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NONCOERCIVE ELLIPTIC BILATERAL VARIATIONAL INEQUALITIES IN THE HOMOGENEOUS SOBOLEV SPACE $D^{1,p}(\mathbb{R}^N)$

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Abstract. In this paper, we prove an existence result for a quasilinear elliptic variational inequality of the form $u \in K \subset V : 0 \in -\Delta_p u + aF(u) + \partial I_K(u) \subset V^*$ in the whole $\mathbb{R}^N$ under bilateral constraints K given by $K = \{v \in V : \phi(x) \le v(x) \le \psi(x) \text{ a.e. in } \mathbb{R}^N\}$, where Δ_p is the p-Laplacian, the underlying solution space V is the homogeneous Sobolev space (also called Beppo-Levi space) $V = D^{1,p}(\mathbb{R}^N)$ with $1 < p < N$, and $I_K : V \to \mathbb{R} \cup \{+\infty\}$ is the indicator functional corresponding to K with its subdifferential ∂I_K. The lower order Nemytskij operator F is generated by a Carathéodory function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, and the measurable and bounded coefficient a is supposed to decay like $|x|^{-(N+\alpha)}$ at infinity. The growth conditions that we impose on f are such that the operator $-\Delta_p + aF : V \to V^*$, in general, is not coercive with respect to K which prevents us from applying standard existence results. Another difficulty, which arises due to the lack of compact embedding of V into $L^q(\mathbb{R}^N)$ spaces, needs to be overcome in an appropriate way. Without assuming additional assumptions such as the existence of sub- and supersolutions, we are able not only to prove the existence of solutions, but also show the compactness of the set of all solutions in V. Finally, an extension of the theory is established, which allows us to deal with noncoercive bilateral variational-hemivariational inequalities in $\mathbb{R}^N$. The proof of our main existence result is based on a modified penalty approach.

Keywords. Beppo-Levi space; Bilateral constraint; Noncoercive variational inequality; Penalty approach; Pseudomonotone operator.

1. Introduction

Let $1 < p < N$, and let $V = D^{1,p}(\mathbb{R}^N)$ be the homogeneous Sobolev space (also called Beppo-Levi space), which is the completion of $C_c^\infty(\mathbb{R}^N)$ (space of infinitely differentiable functions with compact support in $\mathbb{R}^N$) with respect to the norm

$$\|u\|_V = \Big(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \Big)^{\frac{1}{p}}.$$

We denote by K the closed convex subset of V representing bilateral constraints, that is,

$$K = \{v \in V : \phi(x) \le v(x) \le \psi(x) \quad \text{a.e. in } \mathbb{R}^N\}, \tag{1.1}$$

where ϕ and ψ are assumed to belong to V.

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In this paper, we study quasilinear elliptic variational inequalities in the whole $\mathbb{R}^N$ of the form:

$$u \in K : 0 \in -\Delta_p u + aF(u) + \partial I_K(u) \quad \text{in } V^*, \tag{1.2}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian at u, V^* denotes the dual space of V, and I_K is the indicator functional related to K with its subdifferential ∂I_K. The lower order Nemytskij operator is generated by the Carathéodory function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ through $F(u)(x) = f(x,u(x))$, and the measurable and bounded coefficient $a : \mathbb{R}^N \to \mathbb{R}$ is supposed to decay at infinity like $|x|^{-(N+\alpha)}$ with $\alpha > 0$. By definition of the subdifferential ∂I_K, inclusion (1.2) is equivalent to the following variational inequality

$$u \in K : \langle -\Delta_p u + aF(u), v-u\rangle \geq 0, \quad \forall v \in K, \tag{1.3}$$

where $\langle \cdot,\cdot \rangle$ denotes the duality pairing between V^* and V.

Existence results for variational inequalities in the general form

$$u \in K \subset X : \langle Au, v-u\rangle \geq \langle f, v-u\rangle, \quad \forall v \in K, \tag{1.4}$$

on bounded domains $\Omega \subset \mathbb{R}^N$ under some coercivity conditions for the operator $A : X \to X^*$ can be found in standard monographs such as [1–3]. However, if either the domain Ω is unbounded or any variant of coercivity condition on the operator A fails, then solutions may not even exist and specific investigations are required. In this regard, obstacle problems for noncoercive linear operators on bounded domains have been considered in [4]. In [5], under standard pseudomonotonicity and coercivity conditions on the operator $A : X \to X^*$, a result on the convergence of solutions of Galerkin inequalities was proved. Bilateral problems on bounded domains in a Sobolev space setting and their approximation were treated in, e.g., [6, 7]. In [8], a quasilinear noncoercive elliptic obstacle problem on bounded domains in a Sobolev space setting was considered under the additional assumption of the existence of a supersolution.

In this paper, unlike in the papers mentioned above, neither the domain $\Omega = \mathbb{R}^N$ is bounded nor is the leading operator supposed to satisfy a coercivity condition. The main goal of this paper is to prove an existence result for the bilateral variational inequality (1.3) in the whole $\mathbb{R}^N$ without supposing coercivity with respect to K of the nonlinear operator $-\Delta_p + aF : V \to V^*$, and without supposing the existence of sub- and supersolutions for (1.3), which usually is used as a substitute for the lack of coercivity; see e.g., [9]. Another difficulty to overcome in the treatment of (1.3) in the whole $\mathbb{R}^N$ is the lack of compact embedding of $V = D^{1,p}(\mathbb{R}^N)$ into Lebesque spaces $L^q(\mathbb{R}^N)$. Moreover, we prove that the solution set of (1.3) is compact in V. Finally, an extension to noncoercive multi-valued bilateral variational inequalities is provided, which include noncoercive bilateral variational-hemivariational inequalities in the whole $\mathbb{R}^N$ as a special case.

2. Preliminaries

Throughout this paper, we assume $1 < p < N$, and use the following notations: For any $\sigma \in (1,\infty)$, its Hölder conjugate is denoted by σ', i.e., $1/\sigma + 1/\sigma' = 1$, and the $L^\sigma(\mathbb{R}^N)$-norm is denoted by $\|\cdot\|_\sigma$. For normed linear spaces X and Y, $X \hookrightarrow Y$ denotes the continuous embedding, and $X \hookrightarrow\hookrightarrow Y$ stands for the compact embedding of X into Y.

A few words regarding the homogeneous Sobolev space (Beppo-Levi space) $V = D^{1,p}(\mathbb{R}^N)$, which is the underlying solution space, are in order. For the range $1 < p < N$ considered here, this space is a Banach space, which is separable, reflexive, and even uniformly convex;

see [10, Theorem 12.2.3] and [11]. Due to the Gagliardo-Nirenberg-Sobolev inequality, the Beppo-Levi space V is continuously embedded into $L^{p^*}(\mathbb{R}^N)$ with

$$p^* = \frac{Np}{N-p} \quad \text{denoting the critical Sobolev exponent.}$$

Thus V can be characterized as

$$V = \left\{ v \in L^{p^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^p < \infty \right\}.$$

Clearly, $V \subset W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$, where $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ stands for the local Sobolev space in $\mathbb{R}^N$. However, the Sobolev space $W^{1,p}(\mathbb{R}^N)$ is a strict subspace of V, which can be seen by the following example.

Example 2.1. For $p = 2$ and $N = 3$, the function $u(x) = \left(1+|x|^2\right)^{-\frac{1}{2}}$ belongs to $V = D^{1,2}(\mathbb{R}^3)$, but u does not belong to $W^{1,2}(\mathbb{R}^3)$, because $u \notin L^2(\mathbb{R}^3)$, which is easily seen by the following elementary calculation making use of spherical coordinates:

$$\begin{aligned} \int_{\mathbb{R}^3} |u(x)|^2 dx &= \int_{\mathbb{R}^3} \frac{1}{1+|x|^2} dx = c \int_0^\infty \frac{r^2}{1+r^2} dr \\ &\geq c \int_1^\infty \frac{r^2}{1+r^2} dr \geq c \int_1^\infty \frac{r^2}{2r^2} dr = \infty, \end{aligned}$$

where c is some positive constant.

We assume the following assumptions on the coefficient a and the nonlinearity f:

(Ha) The function $a : \mathbb{R}^N \to \mathbb{R}$ is measurable and satisfies the decay property

$$|a(x)| \leq c_a \frac{1}{1+|x|^{N+\alpha}} \quad \text{for a.e. } x \in \mathbb{R}^N,$$

where c_a and α are positive constants, and $|x|$ stands for the Euclidian norm of $x \in \mathbb{R}^N$.

(Hf) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e., $x \mapsto f(x,s)$ is measurable in $\mathbb{R}^N$ for all $s \in \mathbb{R}$, and $s \mapsto f(x,s)$ is continuous in $\mathbb{R}$ for a.e. $x \in \mathbb{R}^N$, and f satisfies the following growth condition

$$|f(x,s)| \leq k(x) + c_f |s|^{p-1}, \ \ \forall s \in \mathbb{R}, \ \text{ and for a.e. } x \in \mathbb{R}^N,$$

where c_f is some positive constant, and $k \in L^{p'}(\mathbb{R}^N, w)$.

The space $L^{p'}(\mathbb{R}^N, w)$ that appears in (Hf) is the weighted Lebesgue space with weight w. In what follows, the weight function $w : \mathbb{R}^N \to \mathbb{R}$ is given by

$$w(x) = \frac{1}{1+|x|^{N+\alpha}} \quad \text{with } \alpha > 0. \tag{2.1}$$

For convenience, let us shortly recall the definition of weighted Lebesgue spaces. Let $1 \leq r < \infty$. We define the weighted Lebesgue space $L^r(\mathbb{R}^N, w)$ by

$$L^r(\mathbb{R}^N, w) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable } : \int_{\mathbb{R}^N} w|u|^r dx < \infty \right\}.$$

For $1 < r < \infty$, $L^r(\mathbb{R}^N, w)$ is a separable and reflexive (and even uniformly convex) Banach space equipped with the norm

$$\|u\|_{r,w} = \Big(\int_{\mathbb{R}^N} w|u|^r \, dx \Big)^{\frac{1}{r}}.$$

One readily verifies that the weight w given by (2.1) belongs to $L^\sigma(\mathbb{R}^N)$ for $1 \le \sigma \le \infty$. For the following embedding result, we refer to [9, Lemma 6.1].

Lemma 2.1. *[9] The embedding $V \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N, w)$ is compact for $1 < q < p^*$, that is, the embedding operator $i_w : V \to L^q(\mathbb{R}^N, w)$ defined by $u \mapsto i_w u = u$ is linear and compact.*

Since the proof of the following lemma on the mapping properties of the Nemytskij operators is quite standard, we omit the proof here.

Lemma 2.2. *Under hypothesis (Hf), the Nemytskij operator F generated by f through $F(u)(x) = f(x, u(x))$ is bounded and continuous from $L^p(\mathbb{R}^N, w)$ to $L^{p'}(\mathbb{R}^N, w)$.*

As a consequence of Lemma 2.1 and Lemma 2.2, we obtain the following result.

Corollary 2.1. *The composed operator $F \circ i_w : V \to L^{p'}(\mathbb{R}^N, w)$ is bounded and completely continuous.*

Let $i_w^* : L^{p'}(\mathbb{R}^N, w) \to V^*$ denote the adjoint operator to i_w given by

$$\eta \in L^{p'}(\mathbb{R}^N, w) : \ \langle i_w^* \eta, \varphi \rangle = \int_{\mathbb{R}^N} w \eta \varphi \, dx, \quad \forall \varphi \in V.$$

Similarly, by means of the coefficient a, we define operators $i_a^* : L^{p'}(\mathbb{R}^N, w) \to V^*$ and $i_{|a|}^* : L^{p'}(\mathbb{R}^N, w) \to V^*$, respectively, through

$$\eta \in L^{p'}(\mathbb{R}^N, w) : \ \langle i_a^* \eta, \varphi \rangle = \int_{\mathbb{R}^N} a \eta \varphi \, dx, \ \text{ and } \ \langle i_{|a|}^* \eta, \varphi \rangle = \int_{\mathbb{R}^N} |a| \eta \varphi \, dx, \quad \forall \varphi \in V.$$

Lemma 2.3. *Let the coefficient a satisfy (Ha), and let w be given by (2.1). Then i_w^*, i_a^*, $i_{|a|}^* : L^{p'}(\mathbb{R}^N, w) \to V^*$ are linear and continuous.*

Proof. Clearly, all three operators are linear. Let us show the boundedness only for i_w^* since the proof of the others can be done in just the same way by taking into account that $|a(x)| \le c_a w(x)$. Given $\eta \in L^{p'}(\mathbb{R}^N, w)$, we obtain by applying Hölder inequality the estimate

$$|\langle i_w^* \eta, \varphi \rangle| \le \int_{\mathbb{R}^N} w |\eta| |\varphi| \, dx = \int_{\mathbb{R}^N} w^{\frac{1}{p'}} |\eta| w^{\frac{1}{p}} |\varphi| \, dx \le \|\eta\|_{p',w} \|\varphi\|_{p,w} \le c \|\eta\|_{p',w} \|\varphi\|_V$$

for all $\varphi \in V$, which proves the boundedness. □

From Corollary 2.1 and Lemma 2.3, we immediately obtain the following result.

Corollary 2.2. *Let (Ha) and (Hf) be satisfied. Then the operator $F_a = aF : V \to V^*$ defined by $u \mapsto F_a(u) = (i_a^* \circ F \circ i_w)(u)$ is bounded and completely continuous. The same holds true for the composed operators $F_{|a|} = |a|F = i_{|a|}^* \circ F \circ i_w : V \to V^*$ and $F_w = wF = i_w^* \circ F \circ i_w : V \to V^*$.*

For the reader's convenience, we recall the following notions.

Definition 2.1. Let X be a real reflexive Banach space, and let $A : X \to X^*$. Then the operator A is called

(i) hemicontinuous iff the real-valued function $t \mapsto \langle A(u+tv), v-u\rangle$ is continuous on $\mathbb{R}$ for all $u, v \in X$;
(ii) monotone (resp. strictly monotone) iff $\langle Au - Av, u-v\rangle \geq$ (resp. > 0) for all $u, v \in X$ with $u \neq v$;
(iii) pseudomonotone iff $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0$ implies $\langle Au, u-w\rangle \leq \liminf_{n\to\infty}\langle Au_n, u_n - w\rangle$ for all $w \in X$;
(iv) coercive iff $\frac{1}{\|u\|_X}\langle Au, u\rangle \to \infty$ as $\|u\|_X \to \infty$;
(v) coercive relative to K (K closed convex subset of X) if there exists a $v_0 \in K$ such that

$$\frac{1}{\|u\|_X}\langle Au, u - v_0\rangle \to \infty \quad \text{as } \|u\|_X \to \infty.$$

Definition 2.2. A bounded, hemicontinuous, and monotone operator $P : V \to V^*$ is called a penalty operator associated with $K \subset V$ if $P(u) = 0 \Longleftrightarrow u \in K$.

Finally, we recall the main theorem on pseudomonotone operators due to Brézis; see, e.g., [12, Theorem 27.A].

Theorem 2.1. *Let X be a real reflexive Banach space, and let $A : X \to X^*$ be a pseudomonotone, bounded, and coercive operator. Then A is surjective, that is,* $\mathrm{range}(A) = X^*$.

Remark 2.1. We remark that, in general, the convex and closed bilateral constraint K given in (1.1) is unbounded in V as this can be seen by the following example: Let $\phi(x) \equiv 0$ and ψ be given by

$$\psi(x) = \begin{cases} |x| & \text{if} \quad 0 \leq |x| \leq 1 \\ 2 - |x| & \text{if} \quad 1 \leq |x| \leq 2 \\ 0 & \text{if} \quad |x| \geq 2. \end{cases}$$

One readily verifies that ϕ and ψ belong to V, and elementary calculations show that the sequence u_n defined by

$$u_n(x) = \begin{cases} |x|^n & \text{if} \quad 0 \leq |x| \leq 1 \\ 2 - |x| & \text{if} \quad 1 \leq |x| \leq 2 \\ 0 & \text{if} \quad |x| \geq 2. \end{cases}$$

belongs to V with $\phi \leq u_n \leq \psi$ and $\|u_n\|_V \to \infty$ as $n \to \infty$.

3. Main Results

The main result of this section reads as follows.

Theorem 3.1. *Assume hypotheses (Ha) and (Hf), and let K be the closed and convex set in $V = D^{1,p}(\mathbb{R}^N)$, $1 < p < N$, given by (1.1) with ϕ, $\psi \in V$. Then bilateral variational inequality (1.3), i.e,*

$$u \in K : \langle -\Delta_p u + aF(u), v - u\rangle \geq 0, \quad \forall v \in K,$$

admits at least one solution.

Before proving Theorem 3.1, we provide various auxiliary results next. Throughout this section, we assume the hypotheses (Ha) and (Hf) are fulfilled.

Lemma 3.1. *The leading differential operator of the variational inequality (1.3) $-\Delta_p + aF : V \to V^*$ is bounded, continuous, and pseudomonotone.*

Proof. The negative p-Laplacian $-\Delta_p : V \to V^*$ is defined by

$$\langle -\Delta_p u, \varphi \rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx, \quad u, \varphi \in V,$$

which yields by applying Hölder inequality that

$$|\langle -\Delta_p u, \varphi \rangle| \le \int_{\mathbb{R}^N} |\nabla u|^{p-1} |\nabla \varphi| \, dx \le \|u\|_V^{p-1} \|\varphi\|_V, \ \forall \varphi \in V.$$

Thus $\|-\Delta_p u\|_{V^*} \le \|u\|_V^{p-1}$, and $-\Delta_p : V \to V^*$ is bounded. To prove the continuity of $-\Delta_p$, let the function $d : \mathbb{R}^N \to \mathbb{R}^N$ be defined by $d(\xi) = |\xi|^{p-2}\xi$, which is continuous and satisfies the growth condition $|d(\xi)| = |\xi|^{p-1}$. Thus the Nemytskij operator D generated by d is bounded and continuous from $[L^p(\mathbb{R}^N)]^N$ to $[L^{p'}(\mathbb{R}^N)]^N$. Let $(u_n) \subset V$ with $u_n \to u$ in V. It follows that

$$\begin{aligned} |\langle -\Delta_p u_n - (-\Delta_p u), \varphi \rangle| &\le \int_{\mathbb{R}^N} ||\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u| |\nabla \varphi| \, dx \\ &\le \| |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \|_{p'} \|\varphi\|_V, \ \forall \varphi \in V, \end{aligned}$$

which implies that

$$\|-\Delta_p u_n - (-\Delta_p u)\|_{V^*} \le \| |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \|_{p'} = \|D(\nabla u_n) - D(\nabla u)\|_{p'} \to 0, \ \text{as } n \to \infty.$$

Since $-\Delta_p : V \to V^*$ is monotone and continuous, it follows from [12, Proposition 27.6] that $-\Delta_p$ is pseudomonotone. In view of Corollary 2.2, we see that $aF : V \to V^*$ is bounded and completely continuous, and hence pseudomonotone; see [12, Proposition 27.6]. Finally, by [12, Proposition 27.6], we see that the sum of pseudomonotone operators is pseudomonotone, which completes the proof. □

Remark 3.1. Under the hypotheses (Ha) and (Hf), the operator $A := -\Delta_p + aF : V \to V^*$ is, in general, neither coercive nor coercive relative to K, which can be demonstrated by a counterexample as follows. Let $v_0 = 0$ and $f(x,s) = -c_f |s|^{p-2}s$ with $c_f > 0$, and $a(x) = c_a w(x)$ with $c_a > 0$. Then f satisfies (Hf), and a fulfills (Ha). Thus we obtain by taking into account $c := c_a c_f > 0$ that

$$\langle Au, u \rangle = \|u\|_V^p - c \int_{\mathbb{R}^N} w|u|^p \, dx = \|u\|_V^p - c\|u\|_{p,w}^p. \tag{3.1}$$

Let $u_0 \in V$ be arbitrarily chosen with $u_0 \neq 0$. Substituting $u = tu_0$ in (3.1) with the real parameter $t > 0$, we obtain

$$\langle Au, u \rangle = \langle A(tu_0), tu_0 \rangle = t^p \|u_0\|_V^p - ct^p \|u_0\|_{p,w}^p = t^p \Big(\|u_0\|_V^p - c\|u_0\|_{p,w}^p \Big),$$

which yields (note: $u = tu_0$ and $\|u\|_V \to \infty \iff t \to +\infty$)

$$\frac{1}{\|u\|_V} \langle Au, u \rangle = t^{p-1} \frac{1}{\|u_0\|_V} \Big(\|u_0\|_V^p - c\|u_0\|_{p,w}^p \Big) \le 0, \ \forall t > 0,$$

provided that $c \ge \frac{\|u_0\|_V^p}{\|u_0\|_{p,w}^p}$, and even $\frac{1}{\|u\|_V} \langle Au, u \rangle \to -\infty$ as $\|u\|_V \to \infty$ if $c > \frac{\|u_0\|_V^p}{\|u_0\|_{p,w}^p}$, which shows that $A := -\Delta_p + aF : V \to V^*$ is not coercive.

The proof of Theorem 3.1 is based on a penalty approach along with a suitable splitting of the penalty parameter. An appropriate penalty operator P is constructed by using the functions ϕ and ψ of the bilateral constraint K as follows:

$$\langle P(u),\varphi\rangle = \int_{\mathbb{R}^N} w\Big([(u-\psi)^+]^{p-1} - [(\phi-u)^+]^{p-1}\Big)\varphi dx, \quad \varphi \in V, \tag{3.2}$$

where $v^+ = \max\{v,0\}$. Let $b:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ be given by

$$b(x,s) = [(s-\psi)^+]^{p-1} - [(\phi-s)^+]^{p-1},$$

which can equivalently be characterized by

$$b(x,s) = \begin{cases} (s-\psi(x))^{p-1} & \text{if} \quad s>\psi(x) \\ 0 & \text{if} \quad \phi(x)\le s\le\psi(x) \\ -(\phi(x)-s)^{p-1} & \text{if} \quad s<\phi(x). \end{cases}$$

One readily verifies that $b:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, which is monotone nondecreasing and satisfies the following growth condition

$$|b(x,s)| \le \beta(x) + c_b|s|^{p-1}, \quad \forall (x,s)\in\mathbb{R}^N\times\mathbb{R},\ c_b>0,$$

where $\beta(x) = c\big(|\psi(x)|^{p-1} + |\phi(x)|^{p-1}\big)$ with some positive constant c, and thus $\beta\in L^{p'}(\mathbb{R}^N,w)$, since $\phi,\ \psi\in L^p(\mathbb{R}^N,w)$. Therefore, b fulfills qualitatively the same regularity and growth conditions like f in (Hf). Hence, by Lemma 2.2, the Nemytskij operator B associated with b through $B(u)(x) = b(x,u(x))$ yields a continuous and bounded mapping from $L^p(\mathbb{R}^N,w)$ to $L^{p'}(\mathbb{R}^N,w)$. In view of (3.2), we see that P can be characterized as $P = wB$ or

$$P = i_w^*\circ B\circ i_w : V\to V^*, \tag{3.3}$$

which according to Corollary 2.2 is a bounded and completely continuous, and thus pseudomonotone operator. Moreover, since $s\mapsto b(x,s)$ is monotone nondecreasing, $P:V\to V^*$ is also a monotone operator.

Lemma 3.2. *If $\phi,\ \psi\in V$, then the operator $P:V\to V^*$ defined in (3.3) is a penalty operator associated with K which, in addition, is pseudomonotone.*

Proof. Taking the above considerations into account, it only remains to verify that

$$P(u) = 0 \Longleftrightarrow u\in K.$$

If $u\in K$, then we have by the definition of the function b that $b(x,u)=0$. Thus $P(u)=0$. To show the converse, let $P(u)=0$, that is, $\langle P(u),\varphi\rangle = 0$ for all $\varphi\in V$. Using the special test function $\varphi = (u-\psi)^+\in V$, we obtain

$$0 = \langle P(u),(u-\psi)^+\rangle = \int_{\mathbb{R}^N} w[(u-\psi)^+]^p\, dx,$$

which implies $(u-\psi)^+ = 0$, i.e., $u\le\psi$ a.e. in $\mathbb{R}^N$. Testing $\langle P(u),\varphi\rangle = 0$ with $\varphi = (\phi-u)^+\in V$ yields

$$0 = \langle P(u),(\phi-u)^+\rangle = -\int_{\mathbb{R}^N} w[(\phi-u)^+]^p\, dx,$$

which implies $(\phi-u)^+ = 0$, i.e., $\phi\le u$ a.e. in $\mathbb{R}^N$. $\square$

While the operator $-\Delta_p + aF : V \to V^*$ is not coercive (see Remark 3.1), we are in a position to show that the perturbed operator $-\Delta_p + aF + \lambda P : V \to V^*$ is coercive for λ large. More precisely, we have the following result.

Lemma 3.3. *The operator $-\Delta_p + aF + \lambda P : V \to V^*$ is bounded, continuous and pseudomonotone, and coercive relative to K for $\lambda > 0$ large enough.*

Proof. In view of Lemma 3.1 and Lemma 3.2, it remains to show that for $\lambda > 0$ large the operator $-\Delta_p + aF + \lambda P : V \to V^*$ is coercive relative to K, that is, there exists a $v_0 \in K$ such that

$$\frac{1}{\|u\|_V}\langle -\Delta_p u + aF(u) + \lambda P(u), u - v_0\rangle \to \infty \quad \text{as } \|u\|_V \to \infty.$$

First, let us estimate the term $\langle P(u), u - v_0\rangle = \langle wB(u), u - v_0\rangle$

$$\begin{aligned}\langle P(u), u - v_0\rangle &= \int_{\mathbb{R}^N} wb(\cdot,u)u\,dx - \int_{\mathbb{R}^N} wb(\cdot,u)v_0\,dx \\ &\geq c_1\|u\|_{p,w}^p - c_2 - \int_{\mathbb{R}^N} w^{\frac{1}{p'}} b(\cdot,u) w^{\frac{1}{p}} v_0\,dx \\ &\geq c_1\|u\|_{p,w}^p - c_2 - c_3\|u\|_{p,w}^{p-1} - c_4. \end{aligned} \tag{3.4}$$

For the term $\langle aF(u), u - v_0\rangle$, we get the estimate

$$\langle aF(u), u - v_0\rangle \geq -c_5\|u\|_{p,w}^p - c_6\|u\|_{p,w}^{p-1} - c_7. \tag{3.5}$$

For $\langle -\Delta_p u, u - v_0\rangle$, we have

$$\langle -\Delta_p u, u - v_0\rangle \geq \|u\|_V^p - c_8\|u\|_V^{p-1} - c_9, \tag{3.6}$$

where c_i are positive constants. Finally, estimates (3.4)–(3.6) yield

$$\begin{aligned}\langle -\Delta_p u + aF(u) + \lambda P(u), u - v_0\rangle \geq\; &\|u\|_V^p - c_8\|u\|_V^{p-1} + (\lambda c_1 - c_5)\|u\|_{p,w}^p \\ &-(\lambda c_3 + c_6)\|u\|_{p,w}^{p-1} - c(\lambda),\end{aligned} \tag{3.7}$$

where $c(\lambda)$ is some positive constant depending on λ. If $\lambda > 0$ is large enough such that $\lambda c_1 - c_5 > 0$, we infer from (3.7) the coercivity of $-\Delta_p + aF + \lambda P : V \to V^*$ relative to K, which completes the proof. $\square$

Remark 3.2. By inspecting the proof of Lemma 3.3, we see that $-\Delta_p + aF + \lambda P : V \to V^*$ is, in particular, coercive in the sense that

$$\frac{1}{\|u\|_V}\langle -\Delta_p u + aF(u) + \lambda P(u), u\rangle \to \infty \quad \text{as } \|u\|_V \to \infty,$$

provided that $\lambda > 0$ is large enough, that is, $\lambda c_1 - c_5 > 0$.

Now we are ready to prove our main result.

Proof of Theorem 3.1 Let $\varepsilon > 0$ be arbitrarily fixed. We consider the penalty equation

$$u \in V : -\Delta_p u + aF(u) + \lambda P(u) + \frac{1}{\varepsilon}P(u) = 0 \quad \text{in } V^*, \tag{3.8}$$

where $\lambda > 0$ is assumed large enough according to Lemma 3.3. Due to Lemma 3.3, we may apply Theorem 2.1 to (3.8), and thus (3.8) admits at least one solution u_ε. Testing (3.8) with $\varphi = u_\varepsilon - v_0$, where $v_0 \in K$, we obtain (note $P(v_0) = 0$)

$$\begin{aligned} 0 &= \frac{1}{\|u_\varepsilon\|_V}\langle -\Delta_p u_\varepsilon + aF(u_\varepsilon) + \lambda P(u_\varepsilon) + \frac{1}{\varepsilon}P(u_\varepsilon), u_\varepsilon - v_0\rangle \\ &\geq \frac{1}{\|u_\varepsilon\|_V}\langle -\Delta_p u_\varepsilon + aF(u_\varepsilon) + \lambda P(u_\varepsilon), u_\varepsilon - v_0\rangle. \end{aligned} \tag{3.9}$$

By Lemma 3.3, we see that $-\Delta_p + aF + \lambda P : V \to V^*$ is coercive relative to K. It follows from (3.9) that $\|u_\varepsilon\|_V \leq c$ for all $\varepsilon > 0$. Since $A := -\Delta_p + aF + \lambda P : V \to V^*$ is, in particular, a bounded operator, we have $\|Au_\varepsilon\|_{V^*} \leq c$ for all $\varepsilon > 0$, which together with (3.8) implies

$$P(u_\varepsilon) = \varepsilon(-Au_\varepsilon) \to 0 \quad \text{in } V^* \quad \text{as } \varepsilon \to 0. \tag{3.10}$$

(Here and in what follows, $c > 0$ stands for some generic constant that may change from line to line.) Thus a sequence (ε_n) and corresponding solutions u_{ε_n} of (3.8) can be chosen such that $\varepsilon_n \downarrow 0$ and $u_n := u_{\varepsilon_n} \rightharpoonup u$ (weakly convergent) in V as $n \to \infty$. Letting $v \in V$ be arbitrarily fixed, we have $\langle P(u_n) - P(v), u_n - v\rangle \geq 0$. By passing to the limit as $n \to \infty$, the weak convergence of (u_n) along with (3.10) results in $\langle P(v), u - v\rangle \leq 0$ for all $v \in V$. In particular, the above inequality holds true for $v = u - tw$ with $t > 0$ and $w \in V$, which gives $\langle P(u - tw), w\rangle \leq 0$ for all $t > 0$, $w \in V$. Passing to the limit as $t \downarrow 0$ in the last inequality, and taking the continuity of P into account, we have $\langle P(u), w\rangle \leq 0$ for all $w \in V$, and hence $P(u) = 0$, which implies $u \in K$, since P is a penalty operator. Testing the equation (3.8) with $\varphi = v - u_n$, where $v \in K$, we obtain by taking into account that P is a monotone operator and $P(v) = 0$ the following inequality

$$\langle -\Delta_p u_n + aF(u_n), v - u_n\rangle = \big(\lambda + \frac{1}{\varepsilon_n}\big)\langle P(v) - P(u_n), v - u_n\rangle \geq 0. \tag{3.11}$$

As (3.11) holds true for the weak limit $v = u \in K$, we infer

$$\limsup_{n\to\infty}\langle -\Delta_p u_n + aF(u_n), u_n - u\rangle \leq 0. \tag{3.12}$$

Since $-\Delta_p + aF : V \to V^*$ is pseudomonotone and $u_n \rightharpoonup u$, it follows from (3.12) that

$$\langle -\Delta_p u + aF(u), u - w\rangle \leq \liminf_{n\to\infty}\langle -\Delta_p u_n + aF(u_n), u_n - w\rangle, \quad \forall w \in V,$$

which is equivalent to

$$\begin{aligned} \langle -\Delta_p u + aF(u), w - u\rangle &\geq -\liminf_{n\to\infty}\langle -\Delta_p u_n + aF(u_n), u_n - w\rangle \\ &\geq \limsup_{n\to\infty}\langle -\Delta_p u_n + aF(u_n), w - u_n\rangle, \quad \forall w \in V. \end{aligned} \tag{3.13}$$

The last inequality holds true, in particular, for $w = v \in K$, which together with

$$P(v) = 0, \text{ and } \langle -\Delta_p u_n + aF(u_n), v - u_n\rangle = \big(\lambda + \frac{1}{\varepsilon_n}\big)\langle P(v) - P(u_n), v - u_n\rangle \geq 0,$$

and (3.13) yields $\langle -\Delta_p u + aF(u), v - u\rangle \geq 0$, for all $v \in K$. This shows that the weak limit $u \in K$ is in fact a solution to bilateral variational inequality (1.3). This completes the proof. □

3.1. **Compactness of the solution set.** Let us denote by $\mathscr{S}$ the set of all solutions of bilateral variational inequality (1.3). Clearly, by Theorem 3.1 we have $\mathscr{S} \neq \emptyset$.

Theorem 3.2. *Assume the hypotheses of Theorem 3.1 hold. Then the solution set $\mathscr{S}$ is compact in V.*

Proof. Let $(u_n) \subset \mathscr{S} \subset K$ be any sequence. We next show the existence of some subsequence $(u_k) \subset (u_n)$ with $u_k \to u$ (strongly convergent in V) and $u \in \mathscr{S}$.

First, we note that $\phi \le u_n \le \psi$. Thus $(\|u_n\|_{p,w})$ is bounded. If $v_0 \in K$ is arbitrarily fixed, then, for any $u_n \in \mathscr{S}$, $\langle -\Delta_p u_n + aF(u_n), v_0 - u_n\rangle \ge 0$ for all n, which yields

$$\langle -\Delta_p u_n, u_n\rangle \le \langle -\Delta_p u_n, v_0\rangle + \langle aF(u_n), v_0 - u_n\rangle,$$

and thus

$$\|u_n\|_V^p \le |\langle -\Delta_p u_n, v_0\rangle| + |\langle aF(u_n), v_0 - u_n\rangle|. \tag{3.14}$$

The first term on the right-hand side of (3.14) can be estimated as follows

$$|\langle -\Delta_p u_n, v_0\rangle| \le \int_{\mathbb{R}^N} |\nabla u_n|^{p-1} |\nabla v_0|\, dx \le \|u_n\|_V^{p-1} \|v_0\|_V. \tag{3.15}$$

For the second term on the right-hand side of (3.14), we have

$$\begin{aligned} |\langle aF(u_n), v_0 - u_n\rangle| &\le \int_{\mathbb{R}^N} |a||F(u_n)|(|v_0| + |u_n|)dx \\ &\le c_a \int_{\mathbb{R}^N} w|F(u_n)|(|v_0| + |u_n|)dx = c_a \int_{\mathbb{R}^N} w^{\frac{1}{p'}} |F(u_n)| w^{\frac{1}{p}} (|v_0| + |u_n|)dx \\ &\le c_a \|F(u_n)\|_{p',w} \big(\|v_0\|_{p,w} + \|u_n\|_{p,w}\big) \le C, \quad \forall n \end{aligned} \tag{3.16}$$

because $(\|u_n\|_{p,w})$ is bounded, and the Nemytskij operator F is a bounded mapping from $L^p(\mathbb{R}^N, w)$ into $L^{p'}(\mathbb{R}^N, w)$. In view of (3.15) and (3.16), we obtain from (3.14) (with $c > 0$ a generic constant) $\|u_n\|_V^p \le c(1 + \|u_n\|_V^{p-1})$ for all n, and thus the boundedness of (u_n) in V, i.e., $\|u_n\|_V \le c$ for all n. Therefore, there exists a weakly convergent subsequence (u_k), that is, $u_k \rightharpoonup u$ in V. Since $u_k \in K$ and K is weakly closed, we obtain $u \in K$. Thus the following inequality holds (note: $u_k \in \mathscr{S}$) $\langle -\Delta_p u_k + aF(u_k), u - u_k\rangle \ge 0$, or equivalently $\langle -\Delta_p u_k, u_k - u\rangle \le \langle aF(u_k), u - u_k\rangle$, which yields

$$0 \le \langle -\Delta_p u_k - (-\Delta_p u), u_k - u\rangle \le \langle aF(u_k), u - u_k\rangle - \langle -\Delta_p u, u_k - u\rangle. \tag{3.17}$$

By Corollary 2.2, $aF : V \to V^*$ is bounded and completely continuous, which along with $u_k \rightharpoonup u$ in V implies that the first term on the right-hand side of (3.17) tends to zero as $k \to \infty$, and clearly the second term tends to zero due $u_k \rightharpoonup u$. Therefore, we obtain from (3.17)

$$0 \le \langle -\Delta_p u_k - (-\Delta_p u), u_k - u\rangle \to 0, \quad \text{as } k \to \infty. \tag{3.18}$$

Further, we have the estimate

$$\begin{aligned}\langle -\Delta_p u_k - (-\Delta_p u), u_k - u\rangle &= \int_{\mathbb{R}^N} \Big(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u|^{p-2}\nabla u\Big)\big(\nabla u_k - \nabla u\big)dx \\ &\geq \int_{\mathbb{R}^N} \Big(|\nabla u_k|^p + |\nabla u|^p\Big)dx \\ &\quad - \int_{\mathbb{R}^N} \Big(|\nabla u_k|^{p-1}|\nabla u| + |\nabla u|^{p-1}|\nabla u_k|\Big)dx \\ &\geq \|u_k\|_V^p + \|u\|_V^p - \|u_k\|_V^{p-1}\|u\|_V - \|u\|_V^{p-1}\|u_k\|_V \\ &= \Big(\|u_k\|_V^{p-1} - \|u\|_V^{p-1}\Big)\Big(\|u_k\|_V - \|u\|_V\Big) \geq 0,\end{aligned}$$

which by using (3.18) implies

$$\Big(\|u_k\|_V^{p-1} - \|u\|_V^{p-1}\Big)\Big(\|u_k\|_V - \|u\|_V\Big) \to 0, \quad \text{as } k \to \infty,$$

and hence it follows $\lim_{k\to\infty}\|u_k\|_V = \|u\|_V$. Since $u_k \rightharpoonup u$ in V, and V is a uniformly convex Banach space, we may apply the Kadec-Klee property (see [13]), which results in the strong convergence $u_k \to u$ in V. The strong convergence allows to pass to the limit as $k \to \infty$ in

$$\langle -\Delta_p u_k + aF(u_k), v - u_k\rangle \geq 0, \ \ v \in K,$$

which shows that the limit u belongs to $\mathscr{S}$, which completes the proof. $\square$

Remark 3.3. The main results obtained in this paper for bilateral variational inequality (1.3) remain true if the p-Laplacian is replaced by a general monotone operator of Leray-Lions type, that is, for the following problem $u \in K : h \in Au + aF(u) + \partial I_K(u)$ in V^* or equivalently

$$u \in K : \langle Au + aF(u), v - u\rangle \geq \langle h, v - u\rangle, \ \ \forall v \in K,$$

where $h \in V^*$, and A is a Leray-Lions operator of the form

$$Au(x) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, \nabla u),$$

whose coefficient a_i satisfy the following conditions:

(A1) each $a_i : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e., $a_i(x,\xi)$ is measurable in $x \in \mathbb{R}^N$ for all $\xi \in \mathbb{R}^N$, and continuous in ξ for a.a. $x \in \mathbb{R}^N$. There exist a constant $c_0 > 0$ and a function $k_0 \in L^{p'}(\mathbb{R}^N)$ such that $|a_i(x,\xi)| \leq k_0(x) + c_0|\xi|^{p-1}$ for a.a. $x \in \mathbb{R}^N$ and for all $\xi \in \mathbb{R}^N$;

(A2) for a.a. $x \in \mathbb{R}^N$ and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$, the following monotonicity holds:

$$\sum_{i=1}^N (a_i(x,\xi) - a_i(x,\xi'))(\xi_i - \xi_i') > 0;$$

(A3) there exists a constant $\nu > 0$ such that for a.a. $x \in \mathbb{R}^N$ and for all $\xi \in \mathbb{R}^N$, the inequality

$$\sum_{i=1}^N a_i(x,\xi)\xi_i \geq \nu|\xi|^p - k_1(x)$$

is satisfied for some function $k_1 \in L^1(\mathbb{R}^N)$.

4. Noncoercive Multi-valued Bilateral Variational Inequalities in $\mathbb{R}^N$

With the notations for K, V, and $L^p(\mathbb{R}^N, w)$ of the preceding sections, in this section, let us consider the following multi-valued bilateral variational inequality in the whole $\mathbb{R}^N$:

$$u \in K \subset V : 0 \in -\Delta_p u + a\hat{F}(u) + \partial I_K(u) \quad \text{in } V^*, \tag{4.1}$$

where $\hat{F}$ is a multi-valued Nemytskij operator generated by the multi-valued function $\hat{f} : \mathbb{R} \to \mathscr{K}(R)$ with

$$\mathscr{K}(R) = \{\text{set of all nonempty, closed intervals in } \mathbb{R}\}.$$

Only for the sake of simplifying our presentation, we assume $\hat{f}$ to be independent of $x \in \mathbb{R}^N$.

Definition 4.1. The function $u \in K$ is called a solution to multi-valued bilateral variational inequality (4.1) if there exists an $\eta \in L^{p'}(\mathbb{R}^N, w)$ such that $\eta \in \hat{F}(u)$ and

$$\langle -\Delta_p u, v-u \rangle + \int_{\mathbb{R}^N} a\eta(v-u)dx \geq 0, \quad \forall v \in K. \tag{4.2}$$

We assume the following hypothesis for the multi-valued function $\hat{f}$.

(H$\hat{f}$) Let $\hat{f} : \mathbb{R} \to \mathscr{K}(R)$ be an upper semicontinuous multi-valued function satisfying the following growth condition ($c_{\hat{f}}$ a positive constant)

$$\sup\{|\eta| : \eta \in \hat{f}(s)\} \leq c_{\hat{f}}\big(1 + |s|^{p-1}\big), \quad \forall s \in \mathbb{R}. \tag{4.3}$$

Remark 4.1. Hypothesis (H$\hat{f}$) implies that $\hat{f}$ is a multi-valued measurable function, and then graph-measurable, which in turn implies that $\hat{f}$ is superpositionally measurable, that is, if $u : \mathbb{R}^N \to \mathbb{R}$ is measurable, then $\hat{f}(u(\cdot)) : \mathbb{R}^N \to \mathscr{K}(\mathbb{R})$ is measurable.

Taking growth condition (4.3) into account, the multi-valued Nemytskij operator $\hat{F}$ generated by $\hat{f}$ and given by

$$\hat{F}(u) = \{\eta : \mathbb{R}^N \to \mathbb{R} \text{ measurable } : \eta(x) \in \hat{f}(u(x)) \text{ for a.e. } x \in \mathbb{R}^N\}$$

provides a well defined mapping from $L^p(\mathbb{R}^N, w)$ into $2^{L^{p'}(\mathbb{R}^N, w)} \setminus \{\emptyset\}$.

Under hypothesis (H$\hat{f}$), one can show that the multi-valued operator $a\hat{F} : V \to 2^{V^*} \setminus \{\emptyset\}$ defined by

$$a\hat{F} = i_a^* \circ \hat{F} \circ i_w : V \to 2^{V^*} \setminus \{\emptyset\}$$

is a bounded and multi-valued pseudomonotone operator; see [9, Chapter 6]. Taking advantage of the theory of multi-valued pseudomonotone operators, and applying the penalty technique developed in the preceding sections, the following existence result can be proved in an analogous way as Theorem 3.1 and Theorem 3.2.

Theorem 4.1. *Assume hypotheses (Ha) and (H$\hat{f}$) hold, and let K be the bilateral constraint. Then multi-valued bilateral variational inequality (4.1) (resp. (4.2)) admits at least one solution. Moreover, the solution set $\mathscr{S}$ of all solutions of (4.1) is compact in V.*

As a special case of Theorem 4.1, we obtain an existence result for the following noncoercive bilateral variational-hemivariational inequality in the whole $\mathbb{R}^N$ of the form:

$$u \in K \subset V : \langle -\Delta_p u, v-u \rangle + \int_{\mathbb{R}^N} a j^o(u; v-u)dx \geq 0, \quad \forall v \in K, \tag{4.4}$$

where $j^o(s;\rho)$ denotes Clarke's generalized directional derivative of the locally Lipschitz function j at $s \in \mathbb{R}$ in the direction $\rho \in \mathbb{R}$; see [14].

Theorem 4.2. *Assume (Ha) with $a(x) \geq 0$ and let $j : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function whose Clarke's generalized gradient $s \mapsto \partial j(s)$ satisfies the following growth condition:*

$$\sup\{|\eta| : \eta \in \partial j(s)\} \leq c_j\big(1+|s|^{p-1}\big), \quad \forall s \in \mathbb{R}. \tag{4.5}$$

Then bilateral variational-hemivariational inequality (4.4) admits at least one solution, and the solution set $\mathscr{S}$ of all solutions of (4.4) is compact in V.

Proof. Define $\hat{f}(s) = \partial j(s)$. By Clarke's calculus (see [14, Chapter 2]), the multi-valued function $s \mapsto \hat{f}(s) \in \mathscr{K}(\mathbb{R})$ and $\hat{f} : \mathbb{R} \to \mathscr{K}(\mathbb{R})$ is upper semicontinuous. Taking (4.5) into account, one sees that $\hat{f} = \partial j$ satisfies (H$\hat{f}$), which allows us to apply Theorem 4.1, and hence the existence of a solution of the following multi-valued bilateral variational inequality: There is an $\eta \in L^{p'}(\mathbb{R}^N, w)$ such that $\eta \in \hat{F}(u)$, that is, $\eta(x) \in \partial j(x)$ for a.e. $x \in \mathbb{R}^N$, and

$$u \in K \subset V : \langle -\Delta_p u, v-u\rangle + \int_{\mathbb{R}^N} a\eta(v-u)dx \geq 0, \quad \forall v \in K. \tag{4.6}$$

Clearly, the bilateral constraint K satisfies the following lattice condition

$$K \vee K \subset K \quad \text{and} \quad K \wedge K \subset K,$$

where

$$K \vee K = \{v \vee w : w, v \in K\} \ \text{ with } \ v \vee w = \max\{v,w\},$$
$$K \wedge K = \{v \wedge w : w, v \in K\} \ \text{ with } \ v \wedge w = \{\min\{v,w\}.$$

As a consequence of [9, Theorem 6.11], we deduce that $u \in K$ is a solution to (4.6) if and only if u is a solution to bilateral variational-hemivariational inequality (4.4), which completes the proof. □

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