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STABILITY ANALYSIS FOR A CONTAMINANT CONVECTION-REACTION-DIFFUSION MODEL OF RECOVERED FRACTURING FLUID

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Abstract. The aim of this paper is to study the stability analysis for a contaminant convection-reaction-diffusion model of the recovered fracturing fluid (RFFM, for short), which couples a nonlinear and nonsmooth stationary incompressible Navier-Stokes equation with a multivalued frictional boundary condition, and a nonlinear reaction-diffusion equation with mixed Neumann boundary conditions. First, we introduce a family of perturbation problems corresponding to (RFFM), and present the variational formulation of perturbation problem which is a perturbation elliptic hemivariational inequality driven by a perturbation nonlinear variational equation. Then, the existence of solutions and the uniform bound of the solution set to the perturbation problem are obtained. Finally, it is established that, as the perturbation parameter tends to zero, the solution set of the perturbation problems converge to the solution set of (RFFM) in the sense of the Kuratowski upper limit. This shows that (RFFM) is stable with respect to the perturbation data.

Keywords. Hemivariational inequality; Kuratowski upper limit; Mosco convergence; Navier-Stokes equation; Recovered fracturing fluid.

1. Introduction

Stability analysis of mathematical models has become an important problem in many fields, such as computation, control theory, and frame theory. The main motivation is that researchers can not expect to know the exact data of the problem in practical applications, so the stability analysis is a key step to evaluate whether a mathematical model is of a good quality. From the view-point of numerical approximation, the stability analysis is essential. Indeed, due to the

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perturbation of coefficients by different factors, stability analysis could help us to study numerical solutions. In the recent years, there have been efforts to establish the stability results for variational problems, optimal control problems, optimization problems, and so on. In particular, Han-Li [1] systematically analyzed the stability of a class of variational-hemivariational inequalities, including the continuous dependence of the solution on the data, and then provided the stability analysis of the solution of inequality problems in contact mechanics with respect to constitutive relations, external forces, constraints, and nonsmooth contact boundary conditions. Chen-Mar [2] proved the Lipschitz continuity of the optimal value, and closedness and upper semicontinuity of the optimal solution set in robust optimization problems with an uncertainty in the constraint. For the more results in this direction, one could refer to [3–8].

The hydraulic fracturing technology has an important role in increasing production of shale gas reservoir, for example, connecting artificial fractures with natural fractures and layered interfaces to form a large-scale fracture network, which is the main channel of shale gas production. In order to obtain the effective network volume and fracture complexity in shale gas reservoir, many scholars paid attention to the flow process of recovered fracturing fluid [9–14], since the flowback data of fracturing fluid carries the characteristic information about effective fracture network. Recently, in [15], the authors studied the flow behavior of the recovered fracturing fluid and the reaction-diffusion phenomenon of contaminants in the wellbore of shale gas reservoir, and applied various constitutive laws, diffusion principles, and friction relations to construct a contaminant convection-reaction-diffusion model of recovered fracturing fluid (recovered fracturing fluid model, RFFM, for short). More precisely, the model introduced in [15] was formulated by the following problem:

Problem 1.1. *Find a velocity field* $\boldsymbol{v}\colon \Omega \to \mathbb{R}^d$, *a pressure* $\pi\colon \Omega \to \mathbb{R}$, *and a concentration* $c\colon \Omega \to \mathbb{R}$ *such that*

$$-\operatorname{Div}\boldsymbol{C}(\boldsymbol{D}(\boldsymbol{v}))+\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\nabla\pi=\boldsymbol{f} \qquad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{v}=0 \qquad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{v}=\boldsymbol{0} \qquad \text{on } \Gamma_0, \tag{1.3}$$

$$\begin{cases} v_\nu=0, \\ -\boldsymbol{\tau}_\tau(\boldsymbol{v})\in\partial j(\boldsymbol{x},c,\boldsymbol{v}_\tau) \end{cases} \qquad \text{on } \Gamma_1, \tag{1.4}$$

$$\begin{cases} v_\nu+\rho\geq 0, \\ \tau_\nu(\boldsymbol{v},\pi)+\phi\geq 0, \\ (v_\nu+\rho)(\tau_\nu(\boldsymbol{v},\pi)+\phi)=0, \\ \boldsymbol{\tau}_\tau(\boldsymbol{v})=\boldsymbol{0}, \end{cases} \qquad \text{on } \Gamma_2, \tag{1.5}$$

$$\begin{cases} v_\nu\geq 0, \\ \tau_\nu(\boldsymbol{v},\pi)=-\varphi, \\ \boldsymbol{\tau}_\tau(\boldsymbol{v})=\boldsymbol{0}, \end{cases} \qquad \text{on } \Gamma_3, \tag{1.6}$$

and

$$\begin{cases} -\operatorname{div}\left(\kappa(\boldsymbol{v})\|\nabla c\|_{\mathbb{R}^d}^{p_2-2}\nabla c\right)+g(\boldsymbol{x},c)+\boldsymbol{v}\cdot\nabla c=0 & \text{in } \Omega, \\ \dfrac{\partial c}{\partial \nu_\kappa}:=\left(\kappa(\boldsymbol{v})\|\nabla c\|_{\mathbb{R}^d}^{p_2-2}\nabla c\right)\cdot\boldsymbol{\nu}=\omega\chi_{\Gamma_2\cup\Gamma_3} & \text{on } \Gamma. \end{cases} \tag{1.7}$$

Here, the boundary Γ of Ω is assumed to be divided into four disjoint and measurable parts Γ_i $(i=0,1,2,3)$ such that $\operatorname{meas}(\Gamma_0)>0$, $\boldsymbol{\nu}$ is the unit outward normal on the boundary Γ, $\chi_{\Gamma_2\cup\Gamma_3}$ is the characteristic function of $\Gamma_2\cup\Gamma_3$, $v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau=\boldsymbol{v}-v_\nu\boldsymbol{\nu}$ represent for the normal and tangential components of velocity field $\boldsymbol{v}$ on Γ, $\tau_\nu(\boldsymbol{v},p)=\boldsymbol{\tau}(\boldsymbol{v},p)\cdot\boldsymbol{\nu}$ and $\boldsymbol{\tau}_\tau(\boldsymbol{v})=\boldsymbol{\tau}(\boldsymbol{v},p)-\tau_\nu(\boldsymbol{v},p)\boldsymbol{\nu}$ are the normal and tangential components to traction vector field $\boldsymbol{\tau}$ on Γ.

The recovered fracturing fluid model couples a nonlinear and nonsmooth stationary incompressible Navier-Stokes equation with a multivalued frictional boundary condition, and a nonlinear reaction-diffusion equation with mixed Neumann boundary condition. In Problem 1.1, condition (1.3) indicates that $\boldsymbol{v}$ satisfies homogeneous Dirichlet condition on Γ_0; condition (1.4) models that there is no inflow and outflow of recovered fracturing fluid, and $\boldsymbol{\tau}_\tau(\boldsymbol{v})$ satisfies a multivalued and nonmonotone friction law on Γ_1; conditions (1.5) reflect that $\boldsymbol{v}$ satisfies a generalized Signorini-type contact condition on Γ_2, and $\rho\geq 0$ is a constant; condition (1.6) describes that the recovered fracturing fluid satisfies the outflow boundary condition on Γ_3; condition $(1.7)_2$ shows that the concentration c satisfies the nonhomogeneous Neumann boundary condition on $\Gamma_2\cup\Gamma_3$.

However, the measurement and acquisition of flowback data of fracturing fluid are often affected by various factors, which may lead to deviations in the identification of fracture parameters by flowback data. Therefore, it is necessary to study the stability analysis of the recovered fracturing fluid model. Essentially speaking, the stability results for the recovered fracturing fluid model could guarantee that the computational implementation of the model is not overly sensitive to possible round-off errors in the data. Based on these motivations, this paper is devoted to the stability of the recovered fracturing fluid model. To be precise, the main purpose of this paper is twofold. The first one is to consider a family of perturbation problems (see Problem 3.1) corresponding to Problem 1.1, and to obtain the existence of weak solutions for the perturbated problem. The second goal is to provide a stability result for Problem 1.1 by using the Mosco convergence approach and the theory of nonsmooth analysis, which reveals that the solution set of Problem 1.1 can be approached in the sense of the Kuratowski upper limit by the solution set of Problem 3.1 when the perturbation parameter ε tends to zero.

This paper is organized as follows. In Section 2, we recall some preliminaries, and present the existence of weak solutions to the recovered fracturing fluid model, Problem 1.1. In Section 3, we introduce a family of perturbation problems (see Problem 3.1) corresponding to Problem 1.1, and deliver its variational formulation, which is a coupled system consisting of an elliptic hemivariational inequality and a nonlinear variational equation with a perturbation parameter ε. Finally, the existence of weak solutions to Problem 3.1 and a stability result to Problem 1.1 are established in Section 4.

2. Preliminaries

In this section, we recall some necessary notations, basic definitions, and a result on the solvability to Problem 1.1 which were recently proved in [15].

Given a normed space X, we denote by $\|\cdot\|_X$ and X^* the norm and its topological dual of X, respectively. In the sequel, we utilize the symbol $\langle\cdot,\cdot\rangle_{X^*\times X}$ to stand for the duality pairing between X^* and X. If no confusion arises, we often skip the subscript. The weak and the strong convergences in X are denoted by "$\rightharpoonup$" and "$\to$", respectively. Furthermore, by $\mathscr{L}(X_1,X_2)$, we denote the space of linear and bounded operators from a normed space X_1 to a normed space X_2 endowed with the operator norm $\|\cdot\|_{\mathscr{L}(X_1,X_2)}$.

Let us recall the definitions concerning the generalized directional derivative and generalized gradient in the sense of Clarke for a locally Lipschitz function; see, e.g., [16–19].

Definition 2.1. Let $J\colon X\to\mathbb{R}$ be a locally Lipschitz function defined on a Banach space X. We define the generalized Clarke directional derivative of J at the point $u\in X$ in the direction $v\in X$ by

$$J^0(u;v)=\limsup_{\lambda\to0^+,\,w\to u}\frac{J(w+\lambda v)-J(w)}{\lambda}.$$

The generalized Clarke subgradient of J at $u\in X$ is a subset in the dual space X^* given by

$$\partial J(u)=\left\{\,\xi\in X^*\mid J^0(u;v)\ge\langle\xi,v\rangle\ \text{ for all } v\in X\,\right\}.$$

Some important properties of the generalized directional derivative and generalized subgradient in the sense of Clarke are selected by the following proposition; see [20, Proposition 3.23].

Proposition 2.1. *Assume that $J\colon X\to\mathbb{R}$ is a locally Lipschitz function. Then the following assertions hold:*

(i) *for every $u\in X$, the function $X\ni v\mapsto J^0(u;v)\in\mathbb{R}$ is positively homogeneous and subadditive, i.e.,*

$$J^0(u;\lambda v)=\lambda J^0(u;v)\ \text{ and }\ J^0(u;v_1+v_2)\le J^0(u;v_1)+J^0(u;v_2)$$

for all $\lambda\ge0$ and $v,u,v_1,v_2\in X$;

(ii) *for each $v\in X$, we have $J^0(u;v)=\max\{\,\langle\xi,v\rangle\mid\xi\in\partial J(u)\,\}$;*

(iii) *the function $X\times X\ni(u,v)\mapsto J^0(u;v)\in\mathbb{R}$ is upper semicontinuous.*

We review the definition of Mosco convergence; see, e.g., [18, Chapter 4.7] and [21].

Definition 2.2. Let X be a Banach space and $\{K_\varepsilon,K\}_{\varepsilon>0}\subset 2^X\setminus\{\emptyset\}$. We say that K_ε converges to K in the sense of Mosco as $\varepsilon\to0$, denoted by $K_\varepsilon\xrightarrow{M}K$, if and only if the following conditions hold:

(i) for each $u\in K$, there exists a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ such that $u_\varepsilon\in K_\varepsilon$ for every $\varepsilon>0$ and $u_\varepsilon\to u$ in X;

(ii) for each sequence $\{u_\varepsilon\}_{\varepsilon>0}$ such that $u_\varepsilon\in K_\varepsilon$ for every $\varepsilon>0$ and $u_\varepsilon\rightharpoonup u$ in X, one has $u\in K$.

To give a result on existence of weak solutions to Problem 1.1, we consider the following spaces

$$\mathscr{E}=\left\{\boldsymbol{v}\in C^\infty(\overline{\Omega};\mathbb{R}^d)\mid\nabla\cdot\boldsymbol{v}=0\text{ in }\Omega,\ \boldsymbol{v}=\boldsymbol{0}\text{ on }\Gamma_0\text{ and }v_\nu=0\text{ on }\Gamma_1\right\},$$

$$E=\text{the closure of }\mathscr{E}\text{ in }W^{1,p_1}(\Omega;\mathbb{R}^d),$$

where $p_1 \geq 2$. The space E is equipped with the standard norm $\|\boldsymbol{v}\| = \|\boldsymbol{v}\|_{W^{1,p_1}(\Omega;\mathbb{R}^d)}$ for $\boldsymbol{v} \in E$, and then it becomes a separable and reflexive Banach space. Invoking Korn's inequality (see [22]), combined with positive measure of Γ_0, we can find a constant $c_K > 0$ such that

$$c_K\|\boldsymbol{v}\|_{W^{1,p_1}(\Omega;\mathbb{R}^d)} \leq \|\boldsymbol{D}(\boldsymbol{v})\|_{L^{p_1}(\Omega;\mathbb{S}^d)} \text{ for all } \boldsymbol{v} \in E,$$

where $\boldsymbol{D}(\boldsymbol{v})$ denotes the deformation tensor of $\boldsymbol{v}$, which means that $\|\boldsymbol{v}\|_E := \|\boldsymbol{D}(\boldsymbol{v})\|_{L^{p_1}(\Omega;\mathbb{S}^d)}$ for $\boldsymbol{v} \in E$ is an equivalent norm of E. Therefore, in what follows, we use $\|\cdot\|_E$ as the norm in the space E, and define the duality brackets for E^* and E as follows

$$\langle \boldsymbol{v}, \boldsymbol{w}\rangle := \int_\Omega \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{w})\, d\boldsymbol{x} \text{ for all } \boldsymbol{v}, \boldsymbol{w} \in E.$$

We introduce the trace operator $\gamma\colon E \to L^{p_1}(\Gamma;\mathbb{R}^d)$, which is continuous and compact; see, e.g., [20, Theorem 2.21]. Its norm is denoted by $\|\gamma\| = \|\gamma\|_{\mathscr{L}(E,L^{p_1}(\Gamma;\mathbb{R}^d))}$. Moreover, we need the admissible set $K \subset E$ of velocity field $\boldsymbol{v}$ given by

$$K := \{\boldsymbol{v} \in E \mid v_\nu \geq -\rho \text{ on } \Gamma_2 \text{ and } v_\nu \geq 0 \text{ on } \Gamma_3\}. \tag{2.1}$$

It should be noted that since $\rho \geq 0$ and $\boldsymbol{0} \in K$, then K is a nonempty, closed, and convex subset of E.

Next, we make the following hypotheses on the data of Problem 1.1.

$\underline{H(\boldsymbol{C})}$: There exists a function $G\colon \Omega \times \mathbb{S}^d \to \mathbb{R}$ such that $\boldsymbol{C}(\boldsymbol{x}, \boldsymbol{D}) = \nabla_D G(\boldsymbol{x}, \boldsymbol{D})$ for all $\boldsymbol{D} \in \mathbb{S}^d$ and a.e. $\boldsymbol{x} \in \Omega$, and the following conditions hold:

(i) $G(\cdot, \boldsymbol{D})$ is measurable in Ω for all $\boldsymbol{D} \in \mathbb{S}^d$;

(ii) $G(\boldsymbol{x}, \cdot)$ is continuously differentiable (i.e., C^1) and strictly convex on $\mathbb{S}^d$ for a.e. $\boldsymbol{x} \in \Omega$ with $G(\boldsymbol{x}, \boldsymbol{0}_{\mathbb{S}^d})$ belonging to $L^1(\Omega)$;

(iii) there exist a function $a_C \in L^{p_1'}(\Omega)_+$ with $p_1 \geq 2$ and a constant $b_C > 0$ satisfying

$$\|\boldsymbol{C}(\boldsymbol{x}, \boldsymbol{D})\|_{\mathbb{S}^d} = \|\nabla_D G(\boldsymbol{x}, \boldsymbol{D})\|_{\mathbb{S}^d} \leq a_C(\boldsymbol{x}) + b_C\|\boldsymbol{D}\|_{\mathbb{S}^d}^{p_1-1}$$

for all $\boldsymbol{D} \in \mathbb{S}^d$ and a.e. $\boldsymbol{x} \in \Omega$;

(iv) the inequality $G(\boldsymbol{x}, \boldsymbol{D}) \geq c_C\|\boldsymbol{D}\|_{\mathbb{S}^d}^{p_1} + d_C(\boldsymbol{x})$ holds for a.e. $\boldsymbol{x} \in \Omega$ and for all $\boldsymbol{D} \in \mathbb{S}^d$ with $d_C \in L^1(\Omega)$ and $c_C > 0$.

$\underline{H(\boldsymbol{f})}$: $\boldsymbol{f} \in L^{p_1'}(\Omega;\mathbb{R}^d)$.

$\underline{H(j)}$: $j\colon \Gamma_1 \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $j(\cdot, s, \boldsymbol{\xi})$ is measurable on Γ_1 for all $s \in \mathbb{R}$ and $\boldsymbol{\xi} \in \mathbb{R}^d$, and $j(\cdot, s, \boldsymbol{0}_{\mathbb{R}^d}) \in L^1(\Gamma_1)$ for all $s \in \mathbb{R}$;

(ii) $j(\boldsymbol{x}, s, \cdot)$ is locally Lipschitz for a.e. $\boldsymbol{x} \in \Gamma_1$ and for all $s \in \mathbb{R}$;

(iii) there exist a function $a_j \in L^{p_1'}(\Gamma_1)_+$ and a constant $b_j \geq 0$ such that

$$\|\partial j(\boldsymbol{x}, s, \boldsymbol{\xi})\|_{\mathbb{R}^d} \leq a_j(\boldsymbol{x}) + b_j\|\boldsymbol{\xi}\|_{\mathbb{R}^d}^{p_1-1} \text{ for a.e. } \boldsymbol{x} \in \Gamma_1, \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d \text{ and } s \in \mathbb{R},$$

where ∂j stands for the generalized Clarke subgradient operator of j with respect to its last variable;

(iv) $\mathbb{R}^{2d+1} \ni (s, \boldsymbol{\xi}, \boldsymbol{\zeta}) \mapsto j^0(\boldsymbol{x}, s, \boldsymbol{\xi}; \boldsymbol{\zeta}) \in \mathbb{R}$ is upper semicontinuous for a.e. $\boldsymbol{x} \in \Gamma_1$, where $j^0(\boldsymbol{x}, s, \boldsymbol{\xi}; \boldsymbol{\zeta})$ is the generalized Clarke directional derivative of $\mathbb{R}^d \ni \boldsymbol{\xi} \mapsto j(\boldsymbol{x}, s, \boldsymbol{\xi}) \in \mathbb{R}$;

(v) either $j(\boldsymbol{x}, s, \cdot)$ or $-j(\boldsymbol{x}, s, \cdot)$ is regular for a.e. $\boldsymbol{x} \in \Gamma_1$ and all $s \in \mathbb{R}$.

$\underline{H(\kappa)}$: $\kappa\colon \Omega\times\mathbb{R}^d\to(0,\infty)$ is such that

(i) $\kappa(\cdot,\boldsymbol{u})$ is measurable on Ω for all $\boldsymbol{u}\in\mathbb{R}^d$;
(ii) $\kappa(\boldsymbol{x},\cdot)$ is continuous on $\mathbb{R}^d$ for a.e. $\boldsymbol{x}\in\Omega$;
(iii) there exist constants $a_\kappa, b_\kappa>0$ such that $0<a_\kappa\le\kappa(\boldsymbol{x},\boldsymbol{u})\le b_\kappa$ for a.e. $\boldsymbol{x}\in\Omega$ and all $\boldsymbol{u}\in\mathbb{R}^d$.

$\underline{H(g)}$: $g\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is such that

(i) $g(\cdot,s)$ is measurable on Ω for all $s\in\mathbb{R}$;
(ii) $g(\boldsymbol{x},\cdot)$ is continuous on $\mathbb{R}$ for a.e. $\boldsymbol{x}\in\Omega$;
(iii) there exist a function $a_g\in L^{p_2'}(\Omega)_+$ and a constant $b_g>0$ such that
$$|g(\boldsymbol{x},s)|\le a_g(\boldsymbol{x})+b_g|s|^{p_2^*-1}\ \text{ for a.e. } \boldsymbol{x}\in\Omega \text{ and all } s\in\mathbb{R},$$
where p_2^* represents the critical exponent of p_2;
(iv) there exist a function $c_g\in L^1(\Omega)$ and a constant $d_g>0$ such that
$$g(\boldsymbol{x},s)s\ge c_g(\boldsymbol{x})+d_g|s|^\theta\ \text{ for a.e. } \boldsymbol{x}\in\Omega \text{ and all } s\in\mathbb{R}$$
with $\theta\ge p_2$.

$\underline{H(\omega)}$: $\omega\in L^{p_2'}(\Gamma_2\cup\Gamma_3)$, $\phi\in L^{p_1'}(\Gamma_2)$, $\varphi\in L^{p_1'}(\Gamma_3)$ and $\rho\ge 0$.

Under hypotheses $H(\boldsymbol{C})$, $H(j)$, $H(\kappa)$, $H(\boldsymbol{f})$, $H(g)$, and $H(\omega)$, the authors in [15, Theorem 5.1] applied a surjectivity theorem for multivalued operators together with an alternative iterative method and the theory of nonsmooth analysis to establish the following existence theorem for the recovered fracturing fluid model (Problem 1.1).

Theorem 2.1. *Assume that $H(\boldsymbol{C})$, $H(j)$, $H(\kappa)$, $H(\boldsymbol{f})$, $H(g)$, and $H(\omega)$ hold. If, in addition, the inequalities*
$$c_C-b_j\|\gamma\|^{p_1}-\delta(p_1)\frac{\rho}{2}\|\gamma\|^2>0\quad\text{and}\quad \min\{a_\kappa,d_g\delta(\theta)\}-\delta(p_2)\frac{\rho}{2}\|\gamma_1\|^2>0,$$
are satisfied, then the recovered fracturing fluid model, Problem 1.1, has at least one weak solution $(\boldsymbol{v},c)\in K\times W^{1,p_2}(\Omega)$, where $\gamma\colon E\to L^{p_1}(\Gamma;\mathbb{R}^d)$, and $\gamma_1: W^{1,p_2}(\Omega)\to L^2(\Gamma)$ are trace operators, and $\delta\colon(0,+\infty)\to\{0,1\}$ is defined by
$$\delta(s)=\begin{cases}1, & \text{if } s=2,\\ 0, & \text{otherwise.}\end{cases}$$

Remark 2.1. It should be pointed out that $(\boldsymbol{v},c)\in K\times W^{1,p_2}(\Omega)$ is a weak solution to the recovered fracturing fluid model, Problem 1.1, if $(\boldsymbol{v},c)$ satisfies the following coupled system
$$\int_\Omega \boldsymbol{C}(\boldsymbol{D}(\boldsymbol{v})):\boldsymbol{D}(\boldsymbol{w}-\boldsymbol{v})\,d\boldsymbol{x}+\int_\Omega(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot(\boldsymbol{w}-\boldsymbol{v})\,d\boldsymbol{x}+\int_{\Gamma_1}j^0(\boldsymbol{x},c,\boldsymbol{v}_\tau;\boldsymbol{w}_\tau-\boldsymbol{v}_\tau)\,d\Gamma$$
$$+\int_{\Gamma_2}\phi(w_\nu-v_\nu)\,d\Gamma+\int_{\Gamma_3}\varphi(w_\nu-v_\nu)\,d\Gamma\ge\int_\Omega\boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{v})\,d\boldsymbol{x}\ \text{ for all }\ \boldsymbol{w}\in K,$$
and
$$\int_\Omega\kappa(\boldsymbol{v})\|\nabla c\|_{\mathbb{R}^d}^{p_2-2}\nabla c\cdot\nabla z\,d\boldsymbol{x}+\int_\Omega g(\boldsymbol{x},c)z\,d\boldsymbol{x}+\int_\Omega(\boldsymbol{v}\cdot\nabla c)\cdot z\,d\boldsymbol{x}=\int_{\Gamma_2\cup\Gamma_3}\omega z\,d\Gamma$$
for all $z\in W^{1,p_2}(\Omega)$.

3. A PERTURBATION SYSTEM

The goal of this section is to introduce a family of perturbation problems corresponding to the recovered fracturing fluid model (Problem 1.1), and to deliver its variational formulation.

Let $\varepsilon > 0$ be a given perturbation parameter. We denote by $\boldsymbol{C}_\varepsilon$, $\boldsymbol{f}_\varepsilon$, j_ε, ρ_ε, ϕ_ε, φ_ε, κ_ε, g_ε, and ω_ε, the corresponding perturbed functions of $\boldsymbol{C}$, $\boldsymbol{f}$, j, ρ, ϕ, φ, κ, g, and ω, respectively. We consider the following perturbed recovered fracturing fluid model.

Problem 3.1. *Find a velocity field $\boldsymbol{v}_\varepsilon \colon \Omega \to \mathbb{R}^d$, a pressure $\pi_\varepsilon \colon \Omega \to \mathbb{R}$, and a concentration $c_\varepsilon \colon \Omega \to \mathbb{R}$ such that*

$$-\operatorname{Div} \boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}_\varepsilon)) + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon + \nabla \pi_\varepsilon = \boldsymbol{f}_\varepsilon \quad \text{in } \Omega, \tag{3.1}$$

$$\nabla \cdot \boldsymbol{v}_\varepsilon = 0 \quad \text{in } \Omega, \tag{3.2}$$

$$\boldsymbol{v}_\varepsilon = \boldsymbol{0} \quad \text{on } \Gamma_0, \tag{3.3}$$

$$\begin{cases} v_{\varepsilon_\nu} = 0, \\ -\boldsymbol{\tau}_{\varepsilon_\tau}(\boldsymbol{v}_\varepsilon) \in \partial j_\varepsilon(\boldsymbol{x}, c_\varepsilon, \boldsymbol{v}_{\varepsilon_\tau}), \end{cases} \quad \text{on } \Gamma_1, \tag{3.4}$$

$$\begin{cases} v_{\varepsilon_\nu} + \rho_\varepsilon \geq 0, \\ \tau_{\varepsilon_\nu}(\boldsymbol{v}_\varepsilon, \pi_\varepsilon) + \phi_\varepsilon \geq 0, \\ (v_{\varepsilon_\nu} + \rho_\varepsilon)(\tau_{\varepsilon_\nu}(\boldsymbol{v}_\varepsilon, \pi_\varepsilon) + \phi_\varepsilon) = 0, \\ \boldsymbol{\tau}_{\varepsilon_\tau}(\boldsymbol{v}_\varepsilon) = \boldsymbol{0}, \end{cases} \quad \text{on } \Gamma_2, \tag{3.5}$$

$$\begin{cases} v_{\varepsilon_\nu} \geq 0, \\ \tau_{\varepsilon_\nu}(\boldsymbol{v}_\varepsilon, \pi_\varepsilon) = -\varphi_\varepsilon, \\ \boldsymbol{\tau}_{\varepsilon_\tau}(\boldsymbol{v}_\varepsilon) = \boldsymbol{0}, \end{cases} \quad \text{on } \Gamma_3, \tag{3.6}$$

and

$$\begin{cases} -\operatorname{div}(\kappa_\varepsilon(\boldsymbol{v}_\varepsilon) \|\nabla c_\varepsilon\|_{\mathbb{R}^d}^{p_2-2} \nabla c_\varepsilon) + g_\varepsilon(\boldsymbol{x}, c_\varepsilon) + \boldsymbol{v}_\varepsilon \cdot \nabla c_\varepsilon = 0 & \text{in } \Omega, \\ \dfrac{\partial c_\varepsilon}{\partial \nu_{\kappa_\varepsilon}} := \left(\kappa_\varepsilon(\boldsymbol{v}_\varepsilon) \|\nabla c_\varepsilon\|_{\mathbb{R}^d}^{p_2-2} \nabla c_\varepsilon\right) \cdot \boldsymbol{\nu} = \omega_\varepsilon \chi_{\Gamma_2 \cup \Gamma_3} & \text{on } \Gamma. \end{cases} \tag{3.7}$$

We suppose that $\boldsymbol{P}$ is the total stress tensor on the boundary Γ. Then $\boldsymbol{\tau}(\boldsymbol{v}, \pi) := \boldsymbol{P} \cdot \boldsymbol{\nu}$ represents for the traction vector of total stress tensor $\boldsymbol{P}$ on the boundary Γ, and $\boldsymbol{P}$ satisfies the following identity (see equation (3.5) in [15]) $\boldsymbol{P} = -\pi \boldsymbol{I} + \boldsymbol{C}(\boldsymbol{D}(\boldsymbol{v}))$. The above equation combined with $\tau_\nu(\boldsymbol{v}, \pi) = \boldsymbol{\tau}(\boldsymbol{v}, \pi) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\tau}_\tau(\boldsymbol{v}) = \boldsymbol{\tau}(\boldsymbol{v}, \pi) - \tau_\nu(\boldsymbol{v}, \pi)\boldsymbol{\nu}$ implies

$$\tau_\nu(\boldsymbol{v}, \pi) = C(\boldsymbol{D}(\boldsymbol{v}))_\nu - \pi \ \text{ and } \ \boldsymbol{\tau}_\tau(\boldsymbol{v}) = \boldsymbol{C}(\boldsymbol{D}(\boldsymbol{v}))_\tau \quad \text{on } \Gamma,$$

where $C(\boldsymbol{D}(\boldsymbol{v}))_\nu := (\boldsymbol{C}(\boldsymbol{D}(\boldsymbol{v})) \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{C}(\boldsymbol{D}(\boldsymbol{v}))_\tau := \boldsymbol{C}(\boldsymbol{D}(\boldsymbol{v}))\boldsymbol{\nu} - C(\boldsymbol{D}(\boldsymbol{v}))_\nu \boldsymbol{\nu}$. Hence, for Problem 3.1, one has

$$\tau_{\varepsilon_\nu}(\boldsymbol{v}, \pi) = C_\varepsilon(\boldsymbol{D}(\boldsymbol{v}))_\nu - \pi \ \text{ and } \ \boldsymbol{\tau}_{\varepsilon_\tau}(\boldsymbol{v}) = \boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}))_\tau \quad \text{on } \Gamma,$$

where $C_\varepsilon(\boldsymbol{D}(\boldsymbol{v}))_\nu := (\boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v})) \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}))_\tau := \boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}))\boldsymbol{\nu} - C_\varepsilon(\boldsymbol{D}(\boldsymbol{v}))_\nu \boldsymbol{\nu}$.

We now introduce appropriate sets of hypotheses.

$\underline{H(\boldsymbol{C}_\varepsilon)}$: There exists a function $G_\varepsilon \colon \Omega \times \mathbb{S}^d \to \mathbb{R}$ such that $\boldsymbol{C}_\varepsilon(\boldsymbol{x}, \boldsymbol{D}) = \nabla_D G_\varepsilon(\boldsymbol{x}, \boldsymbol{D})$ for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{D} \in \mathbb{S}^d$, and the following conditions hold:

(i) $G_\varepsilon(\cdot,\boldsymbol{D})$ is measurable on Ω for all $\boldsymbol{D}\in\mathbb{S}^d$;

(ii) $G_\varepsilon(\boldsymbol{x},\cdot)$ is continuously differentiable (i.e., C^1) and strictly convex on $\mathbb{S}^d$ for a.e. $\boldsymbol{x}\in\Omega$, $G_\varepsilon(\boldsymbol{x},\boldsymbol{0}_{\mathbb{S}^d})$ belongs to $L^1(\Omega)$ and is uniformly bounded with respect to parameter ε;

(iii) there exist a function $a_{C_\varepsilon}\in L^{p_1'}(\Omega)_+$ with $p_1\geq 2$ and a constant $b_{C_\varepsilon}>0$ satisfying

$$\|\boldsymbol{C}_\varepsilon(\boldsymbol{x},\boldsymbol{D})\|_{\mathbb{S}^d}=\|\nabla_D G_\varepsilon(\boldsymbol{x},\boldsymbol{D})\|_{\mathbb{S}^d}\leq a_{C_\varepsilon}(\boldsymbol{x})+b_{C_\varepsilon}\|\boldsymbol{D}\|_{\mathbb{S}^d}^{p_1-1}$$

for a.e. $\boldsymbol{x}\in\Omega$ and all $\boldsymbol{D}\in\mathbb{S}^d$, where $\{a_{C_\varepsilon}\}_{\varepsilon>0}\subset L^{p_1'}(\Omega)_+$ and $\{b_{C_\varepsilon}\}_{\varepsilon>0}\subset(0,+\infty)$ are uniformly bounded with respect to parameter ε;

(iv) the inequality

$$G_\varepsilon(\boldsymbol{x},\boldsymbol{D})\geq c_{C_\varepsilon}\|\boldsymbol{D}\|_{\mathbb{S}^d}^{p_1}+d_{C_\varepsilon}(\boldsymbol{x})$$

holds for a.e. $\boldsymbol{x}\in\Omega$ and all $\boldsymbol{D}\in\mathbb{S}^d$ with $c_{C_\varepsilon}>0$ and $d_{C_\varepsilon}\in L^1(\Omega)$, where $\{d_{C_\varepsilon}\}_{\varepsilon>0}\subset L^1(\Omega)$ and $\{c_{C_\varepsilon}\}_{\varepsilon>0}\subset(0,+\infty)$ are uniformly bounded with respect to parameter ε;

(v) there exist a sequence $\{\alpha_\varepsilon\}\subset(0,+\infty)$ which depends on parameter ε and two nonnegative functions $h_1\colon\mathbb{S}^d\to[0,+\infty)$, $h_2\colon\mathbb{S}^d\times\mathbb{S}^d\to[0,+\infty)$ such that $\alpha_\varepsilon\to 0$ for $\varepsilon\to 0$ and

$$\|\boldsymbol{C}_\varepsilon(\boldsymbol{x},\boldsymbol{D}_1)-\boldsymbol{C}(\boldsymbol{x},\boldsymbol{D}_2)\|_{\mathbb{S}^d}\leq m_C(\alpha_\varepsilon h_1(\boldsymbol{D}_1)+h_2(\boldsymbol{D}_1,\boldsymbol{D}_2))$$

for a.e. $\boldsymbol{x}\in\Omega$ and all $\boldsymbol{D}_1$, $\boldsymbol{D}_2\in\mathbb{S}^d$, where $m_C>0$ and the functions h_1,h_2 satisfy the following conditions

$$h_1(\boldsymbol{D}_1)\leq c_{h_1}+d_{h_1}\|\boldsymbol{D}_1\|_{\mathbb{S}^d}^{p_1-1}\ \text{ and }\ h_2(\boldsymbol{D}_1,\boldsymbol{D}_2)\leq c_{h_2}+d_{h_2}(\|\boldsymbol{D}_1\|_{\mathbb{S}^d}^{p_1-1}+\|\boldsymbol{D}_2\|_{\mathbb{S}^d}^{p_1-1})$$

such that $h_2(\boldsymbol{D}_1,\boldsymbol{D}_1)=0$ and $c_{h_1},c_{h_2},d_{h_1},d_{h_2}>0$.

$\underline{H(\boldsymbol{f}_\varepsilon)}$: $\boldsymbol{f}_\varepsilon\in L^{p_1'}(\Omega;\mathbb{R}^d)$, and $\boldsymbol{f}_\varepsilon\to\boldsymbol{f}$ in $L^{p_1'}(\Omega;\mathbb{R}^d)$ as $\varepsilon\to 0$.

$\underline{H(j_\varepsilon)}$: $j_\varepsilon\colon\Gamma_1\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ is such that

(i) $j_\varepsilon(\cdot,s,\boldsymbol{\xi})$ is measurable on Γ_1 for all $s\in\mathbb{R}$ and $\boldsymbol{\xi}\in\mathbb{R}^d$, $j_\varepsilon(\cdot,s,\boldsymbol{0})\in L^1(\Gamma_1)$ for all $s\in\mathbb{R}$;

(ii) $j_\varepsilon(\boldsymbol{x},s,\cdot)$ is locally Lipschitz for a.e. $\boldsymbol{x}\in\Gamma_1$ and all $s\in\mathbb{R}$;

(iii) there exist a function $a_{j_\varepsilon}\in L^{p_1'}(\Gamma_1)_+$ and a constant $b_{j_\varepsilon}\geq 0$ such that

$$\|\partial j_\varepsilon(\boldsymbol{x},s,\boldsymbol{\xi})\|_{\mathbb{R}^d}\leq a_{j_\varepsilon}(\boldsymbol{x})+b_{j_\varepsilon}\|\boldsymbol{\xi}\|_{\mathbb{R}^d}^{p_1-1}\ \text{ for a.e. } \boldsymbol{x}\in\Gamma_1,\ \text{for all } \boldsymbol{\xi}\in\mathbb{R}^d \text{ and all } s\in\mathbb{R},$$

where ∂j_ε stands for the generalized Clarke subgradient of j_ε with respect to its last variable, $\{a_{j_\varepsilon}\}_{\varepsilon>0}\subset L^{p_1'}(\Gamma_1)_+$ and $\{b_{j_\varepsilon}\}_{\varepsilon>0}\subset(0,+\infty)$ are uniformly bounded with respect to parameter ε;

(iv) $\mathbb{R}^{2d+1}\ni(s,\boldsymbol{\xi},\boldsymbol{\zeta})\mapsto j_\varepsilon^0(\boldsymbol{x},s,\boldsymbol{\xi};\boldsymbol{\zeta})\in\mathbb{R}$ is upper semicontinuous for a.e. $\boldsymbol{x}\in\Gamma_1$, where $j_\varepsilon^0(\boldsymbol{x},s,\boldsymbol{\xi};\boldsymbol{\zeta})$ is the generalized Clarke directional derivative of $\mathbb{R}^d\ni\boldsymbol{\xi}\mapsto j_\varepsilon(\boldsymbol{x},s,\boldsymbol{\xi})\in\mathbb{R}$;

(v) $j_\varepsilon(\boldsymbol{x},s,\cdot)$ or $-j_\varepsilon(\boldsymbol{x},s,\cdot)$ is regular for a.e. $\boldsymbol{x}\in\Gamma_1$ and all $s\in\mathbb{R}$;

(vi) for all $\{\boldsymbol{\xi}_\varepsilon\}\subset\mathbb{R}^d$, $\{\boldsymbol{\zeta}_\varepsilon\}\subset\mathbb{R}^d$ and $\{s_\varepsilon\}\subset\mathbb{R}$ such that $\boldsymbol{\xi}_\varepsilon\to\boldsymbol{\xi}$ and $\boldsymbol{\zeta}_\varepsilon\to\boldsymbol{\zeta}$ in $\mathbb{R}^d$ and $s_\varepsilon\to s$ in $\mathbb{R}$ as $\varepsilon\to 0$, we have

$$\limsup_{\varepsilon\to 0} j_\varepsilon^0(\boldsymbol{x},s_\varepsilon,\boldsymbol{\xi}_\varepsilon;\boldsymbol{\zeta}_\varepsilon)\leq j^0(\boldsymbol{x},s,\boldsymbol{\xi};\boldsymbol{\zeta})\quad\text{for a.e. } \boldsymbol{x}\in\Gamma_1.$$

$\underline{H(\phi_\varepsilon)}$: $\phi_\varepsilon\in L^{p_2'}(\Gamma_2)$, and $\phi_\varepsilon\to\phi$ in $L^{p_2'}(\Gamma_2)$ as $\varepsilon\to 0$.

$\underline{H(\varphi_\varepsilon)}$: $\varphi_\varepsilon\in L^{p_2'}(\Gamma_3)$, and $\varphi_\varepsilon\to\varphi$ in $L^{p_2'}(\Gamma_3)$ as $\varepsilon\to 0$.

$\underline{H(\kappa_\varepsilon)}$: $\kappa_\varepsilon\colon \Omega\times\mathbb{R}^d\to\mathbb{R}$ is such that

(i) $\kappa_\varepsilon(\cdot,\boldsymbol{v})$ is measurable on Ω for all $\boldsymbol{v}\in\mathbb{R}^d$;
(ii) $\kappa_\varepsilon(\boldsymbol{x},\cdot)$ is continuous on $\mathbb{R}^d$ for a.e. $\boldsymbol{x}\in\Omega$;
(iii) there exist constants $a_{\kappa_\varepsilon}, b_{\kappa_\varepsilon}>0$ such that $\inf_{\varepsilon>0} a_{\kappa_\varepsilon}=a_0>0$ and

$$0<a_{\kappa_\varepsilon}\le \kappa_\varepsilon(\boldsymbol{x},\boldsymbol{v})\le b_{\kappa_\varepsilon}\ \text{ for a.e. } \boldsymbol{x}\in\Omega \text{ and all } \boldsymbol{v}\in\mathbb{R}^d;$$

(iv) for all $\{\boldsymbol{v}_\varepsilon\}\subset\mathbb{R}^d$, $\boldsymbol{v}\in\mathbb{R}^d$ with $\boldsymbol{v}_\varepsilon\to\boldsymbol{v}$ as $\varepsilon\to0$, we have

$$\lim_{\varepsilon\to0}\kappa_\varepsilon(\boldsymbol{x},\boldsymbol{v}_\varepsilon)=\kappa(\boldsymbol{x},\boldsymbol{v})\ \text{ for a.e. } \boldsymbol{x}\in\Omega.$$

$\underline{H(g_\varepsilon)}$: $g_\varepsilon\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is such that

(i) $g_\varepsilon(\cdot,s)$ is measurable on Ω for all $s\in\mathbb{R}$;
(ii) $g_\varepsilon(\boldsymbol{x},\cdot)$ is continuous on $\mathbb{R}$ for a.e. $\boldsymbol{x}\in\Omega$;
(iii) there exist a function $a_{g_\varepsilon}\in L^{p_2'}(\Omega)_+$ and a constant $b_{g_\varepsilon}>0$ such that

$$|g_\varepsilon(\boldsymbol{x},s)|\le a_{g_\varepsilon}(\boldsymbol{x})+b_{g_\varepsilon}|s|^{p_2^*-1}\ \text{ for a.e. } \boldsymbol{x}\in\Omega \text{ and all } s\in\mathbb{R},$$

where p_2^* represents the Sobolev critical exponent of p_2, $\{a_{g_\varepsilon}\}_{\varepsilon>0}\subset L^{p_2'}(\Omega)_+$ and $\{b_{g_\varepsilon}\}_{\varepsilon>0}\subset(0,+\infty)$ are uniformly bounded with respect to parameter ε;
(iv) there exist a function $c_{g_\varepsilon}\in L^1(\Omega)$ and a constant $d_{g_\varepsilon}>0$ such that

$$g_\varepsilon(\boldsymbol{x},s)s\ge c_{g_\varepsilon}(\boldsymbol{x})+d_{g_\varepsilon}|s|^\theta\ \text{ for a.e. } \boldsymbol{x}\in\Omega \text{ and all } s\in\mathbb{R}$$

with $\theta\ge p_2$, where $\{c_{g_\varepsilon}\}_{\varepsilon>0}\subset L^1(\Omega)$ and $\{d_{g_\varepsilon}\}_{\varepsilon>0}\subset(0,+\infty)$ are uniformly bounded with respect to parameter ε;
(v) for all $\{c_\varepsilon\}\subset\mathbb{R}$, $c\in\mathbb{R}$ with $c_\varepsilon\to c$ as $\varepsilon\to0$, we have

$$\lim_{\varepsilon\to0}g_\varepsilon(\boldsymbol{x},c_\varepsilon)=g(\boldsymbol{x},c)\ \text{ for a.e. } \boldsymbol{x}\in\Omega.$$

$\underline{H(\omega_\varepsilon)}$: $\omega_\varepsilon\in L^{p_2'}(\Gamma_2\cup\Gamma_3)$, and $\omega_\varepsilon\to\omega$ in $L^{p_2'}(\Gamma_2\cup\Gamma_3)$ as $\varepsilon\to0$.

$\underline{H(\rho_\varepsilon)}$: $\rho_\varepsilon\ge0$, and $\rho_\varepsilon\to\rho$ in $\mathbb{R}$ as $\varepsilon\to0$.

In order to derive the variational formulation of Problem 3.1, we introduce the admissible set $K_\varepsilon\subset E$ of velocity fields given by

$$K_\varepsilon=\{\boldsymbol{v}\in E \mid v_\nu\ge-\rho_\varepsilon \text{ on } \Gamma_2,\ v_\nu\ge0 \text{ on } \Gamma_3\}. \tag{3.8}$$

We can see that K_ε is a nonempty, closed, and convex subset of E due to $\mathbf{0}\in K_\varepsilon$ and $\rho_\varepsilon\ge0$.

Remark 3.1. From the definition of K and K_ε (see (2.1) and (3.8)), we can verify that $K_\varepsilon\xrightarrow{M}K$ as $\varepsilon\to0$ (namely, K_ε converges to K in the sense of Mosco as $\varepsilon\to0$). Indeed, let sequence $\{\boldsymbol{v}_\varepsilon\}_{\varepsilon>0}$ be such that $\boldsymbol{v}_\varepsilon\in K_\varepsilon$ for each $\varepsilon>0$ and $\boldsymbol{v}_\varepsilon\rightharpoonup\boldsymbol{v}$ in E as $\varepsilon\to0$. Then, by the definition of K_ε and the Sobolev embedding theorem, we have $v_\nu\ge-\rho$ on Γ_2 and $v_\nu\ge0$ on Γ_3. Thus $\boldsymbol{v}\in K$. On the other hand, for any $\boldsymbol{v}\in K$, let $\boldsymbol{v}_\varepsilon=\boldsymbol{v}+(\rho-\rho_\varepsilon)_+$, so we can observe that $\boldsymbol{v}_\varepsilon\to\boldsymbol{v}$ in E, $v_{\varepsilon_\nu}\ge-\rho_\varepsilon$ on Γ_2 and $v_{\varepsilon_\nu}\ge(\rho-\rho_\varepsilon)_+\ge0$ on Γ_2, i.e., $\boldsymbol{v}_\varepsilon\in K_\varepsilon$ and $\boldsymbol{v}_\varepsilon\to\boldsymbol{v}$ in E.

Assume that the functions $\boldsymbol{v}_\varepsilon\colon \Omega\to\mathbb{R}^d$, $\pi_\varepsilon\colon \Omega\to\mathbb{R}$, and $c_\varepsilon\colon\Omega\to\mathbb{R}$ are sufficiently smooth and satisfy (3.1)–(3.6). Let $\boldsymbol{w}_\varepsilon\in K_\varepsilon$ be arbitrary fixed. Multiplying (3.1) by $\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon$ and integrating the resulting equality on Ω, we obtain

$$\begin{aligned}&-\int_\Omega \operatorname{Div}\boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}_\varepsilon))\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\boldsymbol{x}+\int_\Omega(\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon)\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\boldsymbol{x}\\&\quad+\int_\Omega\nabla\pi_\varepsilon\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\boldsymbol{x}=\int_\Omega\boldsymbol{f}_\varepsilon\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\boldsymbol{x}.\end{aligned}\tag{3.9}$$

For the pressure π_ε, we apply Green's formula (see [20]) to have

$$\begin{aligned}\int_\Omega\nabla\pi_\varepsilon\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\boldsymbol{x}&=-\int_\Omega\big(\nabla\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\big)\pi_\varepsilon\,d\boldsymbol{x}+\int_\Gamma\pi_\varepsilon\boldsymbol{\nu}\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\Gamma\\&=\int_{\Gamma_2\cup\Gamma_3}\pi_\varepsilon(w_{\varepsilon_\nu}-v_{\varepsilon_\nu})\,d\Gamma,\end{aligned}\tag{3.10}$$

where we used the conditions $\boldsymbol{v}_\varepsilon=\boldsymbol{w}_\varepsilon=\boldsymbol{0}$ on Γ_0, $v_{\varepsilon_\nu}=w_{\varepsilon_\nu}=0$ on Γ_1, and the divergence free condition for $\boldsymbol{v}_\varepsilon$ and $\boldsymbol{w}_\varepsilon$. Also, we use the divergence theorem (see [20]) and the conditions $\boldsymbol{v}_\varepsilon=\boldsymbol{w}_\varepsilon=\boldsymbol{0}$ on Γ_0 to obtain

$$\begin{aligned}&-\int_\Omega\operatorname{Div}\boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}_\varepsilon))\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\boldsymbol{x}\\&=\int_\Omega\boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}_\varepsilon)):\boldsymbol{D}(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\boldsymbol{x}-\int_\Gamma\boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}_\varepsilon))\boldsymbol{\nu}\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\Gamma\\&=\int_\Omega\boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}_\varepsilon)):\boldsymbol{D}(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\boldsymbol{x}-\int_{\Gamma_1\cup\Gamma_2\cup\Gamma_3}\boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}_\varepsilon))\boldsymbol{\nu}\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\Gamma.\end{aligned}\tag{3.11}$$

Combining the boundary conditions $v_{\varepsilon_\nu}=w_{\varepsilon_\nu}=0$ on Γ_1, $\boldsymbol{\tau}_{\varepsilon_\tau}(\boldsymbol{v}_\varepsilon)=\boldsymbol{0}$ on $\Gamma_2\cup\Gamma_3$, with equalities $\tau_{\varepsilon_\nu}(\boldsymbol{v}_\varepsilon,\pi_\varepsilon)=C_\varepsilon(\boldsymbol{D}(\boldsymbol{v}_\varepsilon))_\nu-\pi_\varepsilon$, $\boldsymbol{\tau}_{\varepsilon_\tau}(\boldsymbol{v}_\varepsilon)=\boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}_\varepsilon))_\tau$ on Γ, we obtain

$$\begin{aligned}&-\int_{\Gamma_1\cup\Gamma_2\cup\Gamma_3}\boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}_\varepsilon))\boldsymbol{\nu}\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\Gamma\\&=-\int_{\Gamma_1\cup\Gamma_2\cup\Gamma_3}\big(C_\varepsilon(\boldsymbol{D}(\boldsymbol{v}_\varepsilon))_\nu\cdot(w_{\varepsilon_\nu}-v_{\varepsilon_\nu})+\boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}_\varepsilon))_\tau\cdot(\boldsymbol{w}_{\varepsilon_\tau}-\boldsymbol{v}_{\varepsilon_\tau})\big)\,d\Gamma\\&=-\int_{\Gamma_1}\boldsymbol{\tau}_{\varepsilon_\tau}(\boldsymbol{v}_\varepsilon)\cdot(\boldsymbol{w}_{\varepsilon_\tau}-\boldsymbol{v}_{\varepsilon_\tau})\,d\Gamma-\int_{\Gamma_2\cup\Gamma_3}(\tau_{\varepsilon_\nu}(\boldsymbol{v}_\varepsilon,\pi_\varepsilon)+\pi_\varepsilon)(w_{\varepsilon_\nu}-v_{\varepsilon_\nu})\,d\Gamma.\end{aligned}\tag{3.12}$$

By the boundary condition $-\boldsymbol{\tau}_{\varepsilon_\tau}(\boldsymbol{v}_\varepsilon)\in\partial j_\varepsilon(\boldsymbol{x},c_\varepsilon,\boldsymbol{v}_{\varepsilon_\tau})$ and the definition of the Clarke subgradient, we see that

$$-\int_{\Gamma_1}\boldsymbol{\tau}_{\varepsilon_\tau}(\boldsymbol{v}_\varepsilon)\cdot(\boldsymbol{w}_{\varepsilon_\tau}-\boldsymbol{v}_{\varepsilon_\tau})\,d\Gamma\le\int_{\Gamma_1}j_\varepsilon^0(\boldsymbol{x},c_\varepsilon,\boldsymbol{v}_{\varepsilon_\tau};\boldsymbol{w}_{\varepsilon_\tau}-\boldsymbol{v}_{\varepsilon_\tau})\,d\Gamma.\tag{3.13}$$

We use the boundary conditions $(3.5)_{1-3}$ and $(3.6)_2$ to have

$$
\begin{aligned}
&-\int_{\Gamma_2\cup\Gamma_3}(\tau_{\varepsilon_\nu}(\boldsymbol{v}_\varepsilon,\pi_\varepsilon)+\pi_\varepsilon)(w_{\varepsilon_\nu}-v_{\varepsilon_\nu})\,d\Gamma \qquad (3.14)\\
&=-\int_{\Gamma_2}(\tau_{\varepsilon_\nu}(\boldsymbol{v}_\varepsilon,\pi_\varepsilon)+\phi_\varepsilon)(w_{\varepsilon_\nu}+\rho_\varepsilon)\,d\Gamma+\int_{\Gamma_2}(\tau_{\varepsilon_\nu}(\boldsymbol{v}_\varepsilon,\pi_\varepsilon)+\phi_\varepsilon)(\rho_\varepsilon+v_{\varepsilon_\nu})\,d\Gamma\\
&\quad+\int_{\Gamma_2}\phi_\varepsilon(w_{\varepsilon_\nu}-v_{\varepsilon_\nu})\,d\Gamma-\int_{\Gamma_3}\tau_{\varepsilon_\nu}(\boldsymbol{v}_\varepsilon,\pi_\varepsilon)(w_{\varepsilon_\nu}-v_{\varepsilon_\nu})\,d\Gamma-\int_{\Gamma_2\cup\Gamma_3}\pi_\varepsilon(w_{\varepsilon_\nu}-v_{\varepsilon_\nu})\,d\Gamma\\
&\le\int_{\Gamma_2}\phi_\varepsilon(w_{\varepsilon_\nu}-v_{\varepsilon_\nu})\,d\Gamma+\int_{\Gamma_3}\varphi_\varepsilon(w_{\varepsilon_\nu}-v_{\varepsilon_\nu})\,d\Gamma-\int_{\Gamma_2\cup\Gamma_3}\pi_\varepsilon(w_{\varepsilon_\nu}-v_{\varepsilon_\nu})\,d\Gamma.
\end{aligned}
$$

Inserting (3.10)-(3.14) into (3.9), we deduce

$$
\begin{aligned}
&\int_\Omega \boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}_\varepsilon)):\boldsymbol{D}(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\boldsymbol{x}+\int_\Omega(\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon)\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\boldsymbol{x}+\int_{\Gamma_1}j_\varepsilon^0(\boldsymbol{x},c_\varepsilon,\boldsymbol{v}_{\varepsilon_\tau};\boldsymbol{w}_{\varepsilon_\tau}-\boldsymbol{v}_{\varepsilon_\tau})\,d\Gamma\\
&+\int_{\Gamma_2}\phi_\varepsilon(w_{\varepsilon_\nu}-v_{\varepsilon_\nu})\,d\Gamma+\int_{\Gamma_3}\varphi_\varepsilon(w_{\varepsilon_\nu}-v_{\varepsilon_\nu})\,d\Gamma\ge\int_\Omega \boldsymbol{f}_\varepsilon\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\boldsymbol{x}\ \text{ for all } \boldsymbol{w}_\varepsilon\in K_\varepsilon.
\end{aligned}
$$

On the other hand, we apply Green's formula and the boundary condition $(3.7)_2$ to obtain the following variational equation

$$
\int_\Omega \kappa_\varepsilon(\boldsymbol{v}_\varepsilon)\|\nabla c_\varepsilon\|_{\mathbb{R}^d}^{p_2-2}\nabla c_\varepsilon\cdot\nabla z\,d\boldsymbol{x}+\int_\Omega g_\varepsilon(\boldsymbol{x},c_\varepsilon)z\,d\boldsymbol{x}+\int_\Omega(\boldsymbol{v}_\varepsilon\cdot\nabla c_\varepsilon)\cdot z\,d\boldsymbol{x}=\int_{\Gamma_2\cup\Gamma_3}\omega_\varepsilon z\,d\Gamma
$$

for all $z\in W^{1,p_2}(\Omega)$. Therefore, we obtain the variational formulation of Problem 3.1 as follows.

Problem 3.2. *Find a velocity filed $\boldsymbol{v}_\varepsilon\in K_\varepsilon$ and a concentration field $c_\varepsilon\in W^{1,p_2}(\Omega)$ such that*

$$
\begin{aligned}
&\int_\Omega \boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v}_\varepsilon)):\boldsymbol{D}(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\boldsymbol{x}+\int_\Omega(\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon)\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\boldsymbol{x}+\int_{\Gamma_2}\phi_\varepsilon(w_{\varepsilon_\nu}-v_{\varepsilon_\nu})\,d\Gamma\\
&+\int_{\Gamma_1}j_\varepsilon^0(\boldsymbol{x},c_\varepsilon,\boldsymbol{v}_{\varepsilon_\tau};\boldsymbol{w}_{\varepsilon_\tau}-\boldsymbol{v}_{\varepsilon_\tau})\,d\Gamma+\int_{\Gamma_3}\varphi_\varepsilon(w_{\varepsilon_\nu}-v_{\varepsilon_\nu})\,d\Gamma\ge\int_\Omega \boldsymbol{f}_\varepsilon\cdot(\boldsymbol{w}_\varepsilon-\boldsymbol{v}_\varepsilon)\,d\boldsymbol{x}
\end{aligned}
$$

for all $\boldsymbol{w}_\varepsilon\in K_\varepsilon$ and

$$
\int_\Omega \kappa_\varepsilon(\boldsymbol{v}_\varepsilon)\|\nabla c_\varepsilon\|_{\mathbb{R}^d}^{p_2-2}\nabla c_\varepsilon\cdot\nabla z\,d\boldsymbol{x}+\int_\Omega g_\varepsilon(\boldsymbol{x},c_\varepsilon)z\,d\boldsymbol{x}+\int_\Omega(\boldsymbol{v}_\varepsilon\cdot\nabla c_\varepsilon)\cdot z\,d\boldsymbol{x}=\int_{\Gamma_2\cup\Gamma_3}\omega_\varepsilon z\,d\Gamma
$$

for all $z\in W^{1,p_2}(\Omega)$.

4. Stability Analysis

This section is concerned with the stability analysis of the recovered fracturing fluid model, namely Problem 1.1. More precisely, we are going to establish a result that if a perturbation parameter ε tends to zero, then the solution set of the perturbation problem, Problem 3.1, converges to the solution set of the recovered fracturing fluid model (Problem 1.1) in the sense of Kuratowski upper limit.

The main result is stated in the form of the following theorem.

Theorem 4.1. *Assume that $H(\boldsymbol{C}_\varepsilon)$, $H(\boldsymbol{f}_\varepsilon)$, $H(j_\varepsilon)$, $H(\phi_\varepsilon)$, $H(\varphi_\varepsilon)$, $H(\kappa_\varepsilon)$, $H(g_\varepsilon)$, $H(\omega_\varepsilon)$, and $H(\rho_\varepsilon)$ hold. In addition, we suppose that*

$$
\inf_{\varepsilon>0}\Big(c_{C_\varepsilon}-b_{j_\varepsilon}\|\gamma\|^{p_1}-\delta(p_1)\frac{\rho_\varepsilon}{2}\|\gamma\|^2\Big)>0\quad\text{and}\quad\inf_{\varepsilon>0}\Big(\min\{a_{\kappa_\varepsilon},d_{g_\varepsilon}\delta(\theta)\}-\delta(p_2)\frac{\rho_\varepsilon}{2}\|\gamma_1\|^2\Big)>0.
$$

Then,

(i) *for each* $\varepsilon > 0$, *the perturbation problem (Problem 3.1) has at least one weak solution* $(\boldsymbol{v}_\varepsilon, c_\varepsilon) \in K_\varepsilon \times W^{1,p_2}(\Omega)$;

(ii) *if* $\{\varepsilon_n\} \subset (0,+\infty)$ *is such that* $\varepsilon_n \to 0$ *as* $n \to \infty$, *then it holds*

$$w\text{–}\limsup_{n\to\infty} S_n = s\text{–}\limsup_{n\to\infty} S_n \subset S,$$

where S_n *and* S *are the solution sets of Problems 3.1 and 1.1, respectively,* $w\text{–}\limsup_{n\to\infty} S_n$ *(resp.* $s\text{–}\limsup_{n\to\infty} S_n$*) is the Kuratowski upper limit of the sequence* S_n *with respect to the weak topology (resp. with respect to the strong topology).*

Proof. (i) Arguing as in the proof of Theorem 2.1, it can be obtained directly that, for each perturbation parameter $\varepsilon > 0$, perturbation problem (Problem 3.1) is solvable.

(ii) We consider the nonlinear operators $A\colon E \to E^*$ and $A_\varepsilon\colon E \to E^*$ defined by

$$\langle A\boldsymbol{v}, \boldsymbol{w}\rangle := \int_\Omega \boldsymbol{C}(\boldsymbol{D}(\boldsymbol{v})) : \boldsymbol{D}(\boldsymbol{w})\,d\boldsymbol{x} \ \text{ for all } \boldsymbol{v}, \boldsymbol{w} \in E, \tag{4.1}$$

$$\langle A_\varepsilon\boldsymbol{v}, \boldsymbol{w}\rangle := \int_\Omega \boldsymbol{C}_\varepsilon(\boldsymbol{D}(\boldsymbol{v})) : \boldsymbol{D}(\boldsymbol{w})\,d\boldsymbol{x} \ \text{ for all } \boldsymbol{v}, \boldsymbol{w} \in E, \tag{4.2}$$

and mapping $B[\cdot]\colon E \to E^*$ defined by

$$\langle B(\boldsymbol{v}, \boldsymbol{u}), \boldsymbol{w}\rangle := \int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{w}\,d\boldsymbol{x} =: b(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}) \ \text{ for all } \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w} \in E \tag{4.3}$$

with $B[\boldsymbol{v}] = B(\boldsymbol{v}, \boldsymbol{v})$. Suppose that $(\boldsymbol{v}_n, c_n) := (\boldsymbol{v}_{\varepsilon_n}, c_{\varepsilon_n})$ is a weak solution of the perturbation problem, Problem 3.1, corresponding to $\varepsilon = \varepsilon_n$. Set $K_n = K_{\varepsilon_n}$, $A_n = A_{\varepsilon_n}$, $j_n = j_{\varepsilon_n}$, $\phi_n = \phi_{\varepsilon_n}$, $\varphi_n = \varphi_{\varepsilon_n}$, $\boldsymbol{f}_n = \boldsymbol{f}_{\varepsilon_n}$, $\kappa_n = \kappa_{\varepsilon_n}$, $g_n = g_{\varepsilon_n}$, and $\omega_n = \omega_{\varepsilon_n}$. Then, for each $n \in \mathbb{N}$, one has

$$\begin{aligned}&\langle A_n\boldsymbol{v}_n + B[\boldsymbol{v}_n], \boldsymbol{w}_n - \boldsymbol{v}_n\rangle + \int_{\Gamma_1} j_n^0(\boldsymbol{x}, c_n, \gamma\boldsymbol{v}_n; \gamma(\boldsymbol{w}_n - \boldsymbol{v}_n))\,d\Gamma \\ &+ \int_{\Gamma_2} \phi_n(w_{n_\nu} - v_{n_\nu})\,d\Gamma + \int_{\Gamma_3} \varphi_n(w_{n_\nu} - v_{n_\nu})\,d\Gamma \geq \int_\Omega \boldsymbol{f}_n\cdot(\boldsymbol{w}_n - \boldsymbol{v}_n)\,d\boldsymbol{x}\end{aligned} \tag{4.4}$$

for all $\boldsymbol{w}_n \in K_n$, and

$$\int_\Omega \kappa_n(\boldsymbol{v}_n)\|\nabla c_n\|_{\mathbb{R}^d}^{p_2-2}\nabla c_n\cdot\nabla z\,d\boldsymbol{x} + \int_\Omega g_n(\boldsymbol{x}, c_n)z\,d\boldsymbol{x} + \int_\Omega (\boldsymbol{v}_n\cdot\nabla c_n)\cdot z\,d\boldsymbol{x} = \int_{\Gamma_2\cup\Gamma_3} \omega_n z\,d\Gamma \tag{4.5}$$

for all $z \in W^{1,p_2}(\Omega)$.

We complete the proof in some steps, to underline its crucial moments.

Step 1. *The boundedness of* $\cup_{n\in\mathbb{N}}\{(\boldsymbol{v}_n, c_n)\}$.

We argue by contradiction and assume that $\cup_{n\in\mathbb{N}}\{(\boldsymbol{v}_n, c_n)\}$ is unbounded in $E \times W^{1,p_2}(\Omega)$. Then, without any loss of generality, we may suppose that

$$\|\boldsymbol{v}_n\|_E + \|c_n\|_{W^{1,p_2}(\Omega)} \to \infty \ \text{ as } \ n \to \infty.$$

Since $\boldsymbol{0} \in K_n$, we insert $\boldsymbol{w}_n = \boldsymbol{0}$ into inequality (4.4) to find

$$\begin{aligned}&\langle A_n\boldsymbol{v}_n + B[\boldsymbol{v}_n], \boldsymbol{v}_n\rangle - \int_{\Gamma_1} j_n^0(\boldsymbol{x}, c_n, \gamma\boldsymbol{v}_n; \gamma(\boldsymbol{0} - \boldsymbol{v}_n))\,d\Gamma \\ &\leq \int_\Omega \boldsymbol{f}_n\cdot\boldsymbol{v}_n\,d\boldsymbol{x} - \int_{\Gamma_2} \phi_n v_{n_\nu}\,d\Gamma - \int_{\Gamma_3} \varphi_n v_{n_\nu}\,d\Gamma.\end{aligned} \tag{4.6}$$

From definition of A_ε (see (4.2)) and hypothesis $H(\boldsymbol{C}_\varepsilon)$(ii)(iv), we have

$$\begin{aligned}\langle A_n\boldsymbol{v}_n,\boldsymbol{v}_n\rangle &= \int_\Omega \boldsymbol{C}_n(\boldsymbol{D}(\boldsymbol{v}_n)):\boldsymbol{D}(\boldsymbol{v}_n)\,d\boldsymbol{x} \geq \int_\Omega (G_n(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{v}_n)) - G_n(\boldsymbol{x},\boldsymbol{0}_{\mathbb{S}^d}))\,d\boldsymbol{x} \\ &\geq \int_\Omega \left(c_{C_n}\|\boldsymbol{D}(\boldsymbol{v}_n)\|_{\mathbb{S}^d}^{p_1} + d_{C_n}(\boldsymbol{x}) - |G_n(\boldsymbol{x},\boldsymbol{0}_{\mathbb{S}^d})|\right) d\boldsymbol{x} \\ &\geq c_{C_n}\|\boldsymbol{D}(\boldsymbol{v}_n)\|^{p_1}_{L^{p_1}(\Omega;\mathbb{S}^d)} - \|d_{C_n}\|_{L^1(\Omega)} - \|G_n(\cdot,\boldsymbol{0}_{\mathbb{S}^d})\|_{L^1(\Omega)} \\ &= c_{C_n}\|\boldsymbol{v}_n\|_E^{p_1} - \|d_{C_n}\|_{L^1(\Omega)} - \|G_n(\cdot,\boldsymbol{0}_{\mathbb{S}^d})\|_{L^1(\Omega)}.\end{aligned} \tag{4.7}$$

Next, for each $\boldsymbol{w}\in V$, we use Green's formula and the divergence free condition to deduce

$$\begin{aligned}\langle B[\boldsymbol{w}],\boldsymbol{w}\rangle &= \int_\Omega ((\boldsymbol{w}\cdot\nabla)\boldsymbol{w})\cdot\boldsymbol{w}\,d\boldsymbol{x} = \int_\Omega \sum_{i,j=1}^d w_i\frac{\partial w_j}{\partial x_i}w_j\,d\boldsymbol{x} = \int_\Omega \sum_{i,j=1}^d w_i\frac{\partial}{\partial x_i}(\frac{w_j^2}{2})\,d\boldsymbol{x} \\ &= -\frac{1}{2}\int_\Omega (\nabla\cdot\boldsymbol{w})\sum_{i=1}^d (w_i)^2\,d\boldsymbol{x} + \frac{1}{2}\int_\Gamma w_\nu\sum_{i=1}^d (w_i)^2\,d\Gamma = \frac{1}{2}\int_{\Gamma_2\cup\Gamma_3} w_\nu \sum_{i=1}^d (w_i)^2\,d\Gamma.\end{aligned}$$

Then,

$$\langle B[\boldsymbol{v}_n],\boldsymbol{v}_n\rangle = \frac{1}{2}\int_{\Gamma_2\cup\Gamma_3}(v_n)_\nu\sum_{i=1}^d((v_n)_i)^2\,d\Gamma.$$

The boundary conditions $(3.5)_1$ and $(3.6)_1$ show that

$$\begin{aligned}\langle B[\boldsymbol{v}_n],\boldsymbol{v}_n\rangle &= \frac{1}{2}\int_{\Gamma_2\cup\Gamma_3}(v_n)_\nu\sum_{i=1}^d((v_n)_i)^2\,d\Gamma \geq -\frac{\rho_n}{2}\int_{\Gamma_2}\sum_{i=1}^d((v_n)_i)^2\,d\Gamma \\ &= -\frac{\rho_n}{2}\|\boldsymbol{v}_n\|^2_{L^2(\Gamma_2;\mathbb{R}^d)}.\end{aligned} \tag{4.8}$$

It follows from hypothesis $H(j_\varepsilon)$(ii) and Proposition 2.1(ii) that there exists $\tilde{\boldsymbol{\eta}}\in\partial j_n(\boldsymbol{x},c_n,\gamma\boldsymbol{v}_n)$ such that

$$j_n^0(\boldsymbol{x},c_n,\gamma\boldsymbol{v}_n;\gamma(\boldsymbol{0}-\boldsymbol{v}_n)) = \langle\tilde{\boldsymbol{\eta}},\gamma(\boldsymbol{0}-\boldsymbol{v}_n)\rangle.$$

Combining the latter with the hypothesis $H(j_\varepsilon)$(iii), we have

$$\begin{aligned}&-\int_{\Gamma_1} j_n^0(\boldsymbol{x},c_n,\gamma\boldsymbol{v}_n;\gamma(\boldsymbol{0}-\boldsymbol{v}_n))\,d\Gamma \geq -\int_{\Gamma_1}\|\tilde{\boldsymbol{\eta}}\|_{\mathbb{R}^d}\cdot\|\gamma(\boldsymbol{0}-\boldsymbol{v}_n)\|_{\mathbb{R}^d}\,d\Gamma \\ &\geq -\int_{\Gamma_1}(a_{j_n}(\boldsymbol{x}) + b_{j_n}\|\gamma\boldsymbol{v}_n\|_{\mathbb{R}^d}^{p_1-1})\cdot\|\gamma\boldsymbol{v}_n\|_{\mathbb{R}^d}\,d\Gamma \geq -\|a_{j_n}\|_{L^{p_1'}(\Gamma_1)}\|\gamma\|\|\boldsymbol{v}_n\|_E - b_{j_n}\|\gamma\|^{p_1}\|\boldsymbol{v}_n\|_E^{p_1}.\end{aligned} \tag{4.9}$$

Moreover, by Hölder's inequality and Korn's inequality, we can find a constant $C_1>0$ such that

$$\int_\Omega \boldsymbol{f}_n\cdot\boldsymbol{v}_n\,d\boldsymbol{x} \leq \|\boldsymbol{f}_n\|_{L^{p_1'}(\Omega;\mathbb{R}^d)}\|\boldsymbol{v}_n\|_{L^{p_1}(\Omega;\mathbb{R}^d)} \leq C_1\|\boldsymbol{f}_n\|_{L^{p_1'}(\Omega;\mathbb{R}^d)}\|\boldsymbol{v}_n\|_E, \tag{4.10}$$

and

$$\left\{\begin{aligned} -\int_{\Gamma_2}\phi_n v_{n_\nu}\,d\Gamma &\leq \|\phi_n\|_{L^{p_1'}(\Gamma_2)}\|\boldsymbol{v}_n\|_{L^{p_1}(\Gamma_2;\mathbb{R}^d)} \leq \|\phi_n\|_{L^{p_1'}(\Gamma_2)}\|\gamma\|\|\boldsymbol{v}_n\|_E, \\ -\int_{\Gamma_3}\varphi_n v_{n_\nu}\,d\Gamma &\leq \|\varphi_n\|_{L^{p_1'}(\Gamma_3)}\|\boldsymbol{v}_n\|_{L^{p_1}(\Gamma_3;\mathbb{R}^d)} \leq \|\varphi_n\|_{L^{p_1'}(\Gamma_3)}\|\gamma\|\|\boldsymbol{v}_n\|_E.\end{aligned}\right. \tag{4.11}$$

Taking into account (4.6)–(4.11), we have

$$\left(c_{C_n}-b_{j_n}\|\gamma\|^{p_1}\right)\|\boldsymbol{v}_n\|_E^{p_1}-\frac{\rho_n}{2}\|\gamma\|^2\|\boldsymbol{v}_n\|_E^2-\|d_{C_n}\|_{L^1(\Omega)}-\|G_n(\cdot,\mathbf{0}_{\mathbb{S}^d})\|_{L^1(\Omega)}$$
$$\leq\left(\|a_{j_n}\|_{L^{p_1'}(\Gamma_1)}\|\gamma\|+C_1\|\boldsymbol{f}_n\|_{L^{p_1'}(\Omega;\mathbb{R}^d)}+\|\gamma\|\|\phi_n\|_{L^{p_1'}(\Gamma_2)}+\|\gamma\|\|\varphi_n\|_{L^{p_1'}(\Gamma_3)}\right)\|\boldsymbol{v}_n\|_E.$$

Recall that $\|\boldsymbol{v}_n\|_E\to\infty$ as $n\to\infty$. Passing to the upper limit as $n\to\infty$ in the inequality above, and using the inequalities

$$\begin{cases}\inf_{\varepsilon>0}(c_{C_\varepsilon}-b_{j_\varepsilon}\|\gamma\|^{p_1}-\frac{\rho_\varepsilon}{2}\|\gamma\|^2)>0 & \text{if } p_1=2,\\ \inf_{\varepsilon>0}(c_{C_\varepsilon}-b_{j_\varepsilon}\|\gamma\|^{p_1})>0 & \text{if } p_1>2,\end{cases}$$

we obtain a contradiction, that is,

$$+\infty=\limsup_{n\to\infty}\left((c_{C_n}-b_{j_n}\|\gamma\|^{p_1})\|\boldsymbol{v}_n\|_E^{p_1-1}-\frac{\rho_n}{2}\|\gamma\|^2\|\boldsymbol{v}_n\|_E-\frac{\|d_{C_n}\|_{L^1(\Omega)}+\|G_n(\cdot,\mathbf{0}_{\mathbb{S}^d})\|_{L^1(\Omega)}}{\|\boldsymbol{v}_n\|_E}\right)$$
$$\leq\limsup_{n\to\infty}\left(\|a_{j_n}\|_{L^{p_1'}(\Gamma_1)}\|\gamma\|+C_1\|\boldsymbol{f}_n\|_{L^{p_1'}(\Omega;\mathbb{R}^d)}+\|\gamma\|\|\phi_n\|_{L^{p_1'}(\Gamma_2)}+\|\gamma\|\|\varphi_n\|_{L^{p_1'}(\Gamma_3)}\right)$$
$$<+\infty,$$

which implies that $\{\boldsymbol{v}_n\}$ is uniformly bounded in E.

On the other hand, let us take $z=c_n$ in equation (4.5) to obtain

$$\int_\Omega\kappa_n(\boldsymbol{v}_n)\|\nabla c_n\|_{\mathbb{R}^d}^{p_2-2}\nabla c_n\cdot\nabla c_n\,d\boldsymbol{x}+\int_\Omega g_n(\boldsymbol{x},c_n)c_n\,d\boldsymbol{x}+\int_\Omega(\boldsymbol{v}_n\cdot\nabla c_n)\cdot c_n\,d\boldsymbol{x}\tag{4.12}$$
$$=\int_{\Gamma_2\cup\Gamma_3}\omega_n c_n\,d\Gamma.$$

In virtue of hypotheses $H(\kappa_\varepsilon)$(iii) and $H(g_\varepsilon)$(iv), one has

$$\int_\Omega\kappa_n(\boldsymbol{v}_n)\|\nabla c_n\|_{\mathbb{R}^d}^{p_2-2}\nabla c_n\cdot\nabla c_n\,d\boldsymbol{x}+\int_\Omega g_n(\boldsymbol{x},c_n)c_n\,d\boldsymbol{x}\tag{4.13}$$
$$\geq a_{\kappa_n}\int_\Omega\|\nabla c_n\|_{\mathbb{R}^d}^{p_2}\,d\boldsymbol{x}+\int_\Omega\left(c_{g_n}(\boldsymbol{x})+d_{g_n}|c_n|^\theta\right)\,d\boldsymbol{x}$$
$$\geq a_{\kappa_n}\|\nabla c_n\|_{L^{p_2}(\Omega;\mathbb{R}^d)}^{p_2}+\|c_{g_n}\|_{L^1(\Omega)}+d_{g_n}\|c_n\|_{L^\theta(\Omega)}^\theta.$$

Then, we use the divergence theorem and condition (3.2) to obtain

$$\int_\Omega(\boldsymbol{v}_n\cdot\nabla c_n)\cdot c_n\,d\boldsymbol{x}=\int_\Gamma c_n^2\boldsymbol{v}_n\cdot\boldsymbol{\nu}\,d\Gamma-\int_\Omega c_n^2\operatorname{div}\boldsymbol{v}_n\,d\boldsymbol{x}-\int_\Omega c_n\boldsymbol{v}_n\cdot\nabla c_n\,d\boldsymbol{x}$$
$$=\int_\Gamma c_n^2\boldsymbol{v}_n\cdot\boldsymbol{\nu}\,d\Gamma-\int_\Omega c_n\boldsymbol{v}_n\cdot\nabla c_n\,d\boldsymbol{x}.$$

We combine boundary conditions (3.3), $(3.4)_1$, $(3.5)_1$ and $(3.6)_1$ to see

$$\int_\Omega(\boldsymbol{v}_n\cdot\nabla c_n)\cdot c_n\,d\boldsymbol{x}=\frac{1}{2}\int_{\Gamma_2\cup\Gamma_3}c_n^2\boldsymbol{v}_n\cdot\boldsymbol{\nu}\,d\Gamma\geq-\frac{\rho_n}{2}\int_{\Gamma_2}c_n^2\,d\Gamma\geq-\frac{\rho_n}{2}\|\gamma_1\|^2\|c_n\|_{W^{1,p_2}(\Omega)}^2.\tag{4.14}$$

Applying Hölder's inequality, it follows that

$$\begin{aligned}\int_{\Gamma_2\cup\Gamma_3}\omega_n c_n\,d\Gamma \le&\|\omega_n\|_{L^{p_2'}(\Gamma_2\cup\Gamma_3)}\|c_n\|_{L^{p_2}(\Gamma_2\cup\Gamma_3)}\\ =&\|\omega_n\|_{L^{p_2'}(\Gamma_2\cup\Gamma_3)}\|\gamma_2\|\|c_n\|_{W^{1,p_2}(\Omega)},\end{aligned} \tag{4.15}$$

where γ_2 is the trace operator from $W^{1,p_2}(\Omega)$ to $L^{p_2}(\Gamma)$. Combining this with (4.12)–(4.15), we have

$$\begin{aligned}&a_{\kappa_n}\|\nabla c_n\|^{p_2}_{L^{p_2}(\Omega;\mathbb{R}^d)}+\|c_{g_n}\|_{L^1(\Omega)}+d_{g_n}\|c_n\|^{\theta}_{L^{\theta}(\Omega)}-\frac{\rho_n}{2}\|\gamma_1\|^2\|c_n\|^2_{W^{1,p_2}(\Omega)}\\ \le&\|\omega_n\|_{L^{p_2'}(\Gamma_2\cup\Gamma_3)}\|\gamma_2\|\|c_n\|_{W^{1,p_2}(\Omega)}.\end{aligned} \tag{4.16}$$

Recall that $\|c_n\|_{W^{1,p_2}(\Omega)}\to\infty$ as $n\to\infty$. Passing to the limit as $n\to\infty$ in (4.16), and using the assumption $\theta\ge p_2$ and the inequality

$$\inf_{\varepsilon>0}(\min\{a_{\kappa_\varepsilon},d_{g_\varepsilon}\delta(\theta)\}-\delta(p_2)\frac{\rho_\varepsilon}{2}\|\gamma_1\|^2)>0,$$

we conclude that

$$\begin{aligned}&+\infty\\ &=\lim_{n\to\infty}\frac{1}{\|c_n\|_{W^{1,p_2}(\Omega)}}\left(a_{\kappa_n}\|\nabla c_n\|^{p_2}_{L^{p_2}(\Omega;\mathbb{R}^d)}+d_{g_n}\|c_n\|^{\theta}_{L^{\theta}(\Omega)}-\frac{\rho_n}{2}\|\gamma_1\|^2\|c_n\|^2_{W^{1,p_2}(\Omega)}+\|c_{g_n}\|_{L^1(\Omega)}\right)\\ &\le\|\omega_n\|_{L^{p_2'}(\Gamma_2\cup\Gamma_3)}\|\gamma_2\|,\end{aligned}$$

which triggers a contradiction. Consequently, we conclude that $\{c_n\}$ is uniformly bounded in $W^{1,p_2}(\Omega)$. By using the reflexivity of $E\times W^{1,p_2}(\Omega)$ and passing to a subsequence if necessary, we may suppose that there exist a subsequence of $\{(\boldsymbol{v}_n,c_n)\}$, still denoted by the same way, and a pair of functions $(\boldsymbol{v},c)\in E\times W^{1,p_2}(\Omega)$ such that

$$(\boldsymbol{v}_n,c_n)\rightharpoonup(\boldsymbol{v},c)\ \text{ in } E\times W^{1,p_2}(\Omega)\ \text{ as }\ n\to\infty. \tag{4.17}$$

Step 2. *$(\boldsymbol{v}_n,c_n)$ converges strongly to $(\boldsymbol{v},c)$ in $E\times W^{1,p_2}(\Omega)$.*

Remark 3.1 reveals that $K_\varepsilon\xrightarrow{M}K$ as $\varepsilon\to0$. Invoking the convergence (4.17), we infer that $(\boldsymbol{v},c)\in K\times W^{1,p_2}(\Omega)$. Moreover, for each $n\in\mathbb{N}$, one has

$$\begin{aligned}&\langle A_n\boldsymbol{v}_n+B[\boldsymbol{v}_n],\boldsymbol{w}_n-\boldsymbol{v}_n\rangle+\int_{\Gamma_1}j_n^0(\boldsymbol{x},c_n,\gamma\boldsymbol{v}_n;\gamma(\boldsymbol{w}_n-\boldsymbol{v}_n))\,d\Gamma\\ &+\int_{\Gamma_2}\phi_n(w_{n_\nu}-v_{n_\nu})\,d\Gamma+\int_{\Gamma_3}\varphi_n(w_{n_\nu}-v_{n_\nu})\,d\Gamma\ge\int_\Omega\boldsymbol{f}_n\cdot(\boldsymbol{w}_n-\boldsymbol{v}_n)\,d\boldsymbol{x}\end{aligned} \tag{4.18}$$

for all $\boldsymbol{w}_n\in K_n$, and

$$\int_\Omega\kappa_n(\boldsymbol{v}_n)\|\nabla c_n\|^{p_2-2}_{\mathbb{R}^d}\nabla c_n\cdot\nabla z\,d\boldsymbol{x}+\int_\Omega g_n(\boldsymbol{x},c_n)z\,d\boldsymbol{x}+\int_\Omega(\boldsymbol{v}_n\cdot\nabla c_n)\cdot z\,d\boldsymbol{x}=\int_{\Gamma_2\cup\Gamma_3}\omega_n z\,d\Gamma \tag{4.19}$$

for all $z \in W^{1,p_2}(\Omega)$. By virtue of $K_\varepsilon \xrightarrow{M} K$ as $\varepsilon \to 0$, there exists a sequence $\{\widetilde{\boldsymbol{v}}_n\}$ such that $\widetilde{\boldsymbol{v}}_n \in K_n$ for each $n \in \mathbb{N}$ and $\widetilde{\boldsymbol{v}}_n \to \boldsymbol{v}$ in E. Putting $\boldsymbol{w}_n = \widetilde{\boldsymbol{v}}_n$ into (4.18), it gives

$$\langle B[\boldsymbol{v}_n], \widetilde{\boldsymbol{v}}_n - \boldsymbol{v}_n\rangle + \int_{\Gamma_1} j_n^0(\boldsymbol{x}, c_n, \gamma\boldsymbol{v}_n; \gamma(\widetilde{\boldsymbol{v}}_n - \boldsymbol{v}_n))\, d\Gamma + \int_{\Gamma_2} \phi_n(\widetilde{v}_{n_\nu} - v_{n_\nu})\, d\Gamma \tag{4.20}$$
$$+ \int_{\Gamma_3} \varphi_n(\widetilde{v}_{n_\nu} - v_{n_\nu})\, d\Gamma + \int_\Omega \boldsymbol{f}_n \cdot (\boldsymbol{v}_n - \widetilde{\boldsymbol{v}}_n)\, d\boldsymbol{x} \geq \langle A_n \boldsymbol{v}_n, \boldsymbol{v}_n - \widetilde{\boldsymbol{v}}_n\rangle.$$

It follows from [15, Theorem 5.1] that the operator $B[\cdot]: E \to E^*$ (see (4.3)) is weakly-weakly continuous and

$$\lim_{n\to\infty} \langle B[\boldsymbol{v}_n], \boldsymbol{v}_n\rangle = \langle B[\boldsymbol{v}], \boldsymbol{v}\rangle. \tag{4.21}$$

Then we have

$$\lim_{n\to\infty} \langle B[\boldsymbol{v}_n], \widetilde{\boldsymbol{v}}_n - \boldsymbol{v}_n\rangle = \lim_{n\to\infty} \langle B[\boldsymbol{v}_n] - B[\boldsymbol{v}], \widetilde{\boldsymbol{v}}_n\rangle + \lim_{n\to\infty} \langle B[\boldsymbol{v}], \widetilde{\boldsymbol{v}}_n\rangle - \lim_{n\to\infty} \langle B[\boldsymbol{v}_n], \boldsymbol{v}_n\rangle = 0. \tag{4.22}$$

By the embedding theorem, it holds $\gamma\boldsymbol{v}_n(\boldsymbol{x}) \to \gamma\boldsymbol{v}(\boldsymbol{x})$ and $c_n(\boldsymbol{x}) \to c(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Gamma$. We employ the hypothesis $H(j_\varepsilon)$(vi) and Fatou's lemma to obtain

$$\begin{aligned} \limsup_{n\to\infty} \int_{\Gamma_1} j_n^0(\boldsymbol{x}, c_n, \gamma\boldsymbol{v}_n; \gamma(\widetilde{\boldsymbol{v}}_n - \boldsymbol{v}_n))\, d\Gamma &\leq \int_{\Gamma_1} \limsup_{n\to\infty} j_n^0(\boldsymbol{x}, c_n, \gamma\boldsymbol{v}_n; \gamma\widetilde{\boldsymbol{v}}_n - \gamma\boldsymbol{v}_n)\, d\Gamma \\ &\leq \int_{\Gamma_1} j^0(\boldsymbol{x}, c, \gamma\boldsymbol{v}; \boldsymbol{0})\, d\Gamma = 0. \end{aligned} \tag{4.23}$$

Also, we apply the embedding theorem and hypotheses $H(\phi_\varepsilon)$, $H(\varphi_\varepsilon)$ and $H(\boldsymbol{f}_\varepsilon)$ to obtain

$$\lim_{n\to\infty} \left(\int_{\Gamma_2} \phi_n(\widetilde{v}_{n_\nu} - v_{n_\nu})\, d\Gamma + \int_{\Gamma_3} \varphi_n(\widetilde{v}_{n_\nu} - v_{n_\nu})\, d\Gamma + \int_\Omega \boldsymbol{f}_n \cdot (\boldsymbol{v}_n - \widetilde{\boldsymbol{v}}_n)\, d\boldsymbol{x} \right) = 0. \tag{4.24}$$

From the hypothesis $H(\boldsymbol{C}_\varepsilon)$(v) and the definition of A and A_ε (see (4.1) and (4.2)), we have

$$\begin{aligned} &|\langle A_n \boldsymbol{v}_n - A\boldsymbol{v}_n, \boldsymbol{v}_n - \boldsymbol{v}\rangle| \\ &\leq \int_\Omega \|\boldsymbol{C}_n(\boldsymbol{D}(\boldsymbol{v}_n)) - \boldsymbol{C}(\boldsymbol{D}(\boldsymbol{v}_n))\|_{\mathbb{S}^d} \|\boldsymbol{D}(\boldsymbol{v}_n) - \boldsymbol{D}(\boldsymbol{v})\|_{\mathbb{S}^d}\, d\boldsymbol{x} \\ &\leq \int_\Omega m_C \alpha_n h_1(\boldsymbol{D}(\boldsymbol{v}_n)) \|\boldsymbol{D}(\boldsymbol{v}_n) - \boldsymbol{D}(\boldsymbol{v})\|_{\mathbb{S}^d}\, d\boldsymbol{x} \\ &\leq \int_\Omega m_C \alpha_n (c_{h_1} + d_{h_1} \|\boldsymbol{D}(\boldsymbol{v}_n)\|_{\mathbb{S}^d}^{p_1 - 1}) \|\boldsymbol{D}(\boldsymbol{v}_n) - \boldsymbol{D}(\boldsymbol{v})\|_{\mathbb{S}^d}\, d\boldsymbol{x}, \end{aligned}$$

where $\alpha_n = \alpha_{\varepsilon_n}$. Hence

$$\lim_{n\to\infty} \langle A_n \boldsymbol{v}_n - A\boldsymbol{v}_n, \boldsymbol{v}_n - \boldsymbol{v}\rangle = 0. \tag{4.25}$$

It follows from [15, Lemma 5.2] that operator $A: E \to E^*$ (see (4.1)) is bounded, continuous, maximal monotone and of type (S_+). Passing to the upper limit as $n \to \infty$ in inequality (4.20), and using inequalities (4.22)–(4.25), we find

$$\begin{aligned}
&\limsup_{n\to\infty}\langle A\boldsymbol{v}_n,\boldsymbol{v}_n-\boldsymbol{v}\rangle\\
\le&\limsup_{n\to\infty}\langle A\boldsymbol{v}_n-A_n\boldsymbol{v}_n+A_n\boldsymbol{v}_n,\boldsymbol{v}_n-\widetilde{\boldsymbol{v}}_n\rangle\\
\le&\limsup_{n\to\infty}\Bigg(\langle B[\boldsymbol{v}_n],\widetilde{\boldsymbol{v}}_n-\boldsymbol{v}_n\rangle+\int_{\Gamma_1} j_n^0(\boldsymbol{x},c_n,\gamma\boldsymbol{v}_n;\gamma(\widetilde{\boldsymbol{v}}_n-\boldsymbol{v}_n))\,d\Gamma\\
&+\int_{\Gamma_2}\phi_n(\widetilde{v}_{n_\nu}-v_{n_\nu})\,d\Gamma+\int_{\Gamma_3}\varphi_n(\widetilde{v}_{n_\nu}-v_{n_\nu})\,d\Gamma+\int_\Omega \boldsymbol{f}_n\cdot(\boldsymbol{v}_n-\widetilde{\boldsymbol{v}}_n)\,d\boldsymbol{x}\Bigg)\\
\le&\,0.
\end{aligned}$$

We infer that $\boldsymbol{v}_n\to\boldsymbol{v}$ in E as $n\to\infty$. Besides, we insert $z=c-c_n$ into inequality (4.19) to have

$$\begin{aligned}
&\int_\Omega g_n(\boldsymbol{x},c_n)(c-c_n)\,d\boldsymbol{x}+\int_\Omega(\boldsymbol{v}_n\cdot\nabla c_n)\cdot(c-c_n)\,d\boldsymbol{x}+\int_{\Gamma_2\cup\Gamma_3}\omega_n(c_n-c)\,d\Gamma\\
=&\int_\Omega \kappa_n(\boldsymbol{v}_n)\|\nabla c_n\|_{\mathbb{R}^d}^{p_2-2}\nabla c_n\cdot\nabla(c_n-c)\,d\boldsymbol{x}.
\end{aligned}\tag{4.26}$$

By the Sobolev embedding theorem, we can see that $c_n\to c$ in $L^{p_2}(\Omega)$. Applying hypothesis $H(g_\varepsilon)$(v), we obtain

$$\lim_{n\to\infty}\left(\int_\Omega g_n(\boldsymbol{x},c_n)(c-c_n)\,d\boldsymbol{x}+\int_\Omega(\boldsymbol{v}_n\cdot\nabla c_n)\cdot(c-c_n)\,d\boldsymbol{x}\right)=0.\tag{4.27}$$

It follows from the hypothesis $H(\omega_\varepsilon)$ that

$$\lim_{n\to\infty}\int_{\Gamma_2\cup\Gamma_3}\omega_n(c_n-c)\,d\Gamma=0.\tag{4.28}$$

Passing to upper limit as $n\to\infty$ in inequality (4.26) and using the inequalities (4.27)–(4.28), we have

$$\begin{aligned}
0=&\lim_{n\to\infty}\left(\int_\Omega g_n(\boldsymbol{x},c_n)(c-c_n)\,d\boldsymbol{x}+\int_\Omega(\boldsymbol{v}_n\cdot\nabla c_n)\cdot(c-c_n)\,d\boldsymbol{x}+\int_{\Gamma_2\cup\Gamma_3}\omega_n(c_n-c)\,d\Gamma\right)\\
=&\limsup_{n\to\infty}\int_\Omega\kappa_n(\boldsymbol{v}_n)\|\nabla c_n\|_{\mathbb{R}^d}^{p_2-2}\nabla c_n\cdot\nabla(c_n-c)\,d\boldsymbol{x}\\
\ge&\liminf_{n\to\infty}\int_\Omega\left(\kappa_n(\boldsymbol{v}_n)-\frac{a_0}{2}\right)\|\nabla c_n\|_{\mathbb{R}^d}^{p_2-2}\nabla c_n\cdot\nabla(c_n-c)\,d\boldsymbol{x}\\
&+\limsup_{n\to\infty}\int_\Omega\frac{a_0}{2}\|\nabla c_n\|_{\mathbb{R}^d}^{p_2-2}\nabla c_n\cdot\nabla(c_n-c)\,d\boldsymbol{x}\\
\ge&\liminf_{n\to\infty}\int_\Omega\left(\kappa_n(\boldsymbol{v}_n)-\frac{a_0}{2}\right)\|\nabla c\|_{\mathbb{R}^d}^{p_2-2}\nabla c\cdot\nabla(c_n-c)\,d\boldsymbol{x}\\
&+\limsup_{n\to\infty}\int_\Omega\frac{a_0}{2}\|\nabla c_n\|_{\mathbb{R}^d}^{p_2-2}\nabla c_n\cdot\nabla(c_n-c)\,d\boldsymbol{x}\\
=&\limsup_{n\to\infty}\int_\Omega\frac{a_0}{2}\|\nabla c_n\|_{\mathbb{R}^d}^{p_2-2}\nabla c_n\cdot\nabla(c_n-c)\,d\boldsymbol{x},
\end{aligned}$$

where we used the monotonicity of the p_2-Laplace operator and hypothesis $H(\kappa_\varepsilon)$(v). Combined with (S_+)-property of $-\operatorname{div}(\kappa(\boldsymbol{v})|\nabla c|^{p_2-2}\nabla c)$, we obtain $c_n \to c$ in $W^{1,p_2}(\Omega)$. Therefore, we conclude that $(\boldsymbol{v}_n, c_n) \to (\boldsymbol{v}, c)$ in $E \times W^{1,p_2}(\Omega)$, which means that $w\text{--}\limsup_{n\to\infty} S_n \neq \emptyset$ and $w\text{--}\limsup_{n\to\infty} S_n = s\text{--}\limsup_{n\to\infty} S_n$.

Step 3. $(\boldsymbol{v}, c) \in K \times W^{1,p_2}(\Omega)$ *is also a weak solution of Problem* 1.1.

Let $\boldsymbol{u} \in K$. Because of the convergence $K_\varepsilon \xrightarrow{M} K$ as $\varepsilon \to 0$, by condition (i) of Definition 2.2, there exists a sequence $\{\boldsymbol{u}_n\}$ such that $\boldsymbol{u}_n \in K_n$ and $\boldsymbol{u}_n \to \boldsymbol{u}$ in E as $n \to \infty$. Inserting $\boldsymbol{w}_n = \boldsymbol{u}_n$ in (4.18), we see that

$$\langle A_n \boldsymbol{v}_n + B[\boldsymbol{v}_n], \boldsymbol{u}_n - \boldsymbol{v}_n\rangle + \int_{\Gamma_1} j_n^0(\boldsymbol{x}, c_n, \gamma\boldsymbol{v}_n; \gamma(\boldsymbol{u}_n - \boldsymbol{v}_n))\, d\Gamma \tag{4.29}$$
$$+ \int_{\Gamma_2} \phi_n(u_{n_\nu} - v_{n_\nu})\, d\Gamma + \int_{\Gamma_3} \varphi_n(u_{n_\nu} - v_{n_\nu})\, d\Gamma \geq \int_\Omega \boldsymbol{f}_n \cdot (\boldsymbol{u}_n - \boldsymbol{v}_n)\, d\boldsymbol{x}.$$

Since the operator $A\colon E \to E^*$ is continuous, we deduce from (4.25) that $A_n\boldsymbol{v}_n \to A\boldsymbol{v}$ in E, and

$$\lim_{n\to\infty} \langle A_n \boldsymbol{v}_n, \boldsymbol{u}_n - \boldsymbol{v}_n\rangle = \langle A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}\rangle. \tag{4.30}$$

The weak-weak continuity of B and (4.21) prove that

$$\lim_{n\to\infty} \langle B[\boldsymbol{v}_n], \boldsymbol{u}_n - \boldsymbol{v}_n\rangle = \langle B[\boldsymbol{v}], \boldsymbol{u} - \boldsymbol{v}\rangle. \tag{4.31}$$

By the hypothesis $H(j_\varepsilon)$(vi), we have

$$\limsup_{n\to\infty} \int_{\Gamma_1} j_n^0(\boldsymbol{x}, c_n, \gamma\boldsymbol{v}_n; \gamma(\boldsymbol{u}_n - \boldsymbol{v}_n))\, d\Gamma \leq \int_{\Gamma_1} j^0(\boldsymbol{x}, c, \gamma\boldsymbol{v}; \gamma(\boldsymbol{u} - \boldsymbol{v}))\, d\Gamma. \tag{4.32}$$

We pass to the upper limit as $n \to \infty$ in inequality (4.29), and use hypotheses $H(\phi_\varepsilon)$, $H(\varphi_\varepsilon)$, $H(\boldsymbol{f}_\varepsilon)$ with (4.30)-(4.32) to see that

$$\begin{aligned} 0 \leq \limsup_{n\to\infty} \Big(&\langle A_n \boldsymbol{v}_n + B[\boldsymbol{v}_n], \boldsymbol{u}_n - \boldsymbol{v}_n\rangle + \int_{\Gamma_1} j_n^0(\boldsymbol{x}, c_n, \gamma\boldsymbol{v}_n; \gamma(\boldsymbol{u}_n - \boldsymbol{v}_n))\, d\Gamma \\ &+ \int_{\Gamma_2} \phi_n(u_{n_\nu} - v_{n_\nu})\, d\Gamma + \int_{\Gamma_3} \varphi_n(u_{n_\nu} - v_{n_\nu})\, d\Gamma + \int_\Omega \boldsymbol{f}_n \cdot (\boldsymbol{v}_n - \boldsymbol{u}_n)\, d\boldsymbol{x} \Big) \\ \leq &\langle A\boldsymbol{v} + B[\boldsymbol{v}], \boldsymbol{u} - \boldsymbol{v}\rangle + \int_{\Gamma_1} j^0(\boldsymbol{x}, c, \gamma\boldsymbol{v}; \gamma(\boldsymbol{u} - \boldsymbol{v}))\, d\Gamma \\ &+ \int_{\Gamma_2} \phi(u_\nu - v_\nu)\, d\Gamma + \int_{\Gamma_3} \varphi(u_\nu - v_\nu)\, d\Gamma + \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u} - \boldsymbol{v})\, d\boldsymbol{x}. \end{aligned} \tag{4.33}$$

On the other hand, letting $n \to \infty$ in inequality (4.19) and using the conditions $H(\kappa_\varepsilon)$, $H(g_\varepsilon)$, $H(\omega_\varepsilon)$, we deduce

$$\int_\Omega \kappa(\boldsymbol{v}) \|\nabla c\|_{\mathbb{R}^d}^{p_2-2} \nabla c \cdot \nabla z\, d\boldsymbol{x} + \int_\Omega g(\boldsymbol{x}, c) z\, d\boldsymbol{x} + \int_\Omega (\boldsymbol{v} \cdot \nabla c) \cdot z\, d\boldsymbol{x} - \int_{\Gamma_2 \cup \Gamma_3} \omega z\, d\Gamma = 0.$$

This fact together with (4.33) and the arbitrarity of $\boldsymbol{u} \in K$ implies that $(\boldsymbol{v}, c) \in K \times W^{1,p_2}(\Omega)$ is a weak solution of the recovered fracturing fluid model (Problem 1.1).

Consequently, we conclude that, for each $n \in \mathbb{N}$, if $(\boldsymbol{v}_n, c_n)$ is a weak solution to the perturbed problem (Problem 3.1) with $\varepsilon = \varepsilon_n$, then there exists a subsequence of $\{(\boldsymbol{v}_n, c_n)\}$, still denoted by the same way, and $(\boldsymbol{v}, c) \in E \times W^{1,p_2}(\Omega)$ such that $(\boldsymbol{v}_n, c_n) \to (\boldsymbol{v}, c)$ in $E \times W^{1,p_2}(\Omega)$ as

$n \to \infty$, and $(\boldsymbol{v}, c)$ is a weak solution to the recovered fracturing fluid model (Problem 1.1). Therefore, the stability result $w\text{--}\limsup_{n\to\infty} S_n = s\text{--}\limsup_{n\to\infty} S_n \subset S$ is valid.

□

Remark 4.1. It follows from Theorem 4.1 that if the recovered fracturing fluid model (Problem 1.1) has the unique weak solution, and the perturbed problem (Problem 3.1) also has the unique solution, then the stability result stated in Theorem 4.1 reduces to following one

$$(\boldsymbol{v}_n, c_n) \to (\boldsymbol{v}, c) \text{ in } E \times W^{1,p_2}(\Omega),$$

where $(\boldsymbol{v}_n, c_n)$ and $(\boldsymbol{v}, c)$ are the unique weak solutions of Problems 3.1 and 1.1, respectively.

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REFERENCES

[1] W.M. Han, Y. Li, Stability analysis of stationary variational and hemivariational inequalities with applications, Nonlinear Anal. 50 (2019), 171-191.
[2] T.C.Y. Chan, P.A. Mar, Stability and continuity in robust optimization, SIAM J. Optim. 27 (2017), 817-841.
[3] G.H. Yang, H. Yang, Stability of weakly Pareto-Nash equilibria and Pareto-Nash equilibria for multiobjective population games, Set-Valued Var. Anal. 25 (2017), 427-439.
[4] Y.B. Xiao, M.T. Liu, T. Chen, N.J. Huang, Stability analysis for evolutionary variational-hemivariational inequalities with constraint sets, Sci. China Math. 65 (2022), 1469-1484.
[5] C.J. Fang, W.M. Han, Stability analysis and optimal control of a stationary Stokes hemivariational inequality, Evol. Equ. Control Theory 9 (2020), 995-1008.
[6] J. Gwinner, On a new class of differential variational inequalities and a stability result, Math. Program. 139 (2013), 205-221.
[7] Z.Y. Peng, J.W. Peng, X.J. Long, J.C. Yao, On the stability of solutions for semi-infinite vector optimization problems, J. Glob. Optim. 70 (2018), 55-69.
[8] X. Wang, W. Li, X.S. Li, N.J. Huang, Stability for differential mixed variational inequalities, Optim. Lett. 8 (2014), 1873-1887.
[9] E. Ghanbari, M.A. Abbasi, H. Dehghanpour, D. Bearinger, Flowback volumetric and chemical analysis for evaluating load recovery and its impact on early-time production, Presented at the SPE Unconventional Resources Conference Canada, Calgary, Alberta, Canada, November, SPE-167165-MS, 2013.
[10] M.A. Abbasi, D.O. Ezulike, H. Dehghanpour, R.V. Hawkes, A comparative study of flowback rate and pressure transient behavior in multifractured horizontal wells completed in tight gas and oil reservoirs, J. Nat. Gas Sci. Eng. 17 (2014), 82-93.
[11] A. Alkouh, S. McKetta, R.A. Wattenbarger, Estimation of effective-fracture volume using water-flowback and production data for shale-gas wells, J. Can. Petrol. Technol. 53 (2014), 290-303.
[12] X.X. Dong, W.J. Li, H.H. Wang, Research on convection-reaction-diffusion model of contaminants in fracturing flowback fluid in non-equidistant fractures with arbitrary inclination of shale gas development, J. Petrol. Sci. Eng. 208 (2022), 109479.

[13] A. Zolfaghari, H. Dehghanpour, E. Ghanbari, D. Bearinger, Fracture characterization using flowback salt-concentration transient, SPE J. 21 (2016), 233-244.
[14] F.Y. Zhang, H. Emami-Meybodi, Analysis of early-time production data from multi-fractured shale gas wells by considering multiple transport mechanisms through nanopores, J. Petrol. Sci. Eng. 197 (2021), 108092.
[15] J.X. Cen, S. Migórski, J.C. Yao, S.D. Zeng, Variational-hemivariational system for contaminant convection-reaction-diffusion model of recovered fracturing fluid, Adv. Nonlinear Anal. 13 (2024), 20230141.
[16] F.H. Clarke, Generalized gradients and applications, Trans. Amer. Math. Soc. 205 (1975), 247-262.
[17] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, Interscience, New York, 1983.
[18] Z. Denkowski, S. Migórski, N.S. Papageorgiou, An Introduction to Nonlinear Analysis: Theory, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.
[19] Z. Denkowski, S. Migórski, N.S. Papageorgiou, An Introduction to Nonlinear Analysis: Applications, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.
[20] S. Migórski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Advances in Mechanics and Mathematics 26, Springer, New York, 2013.
[21] U. Mosco, Covergence of convex sets and of solutions variational inequalities, Adv. Math. 3 (1969), 510-585.
[22] S. Migórski, S. Dudek, A class of variational-hemivariational inequalities for Bingham type fluids, Appl. Math. Optimi. 85 (2022), 16.