J. Nonlinear Var. Anal. 9 (2025), No. 1, pp. 31-43
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.9.2025.1.03

LINEAR-IMPLICIT LOCAL ENERGY DISSIPATION-PRESERVING ALGORITHMS FOR THE GRADIENT FLOW SYSTEM

ZHENGUO MU[1,3], YUSHUN WANG[2,3,*]

[1]*College of Science, Nanjing University of Posts and Telecommunications, Nanjing, China*
[2]*School of Mathematical Sciences, Nanjing Normal University, Nanjing, China*
[3]*Jiangsu Key Laboratory for Numerical Simulation of Large Scale Complex Systems for NSLSCS, Nanjing, China*

Abstract. In this paper, we propose two linear-implicit local energy dissipation-preserving algorithms for a gradient flow system. We first prove that the gradient flow system possesses a local energy dissipation law, which is exactly conserved within any local time-space region. We then introduce an auxiliary variable to reformulate the gradient flow system into an equivalent system, which is proven to preserve the local energy dissipation property. To maintain the intrinsic properties as many as possible, two linear-implicit local energy dissipation-preserving algorithms are developed by means of the composition method. Furthermore, we prove that the proposed algorithms adhere to the discrete local energy dissipation laws with the assistance of the Leibnitz rules. Particularly, under appropriate boundary conditions, these innovative algorithms naturally preserve the discrete total mass laws and ensure the global energy stability in the sense of energy decay for the gradient flows. Finally, numerical examples are provided to demonstrate the efficiency of the proposed algorithms and their effectiveness in preserving the energy dissipation laws.

Keywords. Gradient flow system; Global energy stable; Local energy dissipation preserving algorithm; Total mass conservation.

1. INTRODUCTION

With the advances in materials science and fluid dynamics, the dynamics with energy dissipation property were extensively investigated by gradient flow systems. For their applications, we refer to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and the references therein.

In this paper, we are interested in the gradient flow system as follows [11]

$$\begin{cases} \frac{\partial \phi}{\partial t} = \Delta \mu, \quad (\vec{x},t) \in \Omega \times R^+, \\ \mu = \frac{\delta E}{\delta \phi} = F'(\phi) - \kappa \Delta \phi, \end{cases} \tag{1.1}$$

with the initial condition

$$\phi(\vec{x},0) = \phi_0(\vec{x}), \vec{x} \in \Omega = \Omega_x \times \Omega_y = [x_L, x_R] \times [y_L, y_R],$$

*Corresponding author.
E-mail address: mzg@njupt.edu.cn (Z. Mu), wangyushun@njnu.edu.cn (Y. Wang).
Received 7 May 2024; Accepted 25 July 2024; Published online 20 December 2024.

and periodic boundary conditions or $\frac{\partial\phi}{\partial n}|_{\partial\Omega} = \frac{\partial\mu}{\partial n}|_{\partial\Omega} = 0$, in which ϕ represents the concentration field and μ stands for the chemical potential. Here E is a functional of ϕ known as the free energy, typically chosen as $E[\phi(\mathbf{x})] = \int_\Omega \frac{\kappa}{2}|\nabla\phi|^2 + F(\phi)$, where κ denotes the square of the transition-layer thickness of the two phases. Moreover, F is the free energy density, commonly expressed as $F(\phi) = \frac{1}{4}(\phi^2-1)^2$.

One has the following trivial results on system (1.1).

Proposition 1.1. *System* (1.1) *holds the local energy dissipation law* ***LEDL***

$$\partial_t\left(\frac{\kappa}{2}|\nabla\phi|^2 + F(\phi)\right) - \nabla\cdot\left(\kappa\phi_t\nabla\phi + \mu\nabla\mu\right) + |\nabla\mu|^2 = 0. \tag{1.2}$$

Proof. Multiplying the first and the second line of (1.1) with μ and ϕ_t, respectively, one deduces that $\phi_t\mu = \Delta\mu\mu$ and $\mu\phi_t = -\kappa\phi_t\Delta\phi + \phi_t F'(\phi)$. Utilizing the continuous Leibnitz rule, one sees that

$$\partial_t\left(\frac{\kappa}{2}|\nabla\phi|^2 + F(\phi)\right) - \nabla\cdot\left(\kappa\phi_t\nabla\phi + \mu\nabla\mu\right) + |\nabla\mu|^2 = 0.$$

□

Remark 1.1. Given suitable boundary conditions, such as periodic boundary conditions, integrating LEDL (1.2) over the spatial domain leads to the following global energy dissipation law (EDL) $\frac{d}{dt}E[\phi] = -\int_\Omega|\nabla\mu|^2 d\mathbf{x}<0$, where $E[\phi] = \int_\Omega\left(\frac{\kappa}{2}|\nabla\phi|^2 + F(\phi)\right)d\mathbf{x}$ is the global free energy of system (1.1). Furthermore, the gradient flow system (1.1) also conserves the total mass density. By taking the L^2 inner product of the first equation of (1.1) with 1, we obtain

$$\frac{1}{dt}\int_\Omega \phi d\mathbf{x} = \int\Delta\mu d\mathbf{x} = \int_{\partial\Omega}\partial_n\mu(\mathbf{x},t)d\mathbf{x} = 0,$$

where $I(t) = \int_\Omega\phi(\mathbf{x},t)d\mathbf{x}$ is the total mass of the system (1.1).

In view of the wide applications of gradient flow systems, various efficient numerical algorithms were investigated extensively; see, e.g., [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] and the references therein. It is known that the energy dissipation property plays a crucial role in the convergence and existence of numerical solution; see, e.g., [25] and the references therein. Recently, a great deal of attention has been given to various algorithms that adhere to the global energy dissipation law, known as energy stable algorithms.

To the best of our knowledge, the energy stability methods proposed in the previous works can be classified into the following categories, including but not limited to the convex splitting method [26, 27, 28, 29], the stabilizer method [30, 31], the energy quadratization method [32], and the scalar auxiliary variable approach [11]. The convex splitting method proposed by Elliott et al. [26, 27, 28] in 1993 and further developed by Eyre in [29]. The key to this method is to divide the free energy into convex and concave parts, which are dealt with implicit and explicit methods respectively afterwards. Such method, while admitting the energy stable law, leads to a nonlinear system which suffers high computational costs for long time computation. In [30, 31], Shen et al. developed another classical method by introducing a suitable stabilizer. The method has two significant advantages of simplicity and efficiency, but the stabilizer is usually chose empirically, which might fail to preserve the nonlinear energy stability. Unlike the above two methodologies, inspired by seminal work [18, 33], Yang et al. [32] proposed the invariant energy quadratization method (IEQ) for the three-component Cahn-Hilliard phase-field model to obtain linear energy stable algorithms. Subsequently, Shen et al. [11] further popularized

the invariant energy quadratization method (IEQ) to develop the scalar auxiliary variable (SAV) approach. Later on, there has been a great deal of excellent results relevant to the IEQ or the SAV method [34, 35, 36, 37, 38]. We notice that the existing works mainly preserve the global energy stable which defines on the global space region and much more depend on the suitable boundary conditions, or else they will be invalid. In order to reduce numerical methods' dependency on boundary conditions, Wang et al. [39] proposed the concept of local structure-preserving method which is proved to be valid in any local areas or any points in time-space region. In view of its merits, the local structure-preserving method is now under the spotlight; see, e.g, [40, 41]. Meanwhile, we note that the existing local energy dissipation preserving algorithms are mainly implicit which usually needs to solve a nonlinear algebra system. Therefore, these motivate us to explore a novel linear-implicit algorithm preserving local energy dissipation law for the gradient flow system, which inherits the local dissipation law.

The main purpose of this paper is to explore two novel linearly implicit local energy dissipation law preserving algorithms for solving the gradient flow system. To this end, we make use of the composition method and the invariant energy quadratization method to construct expected algorithms. We rigorously prove the proposed algorithms are the local energy dissipation preserving law, which can produce richer information on the discrete system for the gradient flow system rather than the global energy stable preserving algorithms. As given the proper boundary conditions, such as periodic or homogeneous and so on, the local energy dissipation preserving algorithms are to be proven to preserve the global energy dissipation law and the total mass conservation law.

The rest of the paper is organized as follows. In Section 2, we rewrite gradient flow system (1.1) as an equivalent system, which is strictly proven to inherit a modified local energy dissipation law. Meanwhile, some operators and their relevant laws are provided to facilitate follow-up expression and theoretical proof. In Section 3, we present two local energy dissipation preserving algorithms for solving the gradient flow system and prove the preservation of the discrete local energy dissipation law exactly. In particular, given the appropriate boundary conditions, these new algorithms are also proven to preserve the total mass law and the global energy dissipation law. Furthermore, some numerical simulations are carried out to validate the accuracy and efficiency of the proposed numerical algorithms.

2. PRELIMINARIES

In this section, we initially reduce the equation to an equivalent system and correspondingly give the energy dissipation property. Meanwhile, some operators and related properties are enumerated for the convenience of follow-up research expression. Assuming that the function $F(\phi)$ is bounded from below, e.g., there exists a constant $C_0 > 0$, one can introduce an auxiliary variable $q = \sqrt{F(\phi)+C_0}$, and system (1.1) can be recast as

$$\begin{cases} \frac{\partial \phi}{\partial t} = \Delta \mu, \\ \mu = -\kappa \Delta \phi + q \frac{F'(\phi)}{\sqrt{F(\phi)+C_0}}, \\ q_t = \frac{F'(\phi)}{2\sqrt{F(\phi)+C_0}} \cdot \phi_t. \end{cases} \tag{2.1}$$

The corresponding free energy function is as follow

$$E(\phi) = \int_\Omega (\frac{\kappa}{2}|\nabla\phi|^2 + q^2)dxdy.$$

Theorem 2.1. *The transformed system for gradient flow system (2.1) admits the following modified local energy dissipation law* $\partial_t(\frac{\kappa}{2}|\nabla\phi|^2+q^2) - \nabla\cdot(\kappa\phi_t\nabla\phi+\mu\nabla\mu)+|\nabla\mu|^2 = 0$.

Proof. Multiplying the first and the second line of (2.1) with μ and ϕ_t respectively, we obtain

$$\phi_t\mu = \Delta\mu\,\mu \tag{2.2}$$

and

$$\mu\phi_t = -\kappa\phi_t\Delta\phi + \phi_t q\frac{F'(\phi)}{\sqrt{F(\phi)+C_0}}. \tag{2.3}$$

In view of (2.1), we have

$$(q^2)_t = 2qq_t = 2q\frac{F'(\phi)}{2\sqrt{F(\phi)+C_0}}\phi_t. \tag{2.4}$$

By the Leibnitz rule, we obtain from the (2.2), (2.3), and (2.4) that

$$\partial_t(\frac{\kappa}{2}|\nabla\phi|^2+q^2) - \nabla\cdot(\kappa\phi_t\nabla\phi+\mu\nabla\mu)+|\nabla\mu|^2 = 0.$$

This completes this proof. □

Corollary 2.1. *With the periodic boundary conditions, system (2.1) admits the modified global energy dissipative law,*

$$\frac{dE}{dt} = -\int_\Omega \left(|\nabla\mu|^2\right)d\mathbf{x},\ E[\phi] = \int_\Omega\left(\frac{\kappa}{2}|\nabla\phi|^2+q^2\right)d\mathbf{x}.$$

For brevity, some notations are introduced to deal with the discrete systems throughout this paper. Let $t_n = n\Delta t$, $0\le n\le N_t$, where $\Delta t = T/N_t$. The spatial domain $\Omega = [x_L,x_R]\times[y_L,y_R]$ is uniformly partitioned with mesh size $h_x = (x_R-x_L)/N_x, h_y = (y_R-y_L)/N_y$ and

$$\Omega_h = \left\{(x_j,y_k)|x_j = x_L+jh_x, y_k = y_L+kh_y, 0\le j\le N_x, 0\le k\le N_y\right\}.$$

Let the notation $f^n_{j,k}$ represent the approximation value of $f(x_j,y_k,t_n)$, where the index j,k correspond to increments in space and n to increments in time. A grid function $f = \left\{f_{j,k}|j,k\in\mathbb{Z}\right\}$ is called periodic if

$$(x\text{-periodic})\ f_{N_x+j,k} = f_{j,k};\ (y\text{-periodic})\ f_{j,N_y+k} = f_{j,k}.$$

The finite difference operators and the average operators are defined as follows

$$\delta_t^+ f^n_{j,k} = \frac{f^{n+1}_{j,k}-f^n_{j,k}}{\Delta t},\quad \delta_x^+ f^n_{j,k} = \frac{f^n_{j+1,k}-f^n_{j,k}}{h_x},\quad \delta_y^+ f^n_{j,k} = \frac{f^n_{j,k+1}-f^n_{j,k}}{h_y},$$

$$\delta_t f^n_{j,k} = \frac{f^{n+1}_{j,k}-f^{n-1}_{j,k}}{2\Delta t},\quad f^{n+\frac12}_{j,k} = \frac{f^{n+1}_{j,k}+f^{n-1}_{j,k}}{2},\quad \hat{f}^{n+\frac12}_{j,k} = \frac{3f^n_{j,k}-f^{n-1}_{j,k}}{2},$$

and

$$A_t f^n_{j,k} = \frac{f^{n+1}_{j,k}+f^n_{j,k}}{2},\quad A_x f^n_{j,k} = \frac{f^n_{j+1,k}+f^n_{j,k}}{2},\quad A_y f^n_{j,k} = \frac{f^n_{j,k+1}+f^n_{j,k}}{2}.$$

Accordingly, the discrete inner product and norm for tensor valued functions are also defined as

$$(\mathbf{F},\mathbf{G})_h=\sum_{m,n}\sum_{j=0}^{N_x-1}\sum_{k=0}^{N_y-1}(\mathbf{F}_{m,n})_{j,k}(\mathbf{G}_{m,n})_{j,k}h_xh_y,\ \|\mathbf{F}\|_h=(\mathbf{F},\mathbf{F})_h^{\frac{1}{2}}.$$

These operators satisfy the following discrete rules (as [41])

(1) Commutative law:

$$\delta_\alpha\delta_\beta f^n_{j,k}=\delta_\beta\delta_\alpha f^n_{j,k},\quad A_\alpha A_\beta f^n_{j,k}=A_\beta A_\alpha f^n_{j,k},\quad \delta_\alpha A_\beta f^n_{j,k}=A_\beta\delta_\alpha f^n_{j,k},\quad \alpha,\beta\in\{x,y,t\},$$

(2) Discrete Leibnitz rule:

$$\begin{aligned}
&\delta_x(f^n_{j-1,k}\cdot g^n_{j,k})=f^n_{j,k}\cdot\delta_x g^n_{j,k}+\delta_x f^n_{j-1,k}\cdot g^n_{j,k},\\
&\delta_x(f^n_{j,k}\cdot g^n_{j,k})=f^n_{j,k}\cdot\delta_x g^n_{j,k}+\delta_x f^n_{j,k}\cdot g^n_{j+1,k},\\
&\delta_x(f^n_{j,k}\cdot g^n_{j,k})=A_x f^n_{j,k}\cdot\delta_x g^n_{j,k}+\delta_x f^n_{j,k}\cdot A_x g^n_{j,k}.
\end{aligned}$$

For operator δ_y and δ_t, we have a series of analogous discrete Leibnitz rules.

(3) Chain rule:

$$\delta_\alpha F(u^n)=\delta_u F(u^n)\delta_\alpha u^n+o(\Delta\alpha),\ \alpha\in\{x,y,t\}.$$

The properties of these operators play a crucial role in proving the local structure property of the algorithm.

3. Main results

Now, we are ready to give our main results, that is, two linear-implicit local energy dissipation preserving algorithms for the gradient flow system are developed by the concatenating method.

3.1. Linear-implicit local energy dissipation preserving algorithm I (LILEDP-I).

We first introduce the intermediate variables $\mathbf{a}=\nabla\mu$ and $\mathbf{b}=\nabla\phi$, and rewrite system (2.1) into the following first-order partial differential equation system

$$\begin{cases}
\phi_t=\nabla\cdot\mathbf{a},\\
\nabla\mu=\mathbf{a},\\
\nabla\phi=\mathbf{b},\\
\mu=-\kappa\nabla\cdot\mathbf{b}+q\frac{F'(\phi)}{\sqrt{F(\phi)+C_0}},\\
q_t=\frac{F'(\phi)}{2\sqrt{F(\phi)+C_0}}\cdot\phi_t,
\end{cases}\tag{3.1}$$

which admits the following LEDL

$$\partial_t\left(\frac{\kappa}{2}|\nabla\phi|^2+q^2\right)-\nabla\cdot\left(\kappa\phi_t\mathbf{b}+\mu\mathbf{a}\right)+|\nabla\mu|^2=0.$$

Applying the leap frog method and the discrete variational derivative method in space and time for system (3.1), we have

$$\begin{cases} \delta_t \phi_{j,k}^n = \nabla_h^+ \cdot (\mathbf{a}_{j,k}^{n+\frac{1}{2}}), \\ \nabla_h^- \mu_{j,k}^n = \mathbf{a}_{j,k}^n, \\ \nabla_h^- \phi_{j,k}^n = \mathbf{b}_{j,k}^n, \\ \mu_{j,k}^{n+\frac{1}{2}} = -\kappa \nabla_h^+ \cdot (\mathbf{b}_{j,k}^{n+\frac{1}{2}}) + q_{j,k}^{n+\frac{1}{2}} \cdot \frac{F'(\hat{\phi}_{j,k}^{n+\frac{1}{2}})}{\sqrt{F(\hat{\phi}_{j,k}^{n+\frac{1}{2}})+C_0}}, \\ \delta_t q_{j,k}^n = \frac{F'(\hat{\phi}_{j,k}^{n+\frac{1}{2}})}{2\sqrt{F(\hat{\phi}_{j,k}^{n+\frac{1}{2}})+C_0}} \cdot \delta_t \phi_{j,k}^n, \end{cases} \tag{3.2}$$

where

$$\delta_t \phi_{j,k}^n = \frac{\phi_{j,k}^{n+1} - \phi_{j,k}^{n-1}}{2\Delta t}, \quad \phi_{j,k}^{n+\frac{1}{2}} = \frac{\phi_{j,k}^{n+1} + \phi_{j,k}^{n-1}}{2}, \quad \nabla_h^+ = \begin{pmatrix} \delta_x^+ \\ \delta_y^+ \end{pmatrix}, \quad \nabla_h^- = \begin{pmatrix} \delta_x^- \\ \delta_y^- \end{pmatrix}.$$

By eliminating the auxiliary variables, algorithm (3.2) can be recombined into a single equation

$$\delta_t \phi_{j,k}^n = -\kappa \Delta_h^2 \phi_{j,k}^{n+\frac{1}{2}} + \Delta_h \left(\left(q_{j,k}^{n-1} + \frac{F'(\hat{\phi}_{j,k}^{n+\frac{1}{2}})}{4\sqrt{F(\hat{\phi}_{j,k}^{n+\frac{1}{2}})+C_0}} \cdot (\phi_{j,k}^{n+1} - \phi_{j,k}^{n-1}) \right) \cdot \frac{F'(\hat{\phi}_{j,k}^{n+\frac{1}{2}})}{\sqrt{F(\hat{\phi}_{j,k}^{n+\frac{1}{2}})+C_0}} \right). \tag{3.3}$$

Next, we analyze the properties of the above algorithm.

Theorem 3.1. *Algorithm (3.2) satisfies the following discrete modified LEDL*

$$\delta_t \left(\frac{\kappa}{2} |\nabla_h^+ \phi_{j,k}^n|^2 + (q_{j,k}^n)^2 \right) - \nabla_h^+ \cdot \left(\nabla_h^- \mu_{j,k}^{n+\frac{1}{2}} \cdot \mu_{j,k}^{n+\frac{1}{2}} + \kappa \delta_t \phi_{j,k}^n \cdot \nabla_h^- \phi_{j,k}^{n+\frac{1}{2}} \right) = -|\nabla_h^+ (\mu_{j,k}^{n+\frac{1}{2}})|^2, \tag{3.4}$$

where

$$\nabla_h^+ \cdot (f_{j,k}^n \cdot \mathbf{a}_{j,k}^n) := \delta_x^+ (f_{j,k}^{n+\frac{1}{2}} \cdot a1_{j,k}^n) + \delta_y^+ (f_{j,k}^{n+\frac{1}{2}} \cdot a2_{j,k}^n), \ \mathbf{a} = (a_1, a_2)^T,$$

and

$$\nabla_h^+ \cdot (\mathbf{a}_{j,k}^n f_{j,k}^n + \mathbf{b}_{j,k}^n g_{j,k}^n) := \nabla_h^+ \cdot (f_{j,k}^n \mathbf{a}_{j,k}^n) + \nabla_h^+ \cdot (g_{j,k}^n \mathbf{b}_{j,k}^n).$$

Proof. For the sake of convenience, after eliminating the intermediate variables, algorithm (3.2) is rewritten as follow

$$\begin{cases} \delta_t \phi_{j,k}^n = \nabla_h^+ \cdot (\nabla_h^- \mu_{j,k}^{n+\frac{1}{2}}), \\ \mu_{j,k}^{n+\frac{1}{2}} = -\kappa \nabla_h^+ \cdot (\nabla_h^- \phi_{j,k}^{n+\frac{1}{2}}) + q_{j,k}^{n+\frac{1}{2}} \cdot \frac{F'(\hat{\phi}_{j,k}^{n+\frac{1}{2}})}{\sqrt{F(\hat{\phi}_{j,k}^{n+\frac{1}{2}})+C_0}}, \\ \delta_t q_{j,k}^n = \frac{F'(\hat{\phi}_{j,k}^{n+\frac{1}{2}})}{2\sqrt{F(\hat{\phi}_{j,k}^{n+\frac{1}{2}})+C_0}} \delta_t \phi_{j,k}^n. \end{cases} \tag{3.5}$$

Multiplying each these three equations by $\mu_{j,k}^{n+\frac{1}{2}}$, $-\delta_t\phi_{j,k}^n$, and $q_{j,k}^{n+\frac{1}{2}}$, respectively, we obtain

$$\begin{cases} \mu_{j,k}^{n+\frac{1}{2}}\cdot\delta_t\phi_{j,k}^n = \delta_x^+(\delta_x^+\mu_{j-1,k}^{n+\frac{1}{2}}\cdot\mu_{j,k}^{n+\frac{1}{2}}) + \delta_y^+(\delta_y^+\mu_{j,k-1}^{n+\frac{1}{2}}\cdot\mu_{j,k}^{n+\frac{1}{2}}) - |\nabla_h^+(\mu_{j,k}^{n+\frac{1}{2}})|^2, \\ -\mu_{j,k}^{n+\frac{1}{2}}\cdot\delta_t\phi_{j,k}^n = -\kappa\nabla_h^+\cdot(\nabla_h^-\phi_{j,k}^{n+\frac{1}{2}})\cdot\delta_t\phi_{j,k}^n + q_{j,k}^{n+\frac{1}{2}}\cdot\frac{F'(\hat{\phi}_{j,k}^{n+\frac{1}{2}})}{\sqrt{F(\hat{\phi}_{j,k}^{n+\frac{1}{2}})+C_0}}\cdot\delta_t\phi_{j,k}^n, \\ \delta_t q_{j,k}^n\cdot q_{j,k}^{n+\frac{1}{2}} = \frac{F'(\hat{\phi}_{j,k}^{n+\frac{1}{2}})}{2\sqrt{F(\hat{\phi}_{j,k}^{n+\frac{1}{2}})+C_0}}\delta_t\phi_{j,k}^n\cdot q_{j,k}^{n+\frac{1}{2}}. \end{cases} \tag{3.6}$$

Adding the above two equations, we obtain by (3.6) that

$$\begin{aligned} &\kappa\nabla_h^+\cdot(\nabla_h^-\phi_{j,k}^{n+\frac{1}{2}})\cdot\delta_t\phi_{j,k}^n - \delta_x^+(\delta_x^-\mu_{j,k}^{n+\frac{1}{2}}\cdot\mu_{j,k}^{n+\frac{1}{2}}) - \delta_y^+(\delta_y^-\mu_{j,k}^{n+\frac{1}{2}}\cdot\mu_{j,k}^{n+\frac{1}{2}}) - 2\delta_t q_{j,k}^n\cdot q_{j,k}^{n+\frac{1}{2}} \\ &= -|\nabla_h^+(\mu_{j,k}^{n+\frac{1}{2}})|^2. \end{aligned}$$

It follows from the Leibnitz rules that

$$\delta_t\left(\frac{\kappa}{2}|\nabla_h^+\phi_{j,k}^n|^2 + (q_{j,k}^n)^2\right) - \nabla_h^+\cdot(\nabla_h^-\mu_{j,k}^{n+\frac{1}{2}}\cdot\mu_{j,k}^{n+\frac{1}{2}}) - \nabla_h^+\cdot\left(\kappa\delta_t\phi_{j,k}^n\cdot\nabla_h^-\phi_{j,k}^{n+\frac{1}{2}}\right) = -|\nabla_h^+(\mu_{j,k}^{n+\frac{1}{2}})|^2,$$

which leads to (3.4). □

Theorem 3.2. *Under the periodic boundary conditions, algorithm* (3.2) *conserves the discrete modified MCL*

$$\left(\phi^{n+1},1\right)_h = \left(\phi^n,1\right)_h, \tag{3.7}$$

and the discrete modified EDL

$$\delta_t E_h^n + \|\nabla_h^+(\mu^{n+\frac{1}{2}})\|_h^2 = 0, \tag{3.8}$$

where

$$E_h^n = \frac{\kappa}{2}\|\nabla_h^+\phi^n\|_h^2 + \left((q(\phi^n))^2,1\right)_h.$$

Proof. Summing the first equation of algorithm (3.2) and discrete LEDL (3.4) over all index j and k directly, respectively, and then combining the periodic boundary conditions, we obtain (3.7) and (3.8). This completes the proof. □

Remark 3.1. If the terms $F(\hat{\phi}_{j,k}^{n+\frac{1}{n}})$ and $F'(\hat{\phi}_{j,k}^{n+\frac{1}{n}})$ in (3.2) or (3.3) are replaced by $F(\phi_{j,k}^n)$ and $F'(\phi_{j,k}^n)$ separately, the corresponding algorithm still remains the corresponding local energy dissipation law. If these terms are replaced by $F(\phi_{j,k}^{n+\frac{1}{2}})$ and $F'(\phi_{j,k}^{n+\frac{1}{2}})$, the corresponding algorithm becomes an implicit locally structure-preserving algorithm.

3.2. **Linear-implicit local energy dissipation preserving algorithm II (LILEDP-II).** For (2.1), applying the linear-implicit Crank-Nicolson method in time and the leap-frog algorithm

in space, we obtain

$$\begin{cases} \delta_t^+ \phi_{j,k}^n = \Delta_h A_t \mu_{j,k}^n, \\ A_t \mu_{j,k}^n = -\kappa \Delta_h A_t \phi_{j,k}^n + A_t q_{j,k}^n \cdot \dfrac{F'(\hat{\phi}_{j,k}^{n+\frac{1}{2}})}{\sqrt{F(\hat{\phi}_{j,k}^{n+\frac{1}{2}}) + C_0}}, \\ \delta_t^+ q_{j,k}^n = \dfrac{F'(\hat{\phi}_{j,k}^{n+\frac{1}{2}})}{2\sqrt{F(\hat{\phi}_{j,k}^{n+\frac{1}{2}}) + C_0}} \delta_t^+ \phi_{j,k}^n, \end{cases} \tag{3.9}$$

where

$$A_t \phi_{j,k}^n = \frac{1}{2}(\phi_{j,k}^n + \phi_{j,k}^{n+1}),\ \hat{\phi}_{j,k}^{n+\frac{1}{2}} = \frac{3\phi_{j,k}^n - \phi_{j,k}^{n-1}}{2},\ \Delta_h f_{j,k} := \delta_x^2 f_{j-1,k} + \delta_y^2 f_{j,k-1}.$$

By eliminating the auxiliary variables, algorithm (3.9) can be recombined into a single equation

$$\delta_t^+ \phi_{j,k}^n = -\kappa \Delta_h^2 A_t \phi_{j,k}^n + \Delta_h \left(\left(q_{j,k}^n + \frac{F'(\hat{\phi}_{j,k}^{n+\frac{1}{2}})}{4\sqrt{F(\hat{\phi}_{j,k}^{n+\frac{1}{2}}) + C_0}} (\phi_{j,k}^{n+1} - \phi_{j,k}^n) \right) \cdot \frac{F'(\hat{\phi}_{j,k}^{n+\frac{1}{2}})}{\sqrt{F(\hat{\phi}_{j,k}^{n+\frac{1}{2}}) + C_0}} \right). \tag{3.10}$$

Next, we analyze properties of algorithm (3.9).

Theorem 3.3. *Algorithm (3.9) possesses the following discrete modified LEDL*

$$\delta_t^+ \left(\frac{\kappa}{2} |\nabla_h^+ \phi_{j,k}^n|^2 + (q_{j,k}^n)^2 \right) - \nabla_h^+ \cdot \left(\kappa \delta_t^+ \phi_{j,k}^n \cdot \nabla_h^- A_t \phi_{j,k}^n + A_t \mu_{j,k}^n \cdot \nabla_h^- A_t \mu_{j,k}^n \right) = -|\nabla_h^+ A_t \mu_{j,k}^n|^2, \tag{3.11}$$

where

$$\nabla_h^- := \begin{pmatrix} \delta_x^- \\ \delta_y^- \end{pmatrix}.$$

Proof. Multiplying the three equations in (3.9) by $A_t \mu_{j,k}^n, \delta_t \phi_{j,k}^n$, and $A_t q_{j,k}^n$, respectively, we obtain

$$\begin{cases} A_t \mu_{j,k}^n \cdot \delta_t^+ \phi_{j,k}^n = A_t \mu_{j,k}^n \cdot \Delta_h A_t \mu_{j,k}^n, \\ -A_t \mu_{j,k}^n \cdot \delta_t^+ \phi_{j,k}^n = \kappa \Delta_h A_t \phi_{j,k}^n \cdot \delta_t^+ \phi_{j,k}^n - A_t q_{j,k}^n \dfrac{F'(\hat{\phi}_{j,k}^{n+\frac{1}{2}})}{\sqrt{F(\hat{\phi}_{j,k}^{n+\frac{1}{2}}) + C_0}} \cdot \delta_t^+ \phi_{j,k}^n, \\ A_t q_{j,k}^n \cdot \delta_t^+ q_{j,k}^n = \dfrac{F'(\hat{\phi}_{j,k}^{n+\frac{1}{2}})}{2\sqrt{F(\hat{\phi}_{j,k}^{n+\frac{1}{2}}) + C_0}} \delta_t^+ \phi_{j,k}^n \cdot A_t q_{j,k}^n. \end{cases}$$

Adding the above two equations and together with the third equation, we have

$$A_t \mu_{j,k} \cdot \Delta_h A_t \mu_{j,k} + \kappa \Delta_h A_t \phi_{j,k}^n \cdot \delta_t^+ \phi_{j,k}^n - 2 A_t q_{j,k}^n \cdot \delta_t^+ q_{j,k}^n = 0,$$

$$\nabla_h^+ \cdot (\nabla_h^- A_t \mu_{j,k}^n \cdot A_t \mu_{j,k}^n + \kappa \nabla_h^- A_t \phi\, j, k^n \cdot \delta_t^+ \phi_{j,k}^n) - \delta_t^+ (\frac{\kappa}{2} |\nabla_h^+ \phi_{j,k}^n|^2) - \delta_t^+ (q_{j,k}^n)^2 = |\nabla_h^+ A_t \mu_{j,k}^n|^2.$$

Thanks to the following Leibnitz rules, we obtain

$$\delta_x (f_{j-1,k}^n \cdot g_{j,k}^n) = f_{j,k}^n \cdot \delta_x g_{j,k}^n + \delta_x f_{j-1,k}^n \cdot g_{j,k}^n,$$
$$\delta_t (f_{j,k}^n \cdot g_{j,k}^n) = A_t f_{j,k}^n \cdot \delta_t g_{j,k}^n + \delta_t f_{j,k}^n \cdot A_t g_{j,k}^n,$$

The discrete modified LEDL is derived

$$\delta_t^+\left(\frac{\kappa}{2}|\nabla_h^+\phi_{j,k}^n|^2+(q_{j,k}^n)^2\right)-\nabla_h^+\cdot\left(\kappa\delta_t^+\phi_{j,k}^n\cdot\nabla_h^-A_t\phi_{j,k}^n+A_t\mu_{j,k}^n\cdot\nabla_h^-A_t\mu_{j,k}^n\right)=-|\nabla_h^+A_t\mu_{j,k}^n|^2.$$

This completes the proof. □

Theorem 3.4. *Under the periodic boundary conditions, algorithm (3.9) conserves the following discrete modified MCL*

$$\left(\phi^{n+1},1\right)_h=\left(\phi^n,1\right)_h, \tag{3.12}$$

and the discrete modified EDL

$$\delta_t E_h^n+\|\nabla_h^+(A_t\mu^n)\|_h^2=0, \tag{3.13}$$

where

$$E_h^n=\frac{\kappa}{2}\|\nabla_h^+\phi^n\|_h^2+\left((q(\phi^n))^2,1\right)_h.$$

Proof. Summing the first equation of system (3.9) and discrete modified LEDL (3.11) over all index j and k directly, respectively, and then combining the periodic boundary conditions, we obtain (3.12) and (3.13). This completes the proof. □

3.3. **Numerical validation.** In this section, algorithm (3.3) and algorithm (3.10) are implemented for some benchmark examples to test the accuracy and efficiency. For simplicity, we assume periodic boundary conditions in all numerical experiments and consider the free energy functional with a double-well bulk term. To test the proposed algorithms, we define the discrete mass error and energy error of the algorithm (3.10) by using the following unified formula, respectively,

$$R_M^n=\left|(\phi^n,1)_h-(\phi^0,1)_h\right|,\ E_h^n=\frac{\kappa}{2}\|\nabla_h^+\phi^n\|_h^2+\left((q(\phi^n))^2,1\right)_h.$$

Example 3.3.1. (Accuracy verification) We solve the gradient flows (1.1) with the model parameter $\gamma=1$, $\kappa=1$, $c_0=0.01$ for $(x,y)\in[0,2\pi]^2$, and $0<t<1$. The exact solution of the system modified is obtained by a appropriate forcing function $\phi(x,y,t)=\cos(x)\cos(y)\cos(t)$. Here, we choose the number of the spatial grids as $N_x=N_y=N$, and compare the numerical solution with the exact solution at $T=1$. We compute the discrete L^2 and L^∞ errors of phase variable ϕ by varying the grid size in space and time. According to the Tables 1-4, we clearly observe that the new algorithms are second-order accurate in both time and space. Meanwhile, we keep a recorder of the CPU time used in the computations. The results demonstrate that algorithm LILEDP-II has much higher calculation efficiency than that of algorithm LILEDP-I. The algorithm LILEDP-II has an overwhelming advantage. Thus we mainly conduct numerical experiments for the algorithm LILEDP-II in the following experiments.

TABLE 1. Mesh refinement test of the algorithm LILEDP-I at $T=1$ with fixed $\Delta t=0.001$.

N	L^∞-error	L^2-error	L^∞-order	L^2-order	CPU(s)
11	3.39e-02	1.047e-01	-	-	13.69
33	3.70e-03	1.16e-02	2.0002	2.0073	16.27
99	4.15e-04	1.30e-03	1.9995	2.0001	23.91
297	4.64e-05	1.01e-04	1.9933	1.9939	83.53

TABLE 2. Mesh refinement test of the algorithm LILEDP-I at $T = 1$ with fixed $N = 1001$.

Δt	L^∞-error	L^2-error	L^∞-order	L^2-order	CPU(s)
0.2	2.08e-02	5.28e-02	-	-	8.22
0.1	3.40e-03	1.22e-02	2.6264	2.1107	13.18
0.05	9.56e-04	3.00e-03	1.8158	2.0093	22.98
0.025	2.29e-04	7.66e-04	2.0654	1.9863	41.89

TABLE 3. Mesh refinement test of the algorithm LILEDP-II at $T = 1$ with fixed $\Delta t = 0.001$.

N	L^∞-error	L^2-error	L^∞-order	L^2-order	CPU(s)
11	3.39e-02	1.05e-01	-	-	0.59
33	3.70e-03	1.16e-02	2.0073	2.0003	1.09
99	4.15e-04	1.30e-03	2.0005	1.9999	4.96
297	4.62e-05	1.44e-04	1.9972	1.9967	25.18

TABLE 4. Mesh refinement test of the algorithm LILEDP-II at $T = 1$ with fixed $N = 1001$.

Δt	L^∞-error	L^2-error	L^∞-order	L^2-order	CPU(s)
0.2	7.50e-03	2.52e-02	-	-	4.29
0.1	1.70e-03	6.10e-03	2.1524	2.0436	6.69
0.05	4.00e-04	1.50e-03	2.0024	2.0174	11.71
0.025	1.00e-04	4.00e-04	1.9664	1.9773	21.25

Example 3.3.2. We simulate the system (1.1) with the double well free energy to investigate coalescence of two drops on $[0,1]^2$, starting from

$$\phi(x,y,0) = \begin{cases} \tanh\big((0.2-\delta_1)/r\big), & \delta_1 < 0.2+r, \\ \tanh\big((0.2-\delta_2)/r\big), & \delta_2 < 0.2+r, \\ -1, & \text{other}, \end{cases}$$

where $\delta_1 = \sqrt{(x-0.3+r)^2+(y-0.5)^2}$, $\delta_2 = \sqrt{(x-0.7-r)^2+(y-0.5)^2}$, and $r = 0.01$. We choose $\gamma = 0.1$, $\kappa = 1.0e-5$, and discretize the space with 113 modes.

Figure 1 displays the evolution of drops obtained by using the algorithm LILEDP-II with $\Delta t = 10^{-5}$. The numerical solution at $t = 0,1,3,5,10,30$ are shown. As can be seen from Figure 1, the two equal-sized circular drops coalesce quickly in the early stage of the evolution, and slowly later until they eventually merge into a single one. The errors in the total mass and the energy change with respect to time are plotted in Figure 2. It can easily be seen that the mass is numerically conservative, while energy decays with the same trajectory. Through numerical experiments, we observe that the algorithm LILEDP-II demonstrates efficiency and effectiveness in preserving energy dissipation law.

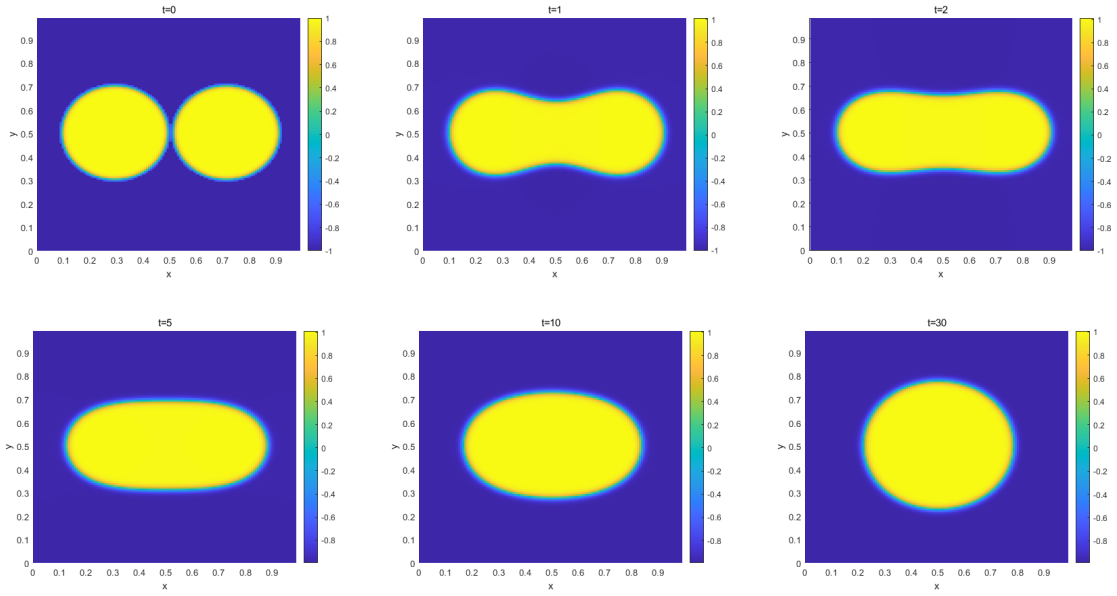
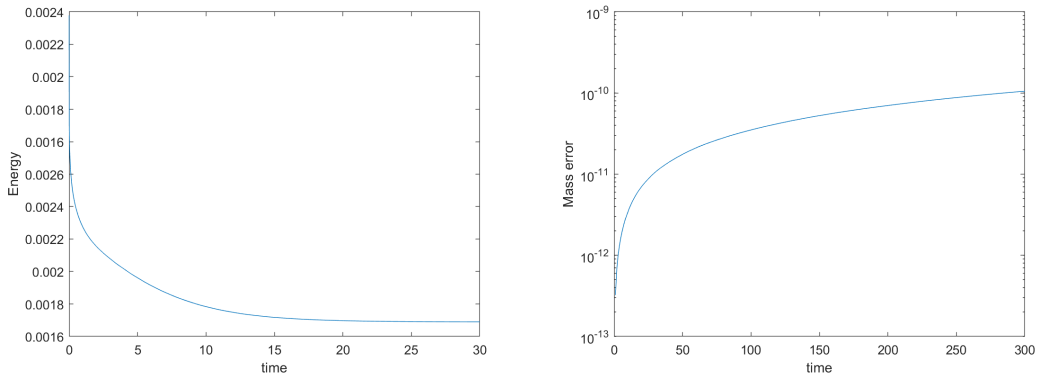


FIGURE 1. Coalescence of two drops simulated using LILEDP-II with $N = 113$, $\Delta t = 10^{-5}$.



(a) (b)

FIGURE 2. The evolution of mass and energy in the drop coalescence example using the LILEDP-II with $N = 113$, $\Delta t = 10^{-5}$. (a) Mass error. (b) The evolution of energy.

.

Acknowledgments

The authors would like to express sincere gratitude to the editors and referees for their insightful feedback and constructive suggestions, which significantly contributed to enhancing the quality of this paper. This work is partially supported by the National Science Foundation of China (Grant No.12171245) and the Foundation of Jiangsu Key Laboratory for Numerical Simulation of Large Scale Complex Systems (201903).

REFERENCES

[1] S.M. Allen, J.W. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, Acta Metall. 27 (1979), 1085-1095.

[2] D.M. Anderson, G.B. McFadden, A.A. Wheeler, Diffuse-interface methods in fluid mechanics, Annu. Rev. Fluid Mech. 30 (1998), 139-165.

[3] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system, I: Interfacial free energy, J. Chem. Phys. 28 (1958), 258.

[4] J.W. Cahn, Free energy of a nonuniform system. II. Thernodynamic basis, J. Chem. Phys. 30 (1959), 1121-1124.

[5] M. Doi, S.F. Edwards, The Theory of Polymer Dynamics, vol. 73, Oxford University Press, 1988.

[6] K. Elder, M. Katakowski, M. Haataja, M. Grant, Modeling elasticity in crystal growth, Phys. Rev. Lett. 88 (2002), 245701.

[7] M.E. Gurtin, D. Polignone, J. Vinals, Two-phase binary fluids and immiscible fluids described by an order parameter, Math. Models Methods Appl. Sci. 6 (1996), 815-831.

[8] D.Z. Han, A. Brylev, X.F. Yang, Z.J. Tan, Numerical analysis of second order, fully discrete energy stable schemes for phase field models of two phase incompressible flows, J. Sci. Comput. 70 (2017), 965-989.

[9] Ryo Kobayashi, Modeling and numerical simulations of dendritic crystal growth, Physica D 63 (1993), 410-423.

[10] F. Leslie, Theory of flow phenomena in liquid crystals, Adv. Liq. Cryst. 4 (1979), 1-81.

[11] J. Shen, J. Xu, J. Yang, The scalar auxiliary variable (SAV) approach for gradient flows, J. Comput. Phys. 353 (2018), 407-416.

[12] F. Chen, J. Shen, Efficient energy stable schemes with spectral discretization in space for anisotropic Cahn-Hilliard systems, Commun. Comput. Phys. 13 (2013), 1189-1208.

[13] A. Christlieb, J. Jones, K. Promislow, B. Wetton, M. Willoughby, High accuracy solutions to energy gradient flows from material science models, J. Comput. Phys. 257 (2014), 193-215.

[14] K.L. Cheng, C. Wang, S.M. Wise, X.Y. Yue, Asecond-order, weakly energy-stable pseudo-spectral scheme for the Cahn-Hilliard equation and its solution by the homogeneous linear iteration method, J. Sci. Comput. 69 (2016), 1083-1114.

[15] Q. Du, X. Feng, The phase field method for geometric moving interfaces and their numerical approximations, Handbook Numer. Anal. 21 (2020), 425-508.

[16] C. Wang, X. Wang, S. Wise, Unconditionally stable schemes for equations of thin film epitaxy, Discrete Contin. Dyn. Syst. 28 (2010), 405-423.

[17] C. Wang, S. Wise, An energy stable and convergent finite-difference scheme for the modified phase field crystal equation, SIAM J. Numer. Anal. 49 (2011), 945-969.

[18] F. Guillen-Gonzailez, G. Tierra, On linear schemes for a Cahn-Hilliard diffuse interface model, J. Comput. Phys. 234 (2013), 140-171.

[19] I. Romero, Thermodynamically consistent time-stepping algorithms for non-linear thermomechanical systems, Int. J. Numer. Meth. Eng. 79 (2009), 706-732.

[20] W.M. Feng, P. Yu, S.Y. Hu, Z.K. Liu, Q. Du, L.Q. Chen, A Fourier spectral moving mesh method for the Cahn-Hilliard equation with elasticity, Commun. Comput. Phys. 5 (2009), 582-599.

[21] Y.N. He, Y.X. Liu, T. Tang, On large time-stepping methods for the Cahn-Hilliard equation, Appl. Numer. Math. 57 (2007), 616-628.

[22] L.L. Ju, J. Zhang, Q. Du, Fast and accurate algorithms for simulating coarsening dynamics of Cahn-Hilliard equations, Comput. Mater. Sci. 108 (2015), 272-282.

[23] C. Liu, J. Shen, A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier-spectral method, Physica D. 179 (2003), 211-228.

[24] J. Shen, X.F. Yang, Numerical approximations of Allen-Cahn and Cahn-Hilliard equations, Discrete Contin, Dyn. Syst. 28 (2012), 1669-1691.

[25] D. Li, Z.H. Qiao, On second order semi-implicit Fourier spectral methods for 2d Cahn-Hilliard equations, J Sci. Comput. 70 (2017), 1-41.

[26] C.M. Elliott, D.A. French, Numerical studies of the Cahn-Hilliard equation for phase separation, IMA J. Appl. Math. 38 (1987), 97-128.
[27] C.M. Elliott, Z. Sonqmu, On the Cahn-Hilliard equation, Arch. Rat. Meth. Anal. 96 (1986), 339-357.
[28] C.M. Elliott, A.M. Stuart, The global dynamics of discrete semilinear parabolic equations, SIAM J. Numer. Anal. 30 (1993), 1622-1663.
[29] D.J. Eyre, Unconditionally Gradient Stable Time Marching the Cahn-Hilliard Equation, Mrs Proceedings, 529 (1998), 39-46.
[30] J. Zhu, L. Chen, J. Shen, V. Tikare, Coarsening kinetics from a variable mobility Cahn-Hilliard equation application of semi-implicit Fourier spectral method, Phys. Rev. E 60 (1999), 3564-3572
[31] J. Shen, X.F. Yang, Numerical approximations of Allen-Cahn and Cahn-Hilliard equations, Discrete Contin. Dyn. Syst., Ser. A 28 (2010), 1669-1691.
[32] X.F. Yang, J. Shen, Q. Wang, Numerical approximations for a three-component cahn-hilliard phase-field model based on the invariant energy quadratization method, Math. Models Meth. Appl. Sci. 27 (2017), 1993-2030.
[33] S. Badia, F. Guillén-González, J.V. Gutiérrez-Santacreu, Finite element approximation of nematic liquid crystal flows using a saddle-point structure, J. Comput. Phys. 230 (2011), 1686-1706.
[34] Q. Cheng, C. Liu, J.Shen, A new Lagrange multiplier approach for gradient flows, Comput. Meth. Appl. Mech. Eng. 367 (2020), 113070.
[35] Y.Z. Gong, J. Zhao, Energy-stable Runge-Kutta schemes for gradient flow models using the energy quadratization approach, Appl. Math. Lett. 94 (2019), 224-231.
[36] Y.Z. Gong, J. Zhao, Q. Wang, Arbitrarily high-order unconditionally energy stable SAV schemes for gradient flow models, Comput. Phys. Commun. 249 (2020), 107033.
[37] J. Shen, J. Xu, Convergence and error analysis for the scalar auxiliary variable (SAV) schemes to gradient flows, SIAM J. Numer. Anal. 56 (2018), 2895-2912.
[38] J. Shen, J. Xu, J. Yang, A new class of efficient and robust energy stable schemes for gradient flows, SIAM Rev. 61 (2019), 474-506.
[39] Y.S. Wang, B.Wang, M.Z. Qin, Local structure-preserving algorithms for partial differential equations, Sci.China Ser. A 51 (2008), 2115-2136.
[40] J. Cai, Y. Wang, H. Liang, Local energy-preserving and momentum-preserving algorithms for coupled nonlinear Schrödinger system, J. Comput. Phys. 239 (2013), 30-50.
[41] Z. Mu, Y. Gong, W. Cai, Y. Wang, Efficient local energy dissipation preserving algorithms for the Cahn-Hilliard equation, J. Comput. Phys. 374 (2018), 654-667.