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CONVEX SETS APPROXIMABLE AS THE SUM OF A COMPACT SET AND A CONE

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Dedicated to the memory of Professor Dr. Alfred Göpfert

Abstract. The class of convex sets that admit approximations as Minkowski sum of a compact convex set and a closed convex cone in the Hausdorff distance is introduced. These sets are called approximately Motzkin-decomposable and generalize the notion of Motzkin-decomposability, i.e., the representation of a set as the sum of a compact convex set and a closed convex cone. We characterize these sets in terms of their support functions and show that they coincide with hyperbolic sets, i.e., convex sets contained in the sum of their recession cone and a compact convex set if their recession cones are polyhedral but are more restrictive in general. In particular, we prove that a set is approximately Motzkin-decomposable if and only if its support function has a closed domain relative to which it is continuous.

Keywords. Approximation; Closed convex sets; Motzkin decomposition; Support function; Unbounded convex sets.

1. Introduction

Consider the closed convex sets $C^1 = \left\{x \in \mathbb{R}^2 \,\middle|\, x_1x_2 \geqslant 1, x_1 \geqslant 0\right\}$ and $C^2 = \left\{x \in \mathbb{R}^2 \,\middle|\, x_2 \geqslant x_1^2\right\}$ with their recession cones $0^+C^1 = \left\{x \in \mathbb{R}^2 \,\middle|\, x \geqslant 0\right\}$ and $0^+C^2 = \left\{x \in \mathbb{R}^2 \,\middle|\, x_1 = 0, x_2 \geqslant 0\right\}$. For $r > 0$ we understand a truncation C^i_r of these sets to be $C^i_r = \left(C^i \cap rB\right) + 0^+C^i$, $i = 1,2$, where B denotes the Euclidean unit ball. That is, we restrict C^i to a compact subset of itself given by a ball of radius r and add the recession cone afterwards, yielding a subset of C^i. This situation is depicted in Figure 1. Clearly, the truncations C^1_r of C^1 converge to C^1 with respect to the Hausdorff distance if r tends to $+\infty$. However, the same is not true for C^2 because, for every $r > 0$, the Hausdorff distance between C^2_r and C^2 is infinite. Note that there is a geometric difference between C^1 and C^2: $C^1 \subseteq \{0\} + 0^+C^1$, but there does not exist a compact set M satisfying $C^2 \subseteq M + 0^+C^2$. In view of this property, C^1 is called hyperbolic, cf. [1, 13]. In this work we investigate the following questions regarding a closed convex set $C \subseteq \mathbb{R}^n$:

(1) What characterizes the property that C can be approximated by truncations C_r in the above sense?
(2) How is this property related to hyperbolicity of C? Do both properties coincide (under certain assumptions)?

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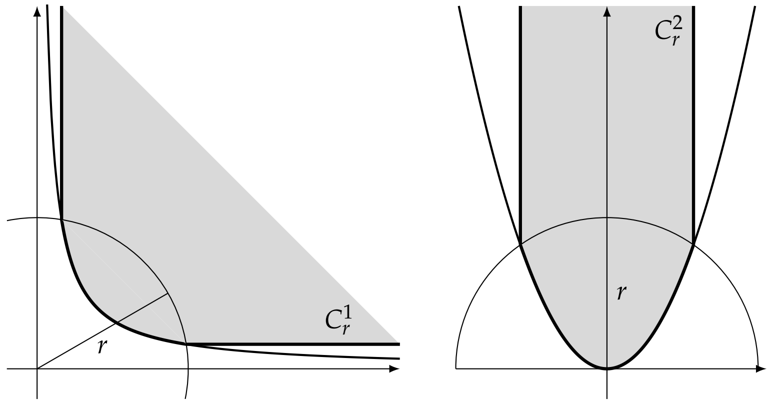

FIGURE 1. When can C be approximated by truncations?

This is motivated by the observation that conducting analysis on C presents various challenges in the presence of unboundedness. The above example demonstrates that the Hausdorff distance is only limitedly suitable for the comparison of unbounded sets and the recession cone may only partly capture the asymptotic behavior of C. The latter is evident from the fact that $0^+C = 0^+C_r$ whenever C_r is nonempty. Hence, from a practical perspective, it seems favourable to reduce investigations to a bounded subset of C.

What needs to be adressed is whether it is possible to select a subset a priori in such a way as to not neglect important information about C. One class of convex sets that permit such a choice are called M-decomposable sets and have been studied by various authors over the last decade [5, 7, 6, 17]. These are the sets that can be represented as the Minkowski sum of a compact convex set and a closed convex cone. They have been characterized in different ways, e.g. in terms of support functions and via an associated vector optimization problem and its set of efficient points [5], with regard to their extreme points [7] and in the sense of intersections of sets with certain hyperplanes [6]. Nevertheless, imposing M-decomposability on C is a strong assumption. For example, both of the sets C^1 and C^2 above are not M-decomposable. However, their truncations always are. Hence, the first question might be restated as:

1^*. When can a closed convex set C be approximated by M-decomposable sets?

To address these questions we introduce and study the class of approximately M-decomposable sets as those closed convex sets that can be approximated by M-decomposable sets in the Hausdorff distance. Our main result is a characterization of this class in terms of support functions and their demarcation from M-decomposable and hyperbolic sets. In the following section we provide preliminary definitions and notation. Section 3 contains the definition of approximately M-decomposable sets and the main results of this manuscript. In particular we show that approximately M-decomposable sets fit between M-decomposable and hyperbolic sets. They coincide with the latter given their recession cones are polyhedral. Moreover, we prove that a set is approximately M-decomposable if and only if its support function has a closed domain relative to which it is continuous.

2. Preliminaries

Given sets M, $P \subseteq \mathbb{R}^n$, and a scalar $\alpha \in \mathbb{R}$, we denote by $\operatorname{cl} M$, $\operatorname{int} M$, $\operatorname{relint} M$, $M+P$, and αM the *closure*, *interior*, *relative interior* of M, the *Minkowski sum* of M and P and the *dilation* of M by α, respectively. The *Euclidean norm* of a vector $x \in \mathbb{R}^n$ and the *Euclidean unit ball* are expressed as $\|x\|$ and B, respectively. A set $C \subseteq \mathbb{R}^n$ is called *convex* if, for every $x, y \in C$ and scalar $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in C$ holds. A *cone* is a set C such that for every $x \in C$ and $\mu \geqslant 0$ it holds $\mu x \in C$. The *recession cone* of a convex set $C \subseteq \mathbb{R}^n$ is defined as the set $\{d \in \mathbb{R}^n \mid \forall\, x \in C, \mu \geqslant 0\colon x + \mu d \in C\}$ and denoted by 0^+C. If C is closed, so is 0^+C. The *polar* C° of C is the set of linear functions that are bounded above on C by 1, i.e. $C^\circ = \left\{y \in \mathbb{R}^n \,\middle|\, \forall\, x \in C\colon y^\mathsf{T} x \leqslant 1\right\}$. If C is a cone, then $C^\circ = \left\{y \in \mathbb{R}^n \,\middle|\, \forall\, x \in C\colon y^\mathsf{T} x \leqslant 0\right\}$. Note that C° is always closed. Given nonempty sets C^1, $C^2 \in \mathbb{R}^n$ the *Hausdorff distance* $d_\mathsf{H}\left(C^1, C^2\right)$ between C^1 and C^2 is defined as

$$d_\mathsf{H}\left(C^1, C^2\right) = \max\left\{\sup_{x \in C^1} \inf_{y \in C^2} \|x-y\|,\ \sup_{x \in C^2} \inf_{y \in C^1} \|x-y\|\right\}.$$

Note that $d_\mathsf{H}\left(C^1, C^2\right)$ may be infinite if any of the sets is unbounded. However, it is well known that $d_\mathsf{H}(\cdot,\cdot)$ defines a metric on the class of nonempty compact subsets of $\mathbb{R}^n$ [9]. Moreover, it can equivalently be expressed as

$$d_\mathsf{H}\left(C^1, C^2\right) = \inf\left\{\varepsilon > 0 \,\middle|\, C^1 \subseteq C^2 + \varepsilon B,\ C^2 \subseteq C^1 + \varepsilon B\right\}, \tag{2.1}$$

see [8].

A function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called *convex* if $f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y)$ holds for every x, $y \in \mathbb{R}^n$ and $\lambda \in [0,1]$. The *domain* $\operatorname{dom} f$ of f is the set of points at which f is finite, i.e., $\operatorname{dom} f = \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$. To every f, we assign a *conjugate* function f^* defined by $f^*(y) = \sup\left\{y^\mathsf{T} x - f(x) \,\middle|\, x \in \mathbb{R}^n\right\}$. The *support function* σ_C of a nonempty, closed, and convex set C is given as $\sigma_C(d) = \sup\left\{d^\mathsf{T} x \,\middle|\, x \in C\right\}$. Its conjugate is called *indicator function* of C and denoted δ_C.

3. Approximately M-Decomposable Sets

In this section, we give an answer to the questions posed in the introduction. We understand a truncation of C to be the sum of a compact subset of C and 0^+C in the following sense.

Definition 3.1. Let $C \subseteq \mathbb{R}^n$ be nonempty closed and convex. The set $C_r = (C \cap rB) + 0^+C$ is called a *truncation* of C of radius $r > 0$.

Clearly, $C_r \subseteq C$ for all $r > 0$. Studying sets that are approximately tructations of themselves is motivated by the fact that, given such a set, it suffices to conduct analyses, such as minimizing some function over the set, on a compact subset to obtain good results in an approximate sense. To this end, we introduce the class of approximately M(otzkin)-decomposable convex sets.

Definition 3.2. A nonempty, closed, and convex set $C \subseteq \mathbb{R}^n$ is called *approximately M-decomposable* if, for every $\varepsilon > 0$, there exists $r > 0$ such that $C \subseteq C_r + \varepsilon B$.

By Equation (2.1), Definition 3.2 is equivalent to the existence of a truncation $C_{r(\varepsilon)}$ such that the Hausdorff distance between $C_{r(\varepsilon)}$ and C is at most ε. The designation approximately M-decomposable is derived from the stronger property of M-decomposability.

Definition 3.3 (see [5])**.** A nonempty, closed, nad convex set $C \subseteq \mathbb{R}^n$ is called *M(otzkin)-decomposable* if there exists a compact convex set $M \subseteq \mathbb{R}^n$ such that $C = M + 0^+C$. Such set M is called a compact component of C.

If C is closed and convex, then every nonempty trunctation C_r of C is M-decomposable with compact component $C \cap rB$. Moreover, if C is M-decomposable, there exist truncations of C that coincide with C. If M is any compact component of C, then $C_r = C$ for all $r \geqslant \max\{\|x\| \,|\, x \in M\}$. In particular, C is approximately M-decomposable in this case. A relaxation of the condition of M-decomposability leads to hyperbolic sets.

Definition 3.4 (cf. [1, 13])**.** A nonempty, closed, and convex set $C \subseteq \mathbb{R}^n$ is called *hyperbolic* if there exists a compact convex set $M \subseteq \mathbb{R}^n$ such that $C \subseteq M + 0^+C$.

We point out that hyperbolic sets also exist under the terms 0^+C-bounded [12] and self-bounded [3] sets in the literature. Moreover, it is known that C is hyperbolic if and only if $\operatorname{dom} \sigma_C$ is closed; see [1, Proposition 5].

Clearly, every M-decomposable set C is also hyperbolic, where the set M in Definition 3.4 may be chosen as any compact component of C, but the converse does not hold as is seen from the example $C^1 = \left\{x \in \mathbb{R}^2 \,\middle|\, x_1x_2 \geqslant 1, x_1 \geqslant 0\right\}$ and the fact $0^+C^1 = \left\{x \in \mathbb{R}^2 \,\middle|\, x \geqslant 0\right\}$; see Section 1 and Figure 1. The set C^1 is the epigraph of the function $x \mapsto x^{-1}$ restricted to the nonnegative real line. It is not difficult to see that this set is approximately M-decomposable. For $\varepsilon > 0$ set $r^2 \geqslant \varepsilon^2 + \varepsilon^{-2}$. Example 3.1 below shows that not every hyperbolic set is approximately M-decomposable. However, the converse statement holds, i.e., the class of hyperbolic sets is larger in general.

Proposition 3.1. *Let $C \subseteq \mathbb{R}^n$ be approximately M-decomposable. Then C is hyperbolic.*

Proof. Let $\varepsilon > 0$. Then there exists $r > 0$ such that $C \subseteq C_r + \varepsilon B$. Now,

$$C_r + \varepsilon B = (C \cap rB) + \varepsilon B + 0^+C$$

and $(C \cap rB) + \varepsilon B$ is compact and convex. Thus C is hyperbolic. □

In order to characterize approximately M-decomposable sets, we need another expression for the Hausdorff distance between certain closed convex sets. The following is a generalization of a well-known result for compact convex sets, see, e.g., [8, Proposition 6.3].

Proposition 3.2. *Let C^1, $C^2 \subseteq \mathbb{R}^n$ be nonempty closed convex sets whose support functions have identical domains D, i.e., $D = \operatorname{dom} \sigma_{C^1} = \operatorname{dom} \sigma_{C^2}$. Then*

$$d_{\mathsf{H}}\left(C^1, C^2\right) = \sup_{d \in D \cap B} \left|\sigma_{C^1}(d) - \sigma_{C^2}(d)\right|.$$

Furthermore, $d_{\mathsf{H}}\left(C^1, C^2\right) = +\infty$ whenever $\operatorname{dom} \sigma_{C^1} \neq \operatorname{dom} \sigma_{C^2}$.

Proof. Since C^1 and C^2 are nonempty, D is also nonempty because $\sigma_{C^1}(0) = \sigma_{C^2}(0) = 0$, i.e. $0 \in D$. Denote by $d\left(x, C^1\right)$ the Euclidean distance from x to C^1, i.e., $d\left(x, C^1\right) = \inf_{y \in C^1} \|x - y\|$. Then $d_{\mathsf{H}}\left(C^1, C^2\right)$ can be written as $\max\left\{\sup_{x \in C^1} d\left(x, C^2\right),\, \sup_{x \in C^2} d\left(x, C^1\right)\right\}$. Moreover, it holds $d\left(x, C^1\right) = \inf_{y \in \mathbb{R}^n} \left(\|x - y\| + \delta_{C^1}(y)\right)$. According to [15, Theorem 16.4], the conjugate of

$d\left(\cdot,C^1\right)$ is thus given as $d^*(\cdot,C^1)=\delta_B+\sigma_{C^1}$. Since $d\left(\cdot,C^1\right)$ is closed proper and convex, then it holds

$$\begin{aligned} d\left(x,C^1\right) &= d^{**}(x,C^1) \\ &= \sup_{d\in\mathbb{R}^n}\left(x^\mathsf{T}d-\delta_B(d)-\sigma_{C^1}(d)\right) \\ &= \sup_{d\in D\cap B}\left(x^\mathsf{T}d-\sigma_{C^1}(d)\right). \end{aligned}$$

The first equation is due to [15, Theorem 12.2] and the third holds because $\operatorname{dom}\sigma_{C^1}=D$. Analogously, one has $d\left(x,C^2\right)=\sup_{d\in D\cap B}\left(x^\mathsf{T}d-\sigma_{C^2}(d)\right)$. Combining both expressions yields

$$\begin{aligned} d_\mathsf{H}\left(C^1,C^2\right) &= \max\left\{\sup_{x\in C^2}\sup_{d\in D\cap B}\left(x^\mathsf{T}d-\sigma_{C^1}(d)\right),\ \sup_{x\in C^1}\sup_{d\in D\cap B}\left(x^\mathsf{T}d-\sigma_{C^2}(d)\right)\right\} \\ &= \sup_{d\in D\cap B}\max\left\{\sigma_{C^2}(d)-\sigma_{C^1}(d),\ \sigma_{C^1}(d)-\sigma_{C^2}(d)\right\} \\ &= \sup_{d\in D\cap B}\left|\sigma_{C^1}(d)-\sigma_{C^2}(d)\right|. \end{aligned}$$

For the second part assume, without loss of generality, $d\in\operatorname{dom}\sigma_{C^1}\setminus\operatorname{dom}\sigma_{C^2}$, $\|d\|=1$. Then, for every $k\in\mathbb{N}$, there exists $y_k\in C^2$ such that $d^\mathsf{T}y_k>k$. Denote the set $\left\{x\in\mathbb{R}^n \,\middle|\, d^\mathsf{T}x\leqslant\sigma_{C^1}(d)\right\}$ by H. Since $C^1\subseteq H$, it holds $d\left(y_k,C^1\right)\geqslant d\left(y_k,H\right)=d^\mathsf{T}y_k-\sigma_{C^1}(d)>k-\sigma_{C^1}(d)$ for all $k\geqslant\sigma_{C^1}(d)$. Taking the limit $k\to+\infty$ yields the result. □

Remark 3.1. Note that, even if $\operatorname{dom}\sigma_{C^1}=\operatorname{dom}\sigma_{C^2}$, it is possible that $d_\mathsf{H}\left(C^1,C^2\right)=+\infty$. Consider, for example, the sets $C^1=\left\{x\in\mathbb{R}^2 \,\middle|\, x_2\geqslant x_1^2\right\}$ and $C^2=2C^1$. The domain of their support functions is the non-closed set $\left\{x\in\mathbb{R}^2 \,\middle|\, x_2<0\right\}\cup\{0\}$. For $d_k=(1,\,-1/k)^\mathsf{T}$ and one has $\sigma_{C^1}(d_k)=k/2$ and $\sigma_{C^2}(d_k)=k$, i.e.,

$$\sigma_{C^2}\left(\frac{d_k}{\|d_k\|}\right)-\sigma_{C^1}\left(\frac{d_k}{\|d_k\|}\right)=\frac{k^2}{2\sqrt{k^2+1}}$$

which is unbounded in $k\in\mathbb{N}$.

We have already seen that not every closed convex set C is approximately M-decomposable, i.e., can be approximated by truncations in the Hausdorff distance. However, truncations always approximate C in a weaker sense.

Definition 3.5 (cf. [16]). A sequence of $\left\{C^k\right\}_{k\in\mathbb{N}}$ of closed subsets of $\mathbb{R}^n$ is said to *converge in the sense of Painlevé-Kuratowski* or *PK-converge* to a closed set $C\subseteq\mathbb{R}^n$ if

$$C=\left\{x\in\mathbb{R}^n \,\middle|\, \lim_{k\to+\infty}d\left(x,C^k\right)=0\right\}.$$

For bounded sequences PK-convergence and convergence with respect to the Hausdorff distance coincide, but the concepts are distinct in general; see [16].

Lemma 3.1. *Let $C\subseteq\mathbb{R}^n$ be nonempty, closed, and convex. Then $\{C_r\}_{r\in\mathbb{N}}$ PK-converges to C.*

Proof. Let $x\in C$. Then $x\in C_r$ for all $r\geqslant\|x\|$. Hence, $\lim_{r\to+\infty}d\left(x,C_r\right)$ exists and is zero. On the contrary, let $x\notin C$. Since C is closed, $d\left(x,C\right)>0$. Moreover, $d\left(x,C_r\right)\geqslant d\left(x,C\right)$ holds for all $r\in\mathbb{N}$ because $C_r\subseteq C$. Thus $x\notin\left\{x\in\mathbb{R}^n \,\middle|\, \lim_{k\to+\infty}d\left(x,C^k\right)=0\right\}$. □

We are now ready to prove the main result of this work.

Theorem 3.1. *Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed, and convex set. The following are equivalent:*

(i) C is approximately M-decomposable,

(ii) $\operatorname{dom} \sigma_C$ is closed and σ_C is continuous relative to $\operatorname{dom} \sigma_C$.

Moreover, the set $\operatorname{dom} \sigma_C$ in (ii) equals $(0^+C)^\circ$.

Proof. Assume that C is approximately M-decomposable, i.e.,

$$\forall\, \varepsilon > 0\, \exists\, r \geqslant 0\colon C_r \subseteq C \subseteq C_r + \varepsilon B. \tag{3.1}$$

For the remainder of the proof, we assume that ε and r are fixed such that the inclusions in (3.1) hold. In particular, $C_r + \varepsilon B$. Consequently, C_r are nonempty. It follows

$$\sigma_{C_r} \leqslant \sigma_C \leqslant \sigma_{C_r+\varepsilon B} \tag{3.2}$$

for the support functions of C_r, C, and $C_r + \varepsilon B$; see [15, Corollary 13.1.1]. Since all three sets have the same recession cone 0^+C, it follows from [15, Corollary 14.2.1] that

$$\operatorname{cl} \operatorname{dom} \sigma_{C_r} = \operatorname{cl} \operatorname{dom} \sigma_C = \operatorname{cl} \operatorname{dom} \sigma_{C_r+\varepsilon B} = \left(0^+C\right)^\circ.$$

However, for $d \in (0^+C)^\circ$, one has

$$\sigma_{C_r+\varepsilon B}(d) = \sigma_{(C\cap rB)+\varepsilon B}(d) < +\infty.$$

Thus $\operatorname{dom} \sigma_{C_r} = \operatorname{dom} \sigma_C = \operatorname{dom} \sigma_{C_r+\varepsilon B} = (0^+C)^\circ$ due to Inequality (3.2).

The support functions σ_{C_r} and $\sigma_{C\cap rB}$ coincide on $(0^+C)^\circ$. Since $C \cap rB$ is compact, its support function is finite everywhere and therefore continuous according to [15, Corollary 10.1.1]. Thus σ_{C_r} is continuous relative to its domain $(0^+C)^\circ$. Using the expression for the Hausdorff distance, we see from Equation (2.1) that (3.1) is equivalent to

$$\forall\, \varepsilon > 0\, \exists\, r \geqslant 0\colon d_{\mathsf{H}}(C_r, C) \leqslant \varepsilon. \tag{3.3}$$

Since

$$r \leqslant \bar{r} \implies C_r \subseteq C_{\bar{r}} \subseteq C, \tag{3.4}$$

Statement (3.3) is equivalent to $\lim_{r\to+\infty} d_{\mathsf{H}}(C_r, C) = 0$. Applying Proposition 3.2 gives

$$\lim_{r\to+\infty} \sup_{d\in(0^+C)^\circ\cap B} |\sigma_C(d) - \sigma_{C_r}(d)| = 0. \tag{3.5}$$

Thus σ_{C_r} converges uniformly to σ_C on $(0^+C)^\circ$. As σ_{C_r} is continunous on its domain, the Uniform Limit Theorem [14, Theorem 21.6] yields that σ_C is continuous on its domain as well.

Now, we assume that $\operatorname{dom} \sigma_C$ is closed and $\sigma_C\colon \operatorname{dom} \sigma_C \to \mathbb{R}$ is continuous. Again, $\operatorname{dom} \sigma_C = (0^+C)^\circ$ by [15, Corollary 14.2.1]. We show that σ_{C_r} converges pointwise to σ_C on $(0^+C)^\circ$ as $r \to +\infty$. Property (3.4) implies that $\lim_{r\to+\infty} \sigma_{C_r}(d)$ exists for all $d \in (0^+C)^\circ$ and is finite. In particular, it holds $\lim_{r\to+\infty} \sigma_{C_r}(d) \leqslant \sigma_C(d)$. Assume σ_{C_r} does not converge to pointwise to σ_C on $(0^+C)^\circ$, i.e., there exists $\bar{d} \in (0^+C)^\circ$ for which the inequality holds strictly, which implies $\bar{d} \neq 0$. Let $\gamma = \lim_{r\to+\infty} \sigma_{C_r}(\bar{d})$. Then $C_r \subseteq H := \left\{x \in \mathbb{R}^n \,\middle|\, \bar{d}^{\mathsf{T}} x \leqslant \gamma\right\}$ for every $r \in \mathbb{N}$ and there exists $\bar{x} \in C$ such that $\bar{d}^{\mathsf{T}} \bar{x} > \gamma$. It holds

$$d(\bar{x}, C_r) \geqslant d(\bar{x}, H) = \frac{\bar{d}^{\mathsf{T}} \bar{x} - \gamma}{\|\bar{d}\|} > 0,$$

which is a contradiction because $\{C_r\}_{r\in\mathbb{N}}$ PK-converges to C by Lemma 3.1, i.e., C can be expressed as $\{x \in \mathbb{R}^n \,|\, \lim_{r\to+\infty} d(x, C_r) = 0\}$. Thus σ_{C_r} converges pointwise to σ_C on $(0^+C)^\circ$.

Since σ_C is continuous and σ_{C_r} is monotonically non-decreasing in r, Dini's theorem [10, Theorem 12.1] yields that the convergence is uniform on $(0^+C)^\circ \cap B$, i.e., Equation (3.5) holds. Finally, using the equivalence between (3.5), (3.3), and (3.1) proves that C is approximately M-decomposable. □

Remark 3.2. In the above proof, we have demonstrated the pointwise convergence of the support functions of truncations C_r to the support function of C on $(0^+C)^\circ$ by using the PK-convergence of $\{C_r\}_{r\in\mathbb{N}}$. We point out that this may also be achieved by using the different concept of $\mathscr{C}$-convergence introduced in [11]. To this end, one needs to take into account that $\sigma_C = \sup_{r\in\mathbb{N}} \sigma_{C_r}$ and apply [11, Theorem 5.9] and [11, Proposition 5.3 (iv)].

Remark 3.3. In [4], *continuous convex sets* were introduced, which are similar to, but different from, approximate M-decomposable sets. A convex set is called continuous if its support function is continuous everywhere (and not necessarily only relative to its domain). For example, the set C^1 from Section 1 is approximately M-decomposable, but not continuous, whereas C^2 is continuous, but the domain of its support function is not closed. Thus it is not approximately M-decomposable; see [4, p. 1]. Clearly, every compact convex set is both continuous and (approximately) M-decomposable.

Using Theorem 3.1, we can show that hyperbolicity does not imply approximate M-decomposability.

Example 3.1. Let $C^2 = \left\{x \in \mathbb{R}^2 \,\middle|\, x_2 \geqslant x_1^2\right\}$ (Section 1 and Figure 1), and consider the following set
$$C = \left(C^2 \times \{0\}\right) + \operatorname{cl}\operatorname{cone}\left(C^2 \times \{1\}\right)$$
in $\mathbb{R}^3$. By [15, Theorem 8.2], it holds
$$\operatorname{cl}\operatorname{cone}\left(C^2 \times \{1\}\right) = \operatorname{cone}\left(C^2 \times \{1\}\right) \cup \left(0^+C^2 \times \{0\}\right).$$
Now, according to [15, Corollary 9.1.2], C is closed and one has $0^+C = \operatorname{cl}\operatorname{cone}\left(C^2 \times \{1\}\right)$. To see that C is hyperbolic, we choose any $x \in C$. Then $x = c + s$ for some $c \in C^2 \times \{0\}$ and $s \in 0^+C$. For $d = (0,0,-1)^\mathsf{T}$, it holds $x = d + (c-d) + s$. Since $c - d \in 0^+C$, $c - d + s \in 0^+C$ as well. Hence, $C \subseteq \{d\} + 0^+C$.

According to [2, Theorem 3.1], one has
$$(0^+C)^\circ = \operatorname{cl}\operatorname{cone}\left(C^{2^\circ} \times \{-1\}\right)$$
and a simple calculation shows that $C^{2^\circ} = \left\{x \in \mathbb{R}^2 \,\middle|\, x_2 \leqslant -x_1^2/4\right\}$. Therefore,
$$d_n = \begin{pmatrix} n^{-1} \\ -(2n)^{-2} \\ -1 \end{pmatrix} \in \left(0^+C\right)^\circ$$
for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} d_n = d$. Now,
$$\sigma_C(d_n) \geqslant d_n^\mathsf{T} \begin{pmatrix} n \\ n^2 \\ 0 \end{pmatrix} = \frac{3}{4} > 0.$$
However, $\sigma_C(d) \leqslant 0$ because for $x \in C$ it holds $x_3 \geqslant 0$. Hence, σ_C is not continuous at $d \in (0^+C)^\circ$ and by Theorem 3.1 not approximately M-decomposable.

If an additional assumption is made, hyperbolicity and approximate M-decomposability are equivalent.

Corollary 3.1. *Let $C \subseteq \mathbb{R}^n$ be nonempty, closed, and convex, and let 0^+C be polyhedral. Then C is approximately M-decomposable if and only if it is hyperbolic.*

Proof. By Proposition 3.1, the first implication is true regardless of the polyhedrality of 0^+C. Assume that C is hyperbolic, i.e., $C \subseteq M + 0^+C$ for some compact convex set M, which implies $\operatorname{dom}\sigma_C \subseteq (0^+C)^\circ$ and thus actually $\operatorname{dom}\sigma_C = (0^+C)^\circ$ by [15, Corollary 14.2.1]. Since 0^+C is polyhedral, so is $(0^+C)^\circ$. Hence it is locally simplicial; see [15, Theorem 20.5], and [15, Theorem 10.2] implies that σ_C is continuous relative to its domain. In view of Theorem 3.1, we have the desired conclusion immediately. □

The corollary implies that, whenever $C \subseteq \mathbb{R}^2$, it is approximately M-decomposable if and only if it is hyperbolic because every closed convex cone in $\mathbb{R}^2$ is polyhedral. Thus, the set from Example 3.1 is in the lowest possible dimension to demonstrate that hyperbolicity is, in general, not equivalent to approximate M-decomposability.

4. Conclusion

In this apaper, we introduced the class of approximately M-decomposable sets as those closed convex sets that can be approximated arbitrarily well by truncations with respect to the Hausdorff distance. We showed that they generalize the class of M-decomposable sets and are a special case of hyperbolic sets but coincide with neither in general as demonstrated by examples. Furthermore, we characterized the approximately M-decomposable sets in terms of their support functions. Finally, we proved that, when considering only polyhedral recession cones, approximate M-decomposability and hyperbolicity are equivalent.

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