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CHARACTERIZATION OF $\mathscr{E}$-BENSON PROPER EFFICIENT SOLUTIONS OF VECTOR OPTIMIZATION PROBLEMS WITH VARIABLE ORDERING STRUCTURES IN LINEAR SPACES

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Dedicated to the memory of Professor Dr. Goepfert by Göpfert

Abstract. In this paper, using improvement-valued maps, we define two types of $\mathscr{E}$-Benson proper efficient elements for subsets within a linear space under a variable ordering map $\mathscr{C}$. Consequently, we delve into studying two types of $\mathscr{E}$-Benson proper efficient solutions of vector optimization problems under variable ordering structures. We establish relationships among different types of $\mathscr{E}$-Benson proper efficient elements. Furthermore, we demonstrate that the two types of $\mathscr{E}$-Benson proper efficiency, in relation to the ordering map $\mathscr{C}$, not only unify and extend certain notions of (weakly) nondominated elements but also extend some well-known notions of Benson proper efficiency under fixed ordering structures. Lastly, under suitable assumptions, we establish linear scalarization theorems for $\mathscr{E}$-Benson proper efficient solutions of vector optimization problems under variable ordering structures. Several examples are also provided to illustrate the derived results.

Keywords. Vector-valued maps; Variable ordering structures; $\mathscr{E}$-Benson proper efficient solution; Scalarization.

1. INTRODUCTION

The vector optimization theory finds widespread application across various fields, including economic analysis, engineering design, network transportation, healthcare, and more. In vector optimization problems, the ordering cones are typically defined by fixed convex cones. However, since Yu [1] introduced the concept of nondominated elements under a variable ordering structure, the field of vector optimization with variable ordering structures has experienced rapid growth. This development has found applications in diverse areas such as medical engineering, psychology, behavioral sciences, and economic theories (see [2, 3, 4, 5, 6] and the references therein).

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Many scholars have conducted in-depth research into vector optimization problems incorporating variable ordering structures in recent years. Chen [7] introduced the concept of nondominated-like minimal elements within the vector variational inequality. Additionally, Eichfelder [8] presented the notions of nondominated elements and minimal elements within a set possessing variable partial ordering structures. They highlighted that these elements within a set, characterized by variable partial ordering structures, can be interconverted through specific inclusion relations. Eichfelder and Kasimbeyli [9] along with Eichfelder and Gerlach [10] introduced various concepts concerning proper optimal solutions within vector optimization problems featuring variable ordering structures. They provided characterization results through scalarizations. Soleimani [11] and Soleimani and Tammer [12] proposed several notions of approximate solutions for vector optimization problems considering variable ordering structures. They established relationships between different approximate solutions and offered characterizations of nonlinear scalarizations for these solutions in such problems. You and Li [13] introduced a new notion of the approximate nondominated element of a set, replacing a constant-value in [11] with a vector-valued map. Shahbeyk, Soleimani-damaneh, and Kasimbeyli [14] introduced the concepts of Hartley proper nondominated solutions and super nondominated solutions within vector optimization involving a variable ordering structure. They established characterizations using approaches like the augmented dual cone, linear scalarization, and variational analysis tools. Moreover, Zhou et al. [15] introduced a novel notion of Benson nondominated elements within in linear spaces under variable ordering structures. This generalized the concept of Benson proper efficient elements under fixed partial ordering cones to those of variable partial ordering cones.

Meanwhile, Chicco et al. [16] introduced the improvement set in finite dimensional space and proposed the notion of E-optimal point with improvement sets. Recently, there has been a growing interest among scholars in vector optimization involving improvement sets. Gutiérrez et al. [17] expanded the concept of E-efficient points in locally convex topological linear spaces, establishing both linear and nonlinear scalarizations for E-efficient solutions in vector optimization problems. Zhou, Chen, and Yang [18] also contributed by establishing a scalarization result for both E-optimal and weak E-optimal solutions in set-valued optimization problems. In the realm of locally convex topological linear spaces, Zhao and Yang [19] introduced the concepts of E-Benson proper efficient points and E-subconvexlikeness for set-valued maps, laying the groundwork for linear scalarizations of E-Benson proper efficient solutions. Additionally, Gutiérrez et al. [17] introduced the idea of E-Benson proper optimal solutions in vector optimization, providing a scalarization characterization of Benson E-proper optimal solutions in linear spaces. Extending the scope further, Liu [20] incorporated improvement sets with variable ordering structures, introducing the concept of E-optimal elements in vector optimization problems with variable ordering structures. Recognizing a distinction between an E-Benson proper efficient element and an E-optimal element as proposed by Zhao and Yang [19], Liu and Yang [21] introduced the notion of type-II E-Benson proper efficiency and established a nonlinear scalarization characterization for type-II E-Benson proper efficient elements in real normed spaces. However, most of the results about the vector optimization problems with variable ordering structures were studied without improvement sets.

Inspired by the important results in [15, 20, 21], in this study, we investigate two kinds of $\mathscr{E}$-Benson proper efficient solutions of vector optimization problems with variable ordering structures in linear spaces. This paper is organized as follows. Section 2 gives some preliminaries, including some basic notions and lemmas. In Sections 3 and 4, by the improvement-valued maps, we introduce the concepts of three types of Benson proper efficient element of the set with variable ordering structures and discuss relationships among the concepts of the proper efficient element or (weakly) nondomination element of a set with both variable ordering structures and fixed ordering structures. In Sections 5, we establish some necessary conditions of these two types of $\mathscr{E}$-Benson proper efficient solution of vector optimization problems to be an $\mathscr{E}$-optimal solution of the scalarization optimization problem under the suitable assumptions. We also establish some sufficient conditions of these $\mathscr{E}$-Benson proper efficient solutions to be an $\mathscr{E}$-optimal solution or an optimal solution of the scalarization optimization problem under suitable assumptions. Finally, in Section 6, the conclusion and future scope of the paper are given.

2. Preliminaries

Throughout this paper, let X and Y be real linear spaces; C be a nonempty subset of Y; and A and M be two nonempty subsets of X and Y, respectively. Set C is said to be

(i) a cone if $\lambda c \in C$ for any $c \in C$ and $\lambda \geq 0$,
(ii) convex if $\lambda c_1 + (1-\lambda)c_2 \in C$ for any $c_1, c_2 \in C$ and $\lambda \in [0,1]$,
(iii) pointed if $C \cap (-C) = \{0\}$, and
(iv) nontrivial if $C \neq \{0\}$ and $C \neq Y$.

The cone generated by C is defined as $\text{cone}(C) := \{\lambda c | \lambda \geq 0, c \in C\}$. Let Y^* be the algebraic dual space of Y. The algebraic dual cone of C is defined by

$$C^+ := \{\mu \in Y^* | \ \langle y, \mu \rangle \geq 0 \ \forall \ y \in C\};$$

the quasi-interior of C^+ is the set

$$C^{+i} := \{\mu \in Y^* | \ \langle y, \mu \rangle > 0 \ \forall \ y \in C \backslash \{0\}\},$$

where $\langle y, \ \mu \rangle$ denotes the value of the linear functional μ at the point y.

For the nonempty subset M of Y, its (see [22, 23])

(i) associated linear subspace is the set $L(M) := \text{span}(M - M)$,
(ii) algebraic interior is the set $\text{cor}(M) := \{m \in M \mid \forall h \in Y, \ \exists \varepsilon > 0, \ \forall \lambda \in [0, \varepsilon], m + \lambda h \in M\}$,
(iii) relative algebraic interior is the set

$$\text{icr}(M) := \{m \in M \mid \forall h \in L(M), \ \exists \varepsilon > 0, \ \forall \lambda \in [0, \varepsilon], m + \lambda h \in M\},$$

and
(iv) vector closure is the set $\text{vcl}(M) := \{m \in Y \mid \exists h \in Y, \ \forall \varepsilon > 0, \ \exists \lambda \in (0, \varepsilon], m + \lambda h \in M\}$.

The set M is called solid if $\text{cor}(M) \neq \emptyset$ and relatively solid if $\text{icr}(M) \neq \emptyset$.

Lemma 2.1. [23] *If* $C \subseteq Y$ *is a convex cone and* $\text{cor}(C) \neq \emptyset$*, then* $\text{cor}(C) = C + \text{cor}(C)$.

Lemma 2.2. [24] *If* $\text{cor}(M) \neq \emptyset$*, then* $\text{cor}(M) = \text{icr}(M)$.

Throughout the paper, we assume that the variable ordering structure is given by the set-valued map $\mathscr{C} : Y \rightrightarrows Y$ with $\mathscr{C}(y)$ being a nontrivial closed convex pointed cone and $\mathrm{cor}(\mathscr{C}(y)) \neq \emptyset$ for all $y \in Y$.

Definition 2.1. [20] A map $\mathscr{E} : Y \rightrightarrows Y$ is called an improvement-valued map with respect to $\mathscr{C} : Y \rightrightarrows Y$ if, for all $y \in Y$, $\mathscr{E}(y) \neq \emptyset$, $0 \notin \mathscr{E}(y)$, and $\mathscr{E}(y) + \mathscr{C}(y) = \mathscr{E}(y)$.

Remark 2.1. Let E be a nonempty subset of Y, and let C be a nontrivial, closed, convex, and pointed cone in Y. If $\mathscr{E}(y) = E$ for all $y \in Y$, and $\mathscr{C}(y) = C$ for all $y \in Y$, then Definition 2.1 reduces to the definition of the improvement set E in [17].

Remark 2.2. If $\mathscr{E}(y) = \mathscr{C}(y) \backslash \{0\}$ for any $y \in Y$, then $\mathscr{E} : Y \rightrightarrows Y$ is an improvement-valued map with respect to $\mathscr{C} : Y \rightrightarrows Y$.

Inspired by the notion of the Benson proper optimal solution of vector optimization in [25], the following notion of the Benson proper efficient element of M was introduced.

Definition 2.2. [25] An element $\bar{y} \in M$ is called a Benson proper efficient element of M if

$$(-C) \cap \mathrm{vcl}(\mathrm{cone}(M + C - \bar{y})) = \{0\}.$$

Inspired by the notion of the E-Benson proper optimal solution of vector-valued optimization in [17] and the definition of E-Benson proper minimal element of M in a real topological vector space (or a real normal space) in [19] (or [21]), the following notions of Benson proper efficient elements of M in linear spaces were introduced.

Definition 2.3. [17] Let E be an improvement set of Y with respect to C. An element $\bar{y} \in M$ is called type-I E-Benson proper efficient element of M if $(-C) \cap \mathrm{vcl}(\mathrm{cone}(M + E - \bar{y})) = \{0\}$.

Definition 2.4. [17] Let E be an improvement set of Y with respect to C. An element $\bar{y} \in M$ is called type-II E-Benson proper efficient element of M if $(-E) \cap \mathrm{vcl}(\mathrm{cone}(M + C - \bar{y})) = \emptyset$.

Next, we define a nondominated element and an $\mathscr{E}$-nondominated element of M in a linear space.

Definition 2.5. [8] An element $\bar{y} \in M$ is called a nondominated element of M with respect to the set-valued map $\mathscr{C} : Y \rightrightarrows Y$ if there does not exist $y \in M$ such that $\bar{y} \in y + \mathscr{C}(y) \backslash \{0\}$.

Definition 2.6. [20] An element $\bar{y} \in M$ is called an $\mathscr{E}$-nondominated element of M with respect to $\mathscr{C}$ if $\{y - \bar{y}\} \cap (-\mathscr{E}(y)) = \emptyset \ \forall\, y \in M$.

Inspired by the definition of weakly $\mathscr{E}$-nondominated element of M with respect to variable ordering structures in real Banach space from [20], we introduce the following notion of weakly $\mathscr{E}$-nondominated element of M with respect to variable ordering structures in linear spaces.

Definition 2.7. An element $\bar{y} \in M$ is called a weakly $\mathscr{E}$-nondominated element of M with respect to $\mathscr{C}$ if $\{y - \bar{y}\} \cap (-\mathrm{cor}(\mathscr{E}(y))) = \emptyset \ \forall\, y \in M$.

Lemma 2.3. [25] *Let $C \subseteq Y$ be a nontrivial convex cone and $M \subseteq Y$. If $\mathrm{cor} C \neq \emptyset$, then $\mathrm{cor}(M + C) = M + \mathrm{cor} C$.*

Lemma 2.4. [15] *Let I be an index set. If A_i is the subset of Y with $\operatorname{cor}(A_i) \neq \emptyset$ for any $i \in I$, then* $\operatorname{cor}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \operatorname{cor}(A_i)$.

Lemma 2.5. [24, 25] *Let M be a nontrivial convex cone of Y. Then, $\operatorname{cor}(M) \neq \emptyset$ if and only if $\operatorname{cor}(M^+) \neq \emptyset$.*

Lemma 2.6. [25] *Let P and Q be two convex, vectorially closed, and relatively solid subsets of Y, and let Q^+ be solid. If $P \cap Q = \{0\}$, then there exists a linear functional $\mu \in Y^* \backslash \{0\}$ such that $\langle q,\ \mu\rangle \geq 0 \geq \langle p,\ \mu\rangle$ for all $q \in Q$, $p \in P$. Further, if $q \in Q\backslash\{0\}$, then $\langle q,\ \mu\rangle > 0$.*

Lemma 2.7. [26] *Let $P, Q \subseteq Y$ be two convex sets such that $P \neq \emptyset$, $\operatorname{cor}(Q) \neq \emptyset$ and $P \cap \operatorname{cor}(Q) = \emptyset$. Then, there exists a hyperplane separating P and Q in Y.*

Lemma 2.8. *Let Y be a real linear space and $\mathscr{C} : Y \rightrightarrows Y$ be such a set-valued map that $\mathscr{C}(y)$ be a solid cone for any $y \in Y$. If $\mathscr{E}$ is an improvement-valued map with respect to $\mathscr{C}$, then $\mathscr{E}(y)^+ \subseteq \mathscr{C}(y)^+$ for any $y \in Y$.*

Proof. It follows from Definition 2.1 that, for any fixed point $y \in Y$, $\mathscr{E}(y) + \mathscr{C}(y) = \mathscr{E}(y)$. Then, there exists a fixed point $e_0 \in \mathscr{E}(y)$ such that $e_0 + \mathscr{C}(y) \subseteq \mathscr{E}(y)$. For the chosen $y \in Y$, let $\mu \in \mathscr{E}(y)^+$. Then, it follows that $\langle e_0 + c,\ \mu\rangle \geq 0$ for all $c \in \mathscr{C}(y)$. Further, $\langle c,\ \mu\rangle \geq \langle -e_0,\ \mu\rangle$ for all $c \in \mathscr{C}(y)$. Since $\mathscr{C}(y)$ is a cone, we have $\langle c,\ \mu\rangle \geq 0$ for all $c \in \mathscr{C}(y)$, which implies that $\mu \in \mathscr{C}(y)^+$. Thus, for any $y \in Y$, $\mathscr{E}(y)^+ \subseteq \mathscr{C}(y)^+$. □

We undertake the following assumptions in the rest of the paper.

Condition A. The map $\mathscr{C} : Y \rightrightarrows Y$ is such a set-valued map that $\mathscr{C}(y)$ is a nontrivial closed convex pointed and solid cone for any $y \in Y$.

Condition B. The set-valued map $\mathscr{E} : Y \rightrightarrows Y$ is an improvement-valued map with respect to $\mathscr{C} : Y \rightrightarrows Y$.

3. Type-I $\mathscr{E}$-Benson Proper Efficient Elements of M

Zhao and Yang [19] introduced the notion of E-Benson proper efficient elements of M in topological linear spaces. Gutiérrez et al. [17] introduced the notion of the E-Benson proper optimal solution for vector optimization problems in real linear spaces. Inspired by these results, by means of the improvement-valued map $\mathscr{E} : Y \rightrightarrows Y$ with respect to the variable ordering structure $\mathscr{C}$, we introduce a notion of the type-I $\mathscr{E}$-Benson proper efficient element of a set M with respect to the following variable ordering structure.

Definition 3.1. An element $\bar{y} \in M$ is called a type-I $\mathscr{E}$-Benson proper efficient element of M with respect to the ordering map $\mathscr{C}$ if

$$(-\mathscr{C}(y)) \cap \operatorname{vcl}(\operatorname{cone}(\bigcup_{y\in M}(y + \mathscr{E}(y)) - \bar{y})) = \{0\}\ \forall\, y \in M.$$

The set of all type-I $\mathscr{E}$-Benson proper efficient element of M is denoted by $\mathscr{E}_I - O_{BS}^{\mathscr{E}(\cdot)}(M)$.

Theorem 3.1. *If $\bar{y} \in M$ is a type-I $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$, then $\bar{y}$ is a weakly $\mathscr{E}$-nondominated element of M with respect to $\mathscr{C}$.*

Proof. On the contrary, suppose that $\bar{y}$ is not a weakly $\mathscr{E}$-nondominated element of M. Then, there exists $\widehat{y} \in M$ such that

$$\widehat{y} - \bar{y} \in -(\mathrm{cor}(\mathscr{E}(y))). \tag{3.1}$$

As $\bar{y}$ is a type-I $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$,

$$(-\mathscr{C}(y)) \cap \mathrm{vcl}(\mathrm{cone}(\bigcup_{y \in M}(y + \mathscr{E}(y)) - \bar{y})) = \{0\} \ \forall\, y \in M.$$

Evidently, $(-\mathrm{cor}(C(y))) \cap (\bigcup_{y \in M}(y + \mathscr{E}(y)) - \bar{y}) = \emptyset$ for all $y \in M$. It follows that there exists $\widehat{y} \in M$ with $(-\mathrm{cor}(\mathscr{C}(\widehat{y}))) \cap (\widehat{y} + \mathscr{E}(\widehat{y}) - \bar{y}) = \emptyset$. Further,

$$(-\mathrm{cor}(\mathscr{C}(\widehat{y})) - \mathscr{E}(\widehat{y})) \cap \{\widehat{y} - \bar{y}\} = \emptyset \text{ for } \widehat{y} \in M. \tag{3.2}$$

It follows from Lemma 2.1 and Conditions A and B that

$$\mathrm{cor}(\mathscr{E}(y)) = \mathrm{cor}(\mathscr{C}(y) + \mathscr{E}(y)) = \mathrm{cor}(\mathscr{C}(y)) + \mathscr{E}(y) \ \forall\, y \in M. \tag{3.3}$$

By combining (3.2) and (3.3), we see that $(-\mathrm{cor}(\mathscr{E}(\widehat{y})) \cap \{\widehat{y} - \bar{y}\} = \emptyset$ for $\widehat{y} \in M$, which is contradictory to (3.1). Thus $\{\widehat{y} - \bar{y}\} \cap (-\mathrm{cor}(\mathscr{E}(y))) = \emptyset$ for all $y \in M$. Therefore, $\bar{y}$ is a weakly $\mathscr{E}$-nondominated element of M with respect to the ordering map $\mathscr{C}$. □

Remark 3.1. It is worth noting that if $\bar{y}$ is a type-I $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$, then $\bar{y}$ may neither be a nondominated element of M in [8], nor be an $\mathscr{E}$-nondominated element of M with respect to $\mathscr{C}$ in [20].

Remark 3.2. (i) If $\mathscr{E}(y) = C \backslash \{0\}$ and $\mathscr{C}(y) = C$ for any $y \in Y$, then Definition 3.1 of the type-I $\mathscr{E}$-Benson proper efficient points of M with respect to $\mathscr{C}$ reduces to Definition 2.2 of Benson proper efficient element of M.

(ii) If $\mathscr{E}(y) = E$ and $\mathscr{C}(y) = C$ for any $y \in Y$, then Definition 3.1 reduces to Definition 2.3 of type-I E-Benson proper efficient element of M.

Hence, the type-I $\mathscr{E}$-Benson proper efficient elements of M with respect to $\mathscr{C}$ unifies and generalizes the notions of Benson proper efficient elements of M in [19] and [21] in one of the following two ways

(a) replacing the fixed ordering structure with a variable ordering structure;
(b) replacing topological vector spaces or normal spaces with linear spaces.

The following example shows that a type-I $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$ may not be a Benson proper efficient element of M. Hence, Definition 3.1 of the type-I $\mathscr{E}$-Benson proper efficient element of M with respect to the ordering map $\mathscr{C}$ is a true generalization of the Benson proper efficient element of M.

Example 3.1. Let $Y = \mathbb{R}^2$, $M = \{(y_1, y_2) \in \mathbb{R}^2 | (y_1 - 1)^2 + (y_2 - 1)^2 \leq 1\} \cup \{(0,0)\}$, $C = \{(y_1, y_2) \in \mathbb{R}^2 | \ y_2 \geq 0\}$, $\bar{y} = (0,0)$, and the set-valued map $\mathscr{C} : Y \rightrightarrows Y$ be defined by

$$\mathscr{C}(y) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 | \ y_1 \geq 0, \ y_2 \geq 0\} & \text{if } y \in Y \backslash \{(1, \frac{1}{3})\} \\ \{(y_1, y_2) \in \mathbb{R}^2 | \ y_1 \geq 0, \ y_2 = 0\} & \text{if } y = (1, \frac{1}{3}). \end{cases}$$

Consider the set-valued map $\mathscr{E} : Y \rightrightarrows Y$ as

$$\mathscr{E}(y) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 | \ y_1 \geq 2, \ y_2 \geq 0\} & \text{if } y \in Y \backslash \{(1, \frac{1}{3})\} \\ \{(y_1, y_2) \in \mathbb{R}^2 | \ y_1 \geq 0, \ y_2 \geq 2\} & \text{if } y = (1, \frac{1}{3}). \end{cases}$$

It is easy to check that

$$(-\mathscr{C}(y))\cap \text{vcl}(\text{cone}(\bigcup_{y\in M}(y+\mathscr{E}(y))-\bar{y}))=\{0\}\ \forall\, y\in M \tag{3.4}$$

and

$$(-C)\cap \text{vcl}(\text{cone}(M+C-\bar{y}))=\{(y_1,y_2)\in \mathbb{R}^2|\ y_2=0\}\neq\{0\}. \tag{3.5}$$

It follows from (3.4) that $\bar{y}=(0,0)$ is a type-I $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$. However, (3.5) shows that $\bar{y}=(0,0)$ is not a Benson proper efficient element of M with respect to C.

The following example shows that a type-I $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$ may not be a type-I E-Benson proper efficient element of M. Hence, Definition 3.1 is a true generalization of type-I E-Benson proper efficient element of M.

Example 3.2. Consider Example 3.1 and take $E=\{(y_1,y_2)\in\mathbb{R}^2|y_2\geq 3\}$. Then, it is easy to check that

$$(-\mathscr{C}(y))\cap \text{vcl}(\text{cone}(\bigcup_{y\in M}(y+\mathscr{E}(y))-\bar{y}))=\{0\}\ \forall\, y\in M \tag{3.6}$$

and

$$(-C)\cap \text{vcl}(\text{cone}(M+E-\bar{y}))=\{(y_1,y_2)\in \mathbb{R}^2|\ y_2=0\}\neq\{0\}. \tag{3.7}$$

It follows from (3.6) that $\bar{y}=(0,0)$ is a type-I $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$. However, (3.7) shows that $\bar{y}=(0,0)$ is not a type-I E-Benson proper efficient element of M with respect to C.

Remark 3.3. If $\varepsilon\in C\backslash\{0\}$, $\mathscr{E}(y)=\varepsilon+C$ and $\mathscr{C}(y)=C$ for all $y\in Y$, then the type-I $\mathscr{E}$-Benson proper efficiency of M reduces to the ε-Benson proper efficiency of M in [27].

Inspired by [15, Remark 2.8] and the Benson proper optimal solutions of vector optimization problems in [25], we introduce the following $\mathscr{C}$-Benson proper efficient element of M with variable ordering structures that extends the notion of Benson proper efficient elements of M from the case of fixed ordering structure to that of variable ordering structure.

Definition 3.2. An element $\bar{y}\in M$ is called a $\mathscr{C}$-Benson proper efficient element of M if

$$(-\mathscr{C}(y))\cap \text{vcl}(\text{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y}))=\{0\}\ \forall\, y\in M.$$

To illustrate the relationship between the type-I $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$ and the $\mathscr{C}$-Benson proper efficient element of M, we need the following result.

Lemma 3.1. $\text{vcl}(\text{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y}))=\text{vcl}(\text{cone}(\bigcup_{y\in M}(y+\text{cor}\mathscr{C}(y))-\bar{y}))$.

Proof. Since $\text{vcl}(\text{cone}(\bigcup_{y\in M}(y+\text{cor}\mathscr{C}(y))-\bar{y}))\subseteq\text{vcl}(\text{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y}))$, we only need to prove that

$$\text{vcl}(\text{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y}))\subseteq\text{vcl}(\text{cone}(\bigcup_{y\in M}(y+\text{cor}\mathscr{C}(y))-\bar{y})).$$

Let $b \in \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y}))$. Then, by the definition of vector closure of a set, $\exists h \in Y$, $\forall \varepsilon > 0$, $\exists r \in [0,\varepsilon]$ such that

$$b+rh \in \mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y}). \tag{3.8}$$

Then, from the definition of a cone, there exists $\lambda \geq 0$ such that $b+rh \in \lambda(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y})$. Hence, $\exists h \in Y$, $\forall \varepsilon > 0$, there exist $r \in [0,\varepsilon]$, $\lambda \geq 0$ and $\widehat{y} \in M$ for which

$$b+rh \in \lambda(\widehat{y}+\mathscr{C}(\widehat{y})-\bar{y}). \tag{3.9}$$

It follows from (3.9) that there exists $c_1 \in \mathscr{C}(\widehat{y})$ such that $b+rh = \lambda(\widehat{y}+c_1-\bar{y})$. Since $\mathrm{cor}(\mathscr{C}(y)) \neq \emptyset$ for all $y \in Y$, there exists $c_2 \in \mathrm{cor}(\mathscr{C}(\widehat{y}))$ such that

$$b+rh+r\lambda c_2 = \lambda(\widehat{y}+c_1+rc_2-\bar{y}). \tag{3.10}$$

As $\mathrm{cor}(\mathscr{C}(y)) \neq \emptyset$, we have from Lemma 2.1 that

$$\mathrm{cor}(\mathscr{C}(y)) = \mathscr{C}(y)+\mathrm{cor}(\mathscr{C}(y)) \ \forall\, y \in M. \tag{3.11}$$

By (3.10) and (3.11),

$$\lambda(\widehat{y}+c_1+rc_2-\bar{y}) \in \mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y)+\mathrm{cor}\mathscr{C}(y))-\bar{y}) = \mathrm{cone}(\bigcup_{y\in M}(y+\mathrm{cor}\mathscr{C}(y))-\bar{y}). \tag{3.12}$$

Therefore, it follows from (3.8)–(3.12) that $\exists \hat{h} := h+\lambda c_2 \in Y$ with $\lambda \geq 0$, $\forall \varepsilon > 0$, there exists $r \in [0,\varepsilon]$ satisfying $b+r\hat{h} \in \mathrm{cone}(\bigcup_{y\in M}(y+\mathrm{cor}\mathscr{C}(y))-\bar{y})$. Thus, $b \in \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathrm{cor}\mathscr{C}(y))-\bar{y}))$, and hence

$$\mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y})) \subseteq \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathrm{cor}\mathscr{C}(y))-\bar{y})).$$

Hence, the result follows. □

Next, we establish a relationship between type-I $\mathscr{E}$-Benson proper efficient elements of M with respect to $\mathscr{C}$ and the $\mathscr{C}$-Benson proper efficient elements of M.

Theorem 3.2. *Assume that $\mathscr{E}(y) = \mathscr{C}(y)\setminus\{0\}$ for all $y \in Y$. Then, the following three statements are true.*

(i) A point $\bar{y}$ is a type-I $\mathscr{E}$-Benson proper efficient point of M with respect to $\mathscr{C}$ if and only if $\bar{y}$ is a $\mathscr{C}$-Benson proper efficient point of of M.

(ii) If $\bar{y}$ is a type-I $\mathscr{E}$-Benson proper efficiency of M with respect to $\mathscr{C}$, then $\bar{y}$ is a nondominated element of M with respect to $\mathscr{C}$.

(iii) If $\bar{y}$ is a $\mathscr{C}$-Benson proper efficient point of M, then $\bar{y}$ is a nondominated element of M with respect to $\mathscr{C}$.

Proof. (i) From Remark 2.2, we see that $\mathscr{E}: Y \rightrightarrows Y$ is an improvement-valued map with respect to $\mathscr{C}: Y \rightrightarrows Y$. Since $\mathscr{E}(y) = \mathscr{C}(y)\backslash\{0\}$,

$$\begin{aligned} \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{E}(y))-\bar{y})) =& \ \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y)\setminus\{0\})-\bar{y})) \\ \subseteq& \ \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y})). \end{aligned} \tag{3.13}$$

It follows from Lemma 3.1 that

$$\begin{aligned} \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y})) =& \ \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathrm{cor}\mathscr{C}(y))-\bar{y})) \\ \subseteq& \ \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y)\setminus\{0\})-\bar{y})) \\ =& \ \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{E}(y))-\bar{y})). \end{aligned} \tag{3.14}$$

Combining (3.13) with (3.14), we have

$$\mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{E}(y))-\bar{y})) = \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y})).$$

Since $\mathscr{E}(y) = \mathscr{C}(y)\backslash\{0\}$ for all $y \in Y$, the set of type-I $\mathscr{E}$-Benson proper efficient elements of M with respect to $\mathscr{C}$ coincides with the set of $\mathscr{C}$-Benson proper efficient elements of M.

(ii) On the contrary, we assume that $\bar{y}$ is not a nondominated element of M. Then, there exists $\hat{y} \in M$ such that

$$\hat{y} - \bar{y} \in -\mathscr{C}(\hat{y})\backslash\{0\}. \tag{3.15}$$

Because $\bar{y}$ is a type-I $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$,

$$(-\mathscr{C}(y)) \cap \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{E}(y))-\bar{y})) = \{0\} \ \forall\, y \in M.$$

Evidently,

$$(-\mathscr{C}(y)\backslash\{0\}) \cap (\bigcup_{y\in M}(y+\mathscr{E}(y))-\bar{y}) = \emptyset \ \forall y \in M. \tag{3.16}$$

Therefore, for $\hat{y} \in M$, it follows from (3.16) that $(-\mathscr{C}(\hat{y})\backslash\{0\}) \cap (\hat{y}+\mathscr{E}(\hat{y})-\bar{y}) = \emptyset$. Further, $(-\mathscr{C}(\hat{y})\backslash\{0\} - \mathscr{E}(\hat{y})) \cap \{\hat{y}-\bar{y}\} = \emptyset$. Since $\mathscr{E}(y) = \mathscr{C}(y)\backslash\{0\}$ for all $y \in M$,

$$(-\mathscr{C}(\hat{y})\backslash\{0\}) \cap \{\hat{y}-\bar{y}\} = \emptyset,$$

which is contradictory to (3.15). Thus $\{\bar{y}\} \cap (y+\mathscr{C}(y)\backslash\{0\}) = \emptyset \ \forall\, y \in M$, which implies that $\bar{y}$ is a nondominated element of M with respect to the ordering map $\mathscr{C}$.

(iii) Since $\mathscr{E}(y) = \mathscr{C}(y)\backslash\{0\}$ for all $y \in Y$, from the proof of (ii), we can easily see that (iii) holds true.

□

4. TYPE-II $\mathscr{E}$-BENSON PROPER EFFICIENT ELEMENTS OF M

Liu and Yang [21] introduced a new notion called the type-II E-Benson proper minimal element of M in real normed spaces. We extend this concept from fixed ordering structures in real normed spaces to variable ordering structures in real linear space.

Definition 4.1. An element $\bar{y} \in M$ is called a type-II $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$ if

$$(-\mathscr{E}(y)) \cap \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y})) = \emptyset \;\forall\; y \in M.$$

Theorem 4.1. *If $\bar{y} \in M$ is a type-II $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$, then $\bar{y}$ is both an $\mathscr{E}$-nondominated element and a weakly $\mathscr{E}$-nondominated element of M with respect to the ordering map $\mathscr{C}$.*

Proof. On the contrary, we assume that $\bar{y}$ is not an $\mathscr{E}$-nondominated element of M. Thus there exists $\widehat{y} \in M$ such that

$$\widehat{y} - \bar{y} \in -\mathscr{E}(\widehat{y}). \tag{4.1}$$

As $\bar{y}$ is a type-II $\mathscr{E}$-Benson proper efficient element of M, we have

$$(-\mathscr{E}(y)) \cap \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y})) = \emptyset \;\forall\; y \in M.$$

Observe that $(-\mathscr{E}(y)) \cap (\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y}) = \emptyset$ for all $y \in M$, which implies that there exists $\widehat{y} \in M$ such that $(-\mathscr{E}(\widehat{y})) \cap (\widehat{y}+\mathscr{C}(\widehat{y})-\bar{y}) = \emptyset$. Further,$(-\mathscr{C}(\widehat{y}) - \mathscr{E}(\widehat{y})) \cap \{\widehat{y}-\bar{y}\} = \emptyset$. Since $\mathscr{E}(y)$ is an improvement set with respect to $\mathscr{C}$, we have $(-\mathscr{E}(\widehat{y})) \cap \{\widehat{y}-\bar{y}\} = \emptyset$, which is contradictory to (4.1). Thus$\{y-\bar{y}\} \cap (-\mathscr{E}(y)) = \emptyset$ for all $y \in M$, and then $\{y-\bar{y}\} \cap (-\mathrm{cor}\mathscr{E}(y)) = \emptyset$ for all $y \in M$. The last two equations show that $\bar{y}$ is both an $\mathscr{E}$-nondominated element and a weakly $\mathscr{E}$-nondominated element of M with respect to $\mathscr{C}$. □

Remark 4.1. It is worth noting that if $\bar{y}$ is a type-II $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$, then $\bar{y}$ may not be a nondominated element of M in [8] with respect to the ordering map $\mathscr{C}$.

Remark 4.2. If $\mathscr{E}(y) = \mathscr{C}(y)\backslash\{0\}$ for all $y \in Y$, then it is easy to see that $\bar{y}$ is a type-II $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$ which is equivalent to $\bar{y}$, a $\mathscr{C}$-Benson proper efficient element of M.

Remark 4.3. If $\mathscr{E}(y) = E$ and $\mathscr{C}(y) = C$ for all $y \in Y$, then the definition of type II $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$ reduces to that of the type-II E-Benson proper efficient element of M in real linear space (i.e., Definition 2.4), which in turn becomes that of the type-II E-Benson proper minimal element of M in [21] if Y is a real normal space.

The following example shows that a type-II $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$ may be not a type-II E-Benson proper efficient element of M. Hence, the type-II $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$ is true generalization of the type-II E-Benson proper efficient element of M.

Example 4.1. Let $Y=\mathbb{R}^2$, $M=\{(y_1,y_2)\in\mathbb{R}^2|\ y_1=0,\ y_2<0\}\cup\{(y_1,y_2)\in\mathbb{R}^2|\ 0\le y_1\le 1,\ 0\le y_2\le 1\}$, $C=\{(y_1,y_2)\in\mathbb{R}^2|\ y_1\ge 0,\ y_2\ge 0\}$, $E=\{(y_1,y_2)\in\mathbb{R}^2|\ y_1\ge 0,\ y_2\ge 1\}$, and $\bar{y}=(0,0)$. Consider the set-valued map $\mathscr{C}:Y\rightrightarrows Y$ as

$$\mathscr{C}(y)=\begin{cases}\{(y_1,y_2)\in\mathbb{R}^2|\ y_1\ge 0,\ y_2\ge 0\} & \text{if } y\in Y\backslash\{(1,\frac{1}{3})\}\\ \{(y_1,y_2)\in\mathbb{R}^2|\ y_1\ge 0,\ y_2=0\} & \text{if } y=(1,\frac{1}{3}),\end{cases}$$

and $\mathscr{E}:Y\rightrightarrows Y$ as

$$\mathscr{E}(y)=\begin{cases}\{(y_1,y_2)\in\mathbb{R}^2|\ y_1\ge 1,\ y_2\ge 2\} & \text{if } y\in Y\backslash\{(1,\frac{1}{3})\}\\ \{(y_1,y_2)\in\mathbb{R}^2|\ y_1\ge 1,\ 1\le y_2<2\} & \text{if } y=(1,\frac{1}{3}).\end{cases}$$

It is easy to check that

$$(-\mathscr{E}(y))\cap\operatorname{vcl}(\operatorname{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y}))=\emptyset\ \forall\, y\in M \tag{4.2}$$

and

$$(-E)\cap\operatorname{vcl}(\operatorname{cone}(M+C-\bar{y}))=\{(y_1,y_2)\in\mathbb{R}^2|\ y_1=0,\ y_2\le -1\}\neq\emptyset. \tag{4.3}$$

It follows from (4.2) that $\bar{y}=(0,0)$ is a type-II $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$. However, (4.3) shows that $\bar{y}=(0,0)$ is not a type-II E-Benson proper efficient element of M.

The following two examples show that a type-II $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$ may be not a type-I $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$, and a type-I $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$ also may be not a type-II $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$.

Example 4.2. Let $Y=\mathbb{R}^2$, $M=\{(y_1,y_2)\in\mathbb{R}^2|\ 2y_1+y_2\ge 0\}\cup\{(y_1,y_2)\in\mathbb{R}^2|\ 0\le y_1\le 1,\ -1\le y_2\le 0\}$, $C=\{(y_1,y_2)\in\mathbb{R}^2|\ y_1\ge 0,\ y_2\ge 0\}$, $E=\{(y_1,y_2)\in\mathbb{R}^2|\ y_1\ge 0,\ y_2\ge 1\}$ and $\bar{y}=(0,0)$. Consider $\mathscr{C}:Y\rightrightarrows Y$ as

$$\mathscr{C}(y)=\begin{cases}\{(y_1,y_2)\in\mathbb{R}^2|\ y_1>0,\ y_2>0\}\cup\{(0,0)\} & \text{if } y\in Y\backslash\{(1,-\frac{1}{4})\}\\ \{(y_1,y_2)\in\mathbb{R}^2|\ y_1\ge 0,\ y_2=0\} & \text{if } y=(1,-\frac{1}{4}),\end{cases}$$

and $\mathscr{E}:Y\rightrightarrows Y$ as

$$\mathscr{E}(y)=\begin{cases}\{(y_1,y_2)\in\mathbb{R}^2|\ y_1\ge 1,\ y_2\ge 0\} & \text{if } y\in Y\backslash\{(1,-\frac{1}{4})\}\\ \{(y_1,y_2)\in\mathbb{R}^2|\ y_1\ge 0,\ y_2\ge 1\} & \text{if } y=(1,-\frac{1}{4}).\end{cases}$$

It is easy to check that

$$(-\mathscr{C}(y))\cap\operatorname{vcl}(\operatorname{cone}(\bigcup_{y\in M}(y+\mathscr{E}(y))-\bar{y}))=\{0\}\ \forall\, y\in M \tag{4.4}$$

and

$$(-\mathscr{E}(y))\cap\operatorname{vcl}(\operatorname{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y}))=\{(y_1,y_2)\in\mathbb{R}^2|\ y_1=0,\ y_2\le -1\}\neq\emptyset\ \forall\, y\in M. \tag{4.5}$$

It follows from (4.4) that $\bar{y}=(0,0)$ is a type-I $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$. However, (4.5) shows that $\bar{y}=(0,0)$ is not a type-II $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$.

Example 4.3. In Example 4.1, we see that

$$(-\mathscr{E}(y))\cap \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{C}(y))-\bar{y}))=\emptyset\ \forall\, y\in M \tag{4.6}$$

and

$$(-\mathscr{C}(y))\cap \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in M}(y+\mathscr{E}(y))-\bar{y}))=\{(y_1,y_2)\in\mathbb{R}^2 \mid y_1=0,\ y_2\le 0\}\ne\{0\}\ \forall\, y\in M. \tag{4.7}$$

It follows from (4.6) that $\bar{y}=(0,0)$ is a type-II $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$. However, (4.7) shows that $\bar{y}=(0,0)$ is not a type-I $\mathscr{E}$-Benson proper efficient element of M with respect to $\mathscr{C}$.

5. Characterization of Linear Scalarization for (VOP)

In this section, we first introduce two types of $\mathscr{E}$-Benson proper efficient solutions for vector optimization problems with variable ordering structures. Then, we establish some scalarization results for vector optimization problems in the sense of two types of $\mathscr{E}$-Benson proper efficient solutions. Let $\mathscr{C}:Y\rightrightarrows Y$ and $\mathscr{E}:Y\rightrightarrows Y$ be two nonempty set-valued maps, and let $f:X\to Y$ be a vector-valued map. Suppose that $\mathscr{C}(y)$ is a nontrivial solid closed convex pointed cone, and $\mathscr{E}(y)$ is an improvement set with respect to $\mathscr{C}(y)$ for any $y\in Y$. In this paper, we consider the following *vector optimization problem* (VOP):

$$\min\ f(x)\ \text{ subject to }\ x\in A\subseteq X. \tag{VOP}$$

We recall here that the support function of a non-empty closed convex set $M\subseteq Y$ is defined by

$$\sigma_M(\mu):=\sup\{\langle y,\ \mu\rangle : y\in M\},\ \mu\in Y^*.$$

Depending on Definition 3.1, we introduce the following concept of the type-I $\mathscr{E}$-Benson proper efficient solution of (VOP).

Definition 5.1. A point $\bar{x}\in A$ is called a type-I $\mathscr{E}$-Benson proper efficient solution of (VOP) with respect to $\mathscr{C}$ if

$$(-\mathscr{C}(y))\cap \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)}(y+\mathscr{E}(y))-f(\bar{x})))=\{0\}\ \forall\, y\in f(A).$$

Remark 5.1. If $\mathscr{E}(y)=E$ and $\mathscr{C}(y)=C$ for all $y\in Y$, then Definition 5.1 reduces to the definition of the E-Benson proper efficient solution in [17]. Moreover, if $E=C\backslash\{0\}$, then the type-I E-Benson proper optimal solution for vector optimization becomes the Benson proper efficient solution in [25].

Definition 5.2. A point $\bar{x}\in A$ is called a nondominated solution of (VOP) with respect to $\mathscr{C}$ if $f(\bar{x})\notin y+\mathscr{C}(y)\backslash\{0\}$ for all $y\in f(A)$.

Definition 5.3. A point $\bar{x}\in A$ is called an $\mathscr{E}$-nondominated solution of (VOP) with respect to $\mathscr{C}$ if $f(\bar{x})\notin y+\mathscr{E}(y)$ for all $y\in f(A)$.

Definition 5.4. A point $\bar{x}\in A$ is called a weakly $\mathscr{E}$-nondominated solution of (VOP) with respect to $\mathscr{C}$ if $f(\bar{x})\notin y+\mathrm{cor}(\mathscr{E}(y))$ for all $y\in f(A)$.

Definition 5.5. A point $\bar{x} \in A$ is called a $\mathscr{C}$-Benson proper efficient solution of (VOP) with respect to $\mathscr{C}$ if $(-\mathscr{C}(y)) \cap \text{vcl}(\text{cone}(\bigcup_{y\in f(A)} (y+\mathscr{C}(y)) - f(\bar{x}))) = \{0\}$ for all $y \in f(A)$.

From Theorem 3.1, we obtain the following result for (VOP).

Proposition 5.1. *If $\bar{x} \in A$ is a type-I $\mathscr{E}$-Benson proper efficient element of (VOP) with respect to $\mathscr{C}$, then $\bar{x}$ is a weakly $\mathscr{E}$-nondominated element of (VOP) with respect to $\mathscr{C}$.*

From Theorem 3.2, we obtain the following result for (VOP).

Proposition 5.2. *Assume that $E(y) = \mathscr{C}(y)\backslash\{0\}$ for all $y \in Y$. Then, the following three statements are true*

(i) An element $\bar{x} \in A$ is a type-I $\mathscr{E}$-Benson proper efficient solution of (VOP) with respect to $\mathscr{C}$ if and only if $\bar{x}$ is a $\mathscr{C}$-Benson proper efficient solution of (VOP).

(ii) If $\bar{x}$ is a type-I $\mathscr{E}$-Benson proper efficient solution of (VOP) with respect to $\mathscr{C}$, then $\bar{x}$ is a nondominated solution of (VOP) with respect to $\mathscr{C}$.

(iii) If $\bar{x}$ is a $\mathscr{C}$-Benson proper efficient solution of (VOP), then $\bar{x}$ is a nondominated solution of (VOP) with respect to $\mathscr{C}$.

Base on Definition 4.1, we introduce the following concept of type-II $\mathscr{E}$-Benson proper efficient solution of (VOP).

Definition 5.6. A point $\bar{x} \in A$ is called a type-II $\mathscr{E}$-Benson proper efficient solution of (VOP) with respect to $\mathscr{C}$ if $(-\mathscr{E}(y)) \cap \text{vcl}(\text{cone}(\bigcup_{y\in f(A)} (y+\mathscr{C}(y)) - f(\bar{x}))) = \emptyset$ for all $y \in f(A)$.

From Theorem 4.1, we obtain the following result.

Proposition 5.3. *If $\bar{x}$ is a type-II $\mathscr{E}$-Benson proper efficient solution of (VOP) with respect to $\mathscr{C}$, then $\bar{x}$ is both an $\mathscr{E}$-nondominated solution of (VOP) and a weakly $\mathscr{E}$-nondominated solution of (VOP) with respect to $\mathscr{C}$.*

Next, we consider the following scalarization of (VOP) to characterize $\mathscr{E}$-Benson proper optimal solutions:

$$\min \langle f(x), \mu\rangle \text{ subject to } x \in A, \tag{VOP$_\mu$}$$

where $\mu \in Y^* \setminus \{0\}$.

Definition 5.7. A point $\bar{x} \in A$ is called an $\mathscr{E}$-optimal solution of (VOP$_\mu$) if $\langle f(x) - f(\bar{x}), \mu\rangle \geq \sigma_{-\mathscr{E}(y)}(\mu)$ for all $x \in A, y \in f(A)$.

Remark 5.2. If $\mathscr{E}(y) = E$, an $\mathscr{E}$-optimal solution of (VOP$_\mu$) reduces to an E-optimal solution of (VOP$_\mu$).

Next, we provide a necessary condition for a type-I $\mathscr{E}$-Benson proper efficient solution of (VOP), which is also a characterization of type-I $\mathscr{E}$-Benson proper efficient solution in terms of an $\mathscr{E}$-optimal solution of the scalarized problem (VOP$_\mu$).

Theorem 5.1. *Suppose that*

(i) $\bar{x}$ is a type-I $\mathscr{E}$-Benson proper efficient solution of (VOP) with respect to $\mathscr{C}$, and

(ii) $\text{vcl}(\text{cone}(\bigcup_{y\in f(A)} (y+\mathscr{E}(y)) - f(\bar{x})))$ is a nontrival convex subset of Y.

Then, there exists $\mu \in (\mathscr{C}(y))^{+i}$ *such that* $\bar{x}$ *is an* $\mathscr{E}$*-optimal solution of* (VOP$_\mu$).

Proof. Since $\bar{x}$ is a type-I $\mathscr{E}$-Benson proper efficient solution of (VOP) with respect to $\mathscr{C}$, then $\bar{x} \in A$ such that

$$(-\mathscr{C}(y)) \cap \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)} (y+\mathscr{E}(y)) - f(\bar{x}))) = \{0\} \; \forall\, y \in f(A). \tag{5.1}$$

We assert that

$$\mathrm{cor}(\mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)} (y+\mathscr{E}(y)) - f(\bar{x})))) \neq \emptyset. \tag{5.2}$$

Since $\mathscr{E}(y)$ is an improvement set and $\mathrm{cor}\mathscr{C}(y) \neq \emptyset$ for any $y \in f(A)$, it follows from Lemma 2.3 that

$$\mathrm{cor}(y+\mathscr{E}(y)-f(\bar{x})) = \mathrm{cor}(y+\mathscr{E}(y)+\mathscr{C}(y)-f(\bar{x})) = y+\mathscr{E}(y)-f(\bar{x})+\mathrm{cor}\mathscr{C}(y) \neq \emptyset \; \forall\, y \in f(A).$$

Therefore, $\bigcup_{y\in f(A)} (\mathrm{cor}(y+\mathscr{E}(y)-f(\bar{x})) \neq \emptyset$, which together with Lemma 2.4 yields that

$$\mathrm{cor}(\bigcup_{y\in f(A)} (y+\mathscr{E}(y)-f(\bar{x}))) = \bigcup_{y\in f(A)} (\mathrm{cor}(y+\mathscr{E}(y)-f(\bar{x}))) \neq \emptyset.$$

It follows that

$$\mathrm{cor}(\bigcup_{y\in f(A)} (y+\mathscr{E}(y)) - f(\bar{x})) = \mathrm{cor}(\bigcup_{y\in f(A)} (y+\mathscr{E}(y)-f(\bar{x}))) \neq \emptyset.$$

Then, we obtain that $\mathrm{cor}(\mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)} (y+\mathscr{E}(y)) - f(\bar{x})))) \neq \emptyset$. It follows from condition (ii) that $\mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)} (y+\mathscr{E}(y)) - f(\bar{x})))$ is a vectorially closed convex subset of Y. Since $\mathscr{C}(y)$ is solid, we obtain by Lemma 2.5 that

$$\mathrm{cor}(\mathscr{C}(y)^{+}) \neq \emptyset \text{ and } \mathrm{cor}([\mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)} (y+\mathscr{E}(y)) - f(\bar{x})))]^{+}) \neq \emptyset. \tag{5.3}$$

Combining (5.1), (5.2), (5.3) with Lemma 2.6 and Lemma 2.1, we obtain that, for all $y \in f(A)$, there exists $\mu \in Y^*\backslash\{0\}$ such that

$$\langle z,\, \mu\rangle \geq 0 > \langle -c,\, \mu\rangle \; \forall z \in \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)} (y+\mathscr{E}(y)) - f(\bar{x}))) \; \forall\, c \in \mathscr{C}(y)\backslash\{0\}. \tag{5.4}$$

By (5.4), we see that, for all $y \in f(A)$, there exists $\mu \in Y^*\backslash\{0\}$ such that $\mu \in (\mathscr{C}(y))^{+i}$. Obviously,

$$\bigcup_{y\in f(A)} (y+\mathscr{E}(y)) - f(\bar{x}) \subseteq \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)} (y+\mathscr{E}(y)) - f(\bar{x}))). \tag{5.5}$$

It follows from (5.4) and (5.5) that

$$\langle f(x) - f(\bar{x}),\, \mu\rangle \geq \sigma_{-\mathscr{E}(y)}(\mu) \; \forall\, x \in A,\; \forall y \in f(A). \tag{5.6}$$

Thus (5.4) and (5.6) yield that $\bar{x}$ is an $\mathscr{E}$-optimal solution of (VOP$_\mu$). □

Remark 5.3. If $\mathscr{C}(y) = C$ and $\mathscr{E}(y) = E$, then the condition (ii) of Theorem 5.1 reduces to $\mathrm{vcl}(\mathrm{cone}((f(A)+E-f(\bar{x})))$ is a nontrivial convex subset of Y, which is equivalent to that $f - f(\bar{x})$ is v-nearly E-subconvexlike on A in [17]. Therefore, we have the following result by Theorem 5.1.

Corollary 5.1. *If*

(i) $\bar{x}$ *is a type-I* E*-Benson proper efficient solution of* (VOP), *and*

(ii) $f - f(\bar{x})$ *is v-nearly* E*-subconvexlike on* A,

then there exists $\mu \in C^{+i}$ *such that* $\bar{x}$ *is an* E*-optimal solution of* (VOP$_\mu$).

The following example illustrates Theorem 5.1.

Example 5.1. Let $Y = \mathbb{R}^2$ and $A = [0,2] \times [0,2] \subseteq \mathbb{R}^2$. The vector-valued map $f : A \to Y$ is defined as follows:

$$f(x_1,x_2) = \left(\tfrac{3}{2}\sqrt[3]{x_1}, 3x_2^2\right),\ (x_1,x_2) \in A.$$

Consider the set-valued map $\mathscr{C} : Y \rightrightarrows Y$ as

$$\mathscr{C}(y) = \begin{cases} \{(y_1,y_2) \in \mathbb{R}^2 |\ y_1 \geq 0,\ y_2 \geq 0\} & \text{if } y \in Y \setminus \{(1,\frac{1}{3})\} \\ \{(y_1,y_2) \in \mathbb{R}^2 |\ y_1 \geq 0,\ y_2 = 0\} & \text{if } y = (1,\frac{1}{3}), \end{cases}$$

and $\mathscr{E} : Y \rightrightarrows Y$ as

$$\mathscr{E}(y) = \begin{cases} \{(y_1,y_2) \in \mathbb{R}^2 |\ y_1 \geq 1,\ y_2 \geq 0\} & \text{if } y \in Y \setminus \{(1,\frac{1}{3})\} \\ \{(y_1,y_2) \in \mathbb{R}^2 |\ y_1 \geq 0,\ y_2 \geq 1\}, & \text{if } y = (1,\frac{1}{3}) \end{cases}$$

Let $\bar{x} = (0,0)$. Then, $f(\bar{x}) = (0,0)$. Clearly,

$$\operatorname{vcl}(\operatorname{cone}(\bigcup_{y\in f(A)} (y + \mathscr{E}(y)) - f(\bar{x}))) = \{(y_1,y_2) \in \mathbb{R}^2 |\ y_2 \geq 0, y_1 \geq 0\}$$

is a nontrivial convex cone. It is also easy to check that $(0,0)$ is a type-I $\mathscr{E}$-Benson proper efficient solution of (VOP) with respect to $\mathscr{C}$. Note that there exists $\mu = (1,1) \in (\mathscr{C}(y))^{+i} = \{(y_1,y_2) \in \mathbb{R}^2 |\ y_2 > 0,\ y_1 > 0\}$ such that

$$\langle f(x) - f(\bar{x}) + e, \mu\rangle = \langle f(x) + e, \mu\rangle = \langle f(x), \mu\rangle + \langle e,\ \mu\rangle \geq 0\ \forall x \in A,\ \forall e \in \mathscr{E}(y).$$

Hence, $(0,0)$ is an $\mathscr{E}$-optimal element of (VOP$_\mu$).

We now present a sufficient condition for type-I $\mathscr{E}$-Benson proper efficient solutions of (VOP) to be an $\mathscr{E}$-optimal solution of the scalarized problem (VOP$_\mu$) under suitable assumptions.

Theorem 5.2. *Let* $\bar{x} \in A$ *be an* $\mathscr{E}$*-optimal solution of* (VOP$_\mu$) *and*

$$\mu \in (\mathscr{C}(y))^{+i} \text{ for all } y \in f(A). \tag{5.7}$$

Then, $\bar{x}$ *is a type-I* $\mathscr{E}$*-Benson proper efficient solution of* (VOP) *with respect to* $\mathscr{C}$.

Proof. Since $\bar{x}$ is an $\mathscr{E}$-optimal solution of (VOP$_\mu$), then

$$\langle f(x) - f(\bar{x}), \mu\rangle \geq \sigma_{-\mathscr{E}(y)}(\mu)\ \forall x \in A,\ \forall y \in f(A). \tag{5.8}$$

Let

$$m \in \operatorname{vcl}(\operatorname{cone}(\bigcup_{y\in f(A)} (y + \mathscr{E}(y)) - f(\bar{x}))) \cap (-\mathscr{C}(y)). \tag{5.9}$$

Then, there exists $h \in Y$ for which $\forall \lambda > 0$, $\exists \alpha_\lambda \in (0,\lambda]$ such that

$$m + \alpha_\lambda h \in \operatorname{cone}(\bigcup_{y\in f(A)} (y + \mathscr{E}(y)) - f(\bar{x})). \tag{5.10}$$

As $\mathscr{E}(y)$ is an improvement set, for any $z_1 \in \bigcup_{y\in f(A)} (y+\mathscr{E}(y))$, there exist $y_1 \in f(A)$, $e_1 \in \mathscr{E}(y_1)$ and $c_1 \in \mathscr{C}(y_1)$ such that

$$z_1 = y_1 + e_1 + c_1. \tag{5.11}$$

Since $y_1 \in f(A)$, it follows from the condition (5.7) that

$$\mu \in (\mathscr{C}(y_1))^{+i}. \tag{5.12}$$

According to (5.8), (5.9), and (5.12), we obtain

$$\langle z_1 - f(\bar{x}), \mu\rangle = \langle y_1 - f(\bar{x}) + e_1, \mu\rangle + \langle c_1, \mu\rangle \geq 0. \tag{5.13}$$

It follows from (5.11) and (5.13) that

$$\langle z, \mu\rangle \geq 0 \; \forall\, z \in \operatorname{cone}(\bigcup_{y\in f(A)} (y+\mathscr{E}(y)) - f(\bar{x})). \tag{5.14}$$

By (5.10) and (5.14), there exists $h \in Y$, for which $\forall \lambda > 0$, $\exists \alpha_\lambda \in (0,\lambda]$ such that

$$\langle m, \mu\rangle + \alpha_\lambda \langle h, \mu\rangle = \langle m + \alpha_\lambda h, \mu\rangle \geq 0. \tag{5.15}$$

Letting $\lambda \to 0$ in (5.15), we have

$$\langle m, \mu\rangle \geq 0. \tag{5.16}$$

According to (5.9) and condition (5.7), we obtain

$$\langle m, \mu\rangle \leq 0. \tag{5.17}$$

Combining (5.16) with (5.17) yields that $\langle m, \mu\rangle = 0$, which together with condition (5.7) yields $m = 0$ and then

$$\operatorname{vcl}(\operatorname{cone}(\bigcup_{y\in f(A)} (y+\mathscr{E}(y)) - f(\bar{x}))) \cap (-\mathscr{C}(y)) = \{0\} \; \forall\, y \in f(A).$$

Therefore, $(\bar{x}, \bar{y})$ is a type-I $\mathscr{E}$-Benson proper efficient solution of (VOP). □

Corollary 5.2. *Let $\mathscr{C}(y) = C$, $\mathscr{E}(y) = E$, $\bar{x} \in A$ is an E-optimal solution of* (VOP$_\mu$)*, and $\mu \in C^{+i}$ for all $y \in f(A)$. Then, by Theorem 5.2, $\bar{x}$ is a type-I E-Benson proepr efficient solution of* (VOP).

The following example illustrates Theorem 5.2.

Example 5.2. Consider Example 5.1 with $\bar{x} = (0,0) \in A$. Here, there exists $\mu = (1,1) \in (\mathscr{C}(y))^{+i} = \{(y_1,y_2) | y_1 > 0, y_2 > 0\}$ for all $y \in f(A)$. Hence, the condition (5.7) in Theorem 5.2 holds. Obviously, $\langle e, (1,1)\rangle \geq 0$ for all $e \in \mathscr{E}(y)$. Thus

$$\langle (0,0), (1,1)\rangle = 0 \leq \langle f(x) + e, (1,1)\rangle = \langle f(x), (1,1)\rangle + \langle e, (1,1)\rangle \; \forall e \in \mathscr{E}(y), \; \forall x \in A.$$

Therefore, $\bar{x}$ is an $\mathscr{E}$-optimal solution of (VOP$_\mu$). It is easy to check that

$$\operatorname{vcl}(\operatorname{cone}(\bigcup_{y\in f(A)} (y+\mathscr{E}(y)) - f(\bar{x}))) \cap (-\mathscr{C}(y)) = \{0\} \; \forall y \in f(A).$$

Thus, $\bar{x}$ is a type-I $\mathscr{E}$-Benson proper efficient solution of (VOP) with respect to $\mathscr{C}$.

Next, we establish a necessary condition, under a suitable assumption, for type-II $\mathscr{E}$-Benson proper efficient solutions of (VOP), which is also a characterization of type-II $\mathscr{E}$-Benson proper efficient solutions in terms of $\mathscr{E}$-optimal solution of the scalarized problem (VOP$_\mu$).

Theorem 5.3. *Let $\mathscr{E}(y)$ be an nonempty convex set, $\bar{x}$ be a type-II $\mathscr{E}$-Benson proper efficient solution of* (VOP) *with respect to $\mathscr{C}$ and*

$$\mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)}(y+\mathscr{C}(y))-f(\bar{x}))) \text{ is a nontrival convex subset of } Y. \tag{5.18}$$

Then, for any $y\in f(A)$, there exists $\mu\in(\mathscr{E}(y))^{+}\backslash\{0\}$ such that $\bar{x}$ is an $\mathscr{E}$-optimal solution of (VOP_{μ}).

Proof. Since $\bar{x}$ is a type-II $\mathscr{E}$-Benson proper efficient solution of (VOP) with respect to $\mathscr{C}$, then $\bar{x}\in A$ such that

$$(-\mathscr{E}(y))\cap \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)}(y+\mathscr{C}(y))-f(\bar{x})))=\emptyset\ \forall y\in f(A).$$

Obviously,

$$(-\mathscr{E}(y))\cap \mathrm{cor}(\mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)}(y+\mathscr{C}(y))-f(\bar{x}))))=\emptyset\ \forall y\in f(A). \tag{5.19}$$

Since $\mathscr{E}(y)$ is an improvement set and $\mathrm{cor}(\mathscr{C}(y))\neq\emptyset$, we have by Lemma 2.3 that

$$\mathrm{cor}(\mathscr{E}(y))=\mathrm{cor}(\mathscr{E}(y)+\mathscr{C}(y))=\mathscr{E}(y)+\mathrm{cor}(\mathscr{C}(y))\neq\emptyset. \tag{5.20}$$

From condition (5.18), equation (5.20) and Lemma 2.7, we obtain that there exists $\mu\in Y^{*}\backslash\{0\}$ such that

$$\langle z_1,\ \mu\rangle\geq\langle e,\mu\rangle\ \forall z_1\in \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)}(y+\mathscr{C}(y))-f(\bar{x}))),\ \forall e\in-\mathscr{E}(y). \tag{5.21}$$

Clearly,

$$\bigcup_{y\in f(A)}(y+\mathscr{C}(y))-f(\bar{x})\subseteq \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)}(y+\mathscr{C}(y))-f(\bar{x}))). \tag{5.22}$$

It follows from (5.21) and (5.22) that

$$\langle z,\ \mu\rangle\geq\langle e,\mu\rangle\ \forall z\in\bigcup_{y\in f(A)}(y+\mathscr{C}(y))-f(\bar{x})),\ \forall e\in-\mathscr{E}(y). \tag{5.23}$$

Since $0\in\mathscr{C}(y)$, we see that, for $y\in f(A)$, there exists $\mu\in Y^{*}\backslash\{0\}$ such that

$$\langle f(x)-f(\bar{x}),\ \mu\rangle\geq\langle e,\mu\rangle\ \forall e\in-\mathscr{E}(y). \tag{5.24}$$

By (5.24), we have

$$\langle f(x)-f(\bar{x}),\mu\rangle\geq\sigma_{-\mathscr{E}(y)}(\mu)\ \forall x\in A,\ \forall y\in f(x). \tag{5.25}$$

We assert that, for any $y\in f(A)$, there exists $\mu\in Y^{*}\backslash\{0\}$ such that $\mu\in(\mathscr{E}(y))^{+}\backslash\{0\}$. On the contrary, let there exist $e'\in\mathscr{E}(y')$, $y'\in f(A)$ such that

$$\langle e',\mu\rangle<0. \tag{5.26}$$

Then, it follows from (5.21) and $z_1=0$ that $\langle e',\mu\rangle\geq 0$, which contradicts (5.26). Therefore, for any $y\in f(A)$, there exists $\mu\in Y^{*}\backslash\{0\}$ such that $\mu\in\mathscr{E}(y)^{+}\backslash\{0\}$. It follows from (5.25) that $\bar{x}$ is an $\mathscr{E}$-optimal solution of (VOP_{μ}). □

Remark 5.4. Let $\mathscr{C}(y) = C$ and $\mathscr{E}(y) = E$. Then, the condition (5.18) of Theorem 5.3 becomes that $\text{vcl}(\text{cone}((f(A)+C-f(\bar{x})))$ is a nontrival convex subset of Y, which is equivalent to $f - f(\bar{x})$ is nearly C-subconvexlike on A if Y is a topological linear space and set-valued map F is replaced by a vector-valued map f in [28]. Therefore, we obtain the following result by Theorem 5.3.

Corollary 5.3. *Let E be an nonempty convex set If $\bar{x}$ is a type-II E-Benson proper efficient solution of* (VOP) *and* $\text{vcl}(\text{cone}((f(A)+C-f(\bar{x})))$ *is a nontrival convex subset of Y, then there exists $\mu \in E^{+}\backslash\{0\}$ such that $\bar{x}$ is an E-optimal solution of* (VOP_{μ}).

The following example illustrates Theorem 5.3.

Example 5.3. Consider $Y = \mathbb{R}^2$, $A = [0,2]\times[0,2] \subseteq \mathbb{R}^2$, the vector-valued map $f : A \to Y$ as

$$f(x_1,x_2) = \left(\tfrac{1}{2}x_1^3, \tfrac{4}{3}\sqrt[3]{2x_2}\right),\ (x_1,x_2) \in A,$$

the set-valued map $\mathscr{C} : Y \rightrightarrows Y$ as

$$\mathscr{C}(y) = \begin{cases} \{(y_1,y_2) \in \mathbb{R}^2 |\ y_1 \geq 0,\ y_2 \geq 0\} & \text{if } y \in Y \setminus \{(1,\frac{1}{3})\} \\ \{(y_1,y_2) \in \mathbb{R}^2 |\ y_1 \geq 0,\ y_2 = 0\} & \text{if } y = (1,\frac{1}{3}), \end{cases}$$

and $\mathscr{E} : Y \rightrightarrows Y$ as

$$\mathscr{E}(y) = \begin{cases} \{(y_1,y_2) \in \mathbb{R}^2 |\ y_1 \geq 1,\ y_2 \geq 2\} & \text{if } y \in Y \setminus \{(1,\frac{1}{3})\} \\ \{(y_1,y_2) \in \mathbb{R}^2 |\ y_1 \geq 1,\ 1 \leq y_2 < 2\} & \text{if } y = (1,\frac{1}{3}). \end{cases}$$

Let $\bar{x} = (0,0)$. Then, $f(\bar{x}) = (0,0)$. Clearly,

$$\text{vcl}(\text{cone}(\bigcup_{y\in f(A)} (y+\mathscr{C}(y)) - f(\bar{x}))) = \{(y_1,y_2) \in \mathbb{R}^2 |\ y_2 \geq 0, y_1 \geq 0\}$$

is a nontrivial convex cone. It is easy to check that $(0,0)$ is an type-II $\mathscr{E}$-Benson proper efficient solution with respect to $\mathscr{C}$. Here, there exists $\mu = (2,1) \in (\mathscr{E}(y))^{+}\backslash\{0\} = \{(y_1,y_2)|y_1 \geq 0, y_2 \geq 0\}\backslash\{(0,0)\}$ such that

$$\langle f(x) - f(\bar{x}) + e, \mu\rangle = \langle f(x) + e, \mu\rangle = \langle f(x), \mu\rangle + \langle e,\ \mu\rangle \geq 0\ \forall x \in A,\ \forall e \in \mathscr{E}(y).$$

Thus, $(0,0)$ is an $\mathscr{E}$-optimal element of (VOP_{μ}).

Definition 5.8. A point $\bar{x} \in A$ is called an optimal solution of (VOP_{μ}) if $\langle f(\bar{x}), \mu\rangle \leq \langle f(x), \mu\rangle$ for all $x \in A$.

Finally, under suitable assumptions, we present a sufficient condition for a type-II $\mathscr{E}$-Benson proper efficient element of (VOP) to be an optimal solution of the scalarized problem (VOP_{μ}).

Theorem 5.4. *Let*

(i) $\bar{x}$ be an optimal solution of (VOP_{μ}) *and*
(ii) $\mu \in (\mathscr{E}(y))^{+}\backslash\{0\}$ for any $y \in f(A)$.

Then, $\bar{x}$ is a type-II $\mathscr{E}$-Benson proper efficient solution of (VOP) *with respect to $\mathscr{C}$.*

Proof. Because $\bar{x}$ is an optimal solution of (VOP_{μ}), we have

$$\langle f(x), \mu\rangle \geq \langle f(\bar{x}), \mu\rangle\ \forall x \in A. \tag{5.27}$$

We assert that $\bar{x}$ is a type-II $\mathscr{E}$-Benson proper efficient solution of (VOP) with respect to $\mathscr{C}$. Otherwise, there exists $\hat{w}$ such that

$$\hat{w} \in \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)} (y+\mathscr{C}(y)) - f(\bar{x}))) \cap (-\mathscr{E}(y)) \ \forall y \in f(A). \tag{5.28}$$

By (5.28), $\hat{w} \in \mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)} (y+\mathscr{C}(y)) - f(\bar{x})))$. Therefore, there exists $h \in Y$, for which $\forall \lambda > 0$, $\exists \gamma_\lambda \in (0,\lambda]$ such that

$$\hat{w} + \gamma_\lambda h \in \mathrm{cone}(\bigcup_{y\in f(A)} (y+\mathscr{C}(y)) - f(\bar{x})). \tag{5.29}$$

For any $z_1 \in \bigcup_{y\in f(A)} (y+\mathscr{C}(y))$, there exist $y_1 \in f(A)$ and $c_1 \in \mathscr{C}(y_1)$ such that

$$z_1 = y_1 + c_1. \tag{5.30}$$

Because $y_1 \in f(A)$, it follows from condition (ii) that

$$\mu \in (\mathscr{E}(y_1))^+ \backslash \{0\}. \tag{5.31}$$

By Lemma 2.8, we have

$$\mu \in (\mathscr{E}(y_1))^+ \subseteq \mathscr{C}(y_1)^+. \tag{5.32}$$

Combining (5.27), (5.30), and (5.32), we have

$$\langle z_1 - f(\bar{x}), \mu\rangle = \langle y_1 - f(\bar{x}), \mu\rangle + \langle c_1, \mu\rangle \geq 0 \ \forall z_1 \in \bigcup_{y\in f(A)} (y+\mathscr{C}(y)). \tag{5.33}$$

Therefore, we have

$$\langle z, \mu\rangle \geq 0 \ \forall z \in \mathrm{cone}(\bigcup_{y\in f(A)} (y+\mathscr{C}(y)) - f(\bar{x})). \tag{5.34}$$

It follows from (5.29) and (5.34) that

$$\langle \hat{w}, \mu\rangle + \gamma_\lambda \langle h, \mu\rangle = \langle \hat{w} + \gamma_\lambda h, \mu\rangle \geq 0. \tag{5.35}$$

Letting $\lambda \to 0$ in (5.35), we have

$$\langle \hat{w}, \mu\rangle \geq 0. \tag{5.36}$$

Combining (5.28) with condition (ii), we obtain

$$\langle \hat{w}, \mu\rangle < 0. \tag{5.37}$$

As (5.36) and (5.37) are contradictory,

$$\mathrm{vcl}(\mathrm{cone}(\bigcup_{y\in f(A)} (y+\mathscr{C}(y)) - f(\bar{x}))) \cap (-\mathscr{E}(y)) = \emptyset \ \forall y \in f(A). \tag{5.38}$$

Thus, $\bar{x}$ is a type-II $\mathscr{E}$-Benson proper efficient solution of (VOP) with respect to $\mathscr{C}$. □

If we take $\mathscr{C}(y) = C$ and $\mathscr{E}(y) = E$, then by Theorem 5.4, we have the following result.

Corollary 5.4. *If $\bar{x}$ is an optimal solution of* (VOP$_\mu$) *and $\mu \in E^+ \backslash \{0\}$ for any $y \in f(A)$, then $\bar{x}$ is a type-II E-Benson proepr efficient solution of* (VOP).

The following example illustrates Theorem 5.4.

Example 5.4. Consider Example 5.3 and $\bar{x} = (0,0) \in A$. Here,

$$\mu = (1,3) \in (\mathscr{E}(y))^{+} \backslash \{0\} = \{(y_1, y_2) | y_1 \geq 0, y_2 \geq 0\} \backslash \{(0,0)\} \text{ for all } y \in f(A).$$

Hence, the condition (ii) in Theorem 5.4 holds. Obviously, $\langle e, (1,3) \rangle \geq 0 \; \forall e \in \mathscr{E}(y)$. Note that

$$\langle (0,0), (1,3) \rangle = 0 \leq \langle f(x), (1,3) \rangle \; \forall x \in A.$$

Therefore, the condition (i) in Theorem 5.4 holds. It is easy to check that

$$\mathrm{vcl}(\mathrm{cone}(\bigcup_{y \in f(A)} (y + \mathscr{C}(y)) - f(\bar{x}))) \cap (-\mathscr{E}(y)) = \emptyset \; \forall y \in f(A).$$

Thus, $\bar{x}$ is an type-II $\mathscr{E}$-Benson proper efficient solution of (VOP) with respect to $\mathscr{C}$.

6. CONCLUSIONS

In this paper, we introduced three new notions of Benson proper efficiency under variable ordering structures in linear spaces—type I and type II $\mathscr{E}$-Benson proper efficient elements and $\mathscr{C}$-Benson proper efficient elements. It was demonstrated that the definition of type-I $\mathscr{E}$-Benson proper efficiency with respect to the ordering map $\mathscr{C}$ unifies and extends not only some notions of efficiency (weakly $\mathscr{E}$-nondomination element and $\mathscr{C}$-Benson proper efficiency with respect to $\mathscr{C}$) with variable ordering structures but also some notions (E-Benson proper efficiency, ε-Benson proper efficiency,and type-I E-Benson proper efficiency) of proper efficiency with fixed ordering structures. We also reported that the definition of the type-II $\mathscr{E}$-Benson proper efficient element of M with respect to the ordering map $\mathscr{C}$ unify and extend some notions of efficiency of M with variable ordering structures such as an $\mathscr{E}$-nondominated element of M with respect to the ordering map $\mathscr{C}$, a weakly $\mathscr{E}$-nondominated element of M with respect to the ordering map $\mathscr{C}$ and the $\mathscr{C}$-Benson proper efficiency of M with respect to $\mathscr{C}$ but also some notions of the proper efficiency under fixed ordering structures, such as type-II E-Benson proper minimal element of M. The definitions of type I and II of $\mathscr{E}$-Benson proper efficient solutions for vector optimization problems with respect to the ordering map $\mathscr{C}$ in linear space were studied. By the separation theorem of convex sets in real linear spaces, we established conditions for both type-I and type-II $\mathscr{E}$-Benson and $\mathscr{C}$-Benson proper efficient solution of (VOP) to be an $\mathscr{E}$-optimal solution of the scalarized optimization problem (VOP$_\mu$) under the suitable assumptions. We established a sufficient condition of the type-I $\mathscr{E}$-Benson proper efficient solution of (VOP) (resp., the type-II $\mathscr{E}$-Benson proper efficient solution of (VOP) to be an $\mathscr{E}$-optimal solution (resp., an optimal solution) of the scalarization optimization problem (VOP$_\mu$) under the suitable assumptions. In the future, we attempt to generalize the concepts and results of this paper for set optimization problems under variable ordering structures.

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