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THE TWO-GUARD PROBLEM ON CURVININEAR POLYGONS

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Abstract. Given a simple polygon on the plane with two distinct vertices, s and t, the original two-guard problem asks whether there is a route for two guards to simultaneously walk along the two boundary chains from s to t so that they are always mutually visible. We study a generalization of this problem to *curvilinear polygons*, in which the boundary consists of a finite number of curved pieces. We focus on locally-convex polygons, which are polygons with locally convex arcs, and we solve this problem in $\mathcal{O}(n^2)$ time for a curvilinear polygon with n edges, by either constructing a required route or deciding that such route does not exist.

Keywords. Curvilinear polygon; Piecewise locally convex polygon; Two-guard problem; Visibility.

1. Introduction

Visibility is an important concept in road network surveillance, robotics, motion planning, and security [5]. The formal definition of visibility is as follows.

Definition 1.1. Given a set $P \subseteq \mathbb{R}^n$, we say that a point $x \in P$ is visible from a point $y \in P$ if and only if the line segment $\overline{xy} = \{\lambda x + (1-\lambda)y | \lambda \in [0,1]\}$ is entirely contained in P.

Real-life surveillance systems are usually modeled by using simple polygons, which are closed regions in $\mathbb{R}^2$ with the border made up of finite chains of straight line segments. *The two-guard problem* is an important class of visibility problems. This problem asks for a walk of two points (*guards*) on the boundary of a simple polygon P from the starting vertex s to the ending vertex t, one clockwise and one counterclockwise, such that the guards are always mutually visible. A more formal statement of the problem is given in Definition 1.4 below.

Let a polygon P be given by a simple, closed, and polygonal chain. Any two distinct points s and t on its boundary divide the polygonal chain into two subchains, denoted by L and R, corresponding to the clockwise and counter-clockwise walks from s to t, respectively.

Definition 1.2. Given a simple polygon P and its two vertices s and t, a two-guard boundary walk from s to t is a pair (l,r) of continuous functions such that:

(1) $l:[0,1] \to L, r:[0,1] \to R$,
(2) $l(0) = r(0) = s, l(1) = r(1) = t$,

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(3) $l(x)$ is visible from $r(x)$ for all $x \in [0,1]$.

Any line segment $\overline{l(x)r(x)}, x \in [0,1]$ is called a walk line segment of the two-guard walk. The point $r(x)$ is the walk partner of $l(x)$, and vice versa.

Definition 1.3. A two-guard walk from s to t on P is called straight if both l and r are non-decreasing with respect to the s to t orientation of L and R. The polygon P is called walkable from s to t if it admits a straight walk.

Definition 1.4. Given a polygon P and s,t on the boundary of P, the TWO-GUARD problem is to determine if P is walkable from s to t.

The two-guard problem was first introduced by Icking and Klein [8], who developed an $\mathcal{O}(n\log n)$ time algorithm to decide whether P is walkable. Soon after, Heffernan [7] proposed a linear-time algorithm to solve this problem. Later, Tseng et al. [21] proposed an $\mathcal{O}(n\log n)$-time method to compute all pairs (s,t) of vertices such that P is walkable from s to t, and Bhattacharya et al. [2] developed an optimal $\mathcal{O}(n)$ algorithm for the same problem. There has been a considerable amount of research towards generalizing this problem in various directions [24]. Aurenhammer et al. [1] considered the problem of partially walking a non-walkable polygon, which asks how far the two guards can reach from a given source vertex while staying mutually visible. They showed that there can be $\Theta(n)$ maximal walks of this type and all of them can be found in $\mathcal{O}(n\log n)$ time. Crass et al. studied a modified version called ∞-searcher in an open-edge "corridor". Several researchers [3, 14, 15, 16, 18, 23] generalized the *two-guard* problem to the setting of *rooms*, in which a room is a simple polygon with a designated point on its boundary called the *door*. Suzuki and Yamashita [17] formulated a more general framework of *polygon search problems* and [4, 9, 11, 12, 19, 22] contributed to this framework.

The generalizations considered in the literature assume that the polygon is defined by line segments, whereas in real-life applications the boundaries are often curves instead of line segments. Therefore, generalizing the two-guard problem to a polygon with curves as its boundaries is of interest. Such polygons, referred to as *curvilinear polygons*, were first studied in the context of guarding problems by Karavelas [10]. The curvilinear polygons considered in [10] are assumed to be both piecewise locally convex and made up of convex arcs in order to admit triangulation. However, in our setting of the two-guard problem, the curvilinear polygon is only required to be piecewise locally convex.

Let $v_1,...,v_n, n \geq 2$ be a set of points and let $a_1,...,a_n$ be a set of curvilinear smooth Jordan arcs such that a_i has the points v_i and v_{i+1} as endpoints (here and below, all indices are assumed to be taken $\mod n$, so if $i=n$ then $i+1$ should be replaced with 1). Assume that the arcs a_i and a_j $(i \neq j)$ intersect only if $i = j+1$ or $j = i+1$, and they only intersect at v_i or v_{i+1}. A curvilinear polygon P is the closed region of the plane delimited by the arcs $a_i, i = 1,\dots,n$. The points v_i are the vertices of P.

Definition 1.5. A curvilinear polygon P is called a *piecewise locally convex polygon* if for every non-vertex point p on the boundary of P there exists a disk D_p centered at p such that $P \cap D_p$ is a convex set.

Figure 1 shows an example of a piecewise locally convex polygon. Note that one of its arcs is non-convex and the local convexity requirement imposed on non-vertex points in Definition 1.5 is not satisfied in vertex s. In the two-guard problem, we are interested in ensuring visibility

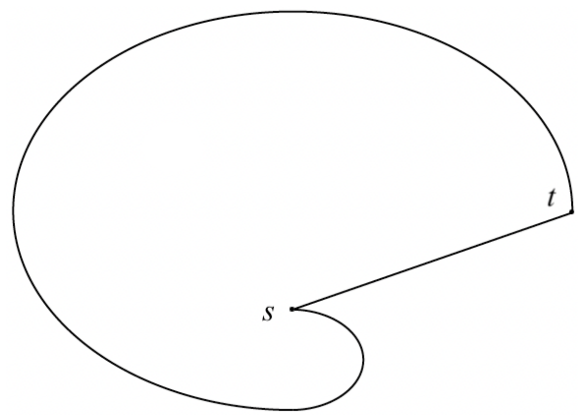

FIGURE 1. Illustration of a piecewise locally convex polygon.

along the guard walks; hence, it is natural to require piecewise local convexity for the conisdered curvilinear polygons, since this property can ensure visibility locally. The two-guard problem in piecewise locally convex polygons is stated as follows.

Definition 1.6. Given a piecewise locally convex polygon P and two points s and t on the boundary of P, the CURVILINEAR TWO-GUARD problem is to determine if P is walkable from s to t.

The remainder of this paper is organized as follows. In Section 2, we analyze the properties of piecewise locally convex polygons essential for solving the curvilinear two-guard problem. In Section 3, we investigate the necessary and sufficient conditions for a piecewise locally convex polygon to be walkable in analogy to the original two-guard problem. In Section 4, we develop an algorithm to construct a solution for the case of walkable curvilinear polygon. Finally, Section 5 concludes this paper and discusses potential future research directions.

2. PROPERTIES OF A PIECEWISE LOCALLY CONVEX POLYGON

In this section, we develop the tools we will use to solve the curvilinear two-guard problem. We will use the following notations, definitions, and results. For $S \subseteq \mathbb{R}^n$, let ∂S denote the boundary of S. By a neighborhood of $x \in \mathbb{R}^n$ we will mean an open ball of positive radius centered at x.

Definition 2.1 ([13])**.** For $S \subseteq \mathbb{R}^n$ and $x \in \partial S$, we say that S is *weakly supported at x locally* if there exists a neighborhood $N(x)$ of x and a linear functional f (f $\not\equiv 0$) such that if $y \in N(x) \setminus \{x\}$ and $f(y) > f(x)$ then $y \notin S$.

Proposition 2.1 (Tietze's Theorem, see p. 110 of [13])**.** *An open connected subset S of $\mathbb{R}^n$ is convex if and only if S is weakly supported locally at each of its boundary points.*

Proposition 2.2 ([20])**.** *Every point on a convex curve γ has a supporting line (supporting hyperplane in $\mathbb{R}^2$). Furthermore, if γ is smooth, then it has tangent line and the tangent line is always a supporting line.*

Given a piecewise locally convex polygon P with the vertices $v_1,\dots,v_n, n \geq 2$ and the arcs $a_1,\dots,a_n$ such that a_i has the points v_i and v_{i+1} as endpoint, we consider v_i and a_i (see Figure 2). We would like to guarantee the existence of directional tangents of the polygon.

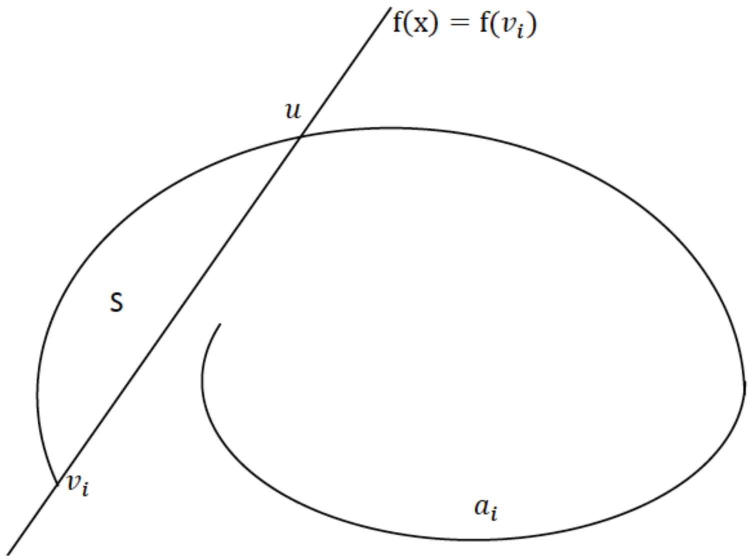

FIGURE 2. Illustration of v_i and a_i.

Proposition 2.3. *Suppose v_i and a_i are as above, then there exists $u \in a_i$ such that the set S made up of the line segment $\overline{v_i u}$ and the segment of a_i between v_i and u (denoted by $a_i[v_i,u]$) is a convex set.*

Proof. Since P is locally convex in all points except for the n points $v_1,\dots,v_n$, there exists $u \in a_i$ such that the open line segment $\overline{v_i u} \subset intP$ and $a_i[v_i,u]$ is entirely contained in one half-plane defined by $f(x) = f(v_i)$ for the functional f corresponding to the line through v_i and u (see Figure 2). Let S be the set defined by the border consisting of $a_i[v_i,u]$ and $\overline{v_i u}$. Then by definition, $int(S)$ is weakly supported at v_i locally with any $N(v_i)$ and f. Since S is locally convex at all points other than v_i, $int(S)$ is weakly supported at its boundary points other than v_i locally by Tietze's Theorem (Proposition 2.1). Now, applying Tietze's Theorem in the opposite direction, $int(S)$ is convex and hence S is convex. □

Proposition 2.3 shows that $a_i[v_i,u]$ is a convex arc, so the tangent line of $a_i[v_i,u]$ at v_i exists; denote it by $T_{a_i}(v_i)$, as illustrated in Figure 3.

Suppose that we are given a piecewise locally convex polygon P and two distinct points s and t on its boundary. Analogously to the case of a simple polygon, we denote by L and R, respectively, the two oriented subchains formed by the arcs of P along the two alternative paths from s to t.

Let v_{i-1}, v_i, v_{i+1} be three consecutive vertices on L (if $v_i = s$, v_{i-1} is the vertex from R that neighbors s; if $v_i = t, v_{i+1}$ is the vertex from R that neighbors t) and a_{i-1} and a_i are the arcs whose endpoints are v_{i-1},v_i and v_i, v_{i+1}, respectively. Let $T_{a_{i-1}}(v_i)$ and $T_{a_i}(v_i)$ be the tangents at v_i as an endpoint of a_{i-1} and as an endpoint of a_i, respectively.

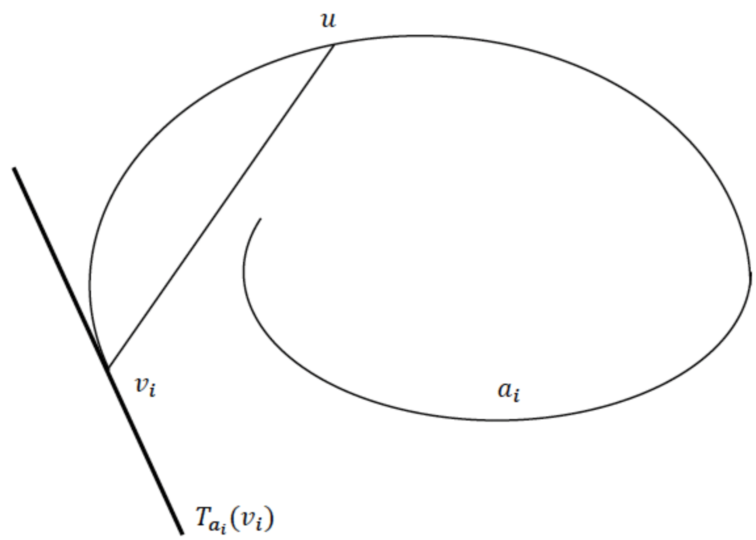

FIGURE 3. Illustration of v_i and a_i.

Obviously, if $T_{a_{i-1}}(v_i)$ is outside P in a neighborhood of v_i, $T_{a_i}(v_i)$ is also outside P in this area, and vice versa. In this case, we call v_i a *straight vertex*. Otherwise, we call v_i a *reflex vertex*.

If v_i is a reflex vertex, let $d^-(v_i)$ be the direction of $T_{a_{i-1}}(v_i)$ from outside of P to inside of P and let $d^+(v_i)$ be the opposite direction of $T_{a_i}(v_i)$. Denote the first intersection point (other than v_i) of the ray originating from v_i in the direction of $d^-(v_i)$ and the boundary of P by $t^-(v_i)$. Analogously, $t^+(v_i)$ is the first intersection point (other than v_i) of the ray from v_i in the direction of $d^+(v_i)$ and P (see Figure 4).

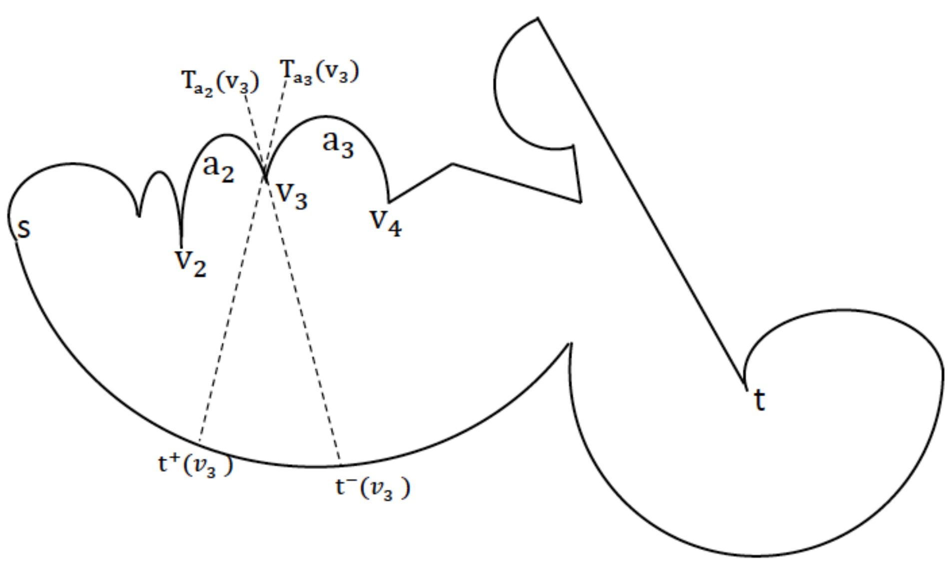

FIGURE 4. Illustration of definitions of $T_{a_{i-1}}(v_i)$, $T_{a_i}(v_i)$, $t^+(v_i)$, and $t^-(v_i)$.

By Proposition 2.2, $T_{a_i}(v_i)$ is a supporting line of S, so S is entirely in one half-plane formed by $T_{a_i}(v_i)$. Hence, a_i in a neighborhood of v_i is entirely in one half-plane formed by $T_{a_i}(v_i)$.

Next lemma shows that for any point u in the piecewise locally convex polygon P that lies in the different half-plane, we can find a point w on a_i in the neighborhood of v_i such that u and w are not mutually visible. For simplicity, in this lemma we suppose that a_i is entirely in one half-plane formed by $T_{a_i}(v_i)$, but it is easy to see the lemma is still correct in general case. Let a_i, v_i, $T_{a_i}(v_i)$, and $t^+(v_i)$ be as defined above; see Figure 5 for an illustration.

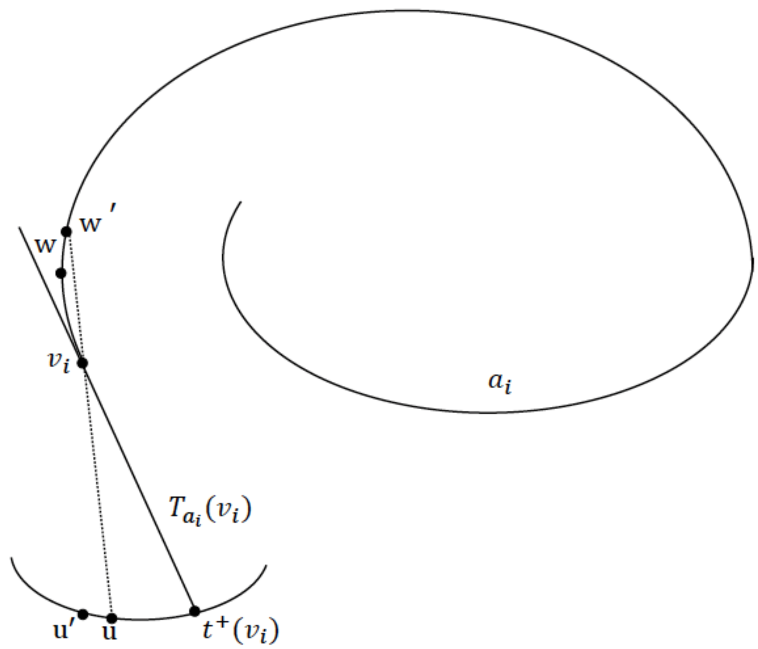

FIGURE 5. An illustration to Lemma 2.1.

Lemma 2.1. *(1) Suppose that u is on the boundary of P and it is in the different from a_i half-plane formed by $T_{a_i}(v_i)$. Then there exists $w \in a_i$ such that u and w are mutually invisible. Furthermore, if u' is also on the boundary of P and the order is u', u and $t^+(v_i)$, then w and u' are mutually invisible. (2) Similarly, if u is on the boundary of P and it is in the different from a_{i-1} half-plane formed by $T_{a_{i-1}}(v_i)$, then there exists $w \in a_{i-1}$ such that u and w are invisible. Furthermore, if u' is also on the boundary of P and the order is $t^-(v_i)$, u and u', then w and u' are invisible.*

Proof. Since $T_{a_i}(v_i)$ is a tangent line of a_i, the line through u and v_i should intersect a_i in another point; suppose it is w'. Since u is in the different from a_i half-plane formed by $T_{a_i}(v_i)$, every line segment from u to a point on a_i in P must cross the line segment $T_{a_i}(v_i)$ between v_i to $t^+(v_i)$. Thus, any point $w \in a_i(v_i, w')$ is not visible from u as the line segment between u and w cannot cross the line segment $T_{a_i}(v_i)$ between v_i and $t^+(v_i)$. Furthermore, if u' is also on the boundary of P and the order is u', u and $t^+(v_i)$, if w is visible from u', then w is also visible from $t^+(v_i)$, and it is easy to show that the set made of line segments $l[w,u']$, $l[w,t^+(v_i)]$ and the boundary of P from u' to $t^+(v_i)$ is convex. So, w should be visible from u, and this is a contradiction. Thus the first statement is true. The proof of the second statement is similar. □

3. NECESSARY AND SUFFICIENT CONDITIONS FOR WALKABILITY

In this section, we develop a necessary and sufficient condition for a piecewise locally convex polygon P to be walkable. As before, we denote by L and R the two sub-chains of the boundary of a piecewise locally convex polygon P corresponding to the clockwise and counter-clockwise walks from s to t, respectively. We use the notation $p < q$ to indicate that p is visited before q when walking from s to t along L or R. In addition, $L_{<p}$ ($L_{>p}$) represents the parts of L preceding (following) p on the walk from s to t. $R_{<p}$ and $R_{>p}$ are defined likewise.

The necessary conditions are summarized in the following lemma.

Lemma 3.1. *If one of the following conditions is satisfied for reflex vertices p, q of P, then P is not walkable.*

(1) $p > t^-(p) \in L$ *or* $p < t^+(p) \in L$ *or* $p < t^+(p) \in R$ *or* $p > t^-(p) \in R$.
(2) $p \in L, q \in R, q < t^+(p) \in R, p < t^+(q) \in L$ *or* $p \in L, q \in R, q > t^-(p) \in R, p > t^-(q) \in L$.
(3) $p, q \in L, p < q, t^-(q) < t^+(p) \in R$ *or* $p, q \in R, q < p, t^-(p) < t^+(q) \in L$.

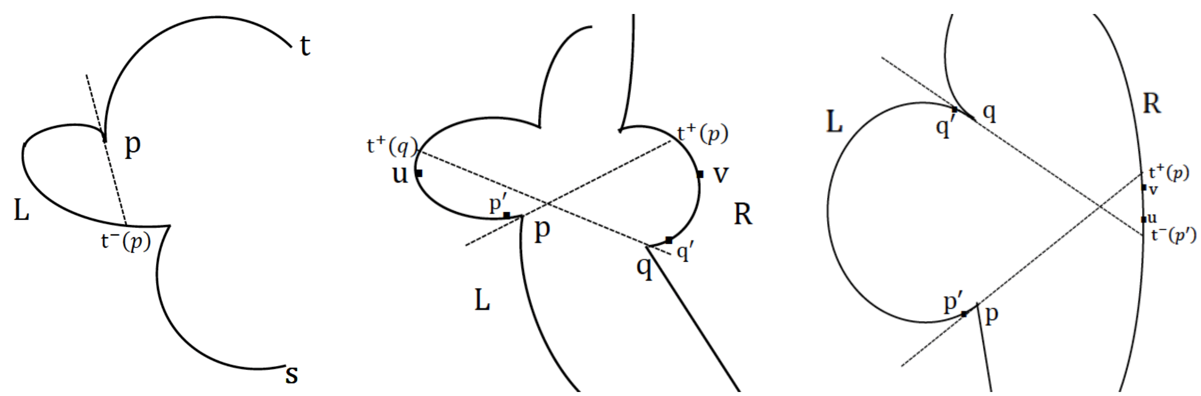

FIGURE 6. Illustration of the three cases considered in Lemma 3.1.

Proof. If case 1 applies, W.L.O.G., we suppose the first alternative holds, see the left image in Figure 6. Denote the boundary curve between p and $t^-(p)$ by a, then by Lemma 2.1, for any point $u \in R$, there exists $w \in a$ such that w is invisible from u or any $u' > u$ in R. So P is not walkable.

If case 2 applies, W.L.O.G., we suppose the first alternative holds, see the middle image in Figure 6. Since $q < t^+(p) \in R$, we can choose $v \in R$ with $q < v < t^+(p)$. By Lemma 2.1, there exists $p' > p$ such that p' is not visible form v and any point in $R_{<v}$. So, any walking partner $\bar{p}$ of p' must satisfy $\bar{p} > v$. Symmetrically, choose $u \in L$ with $p' < u < t^+(q)$, we can find q' with $q' < v$ (so $q' < \bar{p}$) whose walking partner $\bar{q}$ must satisfy $\bar{q} > u > p'$. If P is walkable, $\bar{q} > p'$ implies that a walk must visit p' before q' while $\bar{p} > q'$ implies that a walk must visit q' before p'. We obtained a contradiction. So P is not walkable.

If case 3 applies, again W.L.O.G. we suppose the first alternative holds, see the right image in Figure 6. Choose $u, v \in R$ with $t^-(q) < u < v < t^+(p)$. As before, there exists p' whose walk partner $\bar{p} > v$ and q' whose walk partner $\bar{q} < u$. So $\bar{q} < \bar{p}$. But $p < q$, this is a contradiction. So P is not walkable. ☐

To derive the sufficient conditions, we need the following definitions and lemmas.

Definition 3.1. For every reflex vertex p in L, define

- $hiP(p) = min\{q|q \text{ is a vertex in } R, L \ni t^+(q) > p\}$
- $hiS(p) = min\{t^-(p') \in R|p' \text{ is a vertex in } L_{>p}\}$
- $hi(p) = min\{hiP(p), hiS(p), t\}$
- $loP(p) = max\{q|q \text{ is a vertex in } R, L \ni t^-(q) < p\}$
- $loS(p) = max\{t^+(p') \in R|p' \text{ is a vertex in } L_{<p}\}$
- $lo(p) = max\{loP(p), loS(p), s\}$.

Obviously, lo and hi are monotonically increasing functions in vertices of L. Similarly, we can define lo and hi for vertices of R.

The following two lemmas describe an important relationship between lo and hi.

Lemma 3.2.

(1) If $q < lo(p)$ then $hi(q) < p$; if $q > hi(p)$, then $lo(q) > p$.
(2) $p \in [lo(q), hi(q)]$ if and only if $q \in [lo(p), hi(p)]$.

Proof. (1) For the first statement, if $lo(p) = loS(p)$, $q < loS(p)$. By definition of $loS(p)$, $\exists p' < p$, $q < t^+(p') \in R$, so by definition of $hiP(q)$, $hi(q) \leq hiP(q) \leq p' < p$. If $lo(p) = loP(p)$, $q < loP(p)$. So $loP(p) \in L_{>q}$ and $t^-(loP(p)) < p$, so $hi(q) \leq hiS(q) \leq t^-(loP(p)) < p$. The second statement can be proved similarly. (2) If $q \notin [lo(p), hi(p)]$, then $q > hi(p)$ or $q < lo(p)$. By the first statement, $p < lo(q)$ or $p > hi(q)$, and we get contradiction in both cases. □

Lemma 3.3. *If none of the conditions in Lemma 3.1 applies in any vertex p, then $lo(p) \leq hi(p)$ for every vertex p in P.*

Proof. We use contradiction. If $lo(p) > hi(p)$ for some vertex $p \in P$, W.L.O.G., suppose $p \in L$. Then $lo(p) \neq s$ and $hi(p) \neq t$. There are four cases.

(1) $hi(P) = hiP(p)$, $lo(P) = loS(P)$. In this case, let $q = hiP(p) \in R$, then $t^+(q) > p$. $loS(p) = t^+(p')$ for some $p' \in L_p$, so $t^+(p') > q$ and $t^+(q) > p > p'$. The first alternative of condition 2 in Lemma 3.1 applies.
(2) $hi(P) = hiS(p)$, $lo(P) = loS(P)$. In this case, $hiS(p) = t^-(p')$ for some $p' \in L_{>p}$. $loS(p) = t^+(p'')$ for some $p'' \in L_{<p}$. So, $p'' < p'$ and $t^+(p'') > t^-(p')$. The first alternative of condition 3 in Lemma 3.1 applies.
(3) $hi(P) = hiP(p)$, $lo(P) = loP(P)$. In this case, $hiP(p) = q' \in R$ with $t^+(q') > p$. $loP(p) = q \in R$ with $t^-(q) < p$. So, $q' < q$ and $t^-(q) < t^+(q')$. The second alternative of condition 3 in Lemma 3.1 applies.
(4) $hi(P) = hiS(p)$, $lo(P) = loP(P)$. In this case, $hiS(p) = t^-(p')$ for some $p' \in L_{>p}$. $loP(p) = q \in R$ with $t^-(q') < p$. So, $t^-(p') < q$ and $t^-(q') < p < p'$. The second alternative of condition 2 in Lemma 3.1 applies.

So, in general, $lo(p) \leq hi(p)$ for every vertex p in P. □

The following two lemmas explain the reason that we analyze lo and hi. In fact, these concepts play critical roles in checking whether P is walkable.

Lemma 3.4. *Each walk partner of a vertex p is contained in $[lo(p), hi(p)]$.*

Proof. Let $\bar{p}$ be a walk partner of p. We aims to show $lo(p) \leq \bar{p} \leq hi(P)$. If $lo(p) = s$ or $hi(p) = t$, it is trivial. So, the following four situations are remaining.

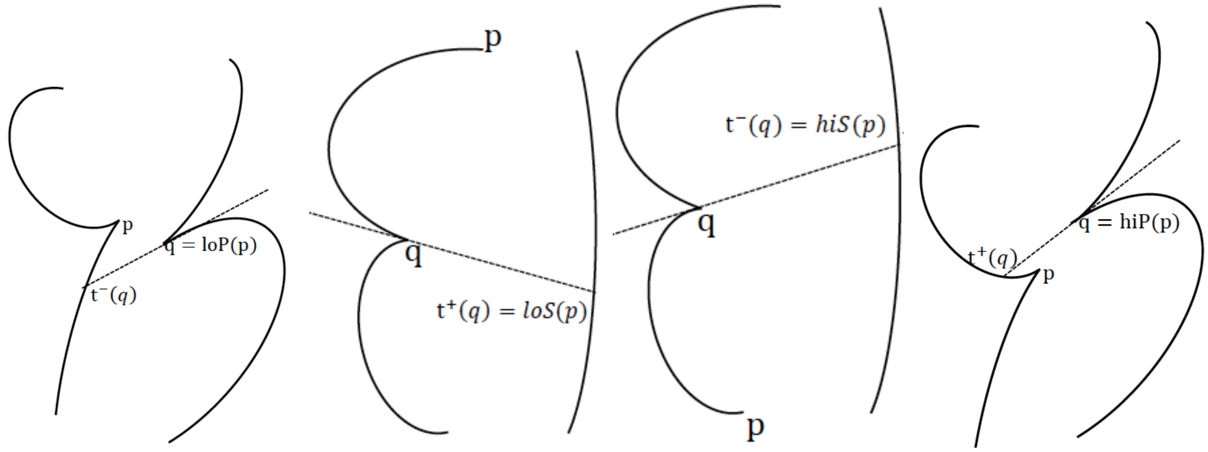

FIGURE 7. Illustration of the four situations in the proof of Lemma 3.4.

(1) $lo(p) = loP(p)$. See the leftmost picture of Figure 7. Suppose $q = loP(p)$. Then by Lemma 2.1, there is a sequence of points $\{q_n\}$ with $R \ni q_n \le q$ and $q_n \to q$, q_n is not visible from $L_{\ge p}$. If $\bar{p} < q$, $\exists q_N$ s.t. $\bar{p} < q_N < q$, then q_N does not have a walk partner.
(2) $lo(p) = loS(p)$. See the second from the left image of Figure 7. Suppose $t^+(q) = loS(p)$. If $\bar{p} < loS(p)$, by Lemma 2.1, there is a sequence of points $\{q_n\}$ with $L \ni q_n \ge q$ and $q_n \to q$, q_n is not visible from $R_{\le \bar{p}}$. Any member of $\{q_n\}$ does not have a walk partner.
(3) $hi(p) = hiS(p)$. See the second from the right picture of Figure 7. Suppose $t^-(q) = hiS(p)$. If $\bar{p} > hiS(p)$, by Lemma 2.1, there is a sequence of points $\{q_n\}$ with $L \ni q_n \le q$ and $q_n \to q$, q_n is not visible from $R_{\ge \bar{p}}$. Any member of $\{q_n\}$ does not have a walk partner.
(4) $hi(p) = hiP(p)$. See right most figure of Fig 7. Suppose $q = hiP(p)$. Then by Lemma 2.1, there is a sequence of points $\{q_n\}$ with $R \ni q_n \ge q$ and $q_n \to q$, q_n is invisible from $L_{\le p}$. If $\bar{p} > q$, $\exists q_N$ s.t. $\bar{p} > q_N > q$, then q_N does not have a walk partner.

□

Lemma 3.5. *Suppose that condition* 1 *in Lemma 3.1 does not apply in any reflex vertex $p \in P$. If $p \in P$ satisfies $lo(p) \le hi(p)$, then $[lo(p), hi(p)]$ is visible from p.*

Proof. Without loss of generality, we suppose $p \in L$.

If $lo(p) = loS(p)$, then there exists $L \ni p' < p$ such that $lo(p) = t^+(p')$, so p' is visible from $lo(p)$. If $lo(p) = loP(p)$, we let $p' = t^-(loP(p))$ and see that $p' < p$ and p' is visible from $lo(p)$. If $lo(p) = s$, we let $p' = s$ and see that p' is visible from $lo(p)$. Thus $\exists L \ni p' < p$, p' is visible from $lo(p)$. Similarly, $\exists L \ni p'' > p$, p'' is visible from $hi(p)$.

If p is not visible from $lo(p)$, the boundary of P must intersect $\overline{plo(p)}$. If $L_{>p''} \cup R_{>hi(p)}$ intersects $\overline{plo(p)}$, it must intersect $\overline{p''hi(p)}$, so p'' is not visible from $hi(p)$, which is a contradiction. If $L_{<p'} \cup R_{<lo(p)}$ intersects $\overline{plo(p)}$, it must intersect $\overline{p'lo(p)}$, so p' is not visible from $lo(p)$, which is also a contradiction.

If $L_{[p',p]}$ intersects $\overline{plo(p)}$, then there is a vertex $p''' \in L_{[p',p]}$ such that $L \ni t^+(q) > p$ or $R \in t^+(q) > lo(p)$, both are contradictions. For the same reason, $L_{[p,p'']}$ does not intersect $\overline{plo(p)}$.

If $R_{[lo(p),hi(p)]}$ intersects $\overline{plo(p)}$, there is a vertex $q \in R_{[lo(p),hi(p)]}$ such that $R \ni t^-(q) < lo(p)$ or $L \in t^-(q) < p$, both are contradictions. So, none of $L \cup R$ intersects $\overline{plo(p)}$, and thus p is visible from $lo(p)$. Similarly, p is visible from $hi(p)$.

For all $q \in [lo(p), hi(p)]$, by definition of lo and hi, we have $t^-(q) \geq p$ and $t^+(q) \leq p$. Now, if the boundary of P intersects $\overline{pq}$, it must intersect one of $\overline{plo(p)}$,$\overline{phi(p)}$,$\overline{t^-(q)q}$ and $\overline{t^+(q)q}$, but all of them cause contradictions. Therefore, $[lo(p), hi(p)]$ is visible from p. □

Now we are ready to present the sufficient condition for P to be walkable.

Lemma 3.6. *If none of the cases in Lemma 3.1 applies, then P is walkable.*

Proof. We show that P is walkable by construction of a straight walk. This task is equivalent to finding a walk instruction that decides the location of two guards at each time moment to keep them visible to each other. First, we partition P into smaller pieces and discuss the walk instruction for each such piece. It is proved in Lemma 3.3 that $lo(p) \leq hi(p)$ for every vertex p. Then it follows by Lemma 3.5 that $[lo(p), hi(p)]$ is visible from p. Choose $lo(p)$ to be a walk partner of p for every vertex p in L. Because lo is monotonically increasing in L, no two walk line segments cross. For every vertex $q \in R$, if it does not have a walk partner yet, then there exist consecutive $p, p' \in L$ with $p < p'$ and $lo(p) < q < lo(p')$. It follows from Lemma 3.2 that $p < hi(p) < p'$. Choose $hi(q)$ to be walk partner of q so that no pairs of $\overline{plo(p)}$ and $\overline{qhi(q)}$ will cross. Since hi is also monotonic, no two walk line segments will cross. Now P is partitioned

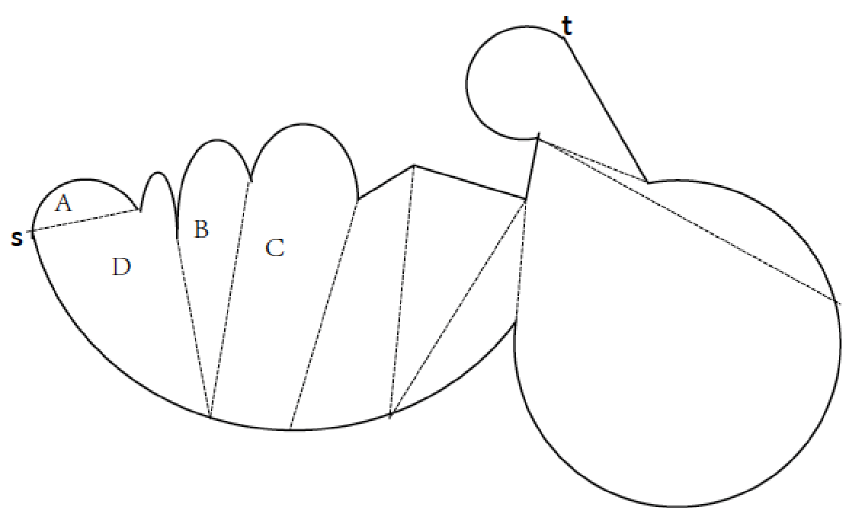

FIGURE 8. Example of a partition constructed in the proof of Lemma 3.6.

into a sequence of lenses (made of two curves), curvilinear triangles (made of three curves), and quadrilaterals (made of four curves). Figure 8 is an example of such a partition. Set A is an example of a lens, set B is an example of a curvilinear triangle, and sets C and D are examples of quadrilaterals. For lens A, it is obvious that one of its vertices must be s or t. In Lemma 2.3, we have already shown that A is convex, so W.L.O.G., assume s is a vertex in A and one curve of A is a part of L. The walk instruction is to keep the guard on R at s while the guard on L moves from s to the other end point.

Next, we need to present a walk instruction in every curvilinear triangle and quadrilateral.

For curvilinear triangles, if one of the three vertices, p, is in L and the other two, q_1 and q_2 ($q_1 < q_2$), are in R, we need to show that $q_1, q_2 \in [lo(p), hi(p)]$. There are three cases.

(1) $q_1 = lo(p)$ and $q_2 = hi(p)$. If $hi(p) < q_2$, then, by Lemma 3.2, $lo(q_2) > p$, which is a contradiction. Thus $lo(p) = q_1 < q_2 \leq hi(p)$, the walk instruction is to keep one guard in p and let the other guard walk from q_1 to q_2.

(2) $q_2 = lo(p)$ and $hi(q_1) = p$. It means $q_1 < lo(p)$. By Lemma 3.2 $hi(q_1) < p$, this case is impossible.

(3) $p = hi(q_1) = hi(q_2)$. If $lo(p) > q_1$, by Lemma 3.2, $hi(q_1) < p = hi(q_1)$, which is a contradiction. If $hi(p) < q_2$, by Lemma 3.2, $lo(q_2) > p = hi(q_2)$, which is also a contradiction. Thus $lo(p) = q_1 < q_2 \leq hi(p)$ for the same reason as in case 1, and we can generate the walk instruction.

If two of the three vertices, p_1 and p_2 ($p_1 < p_2$), are in L and the other one, q, is in R, then $lo(p_1) = lo(p_2) = q$. If $p_1 < lo(q)$, by Lemma 3.2 $hi(p_1) < q = lo(p_1)$, which is a contradiction. If $p_2 > hi(q)$, by Lemma 3.2 $lo(p_2) > q = lo(p_2)$, which is also a contradiction. So, $lo(q) = p_1 < p_2 \leq hi(q)$, then the triangle is convex and it is easy to generate the walk instruction.

Each quadrilateral Q is made up of two consecutive vertices $p < p' \in L$ and two points $q < q' \in R$. Thus, $q = lo(p)$ or $p = hi(q)$; $q' = lo(p')$ or $p' = hi(q')$. If Q is not locally convex in p, then p must be a reflex vertex in P. In this case, if $t^+(p) > q'$, then by definition of $loS(p')$, $lo(p') > q'$. By Lemma 3.2, $hi(q') > p'$. They contradict to both cases of $q' = lo(p')$ or $p' = hi(q')$. Thus, $t^+(p) \leq q'$. Similarly, when Q is not locally convex in either of q, p', q', we have $t^+(q) \leq p'$, $t^-(p') \geq q$, and $t^-(q') \geq p$, respectively.

If Q is not locally convex in both p and q, then $t^+(q) > p$ and $t^+(p) > q$, and case 2 in Lemma 3.1 applies. So, Q must be locally convex in at least one of p, q. Similarly, Q must be locally convex in at least one of p, q. If Q is locally convex in p, q, p', and q', the quadrilateral is convex since it is locally convex in all boundary points. If Q is not locally convex in only one of p, q, p', q', say p, then $q < t^+(p) \leq q'$. The triangle made up of p, q and $t^+(p)$ and the quadrilateral made up of $p, t^+(p), p', q'$ are both convex since it is locally convex in all boundary points. If Q is not locally convex in one of p, q and one of p', q', by symmetry, there are two cases.

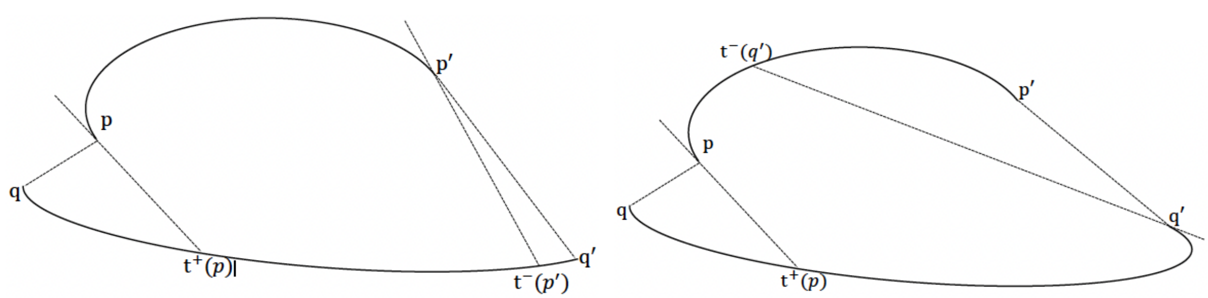

FIGURE 9. An illustration to designing the walk instruction for quadrilaterals.

(1) p, p' are the point of local non-convexity. See Figure 9 (left). Then $q < t^+(p), t^-(p') \leq q'$. If $t^+(p) > t^-(p')$, case 3 in Lemma 3.1 apples, so $t^+(p) \leq t^-(p')$. Then the triangle made up of p, q and $t^+(p)$, the triangle made up of p', q' and $t^-(p')$ and the quadrilateral made up of $t^-(p'), t^+(p), p', q'$ are all convex due to local convexity in all boundary points.

(2) p, q' are the point of local non-convexity. See Figure 9 (right). At that time $q < t^+(p) \leq q'$ and $p < t^+(q) \leq p'$.The triangle made up of p, q and $t^+(p)$, the triangle made up of p', q' and $t^-(q')$ and the quadrilateral made up of $p, t^+(p), p', q'$ are all convex since all boundary points are locally convex.

In each case, we divide Q into at most 3 convex pieces, each of which obviously admits a walk instruction. Putting them together, we get a walk instruction for Q.

Now we generate the walk instruction for every piece, and putting the piece instructions together we get a walk instruction for P, so P is walkable. □

4. Construction of Solutions

In this section, we summarize the results in previous sections and develop an algorithm to check whether a piecewise locally convex polygon is walkable in quadratic time. We also develop an algorithm to generate the walk instruction if the polygon is walkable in quadratic time.

Theorem 4.1. *P is walkable if and only if none the cases in Lemma 3.1 applies. With tangent information of reflex vertices of P at hand, there exists an algorithm running in time $\mathcal{O}(n^2)$ to check whether P is walkable.*

Proof. Combining Lemma 3.1 and Lemma 3.6, we know that P is walkable if and only if none of the cases in Lemma 3.1 applies.

To check the conditions in Lemma 3.1, for each reflex vertex p that is the intersection of boundary curves a and b, it takes $\mathcal{O}(n)$ time to compare the intersection points of $T_a(p)$, $T_b(p)$ and every boundary curve other than a,b to derive $t^-(p)$ and $t^+(p)$ by the similar method as in [6]. So it takes $\mathcal{O}(n^2)$ time to derive $t^-(p)$ and $t^+(p)$ for all reflex vertices.

With information of $t^-(p)$ and $t^+(p)$ for every reflex vertex p, we need $\mathcal{O}(n)$ time to check condition 1 in Lemma 3.1 as we only need to compare p with $t^-(p)$ and $t^+(p)$ for the n reflex vertices. It takes $\mathcal{O}(n^2)$ time to check condition 2 in Lemma 3.1 as we need to compare each pair of p, q with $t^-(p)$, $t^+(p)$, $t^-(q)$ and $t^+(q)$. For the same reason, it takes $\mathcal{O}(n^2)$ time to check condition 3 in Lemma 3.1. So, the total time required to check whether P is walkable is $\mathcal{O}(n^2)$. □

Corollary 4.1. *There is an algorithm running in time $\mathcal{O}(n^2)$ to construct a walk instruction if P is walkable.*

Proof. See Algorithm 1.

Algorithm 1 Construction of a walk instruction.

1: Derive $t^-(p)$ and $t^+(p)$ for every reflex vertex p.
2: Calculate $hi(p)$ and $lo(p)$ for every reflex vertex p.
3: For every reflex vertex $p \in L$, connect p and $lo(p)$; then if for some reflex vertex $q \in R$, q is not connected with any reflex vertex $p \in L$, connect q and $hi(q)$. As a result, P is partitioned into smaller pieces.
4: Construct a walk instruction for every piece.

By Theorem 4.1, Step 1 takes $\mathscr{O}(n^2)$ time. With known $t^-(p)$ and $t^+(p)$ and by definition of lo and hi, it takes $\mathscr{O}(n^2)$ time to complete step 2. Obviously step 3 needs $\mathscr{O}(n)$ time and the resulting small pieces are lenses, curvilinear triangles and quadrilaterals by Lemma 3.6. The total number of small pieces is at most $2n$ and by Lemma 3.6, it takes $\mathscr{O}(1)$ time to construct a walk instruction for every small piece, so the total time required for step 4 is $\mathscr{O}(n)$. Therefore, the time complexity of this algorithm is $\mathscr{O}(n^2)$. The correctness of this algorithm follows directly from Lemma 3.6 and Theorem 4.1. □

5. Conclusion

In this paper, we generalized the two-guard problem from a simple polygon to a piecewise locally convex polygon. By carefully analyzing the properties of piecewise locally convex polygons, we were able to develop tools necessary to solve the two-guard problem on such curvilinear polygons. We presented an algorithm running in quadratic time to decide whether a piecewise locally convex polygon is walkable. In addition, our algorithm generates a valid walk if the polygon is walkable.

There exist linear and $\mathscr{O}(n\log n)$ time algorithms for solving the original two-guard problem but they are not suitable for solving our problem. Instead, our algorithm runs in quadratic time. It is an interesting topic for future research if the running time of our algorithm can be improved from quadratic time to $\mathscr{O}(nlogn)$ time. Such an improvement requires improvement on shortest path queries in a curvilinear polygon, which is itself an interesting problem in computational geometry.

There are many modified versions and generalizations of the two-guard problem, and all of them assume that the polygon is simple, defined by line segments. As our generalization considers curvilinear polygons, it is natural to consider curvilinear polygons in the modified or generalized two-guard problems in the future research. These include the two-guard problem in counter-walk polygons, the two-guard problem in the setting of rooms, and polygon search problems.

Author's Note
This paper is based on results from the first author's dissertation [Y. Wang, Connectivity Constraints in Network Analysis, PhD thesis, Texas A&M University, 2015].

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