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CLOSED CONVEX SETS THAT ARE BOTH MOTZKIN DECOMPOSABLE AND GENERALIZED MINKOWSKI SETS

JUAN ENRIQUE MARTÍNEZ LEGAZ[1,*], CORNEL PINTEA[2]

[1]*Department of Economics and Economic History, Universitat Autònoma de Barcelona and BGSMath, Spain*
[2]*Department of Mathematics, Babeş-Bolyai University, Cluj-Napoca, Romania*

Dedicated to the memory of Prof. Alfred Göpfert

Abstract. We consider and characterize closed convex subsets of the Euclidean space which are simultaneously Motzkin decomposable and generalized Minkowski or, shortly, *MdgM* sets. We also prove the existence of suitably defined fixed points for, possibly multivalued, functions defined on *MdgM* sets along with the existence of classical fixed points for some single valued self functions of *MdgM* sets. The first mentioned type of existence results are based on Kakutani fixed point theorem, and the second type are based both on the Brouwer fixed point theorem and the Banach contraction principle.

Keywords. Fixed points; Generalized Minkowski sets; Motzkin decompositions.

1. Introduction

We prove some fixed point theorems of Brouwer and Kakutani flavor for functions defined on some closed convex, possibly unbounded, sets. The suitable closed convex subsets of the Euclidean space, in this respect, are those that are simultaneously Motzkin decomposable and generalized Minkowski (*MdgM* sets shortly). These sets are proved to be Minkowski sums of compact convex sets and subspaces. We also provide suitable concepts of *fixed* and *strongly-fixed* points for functions defined on such sets along with existence results of such strongly-fixed and fixed points as well. The existence of classical fixed point results proved here is based on the Banach contraction principle for the subspace component.

The paper is divided into three sections. Section 2 is devoted to the characterizations of *MdgM* sets, it starts however with a characterization of Motzkin decomposable sets. In Section 3, the last section, we consider lin-fixed points for, possibly multivalued, functions defined on *MdgM* sets and strongly lin-fixed points for single valued self functions of *MdgM* sets. We prove, based on the Kakutani fixed point theorem, existence results of lin-fixed points for, possibly multi-valued, functions defined on *MdgM* sets. Existence results of classical fixed points are also proved for single-valued self functions of *MdgM* sets, via the Banach contraction principle applied to strongly lin-fixed points sequences.

*Corresponding author.
E-mail address: JuanEnrique.Martinez.Legaz@uab.cat (J. E. Martínez Legaz), cornel.pintea@ubbcluj.ro (C. Pintea).

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We use standard notions and notations in convex analysis. In particular, the recession cone and the linearity space of a nonempty closed convex set $C \subseteq \mathbb{R}^n$ are denoted by

$$0^+(C) := \{d \in \mathbb{R}^n : d + C \subseteq C\}$$

and

$$\text{lin } C := 0^+(C) \cap \left(-0^+(C)\right),$$

respectively; we refer to the classical book [10] for these and other notions from convex analysis. Throughout this paper, given a supplementary subspace $U \subseteq \mathbb{R}^n$ to lin C (i.e., a linear subspace such that $U \oplus \text{lin } C = \mathbb{R}^n$), we denote by $p_U : \mathbb{R}^n \to U$ and $p_{\text{lin}C} : \mathbb{R}^n \to \text{lin } C$ the projections defined by $x = p_U(x) + p_{\text{lin } C}(x)$.

2. *MdgM* Sets and Their Characterizations

In this section, we characterize the *MdgM* sets in various ways. These characterizations involve the total normal, the barrier and the recession cones. We start however with a characterization of the Motzkin decomposable sets, which is used afterwards. The faces of an *MdgM* set are shown to be *MdgM*, too. The class of *MdgM* sets is shown to be closed with respect to Minkowski sums and Cartesian products, although the classes of Motzkin decomposable sets and of generalized Minkowski sets are not.

Definition 2.1. [1] A set $C \subseteq \mathbb{R}^n$ is said to be Motzkin decomposable if there exist a compact convex set K and a closed convex cone D such that $C = K + D$.

Clearly, every Motzkin decomposable set is convex and closed. The term "Motzkin decomposable" is motivated by the fact that Motzkin [9] proved that the solution sets of finite sets of linear inequalities, that is, convex polyhedra, can be decomposed as sums of a convex polytope and a polyhedral convex cone.

The concept of Motzkin decomposability has been extended in several ways. For exampl, Iusem and Todorov [5] introduced the class of OM-decomposable sets, that is, the sets that can be decomposed as the sum of an open bounded convex set and a closed convex cone. It turns out that a set is OM-decomposable if and only if it is convex and open and its closure is Motzkin decomposable. Soltan [11] considered the class of M-polyhedral sets, which are the Minkowski sums of a compact convex set and a polyhedral cone or, in other words, the Motzkin decomposable sets that have a polyhedral recession cone. The stability properties of Motzkin decompositions was studied in [3].

The following result generalizes the first part of [2, Theorem 6], but the proof that we give here is of a different nature.

Theorem 2.1. *Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed, and convex set, and let $U \subseteq \mathbb{R}^n$ be a supplementary subspace to* lin C. *Then C is Motzkin decomposable if and only if $C \cap U$ is Motzkin decomposable. In such a case, every compact component of $C \cap U$ is a compact component of C, too.*

Proof. If $C \cap U$ is Motzkin decomposable, then $C \cap U = K + 0^+(C \cap U)$ for some compact set $K \subseteq \mathbb{R}^n$. Hence,

$$C = C \cap U + \text{lin } C = K + 0^+(C \cap U) + \text{lin } C, \tag{2.1}$$

which shows that C is Motzkin predecomposable, which, by [4, Corollary 9], means that C is actually Motzkin decomposable. In fact, using the closedness of C, we can easily derive from

(2.1) that $C = K + cl\left(0^+(C\cap U) + \text{lin } C\right)$, which shows that K is a compact component of C. Thus this proves the last part of the statement.

Assume now that C is Motzkin decomposable, that is, $C = K + 0^+(C)$ for a compact set $K \subseteq \mathbb{R}^n$. We prove that $C\cap U$ is Motzkin decomposable with compact component $(K + \text{lin } C)\cap U$. To see that the latter set is bounded, let $k \in K$ and $l \in \text{lin } C$ be such that $k + l \in U$. From the equality $k = k + l - l$, we obtain that $p_{\text{lin } C}(k) = -l$. Hence,

$$\|k+l\| \leq \|k\| + \|l\| = \|k\| + \|p_{\text{lin } C}(k)\| \leq \|k\| + \|p_{\text{lin } C}\| \, \|k\| = \left(1 + \|p_{\text{lin } C}\|\right)\|k\|.$$

Since K is bounded, we have that $(K + \text{lin } C)\cap U$ is bounded, too. Thus, to finish the proof, it only remains to observe that $C\cap U = (K + \text{lin } C)\cap U + 0^+(C)\cap U$. The inclusion $\supseteq$ in this equality is immediate, since $K + \text{lin } C + 0^+(C) = K + 0^+(C) = C$. To prove the opposite inclusion, let $c \in C\cap U$, and take $k \in K$ and $d \in 0^+(C)$ such that $c = k + d$. We then have

$$k - p_{\text{lin } C}(k) \in K + \text{lin } C, \; k - p_{\text{lin } C}(k) = p_U(k) \in U$$

and

$$\begin{aligned} p_{\text{lin } C}(k) + d &\in \text{lin } C + 0^+(C) = 0^+(C), \\ p_{\text{lin } C}(k) + d &= p_{\text{lin } C}(k) + c - k = c - p_U(k) \in U, \end{aligned}$$

so that $c = k - p_{\text{lin } C}(k) + p_{\text{lin } C}(k) + d \in (K + \text{lin } C)\cap U + 0^+(C)\cap U$. □

We also need the following result from [2, Theorem 6], and the proof that we give here is completely different and may have an interest in itself.

Theorem 2.2. *Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed, and convex set. If $0^+(C)$ is a linear subspace, then C is Motzkin decomposable.*

Proof. Since $0^+(C) = \text{lin } C$, we have $C = C\cap\left(0^+(C)\right)^\perp + 0^+(C)$. Therefore, to prove that C is Motzkin decomposable, it suffices to see that $C\cap\left(0^+(C)\right)^\perp$ is compact, but this follows from the fact that its recession cone reduces to $\{0\}$:

$$0^+\left(C\cap\left(0^+(C)\right)^\perp\right) = 0^+(C)\cap 0^+\left(\left(0^+(C)\right)^\perp\right) = 0^+(C)\cap\left(0^+(C)\right)^\perp = \{0\}.$$

□

To define the notion of *MdgM* set, we use the concept of generalized Minskowski set. We first recall the notion of Minkowski set.

Definition 2.2. [6] A nonempty, closed, and convex set $C \subseteq \mathbb{R}^n$ is said to be a Minkowski set if it is the convex hull of its extreme points.

The term ”Minkowski set” is due to the fact that it was Minkowski who proved [8] that every compact convex set is the convex hull of its extreme points.

Definition 2.3. [7] A nonempty, closed, and convex set $C \subseteq \mathbb{R}^n$ is said to be a generalized Minkowski set if it is the convex hull of its minimal faces.

Clearly, a Minkowski set is generalized Minkowski, since the minimal faces of a nonempty, closed, and convex set that has extreme points are precisely such extreme points.

Definition 2.4. We say that a nonempty, closed, and convex set $C \subseteq \mathbb{R}^n$ is a *MdgM* set if it is both Motzkin decomposable and a generalized Minkowski set.

The next theorem gives several characterizations of *MdgM* sets, two of them in terms of total normal cones and barrier cones. We recall that the total normal cone of a closed convex set C is

$$N_C(\mathbb{R}^n) := \bigcup_{x\in\mathbb{R}^n} N_C(x),$$

with $N_C(x)$ denoting the normal cone to C at x, defined by

$$N_C(x) := \begin{cases} \{x^* \in \mathbb{R}^n : \langle c - x, x^*\rangle \leq 0, \ \ \forall c \in C\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C, \end{cases}$$

and the barrier cone to C is

$$bar(C) := \left\{ x^* \in \mathbb{R}^n : \sup_{x\in C} \langle x, x^*\rangle < +\infty \right\}.$$

Theorem 2.3. *Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed, and convex set, and let $U \subseteq \mathbb{R}^n$ be a supplementary subspace to* lin C*. Then the following statements are equivalent:*

a) C is MdgM;

b) $C \cap U$ is compact;

c) there exist a compact (convex) set $K \subseteq \mathbb{R}^n$ and a linear subspace $L \subseteq \mathbb{R}^n$ such that $C = K + L$;

d) the total normal cone $N_C(\mathbb{R}^n)$ is a linear subspace;

e) the barrier cone $bar(C)$ is a linear subspace;

f) the recession cone $0^+(C)$ is a linear subspace.

Proof. a) $\Rightarrow$ b). By Theorem 2.1, set $C \cap U$ is Motzkin decomposable. Hence, by [2, Theorem 11], the set of its extreme points is bounded. It follows from [7, Theorem 17] that $C \cap U$ is Minkowski and $C \cap U$ is compact.

b) $\Rightarrow$ c) This is an immediate consequence of the Motzkin decomposition $C = C \cap U + \text{lin}\, C$.

c) $\Rightarrow$ d) By [7, Proposition 13], we have $N_C(\mathbb{R}^n) = (0^+(C))^0 = L^0 = L^\perp$.

d) $\Longrightarrow$ e) Since $ri(bar(C)) \subseteq N_C(\mathbb{R}^n) \subseteq bar(C)$ and $aff(ri(bar(C))) = aff(bar(C))$, we have $N_C(\mathbb{R}^n) \subseteq bar(C) \subseteq aff(bar(C)) = aff(ri(bar(C))) \subseteq aff(N_C(\mathbb{R}^n)) = N_C(\mathbb{R}^n)$. Thus $bar(C) = N_C(\mathbb{R}^n)$.

e) $\Longrightarrow$ f) We have $0^+(C) = (bar(C))^0 = (bar(C))^\perp$.

f) $\Rightarrow$ a) By Theorem 2.2, we have that C is Motzkin decomposable. Moreover, since compact convex sets are Minkowski, we have that C is generalized Minkwoski. □

Remark 2.1. From the equivalence a) $\Leftrightarrow$ f) in Theorem 2.3, it follows that a nonempty, closed, and convex set $C \subseteq \mathbb{R}^n$ is *MdgM* if and only if $0^+(C) = \text{lin}\, C$.

Corollary 2.1. *Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed, and convex set. Then the following statements are equivalent:*

a) C is MdgM;

b) $N_C(\mathbb{R}^n) = (\text{lin}\, C)^\perp$;

c) $N_C(\mathbb{R}^n) = (0^+(C))^\perp$;

d) $bar(C) = (0^+(C))^\perp$;

e) $bar(C) = (\text{lin}\, C)^\perp$;

Proof. a) $\Rightarrow$ b). Since C is Motzkin decomposable, we have $C = K + 0^+(C)$ for some compact set $K \subseteq \mathbb{R}^n$. By Theorem 2.3, implication a) $\Rightarrow$ f), the recession cone $0^+(C)$ is a linear subspace. Hence, by the proof of Theorem 2.3, implication c) $\Rightarrow$ d), we obtain b).

b) $\Rightarrow$ c) and d) $\Rightarrow$ e). By Theorem 2.3, implication d) $\Rightarrow$ f), the recession cone $0^+(C)$ is a linear subspace. Hence $\operatorname{lin} C = 0^+(C) \cap (-0^+(C)) = 0^+(C) \cap 0^+(C) = 0^+(C)$, which, together with b) (with d), yields c (e)).

c) $\Rightarrow$ d). By the proof of Theorem 2.3, implication d) $\Rightarrow$ e), we have $bar(C) = N_C(\mathbb{R}^n)$, which, together with c), yields d).

e) $\Rightarrow$ a). It is an immediate consequence of Theorem 2.3, implication e) $\Rightarrow$ a). □

Corollary 2.2. *Every face of a MdgM set is a MdgM set, too.*

Proof. Let F be a face of C. Using Theorem 2.3, equivalence (a) $\Leftrightarrow$ (f), and [7, Proposition 2.6], we obtain $0^+(F) \subseteq 0^+(C) = \operatorname{lin} C = \operatorname{lin} F \subseteq 0^+(F)$. Hence, $0^+(F) = \operatorname{lin} F$. Therefore, using again Theorem 2.3, equivalence (a) $\Leftrightarrow$ (f), we conclude that F is a *MdgM* set. □

Since the sum of closed convex cones is not necessarily closed, the class of Motzkin decomposable sets is not closed under addition if the dimension of the space is at least 2. On the other hand, the class of generalized Minkowski sets is not closed under addition either. Indeed, the sum of the Minkowski sets $\left\{(x,y) \in \mathbb{R}^2 : x > 0,\ y \geq \frac{1}{x}\right\}$ and $\left\{(x,y) \in \mathbb{R}^2 : x < 0,\ y \geq -\frac{1}{x}\right\}$ is the upper halfplane $\left\{(x,y) \in \mathbb{R}^2 : x < 0\right\}$, which is not even closed. This example also shows that the closure of the sum of two Minkowski sets need not be Minkowski either. However, the class of *MdgM* sets is closed under addition, as the following easy proposition states.

Proposition 2.1. *If the sets $C_i \subseteq \mathbb{R}^n$ $(i = 1,\dots,m)$ are MdgM, then their sum $\sum_{i=1}^m C_i$ is MdgM, too.*

Proof. By Theorem 2.3, implication a) $\Rightarrow$ c), for every $i = 1,\dots,m$, there exists a compact (convex) set $K_i \subseteq \mathbb{R}^n$ and a linear subspace $L_i \subseteq \mathbb{R}^n$ such that $C_i = K_i + L_i$. We then have $\sum_{i=1}^m C_i = \sum_{i=1C}^m K_i + \sum_{i=1}^m L_i$. Hence, using Theorem 2.3, implication c) $\Rightarrow$ a), we deduce that $\sum_{i=1}^m C_i$ is *MdgM*. □

Proposition 2.2. *Let $C_i \subseteq \mathbb{R}^{n_i}$ $(i = 1,\dots,m)$ be nonempty, closed, and convex sets. Then the Cartesian product $\prod_{i=1}^m C_i$ is a MdgM set if and only if C_i is a MdgM set for every $i = 1,\dots,m$.*

Proof. This is an immediate consequence of the obvious equality

$$0^+\left(\prod_{i=1}^m C_i\right) = \prod_{i=1}^m 0^+(C_i),$$

taking into account Theorem 2.3, equivalence (a) $\Leftrightarrow$ (f). □

3. Fixed Point Type Theorems on *MdgM* Sets

In this section, we start by introducing a suitable concept of so-called lin-fixed point for, possibly multivalued, functions defined on a *MdgM* set. Sufficient conditions on such functions are provided in order to obtain existence results of lin-fixed points. We also introduce the notion of strongly lin-fixed point for single valued self functions of *MdgM* sets

Definition 3.1. Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed, and convex set and $S \subseteq C$. We say that $x \in S$ is a lin-fixed point of $F : S \rightrightarrows C$ (of $f : S \longrightarrow C$) if $x \in F(x) + \text{lin}\, C$ (if $x \in f(x) + \text{lin}\, C$, respectively).

Note that $x \in S$ is a lin-fixed point of $F : S \rightrightarrows C$ (of $f : S \longrightarrow C$) if and only if $[x] \in [F(x)]$ (if and only if $[x] = [f(x)]$, respectively), with $[z]$ standing for the equivalence class $z + \text{lin}\, C$ of $z \in C$ in $C/\text{lin}\, C$ and $[F(x)] := \{[y] : y \in F(x)\}$. However, this is not equivalent to saying that $[x]$ is a fixed point for a certain self map on the quotient space $\mathbb{R}^n/\text{lin}\, C$, since, in general, F (f, respectively) does not induce any such map.

We recall that the graph of a set-valued mapping $F : X \rightrightarrows Y$ is the set

$$\textit{graph}\, F := \{(x,y) \in X \times Y : y \in F(x)\}.$$

Theorem 3.1. *Let $C \subseteq \mathbb{R}^n$ be a MdgM set and $U \subseteq \mathbb{R}^n$ be a supplementary subspace to* lin C*. If $F : C \cap U \rightrightarrows C$ has a closed graph and $F(x)$ is nonempty, compact, and convex for all $x \in C \cap U$, then it has a lin-fixed point.*

Proof. For $x \in C \cap U$, we define $(p_U \circ F)(x) := \{p_U(y) : y \in F(x)\}$. For every $y \in C$, we have $y - p_U(y) = p_{\text{lin}\, C}(y) \in \text{lin}\, C$. Hence $p_U(y) \in (y - \text{lin}\, C) \cap U \subseteq C \cap U$. Therefore, $(p_U \circ F)(x) \subseteq C \cap U$ and $p_U \circ F : C \cap U \rightrightarrows C \cap U$ is well defined. Using that F has compact images and a closed graph and p_U is continuous, one can easily prove that $p_U \circ F$ has a closed graph, too. On the other hand, observe that the values that F takes are nonempty and convex, so are the values that $p_U \circ F$ takes. Hence, as $C \cap U$ is compact (by Theorem 2.3, implication a) $\Rightarrow$ b)) and convex, by Kakutani fixed point theorem, there exists a point $x_0 \in C \cap U$ such that $x_0 \in (p_U \circ F)(x_0)$, that is, $x_0 = p_U(y)$ for some $y \in F(x_0)$. Therefore, $x_0 = p_U(y) = y - p_{\text{lin}\, C}(y) \in F(x_0) + \text{lin}\, C$. $\square$

Corollary 3.1. *Let $C \subseteq \mathbb{R}^n$ be a MdgM set and $U \subseteq \mathbb{R}^n$ be a supplementary subspace to* lin C*. If $f : C \cap U \longrightarrow C$ is continuous, then it has a lin-fixed point.*

The classical Kakutani and Brouwer fixed point theorems correspond to the particular cases of Theorem 3.1 and Corollary 3.1, respectively, when C is compact, since then C is a *MdgM* set and, as lin $C = \{0\}$, one necessarily has $U = \mathbb{R}^n$ so that the concepts of lin-fixed point reduce to that of fixed point for a self map.

Corollary 3.2. *Let $C \subseteq \mathbb{R}^n$ be a non compact MdgM set. If $F : C \cap U \rightrightarrows C$ has a closed graph and $F(x)$ is nonempty, compact, and convex for all $x \in C \cap U$, then it has infinitely many lin-fixed points.*

Proof. If $C = \mathbb{R}^n$, then every point in $\mathbb{R}^n$ is obviously a lin-fixed point of F, so we assume that $C \neq \mathbb{R}^n$ and, towards a contradiction, that the set $\Phi \subset C$ of lin-fixed points of F has finite cardinality p, say $\Phi = \{x_1, ..., x_p\}$. Applying a translation if necessary, we can assume that $0 \notin C$. Pick a supplementary subspace U to lin C. We claim that U can be chosen so as to be disjoint from Φ. Suppose it is not. Then, without loss of generality, we can write $U \cap \Phi = \{x_1, ..., x_k\}$ with $k \leq p$. Let $d := \dim \text{lin}\, C$. Since C is not compact, we have $d \geq 1$, and there are d linearly independent vectors $a_1, ..., a_d$ such that $U = \{x \in \mathbb{R}^n : \langle a_i, x\rangle = 0\ (i = 1, ..., d)\}$. For $\varepsilon > 0$, define $U_\varepsilon = \{x \in \mathbb{R}^n : \langle a_i + \varepsilon x_k, x\rangle = 0\ (i = 1, ..., d)\}$. It is easy to see that, for sufficientlt small ε, the subspace U_ε is still a supplementary subspace to lin C, and we are going to prove that we can choose it so as to have $U_\varepsilon \cap \Phi \subsetneq U \cap \Phi$. Indeed, since $x_k \neq 0$ (because $x_k \in \Phi$ and $0 \notin C \supset \Phi$), we clearly have $x_k \notin U_\varepsilon$. Moreover, in case that $k < p$, for $j = k+1, ..., p$, there exists $i_j \in \{1,...,d\}$ such that $\langle a_{i_j}, x_j \rangle \neq 0$ (since $x_j \notin U$), so that, choosing ε in such a way that $\varepsilon \left| \langle x_k, x_j \rangle \right| < \left| a_{i_j}, x_j \right|$ ($j \in \{k+1,...,p\}$), we clearly have $\langle a_{i_j} + \varepsilon x_k, x_j \rangle \neq 0$, which shows that $x_j \notin U_\varepsilon$ ($j = k+1,...,p$), and hence $U_\varepsilon \cap \Phi \subseteq \{x_1,...,x_{k-1}\}$. By repeating this type of construction, after at most $p-k$ steps we end up with a supplementary subspace U' to lin C disjoint with Φ. Then, by Theorem 3.1 applied to the restriction of F to $C \cap U'$, we obtain a lin-fixed point of F belonging to U', but this contradicts the fact that U' is disjoint from Φ. This proves that Φ cannot be finite. □

Corollary 3.3. *Let $C \subseteq \mathbb{R}^n$ be a non compact MdgM set. If $f : C \longrightarrow C$ is continuous, then it has infinitely many lin-fixed points.*

Our next proposition is useful to define the notion of strongly lin-fixed point.

Proposition 3.1. *Let $C \subseteq \mathbb{R}^n$ be a closed convex set and $f : C \longrightarrow C$. For $x \in C$, the following statements are equivalent:*

a) $f^k(x) \in x + \operatorname{lin} C$ for every $k \geq 0$ (with the convention that f^0 is the identity);

b) $f^k(x)$ is a lin-fixed point of f for every $k \geq 0$.

Proof. a) $\Rightarrow$ b). We have $f\left(f^k(x)\right) = f^{k+1}(x) \in x + \operatorname{lin} C = f^k(x) + \operatorname{lin} C$.

b) $\Rightarrow$ a) We proceed by induction. Statement a) for $k = 0$ is trivially satisfied. Let $k > 0$. Then, by b) and the induction hypothesis, we have $f^k(x) = f\left(f^{k-1}(x)\right) \in f^{k-1}(x) + \operatorname{lin} C = x + \operatorname{lin} C$. □

Unlike in the case of classical fixed points, in the case of lin-fixed points the image of one such point need not be a lin-fixed point. We next introduce the class of lin-fixed points whose images are lin-fixed points, too.

Definition 3.2. Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set. We say that $x \in C$ is a strongly lin-fixed point of $f : C \longrightarrow C$ if it satisfies the equivalent conditions a) and b) of Proposition 3.1.

The following proposition gives a simple sufficient condition for every lin-fixed point of f to be a strongly lin-fixed point

Proposition 3.2. *Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed, and convex set, and let $f : C \longrightarrow C$ be such that*

$$f(x + \operatorname{lin} C) \subseteq f(x) + \operatorname{lin} C, \ \forall x \in C. \tag{3.1}$$

Then every lin-fixed point of f is a strongly lin-fixed point of f.

Proof. Let x be a lin-fixed point of f. To prove by induction that it satisfies property b) of Proposition 3.1, it suffices to show that $f(x)$ is a lin-fixed point of f, but this immediately follows from the relations $f(f(x)) \in f(x + \operatorname{lin} C) \subseteq f(x) + \operatorname{lin} C$. □

Remark 3.1. The assumption (3.1) in Proposition 3.2 is not too strong. In fact, there is an easy way to generate a particular class of functions satisfying it: take any supplementary subspace U to lin C, arbitrary $g : C \cap U \to C$, $\tau : C \to \mathbb{R}$ and $h : \operatorname{lin} C \to \operatorname{lin} C$, and define $f = g \circ p_U + \tau \cdot (h \circ p_{\operatorname{lin} C})$. It is easy to see that f satisfies (3.1). Obviously, if we take g, τ and h continuous, then f is also continuous.

Proposition 3.3. *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set, $f : C \longrightarrow C$, and $U \subseteq \mathbb{R}^n$ be a supplementary subspace to* lin C*. Then (3.1) holds if and only if*

$$p_U \circ f \circ p_U = p_U \circ f. \tag{3.2}$$

Proof. If (3.1) holds, then, for every $x \in C$, we have

$$\begin{aligned}(p_U \circ f \circ p_U)(x) &= p_U(f(p_U(x))) = p_U(f(x - p_{\text{lin }C})) \\ &\in p_U(f(x + \text{lin }C)) \subseteq p_U(f(x) + \text{lin }C) \\ &= \{p_U(f(x))\}.\end{aligned}$$

Thus $(p_U \circ f \circ p_U)(x) = p_U(f(x)) = (p_U \circ f)(x)$.

Conversely, if (3.2) holds, then, for every $x \in C$ and $l \in \text{lin }C$, we have

$$f(x+l) = p_U(f(x+l)) + p_{\text{lin }C}(f(x+l)) \tag{3.3}$$

On the other hand, we have

$$\begin{aligned}p_U(f(x+l)) &= (p_U \circ f)(x+l) = (p_U \circ f \circ p_U)(x+l) \\ &= p_U(f(p_U(x+l))) = p_U(f(p_U(x))) \\ &= (p_U \circ f \circ p_U)(x) = (p_U \circ f)(x) = p_U(f(x)) \\ &= f(x) - p_{\text{lin }C}(f(x)).\end{aligned}$$

On combining this chain of equalities with(3.3), we obtain

$$f(x+l) = f(x) - p_{\text{lin }C}(f(x)) + p_{\text{lin }C}(f(x+l)) \in f(x) + \text{lin }C,$$

which proves (3.1). □

Remark 3.2. One can easily observe that Proposition 3.3 remains true if one replaces lin C with an arbitrary linear subspace both in its statement and in condition (3.1).

Theorem 3.2. *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set, and let $f : C \longrightarrow C$ be such that $p_{\text{lin }C} \circ f$ is a contraction. Then, for every strongly lin-fixed point x of f, the sequence $\left(f^k(x)\right)_k$ converges to a fixed point of f.*

Proof. For every $k \geq 1$, since $f^k(x)$ satisfies the property b) of Proposition 3.1, we have $f^{k+1}(x) \in f^k(x) + \text{lin }C$. Hence,

$$\begin{aligned}f^{k+1}(x) - f^k(x) &= p_{\text{lin }C}\left(f^{k+1}(x) - f^k(x)\right) \\ &= p_{\text{lin }C}\left(f^{k+1}(x)\right) - p_{\text{lin }C}\left(f^k(x)\right) \\ &= (p_{\text{lin }C} \circ f)\left(f^k(x)\right) - (p_{\text{lin }C} \circ f)\left(f^{k-1}(x)\right).\end{aligned}$$

Therefore, if $p_{\text{lin }C} \circ f$ is a contraction with constant $\alpha \in (0,1)$, then

$$\left\|f^{k+1}(x) - f^k(x)\right\| \leq \alpha \left\|f^k(x) - f^{k-1}(x)\right\|,$$

from which we can easily deduce that $\|f^{k+1}(x) - f^k(x)\| \leq \alpha^k \|f(x) - x\|$. Thus a standard argument shows that sequence $(f^k(x))_k$ is fundamental and therefore convergent. Its limit is, clearly, a fixed point of f. □

The classical Banach contraction principle, in the case of a function defined on the whole of $\mathbb{R}^n$, is an immediate consequence of Theorem 3.2. Indeed, since lin $\mathbb{R}^n = \mathbb{R}^n$, the mapping $p_{\text{lin }\mathbb{R}^n}$ is the identity, and hence $p_{\text{lin }\mathbb{R}^n} \circ f = f$; on the other hand, it is clear that every point in $\mathbb{R}^n$ is a strongly lin-fixed point of every function. Similarly, one can easily derive Banach principle from the following corollary, from which Brouwer fixed point theorem also follows in a straightforward way; one simply has to observe that (3.1) obviously holds for any function f both when C is compact and when $C = \mathbb{R}^n$.

Corollary 3.4. *Let $C \subseteq \mathbb{R}^n$ be a MdgM set, and let $f : C \longrightarrow C$ be a continuous mapping satisfying (3.1) and such that $p_{\text{lin } C} \circ f$ is a contraction. Then f has a fixed point.*

Proof. By Theorem 3.1, f has a lin-fixed point x. By Proposition 3.2, such a fixed point x is an strongly fixed point of f. Finally, by Theorem 3.2, the sequence $\left(f^k(x)\right)_k$ converges to a fixed point of f, and the statement follows via Theorem 3.1. □

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