchet differentiable at $(x^0,y^0) \in Sol$. Then (x^0,y^0) solves (WVI).*

Proof. Since $(x^0,y^0) \in Sol$ is a weak robust efficient solution in (VP), it follows that there exists no other $(x,y) \in Sol$ satisfying $P(x,y;\overline{\omega}) < P(x^0,y^0;\overline{\omega})$, or

$$P^l(x,y;\overline{\omega}_l) - P^l(x^0,y^0;\overline{\omega}_l) < 0, \quad \forall l = \overline{1,n}. \tag{3.9}$$

Further, since (by hypothesis) $Sol \subset \Phi \times \Psi$ is an invex set, for $\theta \in [0,1]$, we obtain $(z,w) = (x^0,y^0) + \theta \left(\varphi\left(u,x,y,x^0,y^0\right), \chi\left(u,x,y,x^0,y^0\right)\right) \in Sol$. Thus, by using (3.9), we see that there exists no other feasible solution $(x,y) \in Sol$ such that $P(z,w;\overline{\omega}) < P(x^0,y^0;\overline{\omega}_l)$, or, equivalently,

$$P^l(z,w;\overline{\omega}_l) - P^l(x^0,y^0;\overline{\omega}_l) < 0, \quad \forall l = \overline{1,n}. \tag{3.10}$$

Also, since the curvilinear integrals $\int_C \gamma_\xi^l (u, x(u), y(u), \overline{\omega}_l)\, du^\xi$, $l = \overline{1,n}$, are Fréchet differentiable at $(x^0,y^0) \in Sol$ and, we conclude by (3.10) that there exists no other $(x,y) \in Sol$ such that

$$\int_C \left[\frac{\partial \gamma_\xi^l}{\partial x} \left(u, x^0(u), y^0(u), \overline{\omega}_l\right) \varphi + \frac{\partial \gamma_\xi^l}{\partial y} \left(u, x^0(u), y^0(u), \overline{\omega}_l\right) \chi \right] du^\xi < 0,$$

for any $l = \overline{1,n}$. □

Taking into account the weak vector variational commanded inequality (WVI), we see that the following theorem provides a charaterization of weak robust efficient solutions in (VP).

Theorem 3.4. *Consider $(x^0,y^0) \in Sol$ a solution in (WVI), and suppose that each curvilinear integral $\int_C \gamma_\xi^l (u, x(u), y(u), \overline{\omega}_l)\, du^\xi$, $l = \overline{1,n}$, is Fréchet differentiable and pseudoinvex at $(x^0,y^0) \in Sol$ with respect to φ and χ. Then (x^0,y^0) is a weak robust efficient solution in (VP).*

Proof. By means of contradiction, we consider that $(x^0,y^0) \in Sol$ is a solution of (WVI) but it is not a weak robust efficient solution of (VP). In consequence, there exists $(x,y) \in Sol$ such that, for all $l = \overline{1,n}$, $P^l(x,y;\overline{\omega}_l) - P^l(x^0,y^0;\overline{\omega}_l) < 0$. By hypothesis, each curvilinear integral $\displaystyle\int_C \gamma_\xi^l (u, x(u), y(u), \overline{\omega}_l)\, du^\xi$, $l = \overline{1,n}$, is Fréchet differentiable and pseudoinvex at $(x^0,y^0) \in Sol$ with respect to φ and χ. Therefore, we obtain

$$\int_C \left[\frac{\partial \gamma_\xi^l}{\partial x} \left(u, x^0(u), y^0(u), \overline{\omega}_l\right) \varphi + \frac{\partial \gamma_\xi^l}{\partial y} \left(u, x^0(u), y^0(u), \overline{\omega}_l\right) \chi \right] du^\xi < 0,$$

for any $(x,y) \in Sol$ and $l = \overline{1,n}$, which contradicts that $(x^0,y^0) \in Sol$ is a solution of (WVI). □

The next result formulates a sufficient condition for a weak robust efficient solution $(x^0, y^0) \in Sol$ in (VP) to be a robust efficient solution $(x^0, y^0) \in Sol$ in (VP).

Theorem 3.5. *Suppose that $(x^0, y^0) \in Sol$ is a weak robust efficient solution in (VP), and each curvilinear integral $\int_{\mathsf{C}} \gamma_\xi^l (u, x(u), y(u), \overline{\omega}_l)\, du^\xi$, $l = \overline{1,n}$, is Fréchet differentiable and strictly invex at $(x^0, y^0) \in Sol$ with respect to φ and χ and Sol is an invex set with respect to φ and χ. Then (x^0, y^0) is robust efficient solution in (VP).*

Proof. By contradiction, we assume that $(x^0, y^0) \in Sol$ is a weak robust efficient solution in (VP), but $(x^0, y^0) \in Sol$ is not a robust efficient solution in (VP). Thus, there exists $(x, y) \in Sol$ with $P(x, y; \overline{\omega}) \preceq P(x^0, y^0; \overline{\omega})$, or

$$P^l(x, y; \overline{\omega}_l) - P^l(x^0, y^0; \overline{\omega}_l) \leq 0, \ \forall l = \overline{1,n}, \tag{3.11}$$

with $<$ for at least one l. Since each curvilinear integral $\int_{\mathsf{C}} \gamma_\xi^l (u, x(u), y(u), \overline{\omega}_l)\, du^\xi$, $l = \overline{1,n}$, is Fréchet differentiable and strictly invex at $(x^0, y^0) \in Sol$ with respect to φ and χ, we obtain

$$\begin{aligned} &P^l(x, y; \overline{\omega}_l) - P^l(x^0, y^0; \overline{\omega}_l) \\ &> \int_{\mathsf{C}} \left[\frac{\partial \gamma_\xi^l}{\partial x} \left(u, x^0(u), y^0(u), \overline{\omega}_l\right) \varphi + \frac{\partial \gamma_\xi^l}{\partial y} \left(u, x^0(u), y^0(u), \overline{\omega}_l\right) \chi \right] du^\xi, \end{aligned}$$

for any $(x, y) \neq (x^0, y^0) \in Sol$ and $l = \overline{1,n}$, which together with (3.11) yields that there exists $(x, y) \in Sol$ satisfying

$$\int_{\mathsf{C}} \left[\frac{\partial \gamma_\xi^l}{\partial x} \left(u, x^0(u), y^0(u), \overline{\omega}_l\right) \varphi + \frac{\partial \gamma_\xi^l}{\partial y} \left(u, x^0(u), y^0(u), \overline{\omega}_l\right) \chi \right] du^\xi < 0,$$

for all $l = \overline{1,n}$. Consequently, $(x^0, y^0) \in Sol$ is not a solution in (WVI). In accordance with Theorem 3.3, it follows that $(x^0, y^0) \in Sol$ is not a weak robust efficient solution in (VP). □

In the next example, we formulate a concrete problem that can be solved exclusively by using the theoretical results derived in this paper.

Example 3.1. Let us extremize the mechanical work accomplished by the variable force

$$\bar{V}\left(\omega_1 e^{-y(u)} + \frac{1}{2},\ \omega_2 e^{x(u)}\right)$$

to move the application point along a piecewise differentiable curve C, contained in $[0,1]^2 = [0,1] \times [0,1]$ and linking $u_1 = (0,0)$ and $u_2 = (1,1)$, with $(\omega_1, \omega_2) \in [1,2] \times [\frac{1}{2}, 1]$, such that the following controlled dynamic system

$$\frac{\partial x}{\partial u^1}(u) = \frac{\partial x}{\partial u^2}(u) = y(u),$$
$$1 - e^{x(u) + x^2(u)} \leq 0,$$
$$e^{y(u)} - 2 + e^{y^2(u)} \leq 0,$$
$$x|_{u = u_1, u_2} = 0$$

is satisfied with respect to $\varphi = e^{(x^0)^2(u)} - e^{x^2(u)}$, $\forall u \in [0,1]^2 \setminus \{u_1, u_2\}$ and $\varphi = 0$ for $u \in \{u_1, u_2\}$, and $\chi = e^{(y^0)^2(u)} - e^{y^2(u)}$, $\forall u \in [0,1]^2 \setminus \{u_1, u_2\}$ and $\chi = 0$ for $u \in \{u_1, u_2\}$.

In order to solve the above practical problem, we consider $b=n=q=2$, $a=k=1$, $A=[0,1]^2=[0,1]\times[0,1]$, and $W=W_1\times W_2=[1,2]\times[\frac{1}{2},1]$. Also, we assume that $x,y:A\to\mathbb{R}^+$ are piecewise differentiable functions with $\frac{\partial x}{\partial u^1}(u)=\frac{\partial x}{\partial u^2}(u)=y(u)$, $1-e^{x(u)+x^2(u)}\leq 0$, $e^{y(u)}-2+e^{y^2(u)}\leq 0$, $x|_{u=u_1,u_2}=0$, and $\varphi,\chi:A\times\mathbb{R}^4\to\mathbb{R}$ are given by $\varphi=e^{(x^0)^2(u)}-e^{x^2(u)}$, $\forall u\in A\setminus\{u_1,u_2\}$ and $\varphi=0$ for $u\in\{u_1,u_2\}$, and $\chi=e^{(y^0)^2(u)}-e^{y^2(u)}$, $\forall u\in A\setminus\{u_1,u_2\}$ and $\chi=0$ for $u\in\{u_1,u_2\}$. Define the following robust Lagrange type densities $\gamma_\xi=(\gamma_\xi^1,\gamma_\xi^2):A\times\mathbb{R}^2\times W\to\mathbb{R}^2$, where $\xi=\overline{1,2}$ as below

$$\gamma_\xi^1\left(u,x(u),y(u),\omega_1\right)=\omega_1 e^{-y(u)}+\frac{1}{2},\quad \gamma_\xi^2\left(u,x(u),y(u),\omega_2\right)=\omega_2 e^{x(u)}.$$

Now, we consider the following vector variational control problem with partial differential equation constraints and uncertain data

$$(VP1)\quad \min_{(x,y;\omega)}\int_C \gamma_\xi\left(u,x(u),y(u),\omega\right)du^\xi,$$

with

$$\int_C \gamma_\xi\left(u,x(u),y(u),\omega\right)du^\xi=\left(\int_C \gamma_\xi^1\left(u,x(u),y(u),\omega_1\right)du^\xi,\int_C \gamma_\xi^2\left(u,x(u),y(u),\omega_2\right)du^\xi\right)$$
$$=\left(P^1(x,y;\omega_1),P^2(x,y;\omega_2)\right)$$

and subject to the above mentioned constraints. The corresponding robust counterpart is given by

$$(RVP1)\quad \min_{(x,y)}\int_C \max_{\omega\in W}\gamma_\xi\left(u,x(u),y(u),\omega\right)du^\xi,$$

with

$$\int_C \max_{\omega\in W}\gamma_\xi\left(u,x(u),y(u),\omega\right)du^\xi$$
$$=\left(\int_C \max_{\omega_1\in W_1}\gamma_\xi^1\left(u,x(u),y(u),\omega_1\right)du^\xi,\int_C \max_{\omega_2\in W_2}\gamma_\xi^2\left(u,x(u),y(u),\omega_2\right)du^\xi\right)$$
$$=\left(\max_{\omega_1\in W_1}P^1(x,y;\omega_1),\max_{\omega_2\in W_2}P^2(x,y;\omega_2)\right)$$

and subject to the above constraints. Obviously,

$$P(x,y;\omega)=\int_C \gamma_\xi\left(u,x(u),y(u),\omega\right)du^\xi$$
$$=\left(\int_C \gamma_\xi^1\left(u,x(u),y(u),\omega_1\right)du^\xi,\ \int_C \gamma_\xi^2\left(u,x(u),y(u),\omega_2\right)du^\xi\right)$$

is Fréchet differentiable at $(x^0,y^0)=(0,0)$. Moreover, it can be verified that each curvilinear integral $\int_C \gamma_\xi^l\left(u,x(u),y(u),\omega_l\right)du^\xi$, $l=\overline{1,2}$, is invex at $(x^0,y^0)=(0,0)$ with respect to φ and χ.

Further, we can easily see that $(x^0,y^0)=(0,0)$ is a solution for (VI). Indeed, we have

$$\Big(\int_C\left[\frac{\partial\gamma_\xi^1}{\partial x}\left(u,x^0(u),y^0(u),\overline{\omega}_1\right)\varphi+\frac{\partial\gamma_\xi^1}{\partial y}\left(u,x^0(u),y^0(u),\overline{\omega}_1\right)\chi\right]du^\xi,$$

$$\int_C \left[\frac{\partial \gamma_\xi^2}{\partial x} \left(u, x^0(u), y^0(u), \overline{\omega}_2\right) \varphi + \frac{\partial \gamma_\xi^2}{\partial y} \left(u, x^0(u), y^0(u), \overline{\omega}_2\right) \chi \right] du^\xi \Big)$$

$$= \left(\int_C 2e^{-y(u)} \left(e^{y^2(u)} - 1\right) du^\xi, \int_C e^{x(u)} \left(1 - e^{x^2(u)}\right) du^\xi \right) \nleq (0,0),$$

for all piecewise differentiable functions $x, y : A \to \mathbb{R}^+$. Therefore, by Theorem 3.2, we get that (x^0, y^0) is a robust efficient solution of $(VP1)$.

Now, by direct computation, we find

$$\begin{aligned} P^1(x,y;\omega_1) - P^1(x^0,y^0;\omega_1) &= \int_C \gamma_\xi^1 \left(u, x(u), y(u), \omega_1\right) du^\xi - \int_C \gamma_\xi^1 \left(u, x^0(u), y^0(u), \omega_1\right) du^\xi \\ &= \int_C \left(\omega_1 e^{-y(u)} + \frac{1}{2}\right) du^\xi - \int_C \left(\omega_1 + \frac{1}{2}\right) du^\xi \\ &= \int_C \omega_1 \left(e^{-y(u)} - 1\right) du^\xi < 0, \end{aligned}$$

for all piecewise differentiable functions $y : A \to \mathbb{R}^+ \setminus \{0\}$ and $\omega_1 \in [1,2]$.

On the other hand, we have

$$\begin{aligned} & P^2(x,y;\omega_2) - P^2(x^0,y^0;\omega_2) \\ &= \int_C \gamma_\xi^2 \left(u, x(u), y(u), \omega_2\right) du^\xi - \int_C \gamma_\xi^2 \left(u, x^0(u), y^0(u), \omega_2\right) du^\xi \\ &= \int_C \omega_2 \left(e^{x(u)} - 1\right) du^\xi > 0, \end{aligned}$$

for all piecewise differentiable functions $x : A \to \mathbb{R}^+ \setminus \{0\}$ and $\omega_2 \in [\frac{1}{2}, 1]$. Since, for $(x^0, y^0) = (0,0)$, there exists $M = 1$ satisfying

$$\begin{aligned} & \int_C \gamma_\xi^1 \left(u, x^0(u), y^0, \omega_1(u)\right) du^\xi - \int_C \gamma_\xi^1 \left(u, x(u), y(u), \omega_1\right) du^\xi \\ & \leq M \left(\int_C \gamma_\xi^2 \left(u, x(u), y(u), \omega_2\right) du^\xi - \int_C \gamma_\xi^2 \left(u, x^0(u), y^0(u), \omega_2\right) du^\xi \right), \end{aligned}$$

we conclude that $(x^0, y^0) = (0,0)$ is a proper robust efficient solution of $(VP1)$.

4. Conclusions

In this paper, we formulated and proved various connections between the solutions of some robust (weak) vector commanded variational inequalities and (weak, proper) robust efficient solutions associated with the corresponding class of multi-objective variational control problems defined by curvilinear integral type functionals. A very important role in establishing the principal theoretical results was the notion of invex set with respect to some functions. Moreover, the generalized convexity and Fréchet type differentiability hypotheses of the considered functionals played a crucial role. Finally, the applicability and effectiveness of the proposed methods were illustrated by an example from physics.

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