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SUBDIFFERENTIAL CALCULUS AND IDEAL SOLUTIONS FOR SET OPTIMIZATION PROBLEMS

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Abstract. We explore the possibility to derive basic calculus rules for some subdifferential constructions associated to set-valued maps between normed vector spaces. We use these results in order to write optimality conditions for a special kind of solutions for set optimization problems.

Keywords. Ideal solutions; Set-valued maps; Subdifferential calculus; Set optimization.

1. Introduction

The primary aim of this work is to develop some ideas concerning the subdifferential calculus rules for generalized differentiation objects associated to set-valued maps (or multifunctions) which were introduced and only briefly pointed out in [5]. The generalized subgradients under study in this paper were initially designed to deal with set optimization problems and therefore are constructed on the basis of epigraphical set-valued maps, where the epigraphs are defined by means of an ordering cone in the output space. Actually, one of the main tools that we systematically employ for deriving our results is the scalarization of the underlying set-valued maps by elements in the dual cone associated to the ordering cone. By some topological conditions, we reduce, in some relevant cases, the calculus for the subgradients of the set-valued maps with vectorial values to the subgradients of real-valued functions, a case for which many calculus rules are available (see [10]). Besides the Fréchet subdifferential of set-valued maps introduced in [5], we define here a corresponding limiting (or Mordukhovich) subdifferential. For both types of subgradients, we study the (generalized) convex case and sum rules. The calculus that we derive here covers known results from the case of nonsmooth real-valued functions. Moreover, a difference rule for Fréchet subdifferential is derived in a less general situation, but this case is, however, general enough to ensure meaningful necessary optimality conditions for a newly introduced type of solutions for set optimization problems. We notice that the investigation of the main results is based on several statements that could be of some importance for their own: we mention here a "conic" variant of Rådström cancellation law and a penalization procedure. Furthermore, the concept of (sequential) cone compactness of a set seems to be an instrument in various situations that naturally appear in our study.

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The paper is organized as follows. The second section collects some preliminaries concerning scalarization by positive linear continuous functionals and useful (semi) continuity properties of set-valued maps. The main section is the third one. Firstly, we present several calculus rules for already introduced Fréchet subdifferential of set-valued maps between normed vector spaces. As we already mentioned, on the output space, we consider a closed convex pointed cone and we start our development of calculus rules by the case of convex multifunctions (where the convexity is defined by means of the ordering cone). In this basic situation, it becomes apparent that the study requires an adapted version of Rådström cancellation law involving the presence of the ordering cone. Moreover, several concepts related to sets that are used thoroughly afterwards are recalled, the most important one being the sequential compactness with respect to a cone introduced and studied in [4]. Then we present a sum rule in the convex case. Secondly, on the basis of the Fréchet subdifferential, we introduce in the same manner (i.e., by passing to the limit procedure) as in the case of real-valued functions the limiting (or Mordukhovich) subdifferential for which we obtain formulas in the convex case and for the sum. Thirdly, we are interested in some calculus rules concerning the subdifferentials of some special (multi)functions. In particular, we derive an estimation formula for the Fréchet subdifferential of a difference between a general set-valued map and a particular function. The last section applies some of the calculus rules developed in the previous section in order to obtain necessary optimality conditions for a special type of solutions for set optimization problems, which we call ideal solutions. In order to reach the necessary optimality conditions, we display several technical results among which we mention a penalization procedure.

We briefly present the notation we use in this work. Let X,Y be normed spaces over the real field $\mathbb{R}$. The topological dual of X is X^*, the norm is denoted $\|\cdot\|$, and $B(X,Y)$ is the normed vector space of linear bounded operators from X to Y. For $x \in X$ and $\varepsilon > 0$, we put $B(x,\varepsilon)$ for the open ball centered at x with the radius ε. The symbols S_X and D_X stand for the unit sphere and for the closed unit ball, respectively. If $A \subset X$ is a nonempty set, then we denote by $\mathrm{cl}A$, $\mathrm{int}A$, A', $\mathrm{conv}A$, $d(\cdot,A)$ the topological closure, the topological interior, the set of accumulation points, the convex hull, and the associated distance function, respectively. If A,B are nonempty subsets of X, the excess from A to B is $e(A,B) = \sup_{x\in A} d(x,B)$.

2. Preliminaries

In the following, in general, K is supposed to be a pointed closed convex cone in Y and its positive dual cone is denoted by K^+. We consider a set-valued map $F : X \rightrightarrows Y$ and we associate with it the epigraphical set-valued map $\mathrm{Epi}\,F : X \rightrightarrows Y$ given by $\mathrm{Epi}\,F(x) = F(x) + K$. As a standing assumption, we assume that all the values of F that are involved in the discussion are nonempty sets. The following concepts were introduced in [5].

Definition 2.1. Let $F : X \rightrightarrows Y$ and $\bar{x} \in X$. The Fréchet subdifferential of F at $\bar{x}$ is

$$\widehat{\partial} F(\bar{x}) = \left\{ T \in B(X,Y) \mid \lim_{x\to\bar{x}} \frac{e(\mathrm{Epi}\,F(x), \mathrm{Epi}\,F(\bar{x}) + T(x-\bar{x}))}{\|x-\bar{x}\|} = 0 \right\}.$$

Equivalently, $T \in \widehat{\partial} F(\bar{x})$ iff $T \in B(X,Y)$ and

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in B(\bar{x},\delta) : \mathrm{Epi}\,F(x) \subset \mathrm{Epi}\,F(\bar{x}) + T(x-\bar{x}) + \varepsilon \|x-\bar{x}\| D_Y. \tag{2.1}$$

Similarly, we can define the upper subdifferential of F at $\bar{x}$ as follows

$$\widehat{\partial}^{+}F(\bar{x})=\left\{T\in B(X,Y)\mid \lim_{x\to\bar{x}}\frac{e\left(\operatorname{Epi}F(\bar{x})+T(x-\bar{x}),\operatorname{Epi}F(x)\right)}{\|x-\bar{x}\|}=0\right\}.$$

As was remarked in the same work, these notions generalize the respective subdifferentials from the case of scalar functions. Indeed, if $f:X\to\mathbb{R}$ is a function, then relation (2.1) can be equivalently written as

$$\forall\varepsilon>0,\exists\delta>0,\forall x\in B(\bar{x},\delta):f(x)-f(\bar{x})-x^{*}(x-\bar{x})\geq-\varepsilon\|x-\bar{x}\|,$$

i.e., $x^{*}\in\widehat{\partial}f(\bar{x})$, where $\widehat{\partial}f(\bar{x})$ denotes the usual Fréchet subdifferential of f at $\bar{x}$ (see [10]). A similar comment holds for $\widehat{\partial}^{+}f(\bar{x})$.

The next result is based on [5, Lemma 2.5]. For the completeness, we give a direct proof and we point out as well a missing assumption in the mentioned result.

Proposition 2.1. *Let $K\subset Y$ be a pointed closed convex cone, and let A,B be nonempty subsets of Y.*

(i) If $A\subset B+K$, then

$$y^{*}(A)+[0,+\infty)\subset y^{*}(B)+[0,+\infty),\ \forall y^{*}\in K^{+}\setminus\{0\}. \tag{2.2}$$

The converse implication holds provided $B+K$ is convex and closed.

(ii) If $\operatorname{int}K\neq\emptyset$ and $A\subset B+\operatorname{int}K$, then $y^{*}(A)\subset y^{*}(B)+(0,+\infty)$, $\forall y^{*}\in K^{+}\setminus\{0\}$. The converse implication holds provided $B+K$ is convex.

Proof. If $A\subset B+K$, then, for all $a\in A$, there exists $b\in B$ such that $a\in b+K$. Therefore, for all $y^{*}\in K^{+}\setminus\{0\}$, one has $y^{*}(b)\leq y^{*}(a)$, which means that $y^{*}(a)\in y^{*}(b)+[0,\infty)\subset y^{*}(B)+[0,\infty)$. Since a was arbitrarily chosen in A, we have $y^{*}(A)+[0,+\infty)\subset y^{*}(B)+[0,+\infty)$ for all $y^{*}\in K^{+}\setminus\{0\}$. Suppose that $B+K$ is convex and closed. By way of contradiction, we assume that the converse does not hold. Then there exists $a\in A$ such that $a\notin B+K$. The assumptions that we made allow us to strongly separate the point a from the set $B+K$, so one has $y^{*}\in Y^{*}\setminus\{0\}$ and $\alpha\in\mathbb{R}$ such that $y^{*}(a)<\alpha<y^{*}(b+k)$ for all $b\in B$ and $k\in K$. Clearly, this forces $y^{*}\in K^{+}\setminus\{0\}$ and $y^{*}(a)<\alpha<y^{*}(b)$ for all $b\in B$. But, by assumption, there exists $b_{a}\in B$ such that $y^{*}(b_{a})\leq y^{*}(a)$ and we reach a contradiction.

(ii) Since $y^{*}(k)>0$ for all $y^{*}\in K^{+}\setminus\{0\}$ and $k\in\operatorname{int}K$, we easily get the direct implication. For the converse, suppose that $B+K$ is convex and there is $a\in A$ such that $a\notin B+\operatorname{int}K$. Therefore, one gets $y^{*}\in Y^{*}\setminus\{0\}$ such that $y^{*}(a)<y^{*}(b+k)$ for all $b\in B$ and $k\in\operatorname{int}K$. Again, this forces $y^{*}\in K^{+}$ and $y^{*}(a)\leq y^{*}(b)$ for all $b\in B$. But, by assumption, there is $b_{a}\in B$ such that $y^{*}(b_{a})<y^{*}(a)$ and this contradicts the above inequality. □

Example 2.1. Let $Y=\mathbb{R}^{2}$, $A=\{(1,1)\}$, $B=\{(2,0),(0,2)\}$, and $K=\mathbb{R}^{2}_{+}$. Then, for $y^{*}=(y_{1},y_{2})\in K^{+}\setminus\{0\}=\mathbb{R}^{2}_{+}\setminus\{0\}$,

$$y^{*}(1,1)=y_{1}+y_{2}\geq\begin{cases}2y_{1}, & \text{if } y_{2}\geq y_{1}\\ 2y_{2}, & \text{if } y_{1}>y_{2}.\end{cases}$$

Since $2y_{1}$, $2y_{2}\in y^{*}(B)$, we see that the converse in Proposition 2.1 (i) does not hold without additional assumptions.

Remark 2.1. For the converses in (i) and (ii) of the result above, one can take $y^{*}\in S_{Y^{*}}\cap K^{+}$ or only y^{*} in a set of generators of K^{+} without 0.

Now, we recall the notions of lower and upper continuity of a set-valued map (see [7, Definition 2.5.1]) and a generalized Lipschitz condition with respect to K of a set-valued map (see, e.g., [2, 14]).

Definition 2.2. Let $F : X \rightrightarrows Y$ be a set-valued map with nonempty values and $x \in X$. Then F is said to be:

(i) lower continuous at x (in short, l.c.) if, for each open set $V \subset Y$ with $F(x) \cap V \neq \emptyset$, there exists a neighborhood $U \subset X$ of x such that, for each $u \in U$, $F(u) \cap V \neq \emptyset$;

(ii) upper continuous at x (in short, u.c.) if, for each open set $V \subset Y$ with $F(x) \subset V$, there exists a neighborhood $U \subset X$ of x such that, for each $u \in U$, $F(u) \subset V$;

(iii) continuous at x if it is both l.c. and u.c. at x;

(iv) l.c. (u.c., continuous) on a nonempty set $A \subset X$ if it is l.c. (u.c., continuous) at all $x \in A$;

(v) l.c. (u.c., continuous) around x if there exists $\varepsilon > 0$ such that F is l.c. (u.c., continuous) on $B(x, \varepsilon)$.

Definition 2.3. Let $F : X \rightrightarrows Y$ be a set-valued map and $\bar{x} \in X$. One says that F is $K-$Lipschitz around $\bar{x}$ if there exist a neighborhood U of $\bar{x}$, a constant $L > 0$, and an element $e \in K \setminus \{0\}$ such that, for every $x, u \in U$, $F(x) + L\|x-u\| e \subset F(u) + K$.

3. Subdifferential Calculus

We intend to derive some subdifferential calculus rules for the Fréchet subdifferential and then to introduce and study an associated limiting (Mordukhovich) subdifferential.

Let us first remark that if F is a set-valued map from X to $\mathbb{R}$ and has closed, bounded from below values, then taking $f : X \to \mathbb{R}$ given by $f(x) = \min F(x)$, one has, with the natural choice $K = [0, \infty)$, for every $\bar{x} \in X$, that $t \in \widehat{\partial} F(\bar{x})$ if and only if $t \in \widehat{\partial} f(\bar{x})$, and similarly for $\widehat{\partial}^+$. Notice as well that for this kind of set-valued maps, the boundedness from below is required in order to avoid trivial situations in terms of $\widehat{\partial}$ and $\widehat{\partial}^+$. In view of this, we concentrate on $\widehat{\partial}$. The first result that we derive considers a scalarization of a general set-valued map by elements in the positive dual cone in order to obtain the situation described above (that is, the case of set-valued maps taking as values nonempty subsets of $\mathbb{R}$).

Proposition 3.1. *Let $F : X \rightrightarrows Y$ be a set-valued map, $\bar{x} \in X$, and $T \in \widehat{\partial} F(\bar{x})$. Then, for all $y^* \in K^+ \setminus \{0\}$, $y^* \circ T \in \widehat{\partial}(y^* \circ F)(\bar{x})$. If $F(\bar{x}) + K$ is convex and closed, and K is finitely generated, then the converse holds.*

Proof. If $T \in \widehat{\partial} F(\bar{x})$, then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x \in B(\bar{x}, \delta)$,

$$F(x) \subset F(\bar{x}) + T(x-\bar{x}) + \varepsilon \|x-\bar{x}\| D_Y + K.$$

By Proposition 2.1, for all $y^* \in K^+ \setminus \{0\}$, we have

$$(y^* \circ F)(x) \in (y^* \circ F)(\bar{x}) + (y^* \circ T)(x-\bar{x}) + \varepsilon \|x-\bar{x}\| y^*(D_Y) + [0, \infty), \ \forall x \in B(\bar{x}, \delta), \quad (3.1)$$

which implies

$$(y^* \circ F)(x) \in (y^* \circ F)(\bar{x}) + (y^* \circ T)(x-\bar{x}) - \varepsilon \|x-\bar{x}\| \|y^*\| + [0, \infty), \ \forall x \in B(\bar{x}, \delta).$$

It is obvious that this is enough to conclude that $y^* \circ T \in \widehat{\partial}(y^* \circ F)(\bar{x})$.

For the converse, if K is finitely generated, then K^+ is finitely generated (see, e.g., [1, Section 2.3]). Since, in inclusion (2.2) of Proposition 2.1, we can take only elements from a set of generators of K^+ (and this is finite), when we write $y^* \circ T \in \widehat{\partial}(y^* \circ F)(\bar{x})$ for all $\varepsilon > 0$, we obtain the same $\delta > 0$ working for all y^* in that set of generators for which (3.1) holds and we can apply Proposition 2.1. □

Our primary concern is to see that, as in the classical case, the subdifferentials take a special form under convexity assumptions, so we consider now the convex case. Here, the convexity is understood in the generalized sense described in the definition below.

Definition 3.1. Let $A \subset X$ be a nonempty convex set and $G : A \rightrightarrows Y$ be a set-valued map.

(i) One says that G is upper $K-$convex if, for every $x, y \in A$ and every $\lambda \in (0,1)$,

$$\lambda G(x) + (1-\lambda) G(y) \subset G(\lambda x + (1-\lambda) y) + K.$$

(ii) Suppose that K is solid (that is, $\operatorname{int} K \neq \emptyset$). One says that G is strictly upper $K-$convex if, for every $x, y \in A$ and every $\lambda \in (0,1)$,

$$\lambda G(x) + (1-\lambda) G(y) + \operatorname{int} K \subset G(\lambda x + (1-\lambda) y) + \operatorname{int} K.$$

Inspired by [12, Theorem 2.2], we obtain the next result, which is a variant of Rådström cancellation lemma adapted to our setting. We use the following notion: a nonempty set $A \subset Y$ is called $K-$bounded if there is a bounded set $M \subset Y$ such that $A \subset M + K$.

Lemma 3.1. *Suppose that $A, B, C \subset Y$ are nonempty sets such that C is $K-$bounded and $A + C \subset \operatorname{cl}(C + B + K)$. Then $A \subset \operatorname{cl}\operatorname{conv}(B + K)$.*

Proof. The proof is similar to one of the proofs of Rådström cancellation lemma (see, e.g. [12, Theorem 2.2]). Firstly, we observe that it is enough to take A as a singleton, $A = \{a\}$ with $a \in Y$. Then, since $\{a\} + C \subset \operatorname{cl}(C + B + K)$, one has $C \subset \operatorname{cl}(C + B + K) - a = \operatorname{cl}(C + (B - a) + K)$. As the following equalities hold

$$\operatorname{cl}\operatorname{conv}((B-a)+K) = \operatorname{cl}(\operatorname{conv}(B+K) - a) = \operatorname{cl}\operatorname{conv}(B+K) - a,$$

we can even consider $A = \{0\}$.

Therefore, we know that $C \subset \operatorname{cl}(C + B + K)$ and we have to show that $0 \in \operatorname{cl}\operatorname{conv}(B + K)$. Denote by M a bounded set that satisfies $C \subset M + K$. Then, for all natural $n \geq 1$,

$$\begin{aligned} C &\subset \operatorname{cl}(C+B+K) \subset \operatorname{cl}(\operatorname{cl}(C+B+K)+B+K) = \operatorname{cl}(C+B+K+B+K) \subset \ldots \\ &\subset \operatorname{cl}\Big(\underbrace{B+\ldots+B}_{n \text{ times}} + K + C\Big) \subset \operatorname{cl}\Big(\underbrace{B+\ldots+B}_{n \text{ times}} + K + M\Big). \end{aligned}$$

Take $c \in C$. Then there exists $c_n \in B(c, n^{-1})$ and $m_n \in M$ such that, for all n,

$$\frac{1}{n}(c_n - m_n) \in \frac{1}{n}\Big(\underbrace{(B+K)+\ldots+(B+K)}_{n \text{ times}}\Big) \subset \operatorname{conv}(B+K).$$

Since M is bounded, we see that $0 \in \operatorname{cl}\operatorname{conv}(B+K)$, which is the conclusion. □

Remark 3.1. It is obvious that, for $K = \{0\}$, the above result reduces to [12, Theorem 2.2].

Now we present the form of the Fréchet subdifferential for a $K-$convex set-valued map. We recall (see [4]) that a nonempty subset $A \subset Y$ is called $K-$sequentially compact if, for any sequence $(a_n) \subset A$, there exists a sequence $(c_n) \subset K$ such that the sequence $(a_n - c_n)$ has a convergent subsequence towards an element of A. It was proved in [4] that a $K-$compact set (see [9] for the definition) is $K-$sequentially compact and the converse holds provided that K is separable. According to [4], if A is $K-$sequentially compact, then A is also $K-$closed (that is, $A+K$ is closed) and $K-$bounded. We also recall that A is said to be $K-$convex if $A+K$ is a convex set.

Proposition 3.2. *Suppose that F is upper K-convex. Consider $\overline{x} \in X$ and that $F(\overline{x})$ is K-sequentially compact. Moreover, suppose that D_Y is K-closed. Then*

$$\widehat{\partial} F(\overline{x}) = \{T \in B(X,Y) \mid F(x) \subset F(\overline{x}) + T(x-\overline{x}) + K, \ \forall x \in X\}.$$

Proof. The fact that $\widehat{\partial} F(\overline{x})$ includes the right-hand set is clear. Take $T \in \widehat{\partial} F(\overline{x})$. Then, for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that, for all $x \in B(\overline{x}, \delta_\varepsilon)$,

$$F(x) \subset F(\overline{x}) + T(x-\overline{x}) + \varepsilon \|x-\overline{x}\| D_Y + K.$$

For all $x \in X$, there is $\lambda_{\varepsilon,x} \in (0,1)$ such that $(1-\lambda_{\varepsilon,x})\overline{x} + \lambda_{\varepsilon,x} x \in B(\overline{x}, \delta_\varepsilon)$, so we can write, by taking into account the upper $K-$convexity of F, that

$$\begin{aligned}(1-\lambda_{\varepsilon,x}) F(\overline{x}) + \lambda_{\varepsilon,x} F(x) &\subset F((1-\lambda_{\varepsilon,x})\overline{x} + \lambda_{\varepsilon,x} x) + K \\ &\subset F(\overline{x}) + \lambda_{\varepsilon,x} T(x-\overline{x}) + \varepsilon \lambda_{\varepsilon,x} \|x-\overline{x}\| D_Y + K,\end{aligned}$$

whence

$$\begin{aligned}&(1-\lambda_{\varepsilon,x}) F(\overline{x}) + \lambda_{\varepsilon,x} F(\overline{x}) + \lambda_{\varepsilon,x} F(x) + K \\ &\subset F(\overline{x}) + \lambda_{\varepsilon,x} F(\overline{x}) + \lambda_{\varepsilon,x} T(x-\overline{x}) + \varepsilon \lambda_{\varepsilon,x} \|x-\overline{x}\| D_Y + K.\end{aligned}$$

But, the upper $K-$convexity of F implies the convexity of the set $F(\overline{x}) + K$, so

$$F(\overline{x}) + \lambda_{\varepsilon,x} F(x) + K \subset F(\overline{x}) + \lambda_{\varepsilon,x} F(\overline{x}) + \lambda_{\varepsilon,x} T(x-\overline{x}) + \varepsilon \lambda_{\varepsilon,x} \|x-\overline{x}\| D_Y + K,$$

whence

$$F(\overline{x}) + \lambda_{\varepsilon,x} F(x) \subset F(\overline{x}) + \lambda_{\varepsilon,x} F(\overline{x}) + \lambda_{\varepsilon,x} T(x-\overline{x}) + \varepsilon \lambda_{\varepsilon,x} \|x-\overline{x}\| D_Y + K.$$

Now, the set $F(\overline{x})$ is $K-$bounded and the set

$$\lambda_{\varepsilon,x} F(\overline{x}) + \lambda_{\varepsilon,x} T(x-\overline{x}) + \varepsilon \lambda_{\varepsilon,x} \|x-\overline{x}\| D_Y = \lambda_{\varepsilon,x} (F(\overline{x}) + T(x-\overline{x}) + \varepsilon \|x-\overline{x}\| D_Y)$$

is $K-$closed (as a sum between a $K-$sequentially compact set and a $K-$closed one: see [4]) and $K-$convex. Therefore, it follows from Lemma 3.1 that

$$\lambda_{\varepsilon,x} F(x) \subset \lambda_{\varepsilon,x} (F(\overline{x}) + T(x-\overline{x}) + \varepsilon \|x-\overline{x}\| D_Y) + K.$$

Thus $F(x) \subset F(\overline{x}) + T(x-\overline{x}) + \varepsilon \|x-\overline{x}\| D_Y + K$. For fixed x, this is true for all ε. Employing the $K-$closedness of $F(\overline{x}) + T(x-\overline{x})$, we finally deduce $F(x) \subset F(\overline{x}) + T(x-\overline{x}) + K$ for all $x \in X$. Consequently, the equality is proven. □

Corollary 3.1. *If $f : X \to Y$ is an upper $K-$convex function and D_Y is $K-$closed, and $\overline{x} \in X$, then*

$$\widehat{\partial} f(\overline{x}) = \{T \in B(X,Y) \mid f(x) \in f(\overline{x}) + T(x-\overline{x}) + K, \ \forall x \in X\}.$$

In particular, if $Y = \mathbb{R}$ and $K = [0,\infty)$, then this coincides with the classical subdifferential of a convex function.

In the previous results, we use the assumption that $D_Y + K$ is a closed set. Let us mention that this always happens when Y is reflexive. Indeed, in this case, D_Y is weakly compact and K is weakly closed, so the sum $D_Y + K$ is weakly closed, and then (norm-)closed. However, if Y is not reflexive, this can fail to be true, as the next example demonstrates.

Example 3.1. Let $X = c_0$ with its usual (supremum) norm and define $f : c_0 \to \mathbb{R}$,

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} x_n, \ \forall x = (x_n)_{n\geq 1} \in c_0.$$

It is easy to see that $f \in (c_0)^*$, $\|f\| = 1$, and there is no $x \in D_{c_0}$ such that $f(x) = 1$, so $D_{c_0} \subset \{x \in c_0 \mid |f(x)| < 1\}$.

Consider the closed convex pointed cone $A = \left\{x = (x_n)_{n\geq 1} \in c_0 \mid x_1 \leq 0, x_n \geq 0, \ \forall n \geq 2\right\}$, and define $K = A \cap \{x \in c_0 \mid f(x) \geq 0\}$, which is also a closed convex pointed cone. One has that $D_{c_0} + K \subset \{x \in c_0 \mid f(x) > -1\}$. Consider now the following sequences in c_0 : $\left(x^k\right)_{k\geq 1}$, $\left(y^k\right)_{k\geq 1}$ given by

$$x_n^k = \begin{cases} -1, \text{ for } n \in \overline{1,k} \\ 0, \text{ for } n > k \end{cases}$$

and

$$y_n^k = \begin{cases} -1, \text{ for } n = 1 \\ \left(1 + k^{-1}\right), \text{ for } n \in \overline{2,k} \\ 0, \text{ for } n > k. \end{cases}$$

Then $x^k \in D_{c_0}$ for all $k \geq 1$ and $y^k \in K$ for all $k \geq 3$. Indeed, the former inclusion is obvious, while for the latter we have the following computation:

$$\begin{aligned} f\left(y^k\right) &= -\frac{1}{2} + \sum_{n=2}^{k} \left(1 + \frac{1}{k}\right) \frac{1}{2^n} = -\frac{1}{2} + \frac{1}{2}\left(1 + \frac{1}{k}\right)\left(1 - \frac{1}{2^{k-1}}\right) \\ &= \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k}\frac{1}{2^{k-1}} - \frac{1}{2^{k-1}}\right) = \frac{1}{2}\frac{2^{k-1} - k - 1}{k \cdot 2^{k-1}} \geq 0, \ \forall k \geq 3, \end{aligned}$$

so $x^k + y^k \in D_{c_0} + K$ for all $k \geq 3$. One has that

$$x_n^k + y_n^k = \begin{cases} -2, \text{ for } n = 1 \\ k^{-1}, \text{ for } n \in \overline{2,k} \\ 0, \text{ for } n > k. \end{cases}$$

Consequently, $x^k + y^k \to u = (-2, 0, 0, ..., 0...)$, but $f(u) = -1$, whence $u \notin D_{c_0} + K$. This proves that $D_{c_0} + K$ is not closed.

Now, we are concerned with the sum rules for subdifferentials. For this, we use the scalarization procedure described in Proposition 3.1 and then we consider the real-valued functions associated to real-valued set-valued maps, as observed in the opening of this section. Three preliminary remarks, together with a lemma, are in order.

Remark 3.2. Let $F_1, F_2 : X \rightrightarrows Y$ be set-valued maps. Clearly, for every $\overline{x} \in X$,

$$\widehat{\partial} F_1(\overline{x}) + \widehat{\partial} F_2(\overline{x}) \subset \widehat{\partial}(F_1 + F_2)(\overline{x}).$$

Remark 3.3. If $A, B \subset \mathbb{R}$ are nonempty, closed, and bounded from below, then $A \subset B + [0, \infty)$ if and only if $\min A \geq \min B$.

Remark 3.4. If $f : X \to \mathbb{R}$ is a function and $F : X \rightrightarrows \mathbb{R}$ is a set-valued map with closed and bounded from below values, then it is easy to see that the following equivalence holds: $f(x) = \min F(x)$ for every $x \in X$ if and only if $F(x) + [0,\infty) = f(x) + [0,\infty)$ for every $x \in X$.

Lemma 3.2. *Let $f : X \to \mathbb{R}$ be a function, and let $F : X \rightrightarrows \mathbb{R}$ be a set-valued map such that $F(x) + [0,\infty) = f(x) + [0,\infty)$ for all $x \in X$. Let $\bar{x} \in X$.*
(i) If F is u.c. at $\bar{x}$, then f is lower semicontinuous (in short, l.s.c.) at $\bar{x}$;
(ii) If F is l.c. at $\bar{x}$, then f is upper semicontinuous (in short, u.s.c.) at $\bar{x}$.

Proof. (i) Suppose that F is u.c. at $\bar{x}$ and take $\alpha \in \mathbb{R}$ with $\alpha < f(\bar{x})$, i.e., $f(\bar{x}) \in (\alpha,\infty)$. Since $F(\bar{x}) \subset F(\bar{x}) + [0,\infty) = f(\bar{x}) + [0,\infty)$, one has that $F(\bar{x}) \subset (\alpha,\infty)$, whence, by using the upper continuity of F, one sees that there exists a neighborhood U of $\bar{x}$ such that, for all $u \in U$, $F(u) \subset (\alpha,\infty)$. Now, since $f(x) \in F(x) + [0,\infty)$, for every $x \in X$, one obtains that, for all $u \in U$, $f(u) \in (\alpha,\infty)$, i.e., $\alpha < f(u)$, so f is l.s.c. at $\bar{x}$.

(ii) Suppose that F is l.c. at $\bar{x}$ and take $\alpha \in \mathbb{R}$ with $f(\bar{x}) < \alpha$. According to the hypothesis, $f(\bar{x}) \in F(\bar{x}) + [0,\infty)$, so there exists $\bar{y} \in F(\bar{x})$ such that $f(\bar{x}) \geq \bar{y}$. Thus $\bar{y} \in F(\bar{x}) \cap (-\infty,\alpha)$. Further, by using the lower continuity of F, one sees that there exists a neighborhood U of $\bar{x}$ such that, for all $u \in U$, $F(u) \cap (-\infty,\alpha) \neq \emptyset$, so, for all $u \in U$, there exists $v_u \in \mathbb{R}$ such that $v_u < \alpha$ and $v_u \in F(u)$. Since $F(x) \subset f(x) + [0,\infty)$ for every $x \in X$, one has that $v_u \geq f(u)$ for all $u \in U$, whence $f(u) < \alpha$ for all $u \in U$. Thus the conclusion follows immediately. □

Notice that the reverse implications from the above lemma, in general, do not hold as we illustrate in the below example.

Example 3.2. Let $F_1, F_2 : \mathbb{R} \rightrightarrows \mathbb{R}$ be two set-valued maps given by

$$F_1(x) = \begin{cases} \{-1,1\}, \text{ if } x \neq 0, \\ \{0\}, \text{ if } x = 0, \end{cases}$$

and

$$F_2(x) = \begin{cases} \{-1,-\frac{1}{2}\}, \text{ if } x = 0, \\ \{0\}, \text{ if } x \neq 0, \end{cases}$$

and let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be two functions defined by

$$f_1(x) = \begin{cases} -1, \text{ if } x \neq 0, \\ 0, \text{ if } x = 0, \end{cases}$$

and

$$f_2(x) = \begin{cases} -1, \text{ if } x = 0, \\ 0, \text{ if } x \neq 0. \end{cases}$$

It is easy to see that, for $i \in \{1,2\}, F_i(x) + [0,\infty) = f_i(x) + [0,\infty)$, for every $x \in X$, f_1 is u.s.c. at 0, but F_1 is not l.c. at 0 and f_2 is l.s.c. at 0, but F_2 is not u.c. at 0.

Now we are able to present a sum rule for the convex case.

Proposition 3.3. *Let $F_1, F_2 : X \rightrightarrows Y$ be upper $K-$convex set-valued maps with $K-$sequentially compact values. Suppose that F_1 (or F_2) is continuous at some $x \in X$ or X is finite dimensional. Then, for every $\bar{x} \in X$ and $y^* \in K^+ \setminus \{0\}$,*

$$y^*\left(\widehat{\partial}(F_1 + F_2)(\bar{x})\right) \subset \widehat{\partial}(y^* \circ F_1)(\bar{x}) + \widehat{\partial}(y^* \circ F_2)(\bar{x}).$$

Proof. Take $T \in \widehat{\partial}(F_1 + F_2)(\overline{x}) = \widehat{\partial}(\mathrm{Epi}\, F_1 + \mathrm{Epi}\, F_2)(\overline{x})$. According to Proposition 3.1, for all $y^* \in K^+ \setminus \{0\}$,

$$y^* \circ T \in \widehat{\partial}(y^* \circ (\mathrm{Epi}\, F_1 + \mathrm{Epi}\, F_2))(\overline{x}) = \widehat{\partial}(y^* \circ \mathrm{Epi}\, F_1 + y^* \circ \mathrm{Epi}\, F_2)(\overline{x}).$$

For $i \in \{1,2\}$, one has that $y^* \circ \mathrm{Epi}\, F_i$ has as values closed and bounded from below intervals of $\mathbb{R}$. Indeed, $\mathrm{Epi}\, F_i$ has convex values, whence $y^* \circ \mathrm{Epi}\, F_i$ has as values intervals in $\mathbb{R}$. Now, the $K-$sequentially compactness of the values of F_i implies their $K-$boundedness. For all $x \in X$ and $i \in \{1,2\}$, we denote by M_x^i the bounded set with the property $F_i(x) \subset M_x^i + K$. Then one has that, for all $z \in (y^* \circ \mathrm{Epi}\, F_i)(x)$, there exists $y \in M_x^i$ and $k \in K$ such that $z = y^*(y) + y^*(k)$, whence $z \geq -\|y^*\| \|y\|$. Since M_x^i is bounded, one deduces that $(y^* \circ \mathrm{Epi}\, F_i)(x)$ are bounded from below intervals. Finally, we show that these sets are also closed. Take again $x \in X$ and a sequence $(z_n) \subset (y^* \circ \mathrm{Epi}\, F_i)(x)$ such that $z_n \to z$. As above, for all n there is $y_n \in F_i(x)$ and $k_n \in K$ such that $z_n = y^*(y_n) + y^*(k_n)$. By the $K-$sequentially compactness of $F_i(x)$, there is a sequence $(c_n) \subset K$ such that $(y_n - c_n)$ converges towards some $u \in F_i(x)$. Therefore,

$$z_n = y^*(y_n) + y^*(k_n) = y^*(y_n - c_n) + y^*(c_n) + y^*(k_n),$$

so $z_n - y^*(y_n - c_n) \to z - y^*(u) \geq 0$. Thus we obtain that $z \in y^*(u) + [0,\infty) \subset (y^* \circ F_i)(x) + [0,\infty) = (y^* \circ \mathrm{Epi}\, F_i)(x)$, and the claim follows.

Observe now that the values of $y^* \circ (\mathrm{Epi}\, F_1 + \mathrm{Epi}\, F_2)$ share the same properties and take the minimal functions associated to these three set-valued maps, denoted as $f_{F_1+F_2}$, f_{F_1}, and f_{F_2}, respectively. Observe as well that these functions are convex, $f_{F_1+F_2} = f_{F_1} + f_{F_2}$ and at any $x \in X$, the subdifferential of $y^* \circ (\mathrm{Epi}\, F_1 + \mathrm{Epi}\, F_2)$ coincides with the subdifferential of $f_{F_1+F_2}$ and similarly for the other two functions. Then we have that

$$y^* \circ T \in \widehat{\partial}(f_{F_1+F_2})(\overline{x}) = \widehat{\partial}(f_{F_1} + f_{F_2})(\overline{x}).$$

If X is finite dimensional, since f_{F_1} and f_{F_2} are convex, both functions f_{F_1} and f_{F_2} are continuous. If F_1 is continuous at $x \in X$, then $y^* \circ F_1$ is continuous at x, whence, by Lemma 3.2, f_{F_1} is continuous at x. This means that the classical sum rule for the convex subdifferential of real-valued convex functions can be applied, whence we obtain that, for every $\overline{x} \in X$,

$$y^* \circ T \in \widehat{\partial} f_{F_1}(\overline{x}) + \widehat{\partial} f_{F_2}(\overline{x}) = \widehat{\partial}(y^* \circ F_1)(\overline{x}) + \widehat{\partial}(y^* \circ F_2)(\overline{x}).$$

The conclusion follows immediately. □

As it is well known, the Fréchet subdifferential does not enjoy exact sum rules for general functions, so it is not expected to behave differently in this more general setting. For this reason, we introduce now a limiting subdifferential of a set-valued map in a similar manner as in the classical case.

Let X be a Banach space. For $F : X \rightrightarrows Y$ with nonempty values and $\overline{x} \in X$, define

$$\partial F(\overline{x}) = \left\{ T \in B(X,Y) \mid \exists (x_n) \overset{\mathrm{Epi}\, F, e}{\to} \overline{x},\ \exists (T_n) \overset{SOT}{\to} T,\ T_n \in \widehat{\partial} F(x_n),\ \forall n \right\},$$

where $(x_n) \overset{\mathrm{Epi}\, F, e}{\to} \overline{x}$ means that $(x_n) \to \overline{x}$ and $e(\mathrm{Epi}\, F(x_n), \mathrm{Epi}\, F(\overline{x})) \to 0$, and $T_n \overset{SOT}{\to} T$ means the convergence in the strong operator topology (that is, $T_n(x) \to T(x)$ for all $x \in X$).

This concept is a variation of the construction of the classical Mordukhovich subdifferential in the context of Asplund spaces, but in a slightly weaker form, since both the requirements

$e(\operatorname{Epi} F(x_n), \operatorname{Epi} F(\bar{x})) \to 0$ and $e(\operatorname{Epi} F(\bar{x}), \operatorname{Epi} F(x_n)) \to 0$ would lead in the case of real-valued functions to

$$\partial f(\bar{x}) = \left\{x^* \in X^* \mid \exists (x_n) \to \bar{x},\ f(x_n) \to f(\bar{x}),\ \exists (x_n^*) \stackrel{w^*}{\to} x^*,\ x_n^* \in \widehat{\partial} f(x_n),\ \forall n\right\},$$

where by w^* we mean the weak star topology on X^*. For more details, we refer to [10, Theorem 2.34].

Remark 3.5. (i) Clearly, for all x one has $\widehat{\partial} F(x) \subset \partial F(x)$ and $\partial F(x) = \partial(\operatorname{Epi} F)(x)$.

(ii) If F is from X to $\mathbb{R}$ and has closed bounded from below values, then taking $f : X \to \mathbb{R}$ given by $f(x) = \min F(x)$, one has, for every $\bar{x} \in X$, that $t \in \partial F(\bar{x})$ if and only if $t \in \partial f(\bar{x})$.

Clearly, convergence in the strong operator topology implies the convergence in the weak operator topology (that is, $(y^* \circ T_n)$ converges pointwise to $y^* \circ T$ for every $y^* \in Y^*$). It is clear that it reduces to w^* convergence when $Y = \mathbb{R}$.

Proposition 3.4. *Let X be a Banach space, $F : X \rightrightarrows Y$ a set-valued map, $\bar{x} \in X$, and $T \in \partial F(\bar{x})$. Then, for all $y^* \in K^+ \setminus \{0\}$, $y^* \circ T \in \partial(y^* \circ F)(\bar{x})$.*

Proof. Since $T \in \partial F(\bar{x})$, we have that there exist $(x_n) \stackrel{\operatorname{Epi} F,e}{\to} \bar{x}$, $(T_n) \stackrel{SOT}{\to} T$ such that, for all n, $T_n \in \widehat{\partial} F(x_n)$. Following Proposition 3.1, for all n and all $y^* \in K^+ \setminus \{0\}$, $y^* \circ T_n \in \widehat{\partial}(y^* \circ F)(x_n)$. Fix $y^* \in K^+ \setminus \{0\}$. Since $(x_n) \stackrel{\operatorname{Epi} F,e}{\to} \bar{x}$, then $(x_n) \stackrel{\operatorname{Epi}(y^* \circ F),e}{\to} \bar{x}$ and since $(T_n) \stackrel{SOT}{\to} T$, $(y^* \circ T_n) \stackrel{SOT}{\to} y^* \circ T$. This proves that $y^* \circ T \in \partial(y^* \circ F)(\bar{x})$. □

Proposition 3.5. *Suppose that X is a Banach space, and F is upper $K-$convex with $K-$sequentially compact values around $\bar{x} \in X$. Assume that D_Y is $K-$closed. Then,*

$$\partial F(\bar{x}) = \{T \in B(X,Y) \mid F(x) \subset F(\bar{x}) + T(x - \bar{x}) + K,\ \forall x \in X\}.$$

Proof. The inclusion from right to left follows from Remark 3.5 and Proposition 3.2. Take now $T \in \partial F(\bar{x})$. Then there exist $(x_n) \stackrel{\operatorname{Epi} F,e}{\to} \bar{x}$, $(T_n) \stackrel{SOT}{\to} T$ such that, for all n, $T_n \in \widehat{\partial} F(x_n)$. We can suppose, without loss of generality, that $F(x_n)$ is $K-$sequentially compact for all n. According to Proposition 3.2, we have $F(x) \subset F(x_n) + T_n(x - x_n) + K$ for all $x \in X$ and n. Observe that $T_n(x - x_n) \to T(x - \bar{x})$. Indeed, by the Uniform Boundedness Principle, $(\|T_n\|)$ is a bounded sequence (notice that X is a Banach space). Then the inequality

$$\begin{aligned} \|T_n(x - x_n) - T(x - \bar{x})\| &\le \|(T_n - T)(x)\| + \|T_n(x_n) - T(\bar{x})\| \\ &\le \|(T_n - T)(x)\| + \|T_n(x_n - \bar{x})\| + \|T_n(\bar{x}) - T(\bar{x})\| \\ &\le \|(T_n - T)(x)\| + \|T_n\| \|x_n - \bar{x}\| + \|T_n(\bar{x}) - T(\bar{x})\|,\ \forall n \end{aligned}$$

and the facts that $(x_n) \to \bar{x}$ and $T_n \stackrel{SOT}{\to} T$ prove the claim. Take $\varepsilon > 0$. Then, for all $x \in X$ and for all n large enough, $F(x) \subset F(x_n) + T_n(x - x_n) + K \subset F(\bar{x}) + T(x - \bar{x}) + \varepsilon D_Y + K$. Since ε is arbitrary and $F(\bar{x}) + T(x - \bar{x}) + K$ is closed, we obtain the conclusion immediately. □

Proposition 3.6. *Let X be an Asplund space, and let $F_1, F_2 : X \rightrightarrows Y$ be set-valued maps with $K-$sequentially compact values. Take $\bar{x} \in X$. Suppose that F_1 is u.c. around $\bar{x}$, and F_2 is $K-$Lipschitz around $\bar{x}$. Then, for every $y^* \in K^+ \setminus \{0\}$,*

$$y^*(\partial(F_1 + F_2)(\bar{x})) \subset \partial(y^* \circ F_1)(\bar{x}) + \partial(y^* \circ F_2)(\bar{x}).$$

Proof. Take $T \in \partial (F_1 + F_2)(\bar{x}) = \partial (\operatorname{Epi} F_1 + \operatorname{Epi} F_2)(\bar{x})$. By Proposition 3.4, for all $y^* \in K^+ \setminus \{0\}$, one has $y^* \circ T \in \partial (y^* \circ (\operatorname{Epi} F_1 + \operatorname{Epi} F_2))(\bar{x})$. As in the proof of Proposition 3.3, $y^* \circ \operatorname{Epi} F_i$ $(i \in \{1,2\})$ has closed and bounded from below values (subsets of $\mathbb{R}$) and the same can be said for the set-valued map $y^* \circ (\operatorname{Epi} F_1 + \operatorname{Epi} F_2)$. Then we can as well consider the minimal functions associated to these three set-valued maps, denoted as in the mentioned proof. According to Remark 3.5, one has $y^* \circ T \in \partial (f_{F_1+F_2})(\bar{x}) = \partial (f_{F_1} + f_{F_2})(\bar{x})$. Again by Lemma 3.2, since F_1 is u.c. around $\bar{x}$, we see that f_{F_1} is l.s.c. around this point. Similarly, since F_2 is $K-$Lipschitz around $\bar{x}$, we see that f_{F_2} is Lipschitz around this point. Then one can apply the sum rule for the limiting subdifferential (see [10, Theorem 2.33]) to obtain $y^* \circ T \in \partial f_{F_1}(\bar{x}) + \partial f_{F_2}(\bar{x})$, which proves the desired conclusion. □

Finally, we are interested in some statements concerning the subdifferential of some special (multi)functions.

For $e \in K \setminus \{0\}$ we use the notation $K_e^+ = \{y^* \in K^+ \mid y^*(e) \neq 0\}$. Notice that $K_e^+ \neq \emptyset$ (by a standard separation theorem applied to $\{e\}$ and $-K$) and, obviously, if $\operatorname{int} K \neq \emptyset$ and $e \in \operatorname{int} K$, then $K_e^+ = K^+ \setminus \{0\}$.

Remark 3.6. Take in Proposition 3.1 the mapping $F := f$ as a function of the form $\varphi(\cdot)e$, where $\varphi : X \to \mathbb{R}$ and $e \in K \setminus \{0\}$. Therefore, if $t \in \widehat{\partial} f(\bar{x})$, then, for all $y^* \in K_e^+$, $y^*(e)^{-1}(y^* \circ t) \in \widehat{\partial} \varphi(\bar{x})$.

Proposition 3.7. *Let $\varphi : X \to \mathbb{R}$ be a function, $\bar{x} \in X$, and $e \in K \setminus \{0\}$. Denote $f = \varphi(\cdot)e$. Then,*

$$\left\{ x^*(\cdot)e \mid x^* \in \widehat{\partial} \varphi(\bar{x}) \right\} \subset \widehat{\partial} f(\bar{x}).$$

Proof. Let $x^* \in \widehat{\partial} \varphi(\bar{x})$. Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x \in B(\bar{x}, \delta)$,

$$\varphi(x) \in \varphi(\bar{x}) + x^*(x - \bar{x}) + \varepsilon \|x - \bar{x}\| [-1,1] + [0,\infty).$$

Then, for such x,

$$\begin{aligned} \varphi(x)e &\in \varphi(\bar{x})e + x^*(x - \bar{x})e + \varepsilon \|x - \bar{x}\| [-1,1]e + [0,\infty)e \\ &\subset \varphi(\bar{x})e + x^*(x - \bar{x})e + \varepsilon \|x - \bar{x}\| \|e\| D_Y + K, \end{aligned}$$

which is enough to prove that $x^*(\cdot)e \in \widehat{\partial} f(\bar{x})$. □

Proposition 3.8. *Let $e \in K \setminus \{0\}$ and denote $f = \|\cdot\| e$ (that is, $\varphi = \|\cdot\|$ with the above notation). Then, $D_{X^*} e \subset \widehat{\partial} f(0)$ and for all $y^* \in K_e^+$, $y^*\left(\widehat{\partial} f(0)\right) \subset y^*(e) D_{X^*}$.*

Proof. By Proposition 3.7, $D_{X^*} e \subset \widehat{\partial} f(0)$, whence the first inclusion holds. Take $t \in \widehat{\partial} f(0)$. By Remark 3.6, for all $y^* \in K_e^+$, one has $y^*(e)^{-1}(y^* \circ t) \in \widehat{\partial} \varphi(0)$, so there exists $x^*_{y^*} \in D_{X^*}$ such that $y^*(e)^{-1}(y^* \circ t) = x^*_{y^*}$, which implies the conclusion. □

Finally, inspired by [11], we prove a difference rule.

Proposition 3.9. *Let $F : X \rightrightarrows Y$ be a set-valued map, $\varphi : X \to \mathbb{R}$ a function, $\bar{x} \in X$, and $e \in K \setminus \{0\}$. Denote $f = \varphi(\cdot)e$. If $\widehat{\partial} \varphi(\bar{x}) \neq \emptyset$, then*

$$\widehat{\partial}(F - f)(\bar{x}) \subset \bigcap_{\substack{t \in \widehat{\partial} f(\bar{x}) \\ y^* \in K_e^+}} \left(\widehat{\partial} F(\bar{x}) - y^*(e)^{-1}(y^* \circ t)e \right).$$

Proof. Take $T \in \widehat{\partial}(F-f)(\overline{x})$. Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x \in B(\overline{x}, \delta)$,

$$F(x) - \varphi(x)e \subset F(\overline{x}) - \varphi(\overline{x})e + T(x-\overline{x}) + \varepsilon \|x-\overline{x}\| D_Y + K.$$

Since $\widehat{\partial}\varphi(\overline{x}) \neq \emptyset$, by Proposition 3.7, one has $\widehat{\partial} f(\overline{x}) \neq \emptyset$. Take $t \in \widehat{\partial} f(\overline{x})$ and $y^* \in K_e^+$. According to Remark 3.6, one has $(y^*(e))^{-1}(y^* \circ t) \in \widehat{\partial}\varphi(\overline{x})$. Using the variational description of the Fréchet subgradients, and making δ smaller if necessary, one sees that there exists a function s_{y^*}, differentiable at $\overline{x}$, such that $s_{y^*}(\overline{x}) = \varphi(\overline{x})$, $\nabla s_{y^*}(\overline{x}) = y^*(e)^{-1}(y^* \circ t)$ and $s_{y^*}(x) \leq \varphi(x)$ for all $x \in B(\overline{x}, \delta)$. By the differentiability of s_{y^*} at $\overline{x}$, one sees that there exists a function $\alpha : X \to \mathbb{R}$ continuous at $\overline{x}$ and with $\lim_{x \to \overline{x}} \alpha(x) = 0$ such that, for all x,

$$s_{y^*}(x) - s_{y^*}(\overline{x}) = \nabla s_{y^*}(\overline{x})(x-\overline{x}) + \alpha(x)\|x-\overline{x}\|.$$

We can suppose, again by making δ smaller if necessary, that $\alpha(x)e \in \varepsilon D_Y$ for all $x \in B(\overline{x}, \delta)$. We successively obtain, for all $x \in B(\overline{x}, \delta)$,

$$\begin{aligned}
F(x) &\subset F(\overline{x}) + (\varphi(x) - \varphi(\overline{x}))e + T(x-\overline{x}) + \varepsilon\|x-\overline{x}\| D_Y + K \\
&\subset F(\overline{x}) + (s_{y^*}(x) - s_{y^*}(\overline{x}))e + T(x-\overline{x}) + \varepsilon\|x-\overline{x}\| D_Y + K \\
&\subset F(\overline{x}) + (\nabla s_{y^*}(\overline{x})(x-\overline{x}) + \alpha(x)\|x-\overline{x}\|)e + T(x-\overline{x}) + \varepsilon\|x-\overline{x}\| D_Y + K \\
&= F(\overline{x}) + (T(\cdot) + \nabla s_{y^*}(\overline{x})(\cdot)e)(x-\overline{x}) + \varepsilon\|x-\overline{x}\| D_Y + \alpha(x)\|x-\overline{x}\| e + K \\
&\subset F(\overline{x}) + (T(\cdot) + \nabla s_{y^*}(\overline{x})(\cdot)e)(x-\overline{x}) + 2\varepsilon\|x-\overline{x}\| D_Y + K.
\end{aligned}$$

From this, we see that $T + \nabla s_{y^*}(\overline{x})e \in \widehat{\partial} F(\overline{x})$ whence

$$T \in \widehat{\partial} F(\overline{x}) - \nabla s_{y^*}(\overline{x})e = \widehat{\partial} F(\overline{x}) - y^*(e)^{-1}(y^* \circ t)e,$$

and the conclusion follows. □

4. Ideal Solutions

Many authors have discussed optimization problems that are governed by multifunctions, based on well-established relations on sets defined by Kuroiwa (see [8]). We recall one such relation here and a concept of minimality associated with it (see [2]). For two nonempty sets

$$A \preceq_K^l B \iff B \subset A + K.$$

Definition 4.1. Let $F : X \rightrightarrows Y$ be a set-valued map with nonempty values, and let $M \subset X$ be a nonempty set. One says that $\overline{x} \in M$ is a local $l-$minimum for F on M if there exists $\varepsilon > 0$ such that

$$x \in M \cap B(\overline{x}, \varepsilon),\ F(x) \preceq_K^l F(\overline{x}) \Longrightarrow F(\overline{x}) \preceq_K^l F(x).$$

Example 4.1. Notice that, in general, the local $\ell-$minimality of $\overline{x}$ for F does not imply $0 \in \widehat{\partial} F(\overline{x})$. For example, $\overline{x} = 0$ is a $l-$minimum for F on $M = \mathbb{R}$, but $0 \notin \widehat{\partial} F(\overline{x})$, where $K := \mathbb{R}_+^2$, $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$,

$$F(x) = \begin{cases} \{x\} \times [-2, \infty), \text{ if } x > 0, \\ \{0\} \times [0, \infty), \text{ if } x \leq 0. \end{cases}$$

However, if F is from X to $\mathbb{R}$ and has closed, bounded from below values, then taking $f : X \to \mathbb{R}$ given by $f(x) = \min F(x)$, one has that $\ell-$minimality of $\overline{x}$ for F is equivalent to classical minimality of $\overline{x}$ for f, so in this case $0 \in \widehat{\partial} F(\overline{x})$.

We consider now a concept for which the basic Fermat rule takes place.

Definition 4.2. Let $M \subset X$ be a nonempty set. One says that $\bar{x} \in M$ is a local ideal minimum for F on M if there is $\varepsilon > 0$ such that

$$F(\bar{x}) \not\subset F(x) + Y \setminus -K, \; \forall x \in M \cap B(\bar{x}, \varepsilon) \setminus \{\bar{x}\}. \tag{4.1}$$

The global version of the above concept is obtained by taking the open ball $B(\bar{x}, \varepsilon)$ as the whole space X, in which case we omit to write "local". Also, we omit to write "on M" when we study the case without restrictions.

Remark 4.1. Notice that relation (4.1) implies $F(x) \subset F(\bar{x}) + K$ for all $x \in M \cap B(\bar{x}, \varepsilon)$. Consequently, if $\bar{x} \in \operatorname{int} M$ is a local ideal minimum for F, then $0 \in \widehat{\partial} F(\bar{x})$. If $F := f$ is a function, then the above minimality becomes $f(x) - f(\bar{x}) \in K$ for all $x \in M \cap B(\bar{x}, \varepsilon)$, and this is known in vector optimization under the name of ideal minimality.

Lemma 4.1. *Let $A \subset Y$ be a $K-$sequentially compact set, and let $D \subset Y$ be a set such that $A \subset D + Y \setminus -K$. Then there exists $\varepsilon > 0$ such that $A + B(0, \varepsilon) \subset D + Y \setminus -K$.*

Proof. Suppose, by way of contradiction, that the conclusion is not true. Then one can find a sequence $(\rho_n) \to 0$ and $(a_n) \subset A$ such that, for all $n \geq 1$, $a_n + \rho_n \notin D + Y \setminus -K$. Since A is $K-$sequentially compact, there exists a sequence $(c_n) \subset K$ such that a subsequence $(a_{n_k} - c_{n_k})$ converges towards some $a \in A$. Then a belongs to the open set $D + Y \setminus -K$, so for k large enough, $a_{n_k} - c_{n_k} + \rho_{n_k} \in D + Y \setminus -K$, which, together with $Y \setminus -K + K = Y \setminus -K$, implies that $a_{n_k} + \rho_{n_k} \in D + Y \setminus -K$ for large k. This is a contradiction. Hence, the conclusion holds. □

In the following, we present a result which follows a well-established approach known as penalization (as described by Clarke in [3] and extended to vector optimization in [14], and also to set optimization, e.g., in [2]). This approach involves incorporating the distance between the constraint set and the objective map to convert geometric constraints into an unconstrained problem. The Lipschitz assumption used in the below penalization result is consistent with the one employed in previous studies of penalization (see, e.g., [2, 14]) and also it is in the line of the Lipschitz notion given in Definition 2.3.

Proposition 4.1. *Let $M \subset X$ be a nonempty set. Assume that $\bar{x} \in M \cap M'$ and $F(\bar{x})$ is K-sequentially compact. Let $\bar{x}$ be a local ideal minimum for F on M. Suppose that the following generalized Lipschitz condition holds: there exist $e \in K \setminus \{0\}$, $r > 0$, and $\ell > 0$ such that, for all $u \in (X \setminus M) \cap B(\bar{x}, r)$ and $v \in M \cap B(\bar{x}, r)$, $F(u) + \ell \|u - v\| e \subset F(v) + K$. Then $\bar{x}$ is a local ideal minimum on X (that is, without constraints) for the set-valued map $G : X \rightrightarrows Y$, $G(x) = F(x) + \ell d_M(x) e$.*

Proof. Without loss of generality, one can take r as the radius of the ball around $\bar{x}$ where the minimality condition holds. Suppose, by way of contradiction, that there exists $u \in B\left(\bar{x}, 3^{-1} r\right)$ such that $F(\bar{x}) \subset F(u) + \ell d_M(u) e + Y \setminus -K$. Clearly, $u \notin M$. By virtue of Lemma 4.1, there is $\varepsilon > 0$ such that $F(\bar{x}) - \ell \varepsilon e \subset F(u) + \ell d_M(u) e + Y \setminus -K$, that is, $F(\bar{x}) \subset F(u) + \ell (d_M(u) + \varepsilon) e + Y \setminus -K$. On the other hand, by tacking $\delta = \min\left\{\varepsilon, 3^{-1} r\right\}$, we see that there exists $a \in M$ such that $\|u - a\| < d_M(u) + \delta$. Since $\bar{x} \in M'$, we see that exists $a \in M$ such that $a \neq \bar{x}$ and the above inequality holds, so $a \in M \setminus \{\overline{x}\}$ and

$$\begin{aligned}\|a-\overline{x}\| &\leq \|a-u\| + \|u-\overline{x}\| < d_M(u) + \delta + \|u-\overline{x}\| \\ &\leq 2\|u-\overline{x}\| + 3^{-1}r < r,\end{aligned}$$

whence $a \in (M \setminus \{\overline{x}\}) \cap B(\overline{x}, r)$. We have

$$(d_M(u) + \varepsilon - \|u-a\|)e = (d_M(u) + \delta - \|u-a\|)e + (\varepsilon - \delta)e \in K \setminus \{0\},$$

so $(d_M(u) + \varepsilon)e \in \|u-a\|e + K \setminus \{0\}$. Consequently,

$$F(\overline{x}) \subset F(u) + \ell\|u-a\|e + K \setminus \{0\} + Y \setminus -K \subset F(u) + \ell\|u-a\|e + Y \setminus -K.$$

Using the generalized Lipschitz property we consider as hypothesis, we obtain that

$$F(\overline{x}) \subset F(a) + Y \setminus -K,$$

which is a contradiction. Thus we can deduce the desired conclusion immediately. □

The global version reads as follows.

Proposition 4.2. *Let $M \subset X$ be a nonempty set. Assume that $\overline{x} \in M \cap M'$ and $F(\overline{x})$ is K-sequentially compact. Let $\overline{x}$ be an ideal minimum for F on M. Suppose that the following generalized Lipschitz condition holds: there exist $e \in K \setminus \{0\}$ and $\ell > 0$ such that, for all $u \in X \setminus M$ and $v \in M$, $F(u) + \ell\|u-v\|e \subset F(v) + K$. Then $\overline{x}$ is an ideal minimum on X (that is, without constraints) for the set-valued map $G : X \rightrightarrows Y$, $G(x) = F(x) + \ell d_M(x)e$.*

Finally, putting together the local penalization result with some of the subdifferential calculus rules developed before, we write necessary optimality conditions for ideal sharp minimality (for more details, see the comment after the proof).

Proposition 4.3. *Let X be an Asplund space, $M \subset X$ be a nonempty set and $\mu > 0$. Suppose that the following generalized Lipschitz condition holds: there exist $e \in K \setminus \{0\}$, $r > 0$, $\ell > 0$ such that, for all $u \in (X \setminus M) \cap B(\overline{x}, r)$ and $v \in M \cap B(\overline{x}, r)$,*

$$F(u) + \ell\|u-v\|e \subset F(v) + K. \tag{4.2}$$

Take $\overline{x} \in M \cap M'$ as a local ideal minimum for $F(\cdot) - \mu\|\cdot - \overline{x}\|e$ on M. Suppose that F has $K-$sequentially compact values and it is u.c. around $\overline{x}$. Then, for all $y^* \in K_e^+$,

$$\mu y^*(e) D_{X^*} \subset \partial(y^* \circ F)(\overline{x}) + (\ell + \mu) y^*(e) \partial d_M(\overline{x}).$$

Proof. From relation (4.2), one sees that, for all $u \in (X \setminus M) \cap B(\overline{x}, r)$ and $v \in M \cap B(\overline{x}, r)$,

$$F(u) - \mu\|u-\overline{x}\|e + (\ell+\mu)\|u-v\|e \subset F(v) - \mu\|v-\overline{x}\|e + K.$$

that is, the Lipschitz assumption used in the local penalization result from above holds for the set-valued map $x \rightrightarrows F(x) - \mu\|x-\overline{x}\|e$. Therefore, according to Proposition 4.1, $\overline{x} \in M$ is a local ideal minimum for $F(\cdot) - \mu\|\cdot - \overline{x}\|e + (\ell+\mu) d_M(\cdot)e$ on X. By Proposition 3.9 and Remark 4.1, one has

$$\begin{aligned}0 &\in \widehat{\partial}(F(\cdot) + (\ell+\mu) d_M(\cdot)e - \mu\|\cdot - \overline{x}\|e)(\overline{x}) \\ &\subset \bigcap_{\substack{t \in \widehat{\partial}(\mu\|\cdot - \overline{x}\|e)(\overline{x}) \\ y^* \in K_e^+}} \left(\widehat{\partial}(F + (\ell+\mu) d_M e)(\overline{x}) - y^*(e)^{-1}(y^* \circ t)e\right).\end{aligned}$$

Using Proposition 3.6, for all $t \in \widehat{\partial}(\mu \|\cdot - \bar{x}\| e)(\bar{x})$, $y^* \in K_e^+$, and $z^* \in K^+ \setminus \{0\}$, one has

$$z^*\left(y^*(e)^{-1}(y^* \circ t)e\right) \in z^*\left(\widehat{\partial}(F + (\ell+\mu)d_M e)(\bar{x})\right) \subset z^*(\partial(F + (\ell+\mu)d_M e)(\bar{x}))$$
$$\subset \partial(z^* \circ F)(\bar{x}) + (\ell+\mu)z^*(e)\partial d_M(\bar{x}).$$

Taking $z^* = y^*$ and using the first inclusion from Proposition 3.8, one obtains the conclusion immediately. □

Remark 4.2. Observe that the minimality for $F(\cdot) - \mu \|\cdot - \bar{x}\| e$ corresponds to the extension in the current setting of the concept of sharp minimum (see [13] for the scalar case and [6] for the vectorial one).

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