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# NORMALIZED DUALITY MAPPINGS AND PROJECTIONS IN BOCHNER SPACES

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**Abstract.** In the theory of Banach spaces, the normalized duality mapping assumes a pivotal role. The analytic depiction of this mapping holds paramount significance in the associated analysis. Given that Bochner spaces serve as foundational underpinnings in stochastic variational analysis and stochastic optimizations, delving into the analytic representations of the normalized duality mapping becomes imperative, especially in uniformly convex and uniformly smooth Bochner spaces. The study of the analytic representations of normalized duality mapping contributes to our understanding of various geometric properties inherent in Bochner spaces. Leveraging the analytic representation of the normalized duality mapping, we establish and substantiate certain non-convex properties linked to this mapping in uniformly convex and uniformly smooth Bochner spaces.

**Keywords.** Analytic representation; Bochner space; Normalized duality mapping; Uniformly convex; Uniformly smooth.

## 1. INTRODUCTION

Let $X$ be a real Banach space with $X^*$ as its topological dual. We denote the norm in $X$ by $\|\cdot\|_X$ and the norm in $X^*$ by $\|\cdot\|_{X^*}$. The duality pairing between $X$ and $X^*$ is denoted by $\langle\cdot,\cdot\rangle$. We denote the origin in $X$ by $\theta_X$, often dropping the subscript for simplicity. The concept of the duality pairing in a Banach space in geometric form was introduced by Beurling and Livingston [3] in 1962, marking a pioneering contribution to the field. On the other hand, by the Hahn-Banach theorem, there exists at least one $\varphi \in X^*$ such that $\langle \varphi, x\rangle = \|\varphi\|_{X^*}\|x\|_X$. The nomalized duality mapping, which, in general, is a set-valued map $J_X : X \to 2^{X^*}\backslash\{\emptyset\}$, is defined by

$$J_X x = \{\varphi \in X^* : \langle \varphi, x\rangle = \|x\|_X^2 = \|\varphi\|_{X^*}^2\} \text{ for every } x \in X. \tag{1.1}$$

The normalized duality map boasts numerous advantageous properties (see, e.g., [3, 12–14, 16, 17, 21, 24]) several of which we compile for convenient reference in the subsequent section.

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There is an intimate connection between the geometric attributes of a Banach space $X$ and the analytic characteristics of its associated normalized duality mapping $J_X$. This mapping holds substantial importance in both projection theory and approximation theory in Banach spaces.

For example, in [1,2], the authors utilized the normalized duality mapping $J_X$ to formulate generalized projections and generalized metric projections in uniformly convex and uniformly smooth Banach spaces. In [16], Li employed the normalized duality mapping $J_X$ to extend these concepts from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces. Furthermore, Khan, Li, and Reich [14] extended these projection techniques to general Banach spaces by leveraging the normalized duality mapping. The normalized duality mapping has served as a primary tool in various studies, as evidenced in [1,2,5,11,20,21], where it was employed to investigate fixed-point approximation problems and assess the continuity of metric and generalized metric projections within uniformly convex and uniformly smooth Banach spaces. In Hilbert spaces, the metric projection operator adheres to the basic variational principle, which can be considered as the fundamental theorem of projection theory in Hilbert spaces; see [7, Chapter 3]. To extend such a crucial principle to projections in Banach spaces, the normalized duality mapping $J_X$ assumes a pivotal role in establishing basic variational principles for both projection and generalized projections in Banach spaces, as detailed in [1,2,6,10,14,16].

This research focuses on Bochner spaces which are commonly regarded as specialized Banach spaces, and their definitions and basic properties can be found in [9,19,23]. Numerous authors explored the geometric characteristics of Bochner spaces (see, e.g., [4,5,15]), as well as the interconnections between the geometric properties of Bochner spaces and the underlying Banach spaces defining them (see, e.g., [8,15,18,22]). Conversely, Banach spaces can be viewed as specific instances of Bochner spaces concerning certain measure spaces.

Given the pivotal role of the normalized duality mapping in Banach space, particularly its utility in projection theory, approximation theory, and variational inequalities in Bochner spaces, this paper seeks to investigate the properties of the normalized duality mapping in uniformly convex and uniformly smooth Bochner spaces.

The contents of this paper are organized as follows: In Section 2, we provide a review of the properties of the normalized duality mapping in uniformly convex and uniformly smooth Bochner spaces alongside some non-convex properties established in [13]. Additionally, we revisit the definitions and fundamental properties of Bochner spaces, encompassing simple functions in Bochner spaces. Section 3 delves into the analytical representations of the normalized duality mapping in uniformly convex and uniformly smooth Bochner spaces. Section 4 explores various properties and analytical representations of the normalized duality mapping in multiple Bochner spaces. In Section 5, we leverage the normalized duality mapping to examine the geometric properties of both Bochner spaces and multiple Bochner spaces. We employ the representations of the normalized duality mapping in uniformly convex and uniformly smooth Bochner spaces to establish some non-convex properties related to the normalized duality mapping.

## 2. PRELIMINARIES

### 2.1. **The normalized duality map and Projections in Banach spaces.**

We begin with recalling that, in a uniformly convex and uniformly smooth Banach space $X$, the normalized duality mapping $J_X : X \to X^*$ is single-valued, one-to-one and onto, homogeneous, continuous and uniformly continuous on bounded sets; see [7].

Let $X$ be a uniformly convex and uniformly smooth Banach space, and let $C \neq \emptyset$ be a closed, and convex subset of $X$. The metric projection $P_C : X \to C$ is a single-valued map given by

$$\|x - P_C x\|_X \le \|x - z\|_X \quad \text{for all } z \in C.$$

It is known that $P_C : X \to C$ is a continuous map that enjoys the following variational characterization:

$$u = P_C(x) \iff \langle J_X(x-u), u-z\rangle \ge 0 \quad \text{for all } z \in C. \tag{2.1}$$

The generalized projection $\pi_C : X^* \to C$ is a single-valued map that satisfies $V(\psi, \pi_C \psi) = \inf_{y \in C} V(\psi, y)$ for any $\psi \in X^*$, where $V : X^* \times X \to \mathbb{R}$ is a Lyapunov function by the following formula:

$$V(\psi, x) = \|\psi\|_{X^*}^2 - 2\langle \psi, x\rangle + \|x\|_X^2 \quad \text{for any } \psi \in X^*,\ x \in X.$$

The generalized projection $\pi_C : X^* \to C$ enjoys the following variational characterization: For any $\psi \in X^*$ and $y \in C$,

$$y = \pi_C(\psi) \iff \langle \psi - J_X y, y - z\rangle \ge 0 \quad \text{for all } z \in C. \tag{2.2}$$

The generalized metric projection $\Pi_C : X \to C$ is defined by

$$\begin{aligned} \Pi_C x &:= \pi_C(J_X x) \quad \text{for any } x \in X, \\ \pi_C(\varphi) &:= \Pi_C(J_{X^*}\varphi) \quad \text{for any } \varphi \in X^*. \end{aligned}$$

The generalized metric projection $\Pi_C : X \to C$ satisfies the following variational characterization: For any $x \in X$ and $y \in C$,

$$y = \Pi_C(x) \iff \langle J_X x - J_X y, y - z\rangle \ge 0 \quad \text{for all } z \in C. \tag{2.3}$$

In general $\Pi_C \neq P_C$. However, the notions (2.1), (2.2), and (2.3) coincide in a Hilbert space. By the variational characterizations given above, the problems involved with $P_C$, $\pi_C$, and $\Pi_C$ can be converted to variational inequalities, which are easier to solve, in many cases.

We recall that concepts of the generalized projection and the generalized metric projection were introduced by Alber [1] on uniformly convex and uniformly smooth Banach spaces, which have been extended to general Banach spaces; see [14, 17] and the references therein.

### 2.2. **Bochner spaces.**

In this subsection, we recall the definitions and basic properties of Bochner spaces; see, e.g., [4, 5, 7, 9, 10, 18–24] for more details.

Let $(S, \mathscr{A}, \mu)$ be a measure space, which, without any loss generality, is assumed to be positive and complete. Let $X$ be a real uniformly convex and uniformly smooth Banach space with $X^*$ as its topological dual. For any $A \in \mathscr{A}$, and for any $x \in X$, $1_A \otimes x$ denotes the $X$-valued simple function on $S$ with values in $X$ defined, for any $s \in S$, by

$$(1_A \otimes x)(s) = 1_A(s) \otimes x = \begin{cases} x & \text{if } s \in A, \\ \theta & \text{if } s \notin A. \end{cases}$$

where $1_A$ denotes the charateristic function of $A$ on $X$.

For a given integer $n$, let $\{A_1, A_2, \ldots, A_n\}$ be a finite collection of mutually disjoint sets in $\mathscr{A}$ with $0 < \mu(A_i) < \infty$ for all $i = 1, 2, \ldots, n$. Let $\{x_1, x_2, \ldots, x_n\} \subset X$, and let $\{a_1, a_2, \ldots, a_n\}$

be real numbers. Then, $\sum_{i=1}^{n} a_i(1_{A_i} \otimes x_i)$ is called a $\mu-$simple function from $S$ to $X$; see [19, Definition 1.1.13].

**Remark 2.1.** Since the coefficients $\{a_1, a_2, \ldots, a_n\}$ can be included in the points $\{x_1, x_2, \ldots, x_n\}$, it follows that a $\mu$-simple function can have the form $\sum_{i=1}^{n}(1_{A_i} \otimes x_i)$.

For any positive number $p$ with $1 \le p \le \infty$, let $L_p(S;X)$ be the Lebesgue-Bochner function space, called the Bochner space, which is the Banach space of $\mu$-equivalent classes of strongly measurable functions $f : S \to X$ with norm ($f$ takes values in Banach space $X$ as the limit of integrals of simple functions):

$$\|f\|_{L^p(S;X)} := \left( \int_S \|f(s)\|_X^p \, d\mu(s) \right)^{\frac{1}{p}} < \infty, \quad \text{for } 1 \le p < \infty,$$
$$\|f\|_{L_\infty(S;X)} := \text{ess sup} \|f(\cdot)\|_X < \infty.$$

In particular, for $X = \mathbb{R}$, $L_p(S;\mathbb{R})$ is denoted by $L_p(S)$. Next, we list some properties of Bochner integrals and Bochner spaces

- ($B_1$) $\int_S (af + bg)\, d\mu = a \int_S f\, d\mu + b \int_S g\, d\mu$ for every $f, g \in L_p(S, X)$ and $a, b \in \mathbb{R}$;
- ($B_2$) $\|\int_S f\, d\mu\|_X \le \int_S \|f\|_X\, d\mu$ for every $f, g \in L_p(S, X)$;
- ($B_3$) for $p, q \in (1, \infty)$ with $p^{-1} + q^{-1} = 1$, $(L_p(S;X))^* = L_q(S;X^*)$.

The investigation of geometric properties of Bochner spaces has been a subject of examination by various researchers, encompassing aspects such as convexity and smoothness; see, e.g., [5, 7, 9, 16, 18, 22]. Here, we give a compilation of related properties below.

**Theorem 2.1.** *[17] Let $(S, \mathscr{A}, \mu)$ be a measure space, and let $X$ be a Banach space. For any $p$ with $1 < p < \infty$, we have*

$$\begin{aligned} L_p(S;X) \textit{ is uniformly convex} &\iff X \textit{ is uniformly convex}, \\ L_p(S;X) \textit{ is uniformly smooth} &\iff X \textit{ is uniformly smooth}, \\ L_2(S;X) \textit{ is a Hilbert space} &\iff X \textit{ is a Hilbert space}. \end{aligned}$$

We next recall an embedding map of $X$ into $L_p(S;X)$, studied in [17], which is used shortly. For any $A, B \in \mathscr{A}$ with $0 < \mu(A)$, $\mu(B) < \infty$, for any $p, q$ with $1 < p, q < \infty$ and $p^{-1} + q^{-1} = 1$, the function $1_A \otimes x$ holds the following properties:

(a) $\{1_A \otimes x : A \in \mathscr{A}$ with $0 < \mu(A) < \infty$ and $x \in X\} \subseteq L_p(S;X)$;
(b) $\{1_B \otimes \varphi : B \in \mathscr{B}$ with $0 < \mu(A) < \infty$ and $\varphi \in X^*\} \subseteq (L_p(S;X))^* = L_q(S;X^*)$.

**Proposition 2.1.** *[17] Let $(S, \mathscr{A}, \mu)$ be a measure space, and let $X$ be a Banach space. For any arbitrary $A \in \mathscr{A}$ with $0 < \mu(A) < \infty$, for any $x, y \in X$ and for any $1 < p < \infty$, we have*

(a) $\mu(A)^{-\frac{1}{p}}(1_A \otimes x) \in L_p(S;X)$*;*
(b) $\left\| \mu(A)^{-\frac{1}{p}}(1_A \otimes x) \right\|_{L_p(S;X)} = \|x\|_X$*;*
(c) $\left\| \mu(A)^{-\frac{1}{p}}(1_A \otimes x) \pm \mu(A)^{-\frac{1}{p}}(1_A \otimes y) \right\|_{L_p(S;X)} = \|x \pm y\|_X$*;*
(d) *the mapping* $x \to \mu(A)^{-\frac{1}{p}}(1_A \otimes x)$ *(isometric) embeds* $X$ *into* $L_p(S;X)$.

## 3. ANALYTIC REPRESENTATIONS OF NORMALIZED DUALITY MAP IN BOCHNER SPACES

Let $(S,\mathscr{A},\mu)$ be a measure space, and let $X$ be a uniformly convex and uniformly smooth Banach space. For national simplicity, we denote the normalized duality mapping in $X$, $X^*$, $L_p(S;X)$, and $L_q(S;X^*)$ by $J_X$, $J_{X^*}$, $J_p$, and $J_q^*$, respectively. We note that these are all single-valued, one-to-one and onto, and continuous maps.

We recall the following forms of the normalized duality map in two specific cases:

(a) Let $X=\ell_p$ with $1<p<\infty$. For any $x=(t_1,t_2,\dots)\in\ell_p$ with $x\neq\theta$, we have

$$(J_Xx)_n=\frac{|x_n|^{p-1}\operatorname{sign}(x_n)}{\|x\|_{\ell_p}^{p-2}}=\frac{|x_n|^{p-2}x_n}{\|x\|_{\ell_p}^{p-2}}\quad\text{for } n=1,2,\dots.$$

(b) Let $X=L_p(S)$ with $1<p<\infty$. For any $f\in L_p(S)$ with $f\neq\theta$, we have

$$(J_Xf)(s)=\frac{|f(s)|^{p-1}\operatorname{sign}(f(s))}{\|f\|_{L_p(S)}^{p-2}}=\frac{|f(s)|^{p-2}f(s)}{\|f(s)\|_{L_p(S)}^{p-2}}\quad\text{for all } s\in S.$$

The subsequent proposition establishes the relationships between $J_X$ and $J_p$, offering an analytical representation for $J_p$ in the process.

**Proposition 3.1.** *Let $(S,\mathscr{A},\mu)$ be a measure space, let $X$ be a uniformly convex and uniformly smooth Banach space, and let $A\in\mathscr{A}$. Then, for any $x\in X$, with $x\neq 0$, we have*

$$J_X((1_A\otimes x)(s))=(1_A\otimes J_Xx)(s)\quad\textit{for every } s\in S.$$

*Proof.* For any $s\in S$, we have

$$\begin{aligned}J_X((1_A\otimes x)(s))&=\begin{cases}J_Xx & \text{for all } s\in A,\\ \theta & \text{for all } s\notin A,\end{cases}\\ &=(1_A\otimes J_Xx)(s),\end{aligned}$$

which completes the proof. □

By the arguments used in the proof above, we can also establish the following result.

**Proposition 3.2.** *Let $(S,\mathscr{A},\mu)$ be a measure space, and let $X$ be a uniformly convex and uniformly smooth Banach space. Let $\{A_1,A_2,\dots,A_n\}$ be an arbitrary finite collection of mutually disjoint subsets in $\mathscr{A}$, and let $\{x_1,x_2,\dots,x_n\}\subset X$. Then,*

$$J_X\Big(\Big(\sum_{i=1}^{n}1_{A_i}\otimes x_i\Big)(s)\Big)=\sum_{i=1}^{n}(1_{A_i}\otimes J_Xx_i)(s)\quad\textit{for every } s\in S.$$

**Proposition 3.3.** *Let $(S,\mathscr{A},\mu)$ be a measure space, and let $X$ be a uniformly convex and uniformly smooth Banach space. Let $p,q\in(0,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$. Then, for any $f\in L_p(S;X)$, with $f\neq\theta$,*

(a) *$J_pf\in L_q(S;X^*)$;*

(b) *$(J_pf)(s)=\frac{\|f(s)\|_X^{p-2}J_X(f(s))}{\|f\|_{L_p(S;X)}^{p-2}}$ for all $s\in S$.*

*Proof.* Part (a) is evident as $J_pf \in (L_p(S;X))^* = L_q(S;X^*)$. To prove part (b), we calculate

$$\begin{aligned}
\|J_pf\|_{L_q(S;X^*)} &= \left( \int_S \left\| \frac{\|f(s)\|_X^{p-2} J_X(f(s))}{\|f\|_{L_p(S;X)}^{p-2}} \right\|_{X^*}^q d\mu(s) \right)^{\frac{1}{q}} \\
&= \left( \int_S \frac{\|f(s)\|_X^{q(p-2)}}{\|f\|_{L_p(S;X)}^{q(p-2)}} \|J_X(f(s))\|_{X^*}^q \, d\mu(s) \right)^{\frac{1}{q}} \\
&= \left( \int_S \frac{\|f(s)\|_X^{q(p-2)}}{\|f\|_{L_p(S;X)}^{q(p-2)}} \|f(s)\|_X^q \, d\mu(s) \right)^{\frac{1}{q}} \\
&= \left( \int_S \frac{\|f(s)\|_X^{q(p-2)+q}}{\|f\|_{L_p(S;X)}^{q(p-2)}} d\mu(s) \right)^{\frac{1}{q}} \\
&= \left( \|f\|_{L_p(S;X)}^{q(p-2)} \right)^{-\frac{1}{q}} \left( \int_S \|f(s)\|_X^p \, d\mu(s) \right)^{\frac{1}{q}} = \|f\|_{L_p(S;X)}.
\end{aligned}$$

Once again, we have

$$\begin{aligned}
\langle J_pf, f \rangle &= \int_S \left\langle \frac{\|f(s)\|_X^{p-2} J_X(f(s))}{\|f\|_{L^p(S;X)}^{p-2}}, f(s) \right\rangle d\mu(s) \\
&= \int_S \frac{\|f(s)\|_X^{p-2} \langle J_X(f(s)), f(s) \rangle}{\|f\|_{L_p(S;X)}^{p-2}} d\mu(s) \\
&= \int_S \frac{\|f(s)\|_X^{p-2} \|f(s)\|_X^2}{\|f\|_{L_p(S;X)}^{p-2}} d\mu(s) \\
&= \|f\|_{L_p(S;X)}^2.
\end{aligned}$$

Thus $\langle J_pf, f \rangle_p = \|f\|_{L_p(S;X)}^2 = \|J_pf\|_{L_q(S;X)}^2$, which proves the desired claim. □

**Proposition 3.4.** *Let $(S, \mathscr{A}, \mu)$ be a measure space, and let $X$ be a uniformly convex and uniformly smooth Banach space. Let $p, q \in (0, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and let $A \in \mathscr{A}$ with $0 < \mu(A) < \infty$. Then, for any $x \in X$ with $x \neq \theta$,*

$$J_p(1_A \otimes x)(s) = \mu(A)^{\frac{1}{p} - \frac{1}{q}} (1_A \otimes J_X x)(s) \quad \textit{for all } s \in S.$$

*Proof.* We begin by calculating

$$\|1_A \otimes x\|_{L_p(S;X)} = \left( \int_S \|(1_A \otimes x)(s)\|_X^p \, d\mu(s) \right)^{\frac{1}{p}} = \left( \int_A \|x\|_X^p \, d\mu(s) \right)^{\frac{1}{p}} = \|x\|_X \mu(A)^{\frac{1}{p}}.$$

Using part (b) of Proposition 3.3, we have

$$\begin{aligned}
J_p(1_A\otimes x)(s) &= \frac{\|(1_A\otimes x)(s)\|_X^{p-2}J_X((1_A\otimes x)(s))}{\|1_A\otimes x\|_{L_p(S;X)}^{p-2}}\\
&= \begin{cases} \dfrac{\|x\|_X^{p-2}J_X x}{\left(\|x\|_X\mu(A)^{\frac{1}{p}}\right)^{p-2}} & \text{for all } s\in A,\\ \theta & \text{for all } s\notin A,\end{cases}\\
&= \begin{cases} \dfrac{J_X x}{\mu(A)^{\frac{1}{p}(p-2)}} & \text{for all } s\in A,\\ \theta & \text{for all } s\notin A,\end{cases}\\
&= \mu(A)^{\frac{1}{p}-\frac{1}{q}}(1_A\otimes J_X)(s) \quad \text{for all } s\in S,
\end{aligned}$$

and the proof is complete. □

The above result can be extended to all $\mu$-simple functions.

**Proposition 3.5.** *Let $(S,\mathscr{A},\mu)$ be a measure space, and let $X$ be a uniformly convex and uniformly smooth Banach space. Let $p,q\in(0,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$. Then, $J_p$ maps every $\mu$-simple function in $L_p(S;X)$ to a $\mu-$simple function in $L_q(S;X^*)$ with respect to the same partition in $S$. Moreover, for any given $\mu$-simple function $\sum_{i=1}^n(1_{A_i}\otimes x_i)$ in $L_p(S;X)$, we have*

$$J_p\Big(\sum_{i=1}^n(1_{A_i}\otimes x_i)\Big)s) = \frac{\|\sum_{i=1}^n(1_{A_i}\otimes x_i)(s)\|_X^{p-2}}{\left(\sum_{j=1}^n\|x_j\|_X^p\mu(A_j)\right)^{\frac{1}{q}-\frac{1}{p}}}\sum_{i=1}^n(1_{A_i}\otimes J_X x_i)(s) \quad \textit{for all } s\in S.$$

*Proof.* For a given $\mu$-simple function $\sum_{i=1}^n(1_{A_i}\otimes x_i)$ in $L_p(S;X)$, we calculate

$$\begin{aligned}
\left\|\sum_{i=1}^n(1_{A_i}\otimes x_i)(s)\right\|_{L_p(S;X)} &= \left(\int_S\|\sum_{i=1}^n(1_{A_i}\otimes x_i)(s)\|_X^p\,d\mu(s)\right)^{\frac{1}{p}}\\
&= \left(\sum_{i=1}^n\int_S\|(1_{A_i}\otimes x_i)(s)\|_X^p\,d\mu(s)\right)^{\frac{1}{p}}\\
&= \left(\sum_{i=1}^n\int_S\|x_i\|_X^p\,d\mu(s)\right)^{\frac{1}{p}}\\
&= \left(\sum_{i=1}^n\|x_i\|_X^p\mu(A_j)\right)^{\frac{1}{p}}.
\end{aligned}$$

Even though $J_X$ is not a linear operator, by the definition of $\mu$-simple functions, since $\{A_1,A_2,\dots,A_n\}$ is a finite collection of mutually disjoint subsets in $\mathscr{A}$ with $0<\mu(A_i)<\infty$ for all $i=1,2,\dots,n$, by part (b) of Proposition 3.3, for all $s \in S$, by Proposition 3.4, we have

$$\begin{aligned} J_p\Big(\sum_{i=1}^{n}(1_{A_i} \otimes x_i)\Big)(s) &= \frac{\|\sum_{i=1}^n (1_{A_i} \otimes x_i)(s)\|_X^{p-2} J_X\Big(\sum_{i=1}^n (1_{A_i} \otimes x_i)(s)\Big)}{\Big(\|\sum_{j=1}^n (1_{A_j} \otimes x_j)\|_{L_p(S;X)}\Big)^{p-2}} \\ &= \frac{\sum_{i=1}^n \|(1_{A_i} \otimes x_i)(s)\|_X^{p-2} \sum_{i=1}^n J_X\Big(1_{A_i} \otimes x_i)(s)\Big)}{\Big(\|\sum_{j=1}^n (1_{A_j} \otimes x_j)\|_{L_p(S;X)}\Big)^{p-2}} \\ &= \frac{\|\sum_{i=1}^n (1_{A_i} \otimes x_i)(s)\|_X^{p-2} \sum_{i=1}^n (1_{A_i} \otimes J_X x_i)(s)}{\Big(\sum_{i=j}^n \|x_j\|_X^p \mu(A_j)^{\frac{1}{p}}\Big)^{p-2}}, \end{aligned}$$

which completes the proof. □

As mentioned in Section 2, in general, a $\mu$-simple function can be written in the form $\sum_{i=1}^n a_i(1_{A_i} \otimes x_i)$ with real coefficients $a_1, a_2, \dots, a_n$. Otherwise, the real coefficients $a_1, a_2, \dots, a_n$ can be considered included in $x_1, x_2, \dots, x_n$. Therefore, as a consequence of the above result, we obtain the following result.

**Proposition 3.6.** *Let $(S, \mathscr{A}, \mu)$ be a measure space, and let $X$ be a uniformly convex and uniformly smooth Banach space. Let $1 < \{p, q\} < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. For any $f \in L_p(S;X)$, let $\{f_n\}$ be a sequence of $\mu$-simple functions in $L_p(S;X)$ satisfying $f_n \to f$ in $L_p(S;X)$ as $n \to \infty$. Then, $\{J_p f_n\}$ is a sequence of $\mu$-simple functions in $L_q(S;X^*)$ such that $J_p f_n \to J_p f$ in $L_q(S;X^*)$, as $n \to \infty$.*

*Proof.* By Proposition 3.5, we see that $\{J_p f_n\}$ is a sequence of $\mu$-simple functions in $L_q(S;X^*)$. The claim then follows from the continuity of $J_p$. □

## 4. THE NORMALIZED DUALITY MAPPINGS IN MULTIPLE BOCHNER SPACES

In this section, we investigate the characteristics of the normalized duality mapping in specific instances of Bochner spaces denoted as $L_p(S;X)$, where the Banach space $X$ itself is a Bochner space. The examination of these particular Bochner spaces holds significant relevance, especially in the context of stochastic variational inequalities and stochastic optimization theory.

Let $(S, \mathscr{A}, \mu)$ and $(T, \mathscr{B}, \lambda)$ be measure spaces, and let $Y$ be a uniformly convex and uniformly smooth Banach space. Let $p, q, \beta, \xi$ be positive numbers which are greater than 1 and satisfy $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{\beta} + \frac{1}{\xi} = 1$. The Bochner space $L_\beta(T;Y)$ is a uniformly convex and uniformly smooth Banach space, and therefore the Bochner space $L_p(S;L_\beta(T;Y))$ is also a uniformly convex and uniformly smooth space. In the following, the normalized duality maps on $L_\beta(T;Y)$ and $L_p(S;L_\beta(T;Y))$ are denoted by $J_\beta$ and $J_p$, respectively.

We note that, for any $f \in L_p(S;L_\beta(T;Y))$,

$$\|f\|_{L_p(S;L_\beta(T;Y))} = \Big(\int_S \|f(s)\|_{L_\beta(T;Y)}^p d\mu(s)\Big)^{\frac{1}{p}} = \Big(\int_S \Big(\int_T \|f(s)(t)\|_Y^\beta \, d\lambda(t)\Big)^{\frac{p}{\beta}} d\mu(s)\Big)^{\frac{1}{p}}.$$

The following results give an analytic representation of $J_p$ on $L_p(S;L_\beta(T;Y))$.

**Proposition 4.1.** *Let $(S, \mathscr{A}, \mu)$ and $(T, \mathscr{B}, \lambda)$ be measure spaces, and let $Y$ be a uniformly convex and uniformly smooth Banach space. For any $f \in L_p(S, L_\beta(T;Y))$ with $f \neq \theta$, we have*

(a) $J_pf \in L_q(S;L_\xi(T;Y^*))$;

(b) *for every* $s \in S$,

$$(J_pf)(s)(t) = \frac{\|f(s)\|^{p-\beta}_{L_\beta(T;Y)}\|(f(s))(t)\|_Y^{\beta-2}}{\|f\|^{p-2}_{L_p(S,L_\beta(T;Y))}} J_Y(f(s)(t)) \quad \textit{for all } t \in T;$$

(c) *in particular, if* $\beta = p$, *then, for every* $s \in S$,

$$(J_pf)(s)(t) = \frac{\|(f(s))(t)\|_Y^{p-2}}{\|f\|^{p-2}_{L_p(S,L_\beta(T;Y))}} J_Y(f(s)(t)) \quad \textit{for all } t \in T.$$

*Proof.* Part (a) is obvious. We proceed to prove (b). Using Proposition 3.3 repeatedly, we obtain

$$\begin{aligned}
(J_pf)(s)(t) &= \frac{\|f(s)\|^{p-2}_{L_\beta(T;Y)}}{\|f\|^{p-2}_{L_p(S,L_\beta(T;Y))}} J_\beta(f(s))(t) \\
&= \frac{\|f(s)\|^{p-2}_{L_\beta(T;Y)}}{\|f\|^{p-2}_{L_p(S,L_\beta(T;Y))}} \frac{\|f(s)(t)\|_Y^{\beta-2}}{\|f\|^{\beta-2}_{L_p(S,L_\beta(T;Y))}} J_Y(f(s)(t)) \\
&= \frac{\|f(s)\|^{p-\beta}_{L_\beta(T;Y)}\|(f(s))(t)\|_Y^{\beta-2}}{\|f\|^{p-2}_{L_p(S,L_\beta(T;Y))}} J_Y(f(s)(t)) \quad \text{for all } t \in T.
\end{aligned}$$

□

We have the following analogous result.

**Proposition 4.2.** *Let* $(S,\mathscr{A},\mu)$ *and* $(T,\mathscr{B},\lambda)$ *be measure spaces, and let* $Y$ *be a uniformly convex and uniformly smooth Banach space. Let* $A \in \mathscr{A}$ *with* $0 < \mu(A) < \infty$. *For any* $\varphi \in L_\beta(T;Y)$ *with* $\varphi \neq \theta$, $1_A \otimes \varphi$ *is a* $\mu$*-simple function in* $L_p(S;L_\beta(T;Y))$. *Then, for any* $s \in S$ *and* $t \in T$,

$$(J_p(1_A \otimes \varphi)(s))(t) = \mu(A)^{\frac{1}{p}-\frac{1}{q}}(1_A(s) \otimes J_\beta\varphi)(t) = \mu(A)^{\frac{1}{p}-\frac{1}{q}} \frac{\|\varphi(t)\|_Y^{\beta-2}}{\|\varphi\|^{\beta-2}_{L_\beta(T;Y)}}(1_A(s) \otimes J_\beta\varphi(t)).$$

*Proof.* By Propositions 3.3 and 3.4, we have

$$\begin{aligned}
J_p(1_A \otimes \varphi)(s)(t) &= \mu(A)^{\frac{1}{p}-\frac{1}{q}}(1_A(s) \otimes J_\beta\varphi)(t) \quad \text{for all } t \in T, \\
&= \begin{cases} \mu(A)^{\frac{1}{p}-\frac{1}{q}}(J_\beta\varphi)(t) & \text{for all } s \in A, \\ \theta & \text{for } s \notin A, \end{cases} \\
&= \begin{cases} \mu(A)^{\frac{1}{p}-\frac{1}{q}} \frac{\|\varphi(t)\|_Y^{\beta-2} J_Y(\varphi(t))}{\|\varphi\|^{\beta-2}_{L_\beta(T;Y)}} & \text{for all } s \in A, \\ \theta & \text{for } s \notin A, \end{cases} \\
&= \begin{cases} \mu(A)^{\frac{1}{p}-\frac{1}{q}} \frac{\|\varphi(t)\|_Y^{\beta-2} J_Y}{\|\varphi\|^{\beta-2}_{L_\beta(T;Y)}}(1_A(s) \otimes J_Y(\varphi(t))) & \text{for all } s \in A, \\ \theta & \text{for } s \notin A, \end{cases}
\end{aligned}$$

and the proof concludes. □

In view of Propositions 3.4 and 4.2, we have the following.

**Proposition 4.3.** *Let* $(S,\mathscr{A},\mu)$ *and* $(T,\mathscr{B},\lambda)$ *be measure spaces, and let* $Y$ *be a uniformly convex and uniformly smooth Banach space. Then,* $J_p$ *maps every* $\mu$*-simple function in* $L_p(S;L_\beta(T;Y))$ *to* $\mu$*-simple function in* $L_q(S;L_\xi(T;Y^*))$ *with respect to the same partition in* $S$*. Moreover, for an arbitrarily given* $\mu$*-simple function* $\sum_{i=1}^n(1_{A_i}\otimes\varphi_i)$ *in* $L_p(S;L_\beta(T;Y))$*, for every* $s\in S$*, and for every* $t\in T$*, we have*

$$J_p(\sum_{i=1}^n(1_{A_i}\otimes\varphi_i))(s)(t)=\frac{\|\sum_{i=1}^n(1_{A_i}\otimes\varphi_i)(s)\|_{L_\beta(T;Y)}^{p-2}}{(\sum_{j=1}^n\|\varphi_i\|_{L_\beta(T;Y)}^p\mu(A_j))^{\frac{1}{p}-\frac{1}{q}}}\sum_{i=1}^n\frac{\|\varphi_i(t)\|_Y^{\beta-2}}{\|\varphi_i\|_{L_\beta(T;Y)}^{\beta-2}}(1_{A_i}(S)\otimes J_Y(\varphi_i(t))).$$

*Proof.* By Proposition 3.5, we have

$$\begin{aligned}J_p(\sum_{i=1}^n(1_{A_i}\otimes\varphi_i))(s)(t)&=\frac{\|\sum_{i=1}^n(1_{A_i}\otimes\varphi_i)(s)\|_{L_\beta(T;Y)}^{p-2}}{(\sum_{j=1}^n\|\varphi_i\|_{L_\beta(T;Y)}^p\mu(A_j))^{\frac{1}{p}-\frac{1}{q}}}J_\beta(\sum_{i=1}^n(1_{A_i}(s)\otimes\varphi_i))(t)\\&=\frac{\|\sum_{i=1}^n(1_{A_i}\otimes\varphi_i)(s)\|_{L_\beta(T;Y)}^{p-2}}{(\sum_{j=1}^n\|\varphi_i\|_{L_\beta(T;Y)}^p\mu(A_j))^{\frac{1}{p}-\frac{1}{q}}}(J_\beta\sum_{i=1}^n(1_{A_i}(s)\otimes\varphi_i))(t)\\&=\frac{\|\sum_{i=1}^n(1_{A_i}\otimes\varphi_i)(s)\|_{L_\beta(T;Y)}^{p-2}}{(\sum_{j=1}^n\|\varphi_i\|_{L_\beta(T;Y)}^p\mu(A_j))^{\frac{1}{p}-\frac{1}{q}}}\sum_{i=1}^n\frac{\|\varphi_i(t)\|_Y^{\beta-2}}{\|\varphi_i\|_{L_\beta(T;Y)}^{\beta-2}}(1_{A_i}(s)\otimes J_Y(\varphi_i(t))),\end{aligned}$$

and the proof is complete. $\square$

In the above result, if every $\varphi_i$ is a $\lambda$-simple functional in $L_\beta(T;Y)$, then the following result is immediate.

**Proposition 4.4.** *Let* $(S,\mathscr{A},\mu)$ *and* $(T,\mathscr{B},\lambda)$ *be measure spaces, and let* $Y$ *be a uniformly convex and uniformly smooth Banach space. Let* $\{A_1,A_2,\dots,A_n\}$ *be an arbitrary finite collection of mutually disjoint subsets in* $\mathscr{A}$ *with* $0<\mu(A_i)<\infty$*, for* $i=1,2,\dots,n$*. Let* $\{B_1,B_2,\dots,B_n\}$ *be an arbitrary collection of subsets in* $\mathscr{B}$ *(not necessarily disjoint) with* $0<\lambda(B_i)<\infty$*, for* $i=1,2,\dots,n$ *and let* $\{y_1,y_2,\dots,y_n\}\subset Y$*. For every* $s\in S$ *and* $t\in T$*, we have*

$$\begin{aligned}&J_p(\sum_{i=1}^n(1_{A_i}\otimes(1_{B_i}\otimes y_i)))(s)(t)\\&=\frac{(\sum_{i=1}^n\|y_i\|_Y^\beta\lambda(B_i))^{\frac{p-2}{\beta}}}{(\sum_{i=1}^n\|y_i\|_Y^\beta(\lambda(B_i)^{\frac{p}{\beta}}\mu(A_i)^{\frac{1}{q}-\frac{1}{p}}}\sum_{i=1}^n\frac{\|1_{B_i(t)}\otimes y_i\|^{\beta-2}}{(\|y_i\|_Y^\beta\lambda(B_i))^{\frac{\beta-2}{\beta}}}(1_{A_i}(s)\otimes((1_{B_i}(t)\otimes J_Yy_i))).\end{aligned}$$

## 5. SOME GEOMETRIC PROPERTIES OF BOCHNER AND MULTIPLE BOCHNER SPACES

5.1. **Convexity of Bochner and multiple Bochner spaces.** Let $X$ be a uniformly convex and uniformly smooth Banach space. Let $\delta_X$ be the modulus of convexity of $X$ given by

$$\delta_X(\varepsilon)=\inf\left\{1-\frac{1}{2}\|x+y\|_X:\ x,y\in S_X,\ \|x-y\|\geq\varepsilon\right\},\quad \varepsilon\in(0,2],$$

where $S_X$ is the unit ball in $X$.

Let $(S,\mathscr{A},\mu)$ be a measure space and, for $1<p<\infty$, let $L_p(S;X)$ be the uniformly convex and uniformly smooth Banach space. For notational simplicity, we denote the modulus of convexity $\delta_{L_p(S;X)}$ of $L_p(S:X)$ by $\delta_p$. Let $S_p$ be the unit ball of $L_p(S;X)$. Then,

$$\delta_p(\varepsilon)=\inf\left\{1-\frac{1}{2}\|f+g\|_{L_p(S;X)}:\ f,g\in S_p,\ \|f-g\|_{L_p(S;X)}\geq\varepsilon\right\},\quad \varepsilon\in(0,2].$$

It was recently demonstrated in [17] that

$$\delta_p(\varepsilon)\leq\delta_X(\varepsilon)\quad\text{for every }\varepsilon\in(0,2]. \tag{5.1}$$

Let $(S,\mathscr{A},\mu)$ and $(T,\mathscr{B},\lambda)$ be measure spaces, and let $Y$ be a uniformly convex and uniformly smooth Banach space. Let $p,q,\beta,\xi$ be positive numbers, which are greater than 1 and satisfy $\frac{1}{p}+\frac{1}{q}=1$ and $\frac{1}{\beta}+\frac{1}{\xi}=1$. The modulus of convexity of $Y$, $L_\beta(T;Y)$ and $L_p(S;L_\beta(T;Y))$ are denote by $\delta_Y$, $\delta_\beta$, and $\delta_p$, respectively. By applying (5.1) repeatedly, we have

$$\delta_p(\varepsilon)\leq\delta_\beta(\varepsilon)\leq\delta_Y(\varepsilon)\quad\text{for every }\varepsilon\in(0,2]. \tag{5.2}$$

Let $\Gamma_Y$ denote the Figiel's constant of $Y$ which satisfies $1<\Gamma_Y<1.7$. For any $R>0$, and for any $x,y\in Y$, if $\|x\|_Y\leq R$ and $\|y\|_Y\leq R$, with the aid of (5.2), we have

$$\langle J_Yx-J_y,x-y\rangle\geq\frac{R^2}{2\Gamma_Y}\delta_Y\left(\frac{\|x-y\|_Y}{2R}\right)\geq\frac{R^2}{2\Gamma_Y}\delta_\beta\left(\frac{\|x-y\|_Y}{2R}\right)\geq\frac{R^2}{2\Gamma_Y}\delta_p\left(\frac{\|x-y\|_Y}{2R}\right). \tag{5.3}$$

Let $\Gamma_\beta$ and $\Gamma_p$ denote the Figiel's constants of $L_\beta(T;Y)$ and $L_p(S;L_\beta(T;Y))$, respectively. Let $R>0$. For any $f,g\in L_\beta(T;Y)$ with $\|f\|_{L_\beta(T;Y)}\leq R$ and $\|g\|_{L_\beta(T;Y)}\leq R$, we have

$$\langle J_\beta f-J_\beta g,f-g\rangle\geq\frac{R^2}{2\Gamma_\beta}\delta_\beta\left(\frac{\|f-g\|_{L_\beta(T;Y)}}{2R}\right)\geq\frac{R^2}{2\Gamma_\beta}\delta_p\left(\frac{\|f-g\|_{L_\beta(T;Y)}}{2R}\right).$$

For any $\varphi,\psi\in L_p(S;L_\beta(T;Y))$ with $\|\varphi\|_{L_p(S;L_\beta(T;Y))}\leq R$ and $\|\psi\|_{L_p(S;L_\beta(T;Y))}\leq R$, we have

$$\langle J_Y\varphi-J_Y\psi,\varphi-\psi\rangle\geq\frac{R^2}{2\Gamma_p}\delta_p\left(\frac{\|\varphi-\psi\|_{L_p(S;L_\beta(T;Y))}}{2R}\right).$$

5.2. **Smoothness of Bochner and multiple Bochner spaces.** Let $(S,\mathscr{A},\mu)$, $X$, and $L_p(S;X)$ be as in the previous section with $1<p<\infty$. For $\alpha>0$, let $\rho_x$ and $\rho_p$ be the modules of smoothness of the Banach space $X$ and $L_p(S;X)$, given by

$$\rho_X(\alpha):=\sup\left\{\frac{\|x+y\|_X+\|x-y\|_X}{2}-1:\ x,y\in X,\|x\|_X=1,\ \|y\|_X=\alpha\right\},$$

and

$$\rho_p(\alpha):=\sup\{\frac{\|f+g\|_{L_p(S:X)}+\|f-g\|_{L_p(S;X)}}{2}-1:\ x,y\in L_p(S;X),\|x\|_{L_p(S;X)}=1,\ \|y\|_{L_p(S;X)}=\alpha\}.$$

Recently, it was shown in [17] that, for each $\alpha>0$, $\rho_p(\alpha)\geq\rho_X(\alpha)$. Now let $(T,\mathscr{B},\lambda)$ and $Y$ be as in the previous section. Denoting the modules of smoothness of $Y$, $L_\beta(T;Y)$ and $L_p(S;L_\beta(T;Y))$ by $\rho_Y$, $\rho_\beta$, and $\rho_p$, we can show that, for every $\alpha>0$, $\rho_p(\alpha)\geq\rho_\beta(\alpha)\geq\rho_Y(\alpha)$.

Since all the involved spaces are uniformly convex and uniformly smooth Banach spaces, we have the following relationship:

$$\lim_{\alpha\downarrow 0}\frac{\rho_Y(\alpha)}{\alpha}=\lim_{\alpha\downarrow 0}\frac{\rho_\beta(\alpha)}{\alpha}=\lim_{\alpha\downarrow 0}\frac{\rho_p(\alpha)}{\alpha}=0. \tag{5.4}$$

Furthermore, for any $R>0$, we have (see [1])

(a) for any $x,y\in Y$, if $\|x\|_Y\le R$ and $\|y\|_Y\le R$, then

$$\begin{aligned}\|J_Yx-J_Yy\|_{X^*}&\le\frac{R^2}{2\gamma_Y\|x-y\|_Y}\rho_Y\left(\frac{16\Gamma_Y\|x-y\|_Y}{R}\right)\\&\le\frac{R^2}{2\gamma_Y\|x-y\|_Y}\rho_\beta\left(\frac{16\Gamma_Y\|x-y\|_Y}{R}\right)\\&\le\frac{R^2}{2\gamma_Y\|x-y\|_Y}\rho_p\left(\frac{16\Gamma_Y\|x-y\|_Y}{R}\right);\end{aligned}\tag{5.5}$$

(b) for any $f,g\in L_\beta(T;Y)$, with $\|f\|_{L_\beta(T;Y)}\le R$ and $\|g\|_{L_\beta(T;Y)}\le R$, we have

$$\|J_\beta f-J_\beta g\|_{(L_\beta(T;Y))^*}\le\frac{R^2}{2\gamma_\beta\|f-g\|_{L_\beta(T;Y)}}\rho_\beta\left(\frac{16\Gamma_\beta\|f-g\|_{L_\beta(T;Y)}}{R}\right);$$

(c) for any $\varphi,\psi\in L_p(S;L_\beta(T;Y))$, with $\|\varphi\|_{L_p(S;L_\beta(T;Y))}\le R$ and $\|\psi\|_{L_p(S;L_\beta(T;Y))}\le R$, we have

$$\|J_p\varphi-J_p\psi\|_{(L_p(S;L_\beta(T;Y)))^*}\le\frac{R^2}{2\Gamma_p\|\varphi-\psi\|_{L_p(S;L_\beta(T;Y))}}\rho_p\left(\frac{16\Gamma_p\|\varphi-\psi\|_{L_p(S;L_\beta(T;Y))}}{R}\right).$$

5.3. **Connections between convexity and smoothness of Bochner spaces.** In this subsection, we explore the interplay between the convexity and smoothness properties of uniformly convex and uniformly smooth Banach spaces, as well as their counterparts in Bochner spaces and multiple Bochner spaces.

**Proposition 5.1.** *Let $(S,\mathscr{A},\mu)$ and $(T,\mathscr{B},\lambda)$ be measure spaces, and let $Y$ be a uniformly convex and uniformly smooth Banach space $Y$. Let $p,\beta\in(1,\infty)$ be given. Then:*

(a) *for any $x,y\in Y$ with $\|x\|_Y$ and $\|y\|_Y\le R$, the following inequality holds:*

$$\rho_Y\left(\frac{16\Gamma_Y\|x-y\|}{R}\right)\ge\delta_Y\left(\frac{\|x-y\|_Y}{2R}\right); \tag{5.6}$$

(b) *for any $f,g\in L_\beta(T;Y)$ with $\|f\|_{L_\beta(T;Y)}$ and $\|g\|_{L_\beta(T;Y)}\le R$, we have*

$$\rho_\beta\left(\frac{16\Gamma_\beta\|f-g\|_{L_\beta(T;Y)}}{R}\right)\ge\delta_\beta\left(\frac{\|f-g\|_{L_\beta(T;Y)}}{2R}\right);$$

(c) *for any $\varphi,\psi\in L_p(S;L_\beta(T;Y))$ with $\|\varphi\|_{L_p(S;L_\beta(T;Y))}$ and $\|\psi\|_{L_p(S;L_\beta(T;Y))}\le R$, the following inequality holds:*

$$\rho_p\left(\frac{16\Gamma_p\|f-g\|_{L_p(S;L_\beta(T;Y))}}{R}\right)\ge\delta_pL_p(S;L_\beta(T;Y))\left(\frac{\|x-y\|_{L_p(S;L_\beta(T;Y))}}{2R}\right).$$

*Proof.* Under the given conditions, by (5.3) and (5.5), we have

$$\frac{R^2}{2\Gamma_Y}\rho_Y\left(\frac{16\Gamma_Y\|x-y\|_Y}{R}\right) \geq \|J_Yx-J_Yy\|_{Y^*}\|x-y| \geq \langle J_Xx-J_Xy,x-y\rangle$$
$$\geq \frac{R^2}{2\Gamma_Y}\rho_Y\left(\frac{\|x-y\|_Y}{2R}\right),$$

which proves (5.6). Other inequalities follow by similar arguments. □

**Proposition 5.2.** *Let $(S,\mathscr{A},\mu)$ and $(T,\mathscr{B},\lambda)$ be measure spaces, and let $Y$ be a uniformly convex and uniform;y smooth Banach space $Y$. Let $p,\beta\in(1,\infty)$ be given. Then*

$$\lim_{\varepsilon\downarrow 0}\frac{\delta_Y(\varepsilon)}{\varepsilon}=\lim_{\varepsilon\downarrow 0}\frac{\delta_\beta(\varepsilon)}{\varepsilon}=\lim_{\varepsilon\downarrow 0}\frac{\delta_p(\varepsilon)}{\varepsilon}=0.$$

*Proof.* The proof follows from (5.4) and Proposition 5.1. □

5.4. **The basic variational characterizations of the projections in Bochner spaces.** Utilizing the fundamental variational characterizations outlined in Section 2 for metric and generalized metric projection operators within uniformly convex and uniformly smooth Banach spaces, we derive the corresponding fundamental variational characterizations for these operators in uniformly convex and uniformly smooth Bochner spaces, as well as multiple Bochner spaces.

**Theorem 5.1.** *Let $(S,\mathscr{A},\mu)$ and $(T,\mathscr{B},\lambda)$ be measure spaces, and let $Y$ be a uniformly convex and uniformly smooth Banach space. Let $p,q,\beta,\xi$ be positive numbers which are greater than 1 and satisfy $\frac{1}{p}+\frac{1}{q}=1$ and $\frac{1}{\beta}+\frac{1}{\xi}=1$. Let $C$, $D$, and $E$ be nonempty, closed, and convex sets in $Y$, $L_\beta(T;Y)$, and $L_p(S;L_\beta(T;Y))$, Then,*

(a) *for any $x\in Y$, $x^*\in Y^*$ and $y\in C$,*

$$\begin{aligned} y=P_C(x) &\iff \langle J_Y(x-y),y-z\rangle\geq 0 \quad \text{for all } z\in C,\\ y=\pi_C(x^*) &\iff \langle x^*-J_Yy,y-z\rangle\geq 0 \quad \text{for all } z\in C,\\ y=\Pi_C(x) &\iff \langle J_Xx-J_Yy,y-z\rangle\geq 0 \quad \text{for all } z\in C;\end{aligned}$$

(b) *for any $f\in L_\beta(T;Y)$, $f^*\in L_\xi(T;Y^*)$ and $g\in D$,*

$$\begin{aligned} g=P_D(f) &\iff \langle J_\beta(f-g),g-h\rangle\geq 0 \quad \text{for all } h\in D,\\ g=\pi_D(f^*) &\iff \langle f^*-J_\beta g,g-h\rangle\geq 0 \quad \text{for all } h\in D,\\ g=\Pi_D(f) &\iff \langle J_\beta f-J_\beta g,g-h\rangle\geq 0 \quad \text{for all } h\in D;\end{aligned}$$

(c) *for any $\varphi\in L_p(S;L_\beta(T;Y))$, $\varphi^*\in L_q(S;L_{xi}(T;Y^*))$ and $\phi\in E$,*

$$\begin{aligned} \phi=P_E(\varphi) &\iff \langle J_p(\varphi-\phi),\phi-\psi\rangle\geq 0 \quad \text{for all } \psi\in E,\\ \phi=\pi_E(\varphi^*) &\iff \langle \varphi^*-J_p\phi,\phi-\psi\rangle\geq 0 \quad \text{for all } \psi\in E,\\ \phi=\Pi_E(\varphi) &\iff \langle J_p\varphi-J_p\phi,\phi-\psi\rangle\geq 0 \quad \text{for all } \psi\in E.\end{aligned}$$

5.5. **Some non-convex properties related to the normalized duality mapping in uniformly convex and uniformly smooth Bochner spaces.** In [13], the authors established certain non-convex properties associated with the normalized duality mapping and projections $P$, $\pi$, and $\Pi$ in uniformly convex and uniformly smooth Banach spaces. In this section, we extend these results to uniformly convex and uniformly smooth Bochner space $L_p(S;X)$. Specifically, we demonstrate that the normalized duality mapping $J_p$ maintains these non-convex properties in such spaces. Throughout the proofs of the lemmas and propositions in this section, the analytic representations of $J_p$ explored in Sections 3 and 4 play pivotal roles.

We have the following result in this direction.

**Proposition 5.3.** *Let $(S,\mathscr{A},\mu)$ be a measure space, and let $X$ be a uniformly convex and uniformly smooth Banach space. Let $\theta \neq h$ be an arbitrary point in $L_p(S;X)$. Then the set*

$$\{w \in L_p(S;X):\ \langle J_p w, h\rangle \geq 0\} \tag{5.7}$$

*is a closed cone in $L_p(S;X)$ with vertex at $\theta$.*

We construct an example to show that the set define in (5.7) is not convex in general.

**Example 5.1.** We take a measure space $(S,\mathscr{A},\mu)$ with $\mu(S) \geq 3$ and take a uniformly convex and uniformly smooth Banach space $X$ with dimension greater than 1. We consider the uniformly convex and uniformly smooth Bochner space $L_3(S;X)$. Let $A_1, A_2$, and $A_3$ be three arbitrarily chosen mutually disjoint elements in $\mathscr{A}$ with $\mu(A_i) = 1$, and let $x_1, x_2$, and $x_3$ be three linearly independent points in $X$ with $\|x_i\|_i$, for $i = 1,2,3$. We take three $\mu$-simple functional $f, g$ and $h$ in $L_3(S;X)$ such that, for all $s \in S$,

$$\begin{aligned}
f(s) &= (1_{A_1} \otimes (3x_1))(s) + (1_{A_2} \otimes (-2x_2))(s) + (1_{A_3} \otimes (-x_3))(s),\\
g(s) &= (1_{A_1} \otimes (x_1))(s) + (1_{A_2} \otimes (-3x_2))(s) + (1_{A_3} \otimes (2x_3))(s),\\
h(s) &= 25(1_{A_1} \otimes (3x_1))(s) + 37(1_{A_2} \otimes (x_2))(s) + 77(1_{A_3} \otimes (x_3))(s).
\end{aligned}$$

By Proposition 3.5, with $p = 3$ and $q = \frac{3}{2}$, we have

$$\begin{aligned}
(J_3 f)(s) &= \frac{1}{\sqrt[3]{27+8+1}} \left(3(1_{A_1} \otimes J_X(3x_1)) + 2(1_{A_2} \otimes J_X(-2x_2)) + (1_{A_3} \otimes J_X(-x_3))\right)(s)\\
&= \frac{1}{\sqrt[3]{36}} \left(3(1_{A_1} \otimes (3J_X(x_1))) + 2(1_{A_2} \otimes (-2)J_X(x_2)) + (1_{A_3} \otimes (-J_X(x_3)))\right)(s)\\
&= \frac{1}{\sqrt[3]{36}} \left(9(1_{A_1} \otimes J_X(x_1)) - 4(1_{A_2} \otimes J_X(x_2)) - (1_{A_3} \otimes J_X(x_3))\right)(s).
\end{aligned}$$

Analogously, we have

$$(J_3 g)(s) = \frac{1}{\sqrt[3]{36}} \left((1_{A_1} \otimes J_X(x_1)) - 9(1_{A_2} \otimes J_X(x_2))4(1_{A_3} \otimes J_X(x_3))\right)(s).$$

Notice that $\langle J_X x_i, x_i\rangle = 1$ for $i = 1,2,3$. We further compute

$$\begin{aligned}\langle J_3 f, h\rangle &= \frac{1}{\sqrt[3]{36}}\int_{A_1}\langle 9(1_{A_1}\otimes(J_X(x_1))), 25(1_{A_1}\otimes x_1)\rangle(s)\,d\mu(s)\\ &\frac{1}{\sqrt[3]{36}}\int_{A_1}(-4)\langle(1_{A_1}\otimes(J_X(x_1))), 37(1_{A_2}\otimes x_1)\rangle(s)\,d\mu(s)\\ &\frac{1}{\sqrt[3]{36}}\int_{A_1}\langle(-1)(1_{A_1}\otimes(J_X(x_1))), 77(1_{A_1}\otimes x_1)\rangle(s)\,d\mu(s)\\ &= \frac{1}{\sqrt[3]{36}}\int_{A_1}(9)(25)\langle(1_{A_1}\otimes(J_X(x_1))), x_1\rangle(s)\,d\mu(s)\\ &+ \frac{1}{\sqrt[3]{36}}\int_{A_1}(-4)(37)\langle(1_{A_2}\otimes(J_X(x_2))), x_2\rangle(s)\,d\mu(s)\\ &+ \frac{1}{\sqrt[3]{36}}\int_{A_1}\langle(-77)(1_{A_3}\otimes(J_X(x_3))), x_3\rangle(s)\,d\mu(s) = 0.\end{aligned}$$

Analogously, we can similarly calculate $\langle J_3 g, h\rangle = 0$. Hence both $f$ and $g$ are in the set. We take a convex combination as follows $u(s) = \frac{2}{3}f(s) + \frac{1}{3}g(s)$ for all $s \in S$. Then,

$$u(s) = \left(1_{A_1}\otimes\left(\frac{7}{3}x_1\right)\right)(s) + \left(1_{A_2}\otimes\left(-\frac{7}{3}x_2\right)\right)(s) \quad \text{for all } s \in S.$$

By the analogous calculation as above, we have

$$(J_3 u)(s) = \frac{7\sqrt[3]{4}}{6}((1_{A_1}\otimes J_X x_1) + (1_{A_2}\otimes(-J_X x_2)))(s).$$

Similarly, we have

$$\langle J_3 u, h\rangle = \frac{7\sqrt[3]{4}}{6}(25\int_{A_1}\langle J_X x_1, x_1\rangle(s)\,d\mu(s) - 37\int_{A_2}\langle J_X x_2, x_2\rangle(s)\,d\mu(s)) = -\frac{7\sqrt[3]{4}}{3} < 0,$$

which shows that the convex combination $u$ of $f$ and $g$ is not in set defined in (5.7).

**Proposition 5.4.** *Let $(S,\mathscr{A},\mu)$ be a measure space, and let $X$ be a uniformly convex and uniformly smooth Banach space. Let $\theta \neq h$ be an arbitrary point in $L_p(S;X)$. Then $\{w \in L_p(S;X) : \langle J_p w, h\rangle \leq 0\}$ is a closed cone in $L_p(S;X)$ with vertex at $\theta$. However, in general it is not convex.*

**Proposition 5.5.** *Let $(S,\mathscr{A},\mu)$ be a measure space, and let $X$ be a uniformly convex and uniformly smooth Banach space. Let $K$ be a closed cone in $L_p(S;X)$ with vertex at $\theta$. Then, $J_p K$ is a closed cone in $X^*$ with vertex at $J_p\theta = \theta_{X^*}$.*

*Proof.* The proof follows from the fact $J_p$ is continuous and positively homogeneous. □

We construct an example to demonstrate that the convexity of $K$ does not necessarily imply that $J_p K$ is convex.

**Example 5.2.** We take a measure space $(S,\mathscr{A},\mu)$ with $\mu(S) \geq 3$ and take a uniformly convex and uniformly smooth Banach space $X$ with dimension greater than 1. We consider the uniformly convex and uniformly smooth Bochner space $L_3(S;X)$. Let $A_1, A_2$, and $A_3$ be three arbitrarily chosen mutually disjoint elements in $\mathscr{A}$ with $\mu(A_i) = 1$, and let $x_1, x_2$, and $x_3$ be three linearly independent points in $X$ with $\|x_i\|_i$, for $i = 1,2,3$. Let $x_i^* = J_X(x_i)$ with $\|x_i^*\|_{X^*} = \|x_i\|_X = 1$, for $i = 1,2,3$. Take $\varphi \in (L_3(S;X))^*$ as follows

$$\varphi(s) = (1_{A_1} \otimes x_i^*)(s) + (1_{A_2} \otimes x_2^*)(s) + (1_{A_3} \otimes x_3^*)(s) \quad \text{for all } s \in S$$

and define $K := \{w \in L_3(S;X) : \ \langle \varphi, w\rangle = 0\}$. Then, $K$ is a closed and convex cone in $L_3(S;X)$. We will show that $J_3(K)$ is not a convex cone in $(L_3(S;X))^* = L_{3/2}(S;X)$. We take two $\mu-$simple functional $u, v \in L_3(S : X)$ as follows:

$$\begin{aligned} u(a) &= (1_{A_1} \otimes (-x_1))(s) + (1_{A_2} \otimes x_2)(s) \quad \text{for all } s \in S, \\ v(a) &= (1_{A_2} \otimes (-x_2))(s) + (1_{A_3} \otimes x_3)(s) \quad \text{for all } s \in S. \end{aligned}$$

By performing calculations similar to Example 5.1, it can be shown that $\langle \varphi, u\rangle = 0$ and $\langle \varphi, v\rangle = 0$. Therefore, $u, v \in K$. As before, we compute

$$\begin{aligned} (J_3 u)(s) &= \frac{1}{\sqrt[3]{2}}((1_{A_1} \otimes (-x_1^*) + (1_{A_2} \otimes (x_2^*)))(s), \\ (J_3 v)(s) &= \frac{1}{\sqrt[3]{2}}((1_{A_2} \otimes (-x_2^*) + (1_{A_2} \otimes (x_3^*)))(s). \end{aligned}$$

We take a convex combination $\psi$ of $J_3 u$ and $J_3 v$ as follows

$$\psi = \frac{3}{4}J_3 u + \frac{1}{4}J_3 v = \frac{1}{4\sqrt[3]{2}}((1_{A_1} \otimes (-3x_1^*)) + (1_{A_2} \otimes (2x_2^*)) + (1_{A_3} \otimes x_3^*)).$$

Note that $J^*_{\frac{3}{2}}\psi \in L_3(S;X)$. Moreover, $J^*_{\frac{3}{2}} J_3 f = f$ for every $f \in L_3(S;X)$. Using Proposition 3.5, and the fact that $\mu(A_i) = \|x_i^*\|_{X^*} = \|x_i\| = 1$, for $i = 1,2,3$, we have

$$\begin{aligned} J^*_{\frac{3}{2}}\psi &= \frac{1}{4\sqrt[3]{2}} J_{\frac{3}{2}}((1_{A_1} \otimes (-3x_1^*)) + (1_{A_2} \otimes (2x_2^*)) + (1_{A_3} \otimes x_3^*)) \\ &= \frac{\sqrt[3]{3^{3/2} + 2^{3/2} + 1}}{4\sqrt[3]{2}}(\sqrt{3}(1_{A_1} \otimes (-3x_1^*)) + \sqrt{2}(1_{A_2} \otimes (x_2^*)) + (1_{A_3} \otimes x_3^*)). \end{aligned} \tag{5.8}$$

It can be shown that $\langle \varphi, J^*_{\frac{3}{2}}\psi\rangle > 0$, which ensures that $J^*_{\frac{3}{2}} \notin K$. Moreover, $\psi = J_3 J^*_{\frac{3}{2}}\psi \notin J_3(K)$, which proves that $J_3 K$ is not convex.

Finally, we give another related result.

**Proposition 5.6.** *Let $(S, \mathscr{A}, \mu)$ be a measure space, and let $X$ be a uniformly convex and uniformly smooth Banach space. Let $K$ be a closed subset in $L_p(S;X)$. Then the fact that $K$ is a cone with vertex at the origin does not imply that $J_p K$ is a cone.*

*Proof.* We construct a counter example to verify the claim. We adopt the setting of the previous example, and define $K{:=}\{(1-t)v + tu \in L_3(S;X) : \ 0 \leq t < \infty\}$. Thus $K$ is a ray with end points at $v \neq \theta$ and direction $u - v$, which is closed and convex cone with vertex at $v \neq \theta$ in $L_3(S;X)$. We show that $J_3 K$ is not a cone in $L_3(S;X)^*$. Since $J_3 u$ and $J_3 v$ are points in $J_3 K$, we consider the convex combination $\psi = \frac{3}{4}J_3 u + \frac{1}{4}J_3 v$. Observe that $J^*_{\frac{3}{2}} \in L_3(S;X)$ and it satisfies (5.8). Moreover, as in the previous example, we can see that $J^*\frac{3}{2}\psi \notin K$, which implies that $\psi = J(J^*_{\frac{3}{2}}) \notin JK$. This proves that $JK$ cannot be a ray (cone) in $(L_3(S;X))^*$. □

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## References

[1] Y. I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math., vol. 178, pp. 15-50, Dekker, New York, 1996.

[2] Y. Alber, J. L. Li, The connection between the metric and generalized projection operators in Banach spaces, Acta Math. Sin. (Engl. Ser.) 23 (6) (2007), 1109-1120.

[3] A. Beurling, A.E. Livingston, A theorem on duality mappings in Banach spaces, Ark. Mat. 4 (1962), 405-411.

[4] R. D. Bourgin, Geometric aspects of convex sets with the Radon-Nikodym property, Vol. 993 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1983.

[5] C. E. Chidume, J. Li, Projection methods for approximating fixed points of Lipschitz suppressive operators, PanAmer. Math. J. 15 (2005) 29–39.

[6] A. Domokos, Solution sensitivity of variational inequalities, J. Math. Anal. Appl. 230 (1999), 382–389.

[7] J. Gwinner, B. Jadamba, A. A. Khan, F. Raciti, Uncertainty Quantification in Variational Inequalities. Theory, Numerics, and Applications, CRC Press, Boca Raton, FL, 2022.

[8] H. Hudzik, T. R. Landes, Characteristic of convexity of Kothe function spaces, Math. Ann. 294 (1992), 117–124.

[9] T. Hytonen, J. van Neerven, M. Veraar, L. Weis, Analysis in Banach Spaces. Vol. II: Probabilistic Methods and Operator Theory, Springer, Cham, 2017.

[10] T. Ibaraki, W. Takahashi, A new projection and convergence theorems for the projections in Banach spaces, J. Approx. Theory 149 (2007), 1–14.

[11] T. B. Kien, The normalized duality mapping and two related characteristic properties of a uniformly convex Banach space, Acta Math. Vietnam. 27 (2002), 53–67.

[12] A.A. Khan, J. Li, Approximating properties of metric and generalized metric projections in uniformly convex and uniformly smooth Banach spaces. J. Approx. Theory, 297 (2024), 105973.

[13] A.A. Khan, J. Li, Characterizations of the metric and generalized metric projections on subspaces of Banach spaces, J. Math. Anal. Appl. 531 (2024), 127865.

[14] A.A. Khan, J. Li, S. Reich, Generalized projections on general Banach spaces, J. Nonlinear Convex Anal., 24 (2023), 1079-1112.

[15] I. E. Leonard, K. Sundaresan, Geometry of Lebesgue-Bochner function spaces—smoothness, Trans. Amer. Math. Soc. 198 (1974) 229–251.

[16] J. Li, The generalized projection operator on reflexive Banach spaces and its applications, J. Math. Anal. Appl. 306 (2005), 55-71.

[17] J. Li, On geometric properties of Bochner spaces and expected fixed point in Banach spaces, J. Nonlinear Convex Anal. in press.

[18] P.-K. Lin, Kothe-Bochner Function Spaces, Birkhäuser Boston, Inc., Boston, 2004.

[19] J. Mikusinski, The Bochner Integral, Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, Mathematische Reihe, Band 55, Birkhäuser Verlag, Basel-Stuttgart, 1978.

[20] J.-P. Penot, R. Ratsimahalo, Uniform continuity of projection mappings in Banach spaces, Commun. Appl. Nonlinear Anal. 6 (1999), 1–17.

[21] J. Prüss, A characterization of uniform convexity and applications to accretive operators, Hiroshima Math. J. 11 (1981), 229-234.

[22] M. A. Smith, B. Turett, Rotundity in Lebesgue-Bochner function spaces, Trans. Amer. Math. Soc. 257 (1980), 105–118.

[23] K. Sundaresan, The Radon-Nikodym theorem for Lebesgue-Bochner function spaces, J. Funct. Anal. 24 (1977), 276–279.

[24] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.