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LOCAL OPTIMALITY CONDITIONS AND LOCAL SADDLE POINT THEOREMS FOR NONCONVEX ROBUST PROGRAMMING

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Abstract. This paper concerns the study of a class of nonconvex programming problems with data uncertainty in both the objective and constraints. We first introduce two new constraint qualifications in terms of the tangential subdifferential of the involved functions. Under the new constraint qualifications, we provide some necessary and sufficient conditions for KKT-type local optimality conditions to hold. Similarly, local saddle point theorems and local total Lagrange dualities for robust nonconvex programming problems are also given.

Keywords. Constraint qualification; Optimality condition; Robust nonconvex programming; Saddle point theorem; Total duality.

1. INTRODUCTION

The classic convex programming problem with infinite inequality constraints has received much attention and many important results, for example, Farkas lemma, strong duality and total duality, optimality condition, stability and robustness analysis, have been established in the last decades; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the references therein. Especially, as a strengthened condition of the Lagrange multiplier, the KKT-type optimality conditions, and saddle point theorems play important roles in mathematical programming and have been extensively studied by many authors; see, e.g., [3, 4, 9, 10] and the references therein.

As mentioned in [12], the study of mathematical programming problems that are affected by data uncertainty is becoming increasingly important in optimization due to the reality of uncertainty in many real-world optimization problems and the importance of identifying and locating solutions that are immunized against data uncertainty. Thus, mathematical programming problems under uncertainty received much attention; see, e.g., [13, 14, 15, 16, 17, 18, 19, 20, 21] and the references therein. In particular, many authors considered the convex programming problems with data uncertainty in the objective function and/or the constraint functions. By using the properties of epigraphs and subdifferentials of the involved functions, they introduced

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some constraint qualifications. Under these constraint qualifications, they provided some necessary and sufficient conditions for duality results, optimality conditions, and so on to hold, for example, Lagrange duality in [12, 13, 14, 16], characterizations of the solution set in [15, 18], approximate optimality conditions in [22, 23], and saddle point theorems in [19].

Observe that most results in the literature mentioned above were done under the assumption that the involved functions are convex. However, in practical problems, many problems usually involve nonconvex functions. Recently, a lot of attention has been focused on tangent convex programming problems that the objective function and/or the constraint functions are tangent convex. The tangent convex optimization problem has been studied extensively in the literature and a series of meaningful results were obtained, for example, optimality conditions in [24, 25, 26, 27, 28, 29, 30], and characterizations of the solution set in [31, 32, 33]. Especially, in [29], the authors studied a class of semi-infinite programming problems where the objective function and constraint functions, which were assumed to be perturbed by data uncertainty, are tangent convex, and they gave some necessary conditions for KKT-type optimality conditions to hold by using tangential subdifferentials.

Inspired by the works mentioned above, we continue to study the tangent convex programming problem introduced in [29] but in real locally convex Hausdorff topological vector spaces. By using the tangential subdifferential of the involved functions, we introduce some new constraint qualifications. Under those constraint qualifications, some characterizations for the local optimal solution, the saddle point theorem, and the total Lagrangian duality about this tangent convex programming problem under data uncertainty are given. Moreover, applications to robust conic programming are also given. We not only extend and improve some recent known results in [29] but also provide new results as detailed in Section 5.

This paper is organized as follows. The next section contains some necessary notations and preliminary results. In Section 3, some new constraint qualifications are provided and several relationships among them are given. Under those constraint qualifications, some optimality conditions (of KKT-type) for the tangent convex programming problem are established. In Section 4, characterizations for the saddle point theorems and the total Lagrangian dualities are provided. Applications to the robust conic programming problem under data uncertainty are given in the last section, Section 5.

2. Notations and Preliminary Results

Let X be a real locally convex Hausdorff topological vector space, and let X^* denote the dual space of X endowed with the weak*-topology $w^*(X^*,X)$. By $\langle x^*,x\rangle$, we denote the value of the functional $x^*\in X^*$ at $x\in X$; i.e., $\langle x^*,x\rangle = x^*(x)$. We endow $X^*\times\mathbb{R}$ with the product topology of $w^*(X^*,X)$ and the usual Euclidean topology. Let K be a nonempty subset of X. The positive dual cone $K^\oplus$, the indicator function δ_K, and the support function σ_K of K are defined respectively by

$$K^\oplus := \{x^*\in X^* : \langle x^*,x\rangle \geq 0 \quad \text{for each} \quad x\in K\},$$

$$\delta_K(x) := \begin{cases} 0, & x\in K,\\ +\infty, & \text{otherwise,}\end{cases}$$

and

$$\sigma_K(x^*) := \sup_{x\in K}\langle x^*,x\rangle \quad \text{for each} \quad x^*\in X^*.$$

Let $x \in X$. We use $\|x\|$ to denote the norm of x and $\mathbb{B}(x,\varepsilon)$ to denote the neighborhood of x with radius $\varepsilon > 0$, that is,

$$\mathbb{B}(x,\varepsilon) := \{y \in X : \|x-y\| \le \varepsilon\}.$$

Let T be an arbitrary index set. We use $\mathbb{R}^{(T)}$ to denote the space of real tuples $\lambda := (\lambda_t)_{t\in T}$ with only finite many $\lambda_t \neq 0$ and let $\mathbb{R}_+^{(T)}$ denote the nonnegative cone of $\mathbb{R}^{(T)}$, that is,

$$\mathbb{R}_+^{(T)} := \{\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} : \lambda_t \ge 0, t \in T\}.$$

Let $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be a proper function and define the effective domain of f by $\mathrm{dom} f := \{x \in X : f(x) < +\infty\}$ and the subdifferential of f at $x \in \mathrm{dom} f$ by

$$\partial f(x) := \{x^* \in X^* : f(x) + \langle x^*, y-x\rangle \le f(y) \quad \text{for each } y \in X\}. \tag{2.1}$$

Furthermore, the normal cone $N_K(x)$ of a convex set $K \subseteq X$ at the point $x \in K$ is defined by

$$N_K(x) := \partial \delta_K(x) = \{x^* \in X^* : \langle x^*, y-x\rangle \le 0 \text{ for each } y \in K\}.$$

Let $\bar{x}, d \in X$. The directional derivative (or Dini derivative) of f at $\bar{x}$ in direction d is defined by

$$f'(\bar{x};d) := \lim_{\theta \downarrow 0} \frac{f(\bar{x}+\theta d) - f(\bar{x})}{\theta}.$$

We say that f is directionally differentiable at $\bar{x}$ if its directional derivative exists in all directions d; and $f : X \to \overline{\mathbb{R}}$ is tangentially convex at $\bar{x}$ if, for every $d \in X$, $f'(\bar{x};d)$ exists, is finite and the function $f'(\bar{x};d)$ is a convex function of d. Note that $f'(\bar{x};d)$ is positively homogeneous. If f is tangentially convex at $\bar{x}$, then $f'(\bar{x};\cdot)$ is sublinear. The tangential subdifferential of f at $\bar{x}$ is defined by

$$\partial^T f(\bar{x}) := \{x^* \in X^* : f'(\bar{x};d) \ge \langle x^*, d\rangle \quad \text{for all} \quad d \in X\}.$$

It follows that $\partial^T f(\bar{x}) \neq \emptyset$ and $f'(\bar{x};\cdot)$ is the support function of $\partial^T f(\bar{x})$, that is,

$$f'(\bar{x};d) = \max_{\xi \in \partial^T f(\bar{x})} \langle \xi, d\rangle \quad \text{for each} \quad d \in X.$$

If f is a convex function which has an open domain and $\bar{x} \in \mathrm{dom} f$, then f is tangentially convex at $\bar{x}$ and $\partial^T f(\bar{x}) = \partial f(\bar{x})$. If f is Gâteaux differentiable at $\bar{x} \in \mathrm{dom} f$, then f is tangentially convex at $\bar{x}$ and $\partial^T f(\bar{x}) = \{\nabla f(\bar{x})\}$.

Lemma 2.1. [26] *Let f, g be two functions from X to $\overline{\mathbb{R}}$. If f and g are tangentially convex at common point $x \in \mathrm{dom} f \cap \mathrm{dom} g$, then*

$$\partial^T (f+g)(x) = \partial^T f(x) + \partial^T g(x).$$

3. New Constraint Qualifications and Local Optimation Conditions

From now on, let X, Z and $\overline{Z}$ be real locally convex Hausdorff topological vector spaces, $C \subseteq X$ be a nonempty convex set, and T be a nonempty infinite index set. Assume that f is a function from $X \times Z$ to $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ and g_t is a function from $X \times \overline{Z}$ to $\overline{\mathbb{R}}$ for each $t \in T$. We consider the following nonconvex optimization problem under uncertainty:

$$(UP) \qquad \begin{array}{l} \inf \left\{ f(x,u) \right\} \\ \text{s.t. } g_t(x,v_t) \le 0, t \in T, \\ \qquad x \in C, \end{array}$$

where uncertain parameters u and v_t, $t \in T$, belong to the corresponding convex and compact sets $U \subseteq Z$ and $V_t \subseteq \overline{Z}$, $t \in T$, respectively. To address (UP), it typically involves the association with the min-max type robust (worst-case) counterpart:

$$(RP) \qquad \begin{aligned} &\inf \left\{ \max_{u \in U} f(x,u) \right\} \\ &\text{s.t. } g_t(x,v_t) \leq 0, \forall v_t \in V_t, t \in T, \\ &\qquad x \in C, \end{aligned}$$

where one considers the worst case of the uncertain objective w.r.t. the uncertainty set U, and the uncertain constraints are enforced for every possible value of the parameters within the uncertainty set V_t.

To study the necessary and sufficient conditions for optimality conditions and saddle point theorems of the problem (RP), we consider the problem (RP) with linear perturbations:

$$(RP)_p \qquad \begin{aligned} &\inf \left\{ \max_{u \in U} f(x,u) - \langle p, x \rangle \right\} \\ &\text{s.t. } g_t(x,v_t) \leq 0, \forall v_t \in V_t, t \in T, \\ &\qquad x \in C. \end{aligned}$$

In this paper, unless explicity stated otherwise, we always assume that $f(\cdot,u)$ and $g_t(\cdot,v_t)$, $t \in T$, are tangentially convex at each $x \in X$. Denote the feasible set of $(RP)_p$ by

$$A := \{x \in C : g_t(x,v_t) \leq 0, \forall v_t \in V_t, t \in T\}.$$

For a given $x \in A$, we define the index set of all active constraints at x by

$$T(x) := \{t \in T : g_t(x,v_t) = 0, \forall v_t \in V_t\}$$

and the set of active constraint multipliers at x by

$$\Lambda(x) := \{\lambda \in \mathbb{R}_+^{(T)} : \lambda_t g_t(x,v_t) = 0, \forall v_t \in V_t,\ t \in T\}.$$

Definition 3.1. Let $p \in X^*$. It is said that $x \in A$ is a local optimal solution of $(RP)_p$ if there exists $\varepsilon > 0$ such that

$$\max_{u \in U} f(x,u) - \langle p, x \rangle \leq \max_{u \in U} f(y,u) - \langle p, y \rangle \quad \text{for each} \quad y \in \mathbb{B}(x,\varepsilon) \cap A.$$

Proposition 3.1. *Let $p \in X^*$ and $x \in \operatorname{int} A$. Then x is a local optimal solution of $(RP)_p$ if and only if $p \in \partial^T(\max_{u \in U} f(\cdot,u) + \delta_A)(x)$.*

Proof. Let x be a local optimal solution $(RP)_p$. Then, by definition, there exists $\varepsilon > 0$ such that $\mathbb{B}(x,\varepsilon) \subseteq A$ and

$$\max_{u \in U} f(x,u) - \langle p, x \rangle \leq \max_{u \in U} f(z,u) - \langle p, z \rangle \quad \text{for each} \quad z \in \mathbb{B}(x,\varepsilon). \tag{3.1}$$

Let $y \in X$ and $0 < \theta < \frac{\varepsilon}{\|x-y\|}$. Then $x + \theta(y-x) \in \mathbb{B}(x,\varepsilon) \subseteq A$. Thus, $\delta_A(x+\theta(y-x)) = \delta_A(x) = 0$. Hence, by (3.1),

$$\lim_{\theta \downarrow 0} \frac{\max_{u \in U} f(x+\theta(y-x),u) + \delta_A(x+\theta(y-x)) - \max_{u \in U} f(x,u) - \delta_A(x)}{\theta} \geq \langle p, y-x \rangle, \tag{3.2}$$

that is, $p \in \partial^T(\max_{u \in U} f(\cdot,u) + \delta_A)(x)$.

Conversely, Suppose that $p \in \partial^T(\max_{u\in U} f(\cdot,u)+\delta_A)(x)$. Then (3.2) holds for each $y \in X$. Let $y \in X$ be such that $\|y-x\|=1$. Then, by (3.2), there exists $\varepsilon_y > 0$ such that, for each $\theta_y \in (0,\varepsilon_y)$,

$$\frac{\max_{u\in U} f(x+\theta_y(y-x),u)+\delta_A(x+\theta_y(y-x))-\max_{u\in U} f(x,u)-\delta_A(x)}{\theta_y} \geq \langle p, y-x\rangle.$$

Let $\varepsilon := \min_{\|x-y\|=1}\{\varepsilon_y\}$. By the above inequality, for each $\theta \in (0,\varepsilon)$,

$$\max_{u\in U} f(x+\theta(y-x),u)+\delta_A(x+\theta(y-x))-\max_{u\in U} f(x,u)-\delta_A(x) \geq \langle p, \theta(y-x)\rangle.$$

Thus, for each $\theta \in (0,\varepsilon)$ and $z = x+\theta(y-x)$, we see that $z \in \mathbb{B}(x,\varepsilon)$ and

$$\max_{u\in U} f(z,u)+\delta_A(z)-\max_{u\in U} f(x,u) \geq \langle p, z-x\rangle.$$

This implies that x is a local optimal solution of $(RP)_p$. The proof is complete. □

Let $x \in A$, $v = (v_t)_{t\in T} \in V := \prod_{t\in T} V_t$ and $U(x) := \{\bar{u} \in U : f(x,\bar{u}) = \max_{u\in U} f(x,u)\}$. For simplicity, we denote

$$\Omega_1(x) := \bigcup_{u\in U(x)} \partial^T f(\cdot,u)(x) + \bigcup_{\lambda\in\Lambda(x),v\in V} \sum_{t\in T} \lambda_t \partial^T g_t(\cdot,v_t)(x)+N_C(x)$$

and

$$\Omega_2(x) := \bigcup_{(\lambda,u,v)\in\Lambda(x)\times U(x)\times V} \partial^T\left(f(\cdot,u)+\sum_{t\in T}\lambda_t g_t(\cdot,v_t)+\delta_C\right)(x).$$

Then, the following inclusions hold.

Proposition 3.2. *Let $x \in \operatorname{int}A$. Then*

$$\Omega_1(x) \subseteq \Omega_2(x) \subseteq \partial^T(\max_{u\in U} f(\cdot,u)+\delta_A)(x).$$

Proof. Let $p \in \Omega_1(x)$. Then there exists $(\bar{\lambda},\bar{u},\bar{v}) \in \Lambda(x)\times U(x)\times V$ such that

$$p \in \partial^T f(\cdot,\bar{u})(x)+\sum_{t\in T}\bar{\lambda}_t\partial^T g_t(\cdot,\bar{v}_t)(x)+N_C(x),$$

which implies that there exist $\alpha \in \partial^T f(\cdot,\bar{u})(x)$ and $\beta_t \in \partial^T g_t(\cdot,\bar{v}_t)(x), t \in T$ such that $p-\alpha-\sum_{t\in T}\bar{\lambda}_t\beta_t \in N_C(x)$. Thus, by the definitions of tangential subdifferential and normal cone, we have that, for each $y \in C$,

$$\lim_{\theta\downarrow 0}\frac{f(x+\theta(y-x),\bar{u})-f(x,\bar{u})}{\theta} \geq \langle \alpha, y-x\rangle, \tag{3.3}$$

$$\lim_{\theta\downarrow 0}\frac{g_t(x+\theta(y-x),\bar{v}_t)-g_t(x,\bar{v}_t)}{\theta} \geq \langle \beta_t, y-x\rangle, t\in T, \tag{3.4}$$

and

$$\langle p-\alpha-\sum_{t\in T}\bar{\lambda}_t\beta_t, y-x\rangle \leq 0. \tag{3.5}$$

Note that $x \in \text{int}A$. It follows that there exists $\varepsilon > 0$ such that $\mathbb{B}(x,\varepsilon) \subseteq A$. Let $y \in C$ and $0 < \theta < \frac{\varepsilon}{\|x-y\|}$. Then $x+\theta(y-x) \in \mathbb{B}(x,\varepsilon) \subseteq A$. Thus, $\delta_C(x+\theta(y-x)) = \delta_A(x+\theta(y-x)) = \delta_A(x) = \delta_C(x) = 0$. Hence, it follows by (3.3)-(3.5) that

$$\lim_{\theta\downarrow 0} \frac{(f(\cdot,\overline{u}) + \sum_{t\in T} \overline{\lambda}_t g_t(\cdot,\overline{v}_t) + \delta_C)(x+\theta(y-x)) - (f(\cdot,\overline{u}) - \sum_{t\in T} \overline{\lambda}_t g_t(\cdot,\overline{v}_t) + \delta_C)(x)}{\theta} \tag{3.6}$$
$$\geq \langle p, y-x\rangle,$$

which implies that

$$p \in \partial^T (f(\cdot,\overline{u}) + \sum_{t\in T} \overline{\lambda}_t g_t(\cdot,\overline{v}_t) + \delta_C)(x), \tag{3.7}$$

and hence, $p \in \Omega_2(x)$. Therefore, $\Omega_1(x) \subseteq \Omega_2(x)$.

Below we verify that $\Omega_2(x) \subseteq \partial^T(\max_{u\in U} f(\cdot,u) + \delta_A)(x)$. To do this, let $p \in \Omega_2(x)$. Then there exists $(\overline{\lambda},\overline{u},\overline{v}) \in \mathbb{R}_+^{(T)} \times U \times V$ such that

$$\overline{\lambda}_t g_t(x,\overline{v}_t) = 0 \quad \text{for each} \quad t \in T, \tag{3.8}$$

$$f(x,\overline{u}) = \max_{u\in U} f(x,u) \tag{3.9}$$

and (3.7) hold. Note that $x \in \text{int}A$. It follows that there exists $\varepsilon > 0$ such that $\mathbb{B}(x,\varepsilon) \subseteq A$. Let $y \in X$ and $0 \leq \theta \leq \frac{\varepsilon}{\|x-y\|}$. Then, $x+\theta(y-x) \in \mathbb{B}(x,\varepsilon) \subseteq A$ and (3.6) holds by (3.7). Thus, $\delta_C(x+\theta(y-x)) = \delta_A(x+\theta(y-x)) = \delta_A(x) = \delta_C(x) = 0$. Hence,

$$\begin{aligned}
&\lim_{\theta\downarrow 0} \frac{\max_{u\in U} f(x+\theta(y-x),u) + \delta_A(x+\theta(y-x)) - \max_{u\in U} f(x,u) - \delta_A(x)}{\theta} \\
\geq &\lim_{\theta\downarrow 0} \frac{f(x+\theta(y-x),\overline{u}) + \sum_{t\in T} \overline{\lambda}_t g_t(x+\theta(y-x),\overline{v}_t) - f(x,\overline{u}) - \sum_{t\in T} \overline{\lambda}_t g_t(x,\overline{v}_t)}{\theta} \\
\geq &\langle p, y-x\rangle,
\end{aligned}$$

where the first inequality holds by (3.8), (3.9), and $\overline{\lambda}_t g_t(x+\theta(y-x),\overline{v}_t) \leq 0$ for each $t \in T$ and the last inequality holds by (3.6). Thus, by definition, we see that $p \in \partial^T(\max_{u\in U} f(\cdot,u) + \delta_A)(x)$ and hence $\Omega_2(x) \subseteq \partial^T(\max_{u\in U} f(\cdot,u) + \delta_A)(x)$. The proof is complete. □

To establish optimality conditions and saddle point theorems for robust programming problem (RP), we introduce the following robust type constraint qualifications with tangential subdifferential:

$$(RTCQ) \qquad \partial^T(\max_{u\in U} f(\cdot,u) + \delta_A)(x) = \Omega_1(x). \tag{3.10}$$

$$(WRTCQ) \qquad \partial^T(\max_{u\in U} f(\cdot,u) + \delta_A)(x) = \Omega_2(x). \tag{3.11}$$

Remark 3.1. Let $x \in \text{int}A$. Then by Proposition 3.2, we see that (3.10) and (3.11) can be equivalently replaced by

$$\partial^T(\max_{u\in U} f(\cdot,u) + \delta_A)(x) \subseteq \Omega_1(x),$$

and

$$\partial^T(\max_{u\in U} f(\cdot,u) + \delta_A)(x) \subseteq \Omega_2(x).$$

In addition, we also see that the following implication holds:

$$\text{the condition } (RTCQ) \Rightarrow \text{the condition } (WRTCQ).$$

Below we give some local optimality conditions for robust programming problem $(RP)_p$ in terms of the condition $(RTCQ)$.

Theorem 3.1. *Let $\bar{x} \in \operatorname{int} A$. Then the following statements are equivalent.*

(i) *The condition (RTCQ) holds at $\bar{x}$.*

(ii) *For each $p \in X^*$, $\bar{x}$ is a local optimal solution of $(RP)_p$ if and only if there exists $(\bar{\lambda}, \bar{u}, \bar{v}) \in \mathbb{R}_+^{(T)} \times U \times V$ such that*

$$p \in \partial^T f(\cdot, \bar{u})(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t \partial^T g_t(\cdot, \bar{v}_t)(\bar{x}) + N_C(\bar{x}),$$

$$\bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0 \quad \text{for each} \quad t \in T \tag{3.12}$$

and

$$f(\bar{x}, \bar{u}) = \max_{u \in U} f(\bar{x}, u). \tag{3.13}$$

Proof. By Proposition 3.1, (ii) is equivalent to

$$p \in \partial^T (\max_{u \in U} f(\cdot, u) + \delta_A)(\bar{x}) \Leftrightarrow p \in \Omega_1(\bar{x}) \quad \text{for each} \quad p \in X^*,$$

which is equivalent to (i). The proof is complete. □

If $p = 0$, then we obtain by Theorem 3.1 the following corollary straightforwardly.

Corollary 3.1. *Let $\bar{x} \in \operatorname{int} A$. Suppose that the condition (RTCQ) holds at $\bar{x}$. If $\bar{x}$ is a local optimal solution of (RP), then there exists $(\bar{\lambda}, \bar{u}, \bar{v}) \in \mathbb{R}_+^{(T)} \times U \times V$ such that (3.12)-(3.13) hold and*

$$0 \in \partial^T f(\cdot, \bar{u})(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t \partial^T g_t(\cdot, \bar{v}_t)(\bar{x}) + N_C(\bar{x}). \tag{3.14}$$

Remark 3.2. In the case when $X := \mathbb{R}^n$, $Z := \mathbb{R}^m$, $\overline{Z} := \mathbb{R}^q$, $f : \mathbb{R}^n \times U \to \mathbb{R}$, $g_t : \mathbb{R}^n \times V \to \mathbb{R}, t \in T$ are proper functions. Assuming that f is continuous on $\mathbb{R}^n \times U$, $f(x, \cdot)$ is concave on U, and the mapping $x \longmapsto f'(\cdot, u)(x; d)$ is upper semicontinuous on $\mathbb{R}^n$, the authors in [29] obtained a similar result via the following condition

$$N_A(x) \subseteq \bigcup_{\lambda \in \Lambda(x), v \in V} \left(\sum_{t \in T} \lambda_t \partial^T g_t(\cdot, v_t)(x) \right) + N_C(x). \tag{3.15}$$

Note that, in this case, by [29, Proposition 3.1],

$$\partial^T (\max_{u \in U} f(\cdot, u))(x) = \bigcup_{u \in U(x)} \partial^T f(\cdot, u)(x).$$

Let $x \in \operatorname{int} A$. Then, by Lemma 2.1,

$$\partial^T (\max_{u \in U} f(\cdot, u) + \delta_A)(x) = \partial^T (\max_{u \in U} f(\cdot, u))(x) + N_A(x),$$

which together with (3.15) and Remark 3.1 implies that the condition $(RTCQ)$ holds at x. Therefore, Theorem 3.1 extends and improves the corresponding one in [29, Theorem 3.1].

4. Local Saddle Point Theorems

Let $p \in X^*$. Define the Lagrange function L_p on $C \times \mathbb{R}_+^{(T)} \times U \times V$ by

$$L_p(x,\lambda,u,v) := f(x,u) + \sum_{t\in T} \lambda_t g_t(x,v_t) - \langle p,x\rangle$$

for each $(x,\lambda,u,v) \in C \times \mathbb{R}_+^{(T)} \times U \times V$.

Consider the problem $(RP)_p$ and its Lagrange dual problem

$$(RD)_p \qquad \sup_{(\lambda,u,v)\in\mathbb{R}_+^{(T)}\times U\times V} \inf_{x\in C} L_p(x,\lambda,u,v).$$

For simplicity, we denote L_0 by L, that is,

$$L(x,\lambda,u,v) := f(x,u) + \sum_{t\in T} \lambda_t g_t(x,v_t) \quad \text{for each} \quad (x,\lambda,u,v) \in C \times \mathbb{R}_+^{(T)} \times U \times V$$

and denote problem $(RD)_0$ by (RD), that is,

$$(RD) \qquad \sup_{(\lambda,u,v)\in\mathbb{R}_+^{(T)}\times U\times V} \inf_{x\in C} L(x,\lambda,u,v).$$

As usual, we use the notations $v((RP)_p)$ and $v((RD)_p)$ for the optimal values of problems $(RP)_p$ and $(RD)_p$, respectively. Then, the following inequality holds:

$$v((RD)_p) \le v((RP)_p) \quad \text{for each} \quad p \in X^*,$$

that is, the stable weak duality holds between (RP) and (RD).

Let $p \in X^*$. It is said that $(\bar{x},\bar{\lambda},\bar{u},\bar{v}) \in C \times \mathbb{R}_+^{(T)} \times U \times V$ is a local saddle point of L_p if there exists $\varepsilon > 0$ such that, for each $(x,\lambda,u,v) \in (\mathbb{B}(\bar{x},\varepsilon)\cap C) \times \mathbb{R}_+^{(T)} \times U \times V$,

$$L_p(\bar{x},\lambda,u,v) \le L_p(\bar{x},\bar{\lambda},\bar{u},\bar{v}) \le L_p(x,\bar{\lambda},\bar{u},\bar{v}).$$

Theorem 4.1. *Let $\bar{x} \in \mathrm{int}A$. Then the following statements are equivalent.*

(i) *The condition (WRTCQ) holds at $\bar{x}$.*

(ii) *For each $p \in X^*$, if $\bar{x}$ is a local optimal solution of $(RP)_p$, then there exists $(\bar{\lambda},\bar{u},\bar{v}) \in \mathbb{R}_+^{(T)} \times U \times V$ such that $(\bar{x},\bar{\lambda},\bar{u},\bar{v})$ is a local saddle point of L_p.*

(iii) *For each $p \in X^*$, if $\bar{x}$ is a local optimal solution of $(RP)_p$, then there exists $\mathbb{B}(\bar{x},\varepsilon) \subseteq A$ such that*

$$\begin{aligned}\max_{u\in U} f(\bar{x},u) - \langle p,\bar{x}\rangle &= \inf_{x\in\mathbb{B}(\bar{x},\varepsilon)} \{\max_{u\in U} f(x,u) - \langle p,x\rangle\} \\ &= \max_{(\lambda,u,v)\in\mathbb{R}_+^{(T)}\times U\times V} \inf_{x\in\mathbb{B}(\bar{x},\varepsilon)} L_p(x,\lambda,u,v).\end{aligned}$$

Proof. (i)⇒(ii) Suppose that (i) holds. Let $p \in X^*$ and let $\bar{x}$ be a local optimal solution of $(RP)_p$. Then, by Proposion 3.1, we see $p \in \partial^T(\max_{u\in U} f(\cdot,u) + \delta_A)(\bar{x})$. Hence, it follows from condition $(WRTCQ)$ that there exists $(\bar{\lambda},\bar{u},\bar{v}) \in \mathbb{R}_+^{(T)} \times U \times V$ such that (3.12)-(3.13) hold and

$$p \in \partial^T(f(\cdot,\bar{u}) + \sum_{t\in T} \bar{\lambda}_t g_t(\cdot,\bar{v}_t) + \delta_C)(\bar{x}), \tag{4.1}$$

which implies that, for each $y \in X$,

$$\lim_{\theta \downarrow 0} \frac{(f(\cdot,\overline{u}) + \sum_{t\in T} \overline{\lambda}_t g_t(\cdot,\overline{v}_t) + \delta_C)(\overline{x} + \theta(y-\overline{x})) - (f(\cdot,\overline{u}) - \sum_{t\in T} \overline{\lambda}_t g_t(\cdot,\overline{v}_t) + \delta_C)(\overline{x})}{\theta} \geq \langle p, y-\overline{x}\rangle. \tag{4.2}$$

Let $y \in X$ be such that $\|y-\overline{x}\| = 1$. Then, by (4.2), there exists $\varepsilon_y > 0$ such that, for each $\theta_y \in (0,\varepsilon_y)$,

$$\frac{(f(\cdot,\overline{u}) + \sum_{t\in T} \overline{\lambda}_t g_t(\cdot,\overline{v}_t) + \delta_C)(\overline{x} + \theta_y(y-\overline{x})) - (f(\cdot,\overline{u}) - \sum_{t\in T} \overline{\lambda}_t g_t(\cdot,\overline{v}_t) + \delta_C)(\overline{x})}{\theta_y} \geq \langle p, y-\overline{x}\rangle.$$

Let $\varepsilon := \min_{\|y-\overline{x}\|=1}\{\varepsilon_y\}$. From the inequality above, for each $\theta \in (0,\varepsilon)$, one obtains that

$$(f(\cdot,\overline{u}) + \sum_{t\in T} \overline{\lambda}_t g_t(\cdot,\overline{v}_t) + \delta_C)(\overline{x} + \theta(y-\overline{x})) - (f(\cdot,\overline{u}) - \sum_{t\in T} \overline{\lambda}_t g_t(\cdot,\overline{v}_t) + \delta_C)(\overline{x}) \geq \langle p, \theta(y-\overline{x})\rangle.$$

Thus, for each $\theta \in (0,\varepsilon)$ and $x = \overline{x} + \theta(y-\overline{x})$, we see that $x \in \mathbb{B}(\overline{x},\varepsilon)$ and

$$f(x,\overline{u}) + \sum_{t\in T} \overline{\lambda}_t g_t(x,\overline{v}_t) + \delta_C(x) - \langle p,x\rangle \geq f(\overline{x},\overline{u}) + \sum_{t\in T} \overline{\lambda}_t g_t(\overline{x},\overline{v}_t) - \langle p,\overline{x}\rangle.$$

This implies that

$$L_p(x,\overline{\lambda},\overline{u},\overline{v}) \geq L_p(\overline{x},\overline{\lambda},\overline{u},\overline{v}) \quad \text{for each} \quad x \in \mathbb{B}(\overline{x},\varepsilon)\cap C. \tag{4.3}$$

Below we show that

$$L_p(\overline{x},\overline{\lambda},\overline{u},\overline{v}) \geq L_p(\overline{x},\lambda,u,v) \quad \text{for each} \quad (\lambda,u,v) \in \mathbb{R}_+^{(T)} \times U \times V. \tag{4.4}$$

To do this, we find, for each $(\lambda,u,v) \in \mathbb{R}_+^{(T)} \times U \times V$, that

$$\begin{aligned} f(\overline{x},\overline{u}) + \sum_{t\in T} \overline{\lambda}_t g_t(\overline{x},\overline{v}_t) - \langle p,\overline{x}\rangle &= \max_{u\in U} f(\overline{x},u) - \langle p,\overline{x}\rangle \\ &\geq \max_{u\in U} f(\overline{x},u) + \sum_{t\in T} \lambda_t g_t(\overline{x},v_t) - \langle p,\overline{x}\rangle \\ &\geq f(\overline{x},u) + \sum_{t\in T} \lambda_t g_t(\overline{x},v_t) - \langle p,\overline{x}\rangle, \end{aligned}$$

where the first equality holds by (3.12)-(3.13) and the first inequality holds by $\lambda_t g_t(\overline{x},v_t) \leq 0$ for each $t \in T$. Thus, (4.4) holds. This together with (4.3) implies that $(\overline{x},\overline{\lambda},\overline{u},\overline{v})$ is a local saddle point of L_p.

(ii)⇒(i) Suppose that (ii) holds. Let $p \in \partial^T(\max_{u\in U} f(\cdot,u) + \delta_A)(\overline{x})$. Then, by Proposition 3.1, $\overline{x}$ is a local optimal solution of $(RP)_p$. It follows from (ii) that there exists $(\overline{\lambda},\overline{u},\overline{v}) \in \mathbb{R}_+^{(T)} \times U \times V$ such that $(\overline{x},\overline{\lambda},\overline{u},\overline{v})$ is a local saddle point of L_p. This implies that there exists $\mathbb{B}(\overline{x},\varepsilon)$ such that (4.3) and (4.4) hold. Hence, by (4.3), we get that for each $x \in \mathbb{B}(\overline{x},\varepsilon)\cap C$,

$$f(x,\overline{u}) + \sum_{t\in T} \overline{\lambda}_t g_t(x,\overline{v}_t) - f(\overline{x},\overline{u}) - \sum_{t\in T} \overline{\lambda}_t g_t(\overline{x},\overline{v}_t) \geq \langle p, x-\overline{x}\rangle. \tag{4.5}$$

Note that $\bar{x} \in \text{int}A$. It follows that there exists $\varepsilon_1 > 0$ such that $\mathbb{B}(\bar{x}, \varepsilon_1) \subseteq A$. Let $\bar{\varepsilon} = \min\{\varepsilon, \varepsilon_1\}$. Then $\mathbb{B}(\bar{x}, \bar{\varepsilon}) \subseteq \mathbb{B}(\bar{x}, \varepsilon) \cap C$. Let $y \in X$ and let $0 < \theta < \frac{\bar{\varepsilon}}{\|\bar{x}-y\|}$. Then, $\bar{x} + \theta(y - \bar{x}) \in \mathbb{B}(\bar{x}, \bar{\varepsilon})$. Thus, $\delta_C(\bar{x} + \theta(y - \bar{x})) = \delta_C(\bar{x}) = 0$. This together with (4.5) implies that (4.2) holds, so does (4.1). Therefore, to show (i), we only need to show that (3.12) and (3.13) hold. To do this, by (4.4), we see that, for each $(\lambda, u, v) \in \mathbb{R}_+^{(T)} \times U \times V$,

$$f(\bar{x}, u) + \sum_{t \in T} \lambda_t g_t(\bar{x}, v_t) - \langle p, \bar{x} \rangle \leq f(\bar{x}, \bar{u}) + \sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) - \langle p, \bar{x} \rangle. \tag{4.6}$$

Take $u = \bar{u}$ and $\lambda = \widehat{\lambda} \in \mathbb{R}_+^{(T)}$, $v_t = \widehat{v}_t \in V_t$ such that $\widehat{\lambda}_t g_t(\bar{x}, \widehat{v}_t) = 0$ for each $t \in T$. Then $\sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) \geq 0$. Note that $\bar{x} \in \text{int}A$. It follows that $\bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) \leq 0$ for each $t \in T$. Thus, (3.12) holds. Next, we take $\lambda = \widehat{\lambda} \in \mathbb{R}_+^{(T)}$, $v_t = \widehat{v}_t \in V_t$ such that $\widehat{\lambda}_t g_t(\bar{x}, \widehat{v}_t) = 0$ for each $t \in T$. Then $f(\bar{x}, u) \leq f(\bar{x}, \bar{u})$ for each $u \in U$, which implies that (3.13) holds.

(ii)⇒(iii) Suppose that (ii) holds. Let $p \in X^*$, and let $\bar{x}$ be a local optimal solution of $(RP)_p$. Then there exists $\mathbb{B}(\bar{x}, \varepsilon_1) \subseteq A$ such that

$$\max_{u \in U} f(\bar{x}, u) - \langle p, \bar{x} \rangle \leq \max_{u \in U} f(x, u) - \langle p, x \rangle \quad \text{for each} \quad x \in \mathbb{B}(\bar{x}, \varepsilon_1). \tag{4.7}$$

By (ii), we have that there exists $\varepsilon_2 > 0$ such that

$$f(x, \bar{u}) + \sum_{t \in T} \bar{\lambda}_t g_t(x, \bar{v}_t) - \langle p, x \rangle \geq f(\bar{x}, \bar{u}) + \sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) - \langle p, \bar{x} \rangle \tag{4.8}$$

holds for each $x \in \mathbb{B}(\bar{x}, \varepsilon_2) \subseteq C$ and (4.6) holds for each $(\lambda, u, v) \in \mathbb{R}_+^{(T)} \times U \times V$. Hence, by using the same process in (ii)⇒(i), we see that (3.12) and (3.13) hold. This together with (4.8) implies that

$$f(x, \bar{u}) + \sum_{t \in T} \bar{\lambda}_t g_t(x, \bar{v}_t) - \langle p, x \rangle \geq \max_{u \in U} f(\bar{x}, u) - \langle p, \bar{x} \rangle \quad \text{for each} \quad x \in \mathbb{B}(\bar{x}, \varepsilon_2). \tag{4.9}$$

Let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. It follows by (4.9) and (4.7) that

$$\begin{aligned} \inf_{x \in \mathbb{B}(\bar{x}, \varepsilon)} L_p(x, \bar{\lambda}, \bar{u}, \bar{v}_t) &\geq \inf_{x \in \mathbb{B}(\bar{x}, \varepsilon_2)} L_p(x, \bar{\lambda}, \bar{u}, \bar{v}_t) \\ &\geq \max_{u \in U} f(\bar{x}, u) - \langle p, \bar{x} \rangle \\ &= \inf_{x \in \mathbb{B}(\bar{x}, \varepsilon_1)} \{\max_{u \in U} f(x, u) - \langle p, x \rangle\} \\ &= \inf_{x \in \mathbb{B}(\bar{x}, \varepsilon)} \{\max_{u \in U} f(x, u) - \langle p, x \rangle\}. \end{aligned}$$

Thus,

$$\max_{(\lambda, u, v) \in \mathbb{R}_+^{(T)} \times U \times V} \inf_{x \in \mathbb{B}(\bar{x}, \varepsilon)} L_p(x, \lambda, u, v) \geq \inf_{x \in \mathbb{B}(\bar{x}, \varepsilon)} \{\max_{u \in U} f(x, u) - \langle p, x \rangle\}.$$

Next, we show that

$$\max_{(\lambda, u, v) \in \mathbb{R}_+^{(T)} \times U \times V} \inf_{x \in \mathbb{B}(\bar{x}, \varepsilon)} L_p(x, \lambda, u, v) \leq \inf_{x \in \mathbb{B}(\bar{x}, \varepsilon)} \{\max_{u \in U} f(x, u) - \langle p, x \rangle\}. \tag{4.10}$$

To do this, note that, for each $(x,\lambda,u,v)\in\mathbb{B}(\bar{x},\varepsilon)\times\mathbb{R}_+^{(T)}\times U\times V$,

$$\begin{aligned}\max_{u\in U} f(x,u)-\langle p,x\rangle &\geq f(x,u)-\langle p,x\rangle\\ &\geq f(x,u)+\sum_{t\in T}\lambda_t g_t(x,v_t)-\langle p,x\rangle.\end{aligned}$$

This implies that (4.10) holds. Therefore, (iii) is valid.

(iii)⇒(ii) Suppose that (iii) holds. Let $\bar{x}$ be a local optimal solution of $(RP)_p$. Then there exist $\varepsilon>0$ and $(\bar{\lambda},\bar{u},\bar{v})\in\mathbb{R}_+^{(T)}\times U\times V$ such that $\mathbb{B}(\bar{x},\varepsilon)\subseteq A$ and

$$\inf_{x\in\mathbb{B}(\bar{x},\varepsilon)} L_p(x,\bar{\lambda},\bar{u},\bar{v})=\max_{u\in U} f(\bar{x},u)-\langle p,\bar{x}\rangle. \tag{4.11}$$

Thus,

$$f(\bar{x},\bar{u})+\sum_{t\in T}\bar{\lambda}_t g_t(\bar{x},\bar{v}_t)\geq \max_{u\in U} f(\bar{x},u),$$

which together with $\bar{\lambda}_t g_t(\bar{x},\bar{v}_t)\leq 0$ for each $t\in T$ yields that (3.12) and (3.13) hold. Moreover, we note that, for each $(x,\lambda,u,v)\in\mathbb{B}(\bar{x},\varepsilon)\times\mathbb{R}_+^{(T)}\times U\times V$,

$$\begin{aligned}f(x,\bar{u})+\sum_{t\in T}\bar{\lambda}_t g_t(x,\bar{v}_t)-\langle p,x\rangle &\geq \max_{u\in U} f(\bar{x},u)-\langle p,\bar{x}\rangle\\ &= f(\bar{x},\bar{u})+\sum_{t\in T}\bar{\lambda}_t g_t(\bar{x},\bar{v}_t)-\langle p,\bar{x}\rangle\\ &\geq f(\bar{x},u)+\sum_{t\in T}\lambda_t g_t(\bar{x},v_t)-\langle p,\bar{x}\rangle,\end{aligned}$$

where the first inequality holds by (4.11) and the first equality is true due to (3.12) and (3.13), and the last inequality holds by $\lambda_t g_t(\bar{x},v_t)\leq 0$ for each $t\in T$. This implies that $(\bar{x},\bar{\lambda},\bar{u},\bar{v})$ is a local saddle point of L_p. The proof is complete. □

The following example supplies a simple case where Corollary 3.1 and Theorem 4.1 are applicable.

Example 4.1. Let $X=Z=\bar{Z}:=\mathbb{R}$, $C:=[-1,1]$, $U:=[0,1]$, and $V_t:=[1,1+t]$, $t\in T:=[0,1]$. Consider the functions

$$f(x,u):=\begin{cases}-x^2-2x-u^2, & \text{if } x\leq 0,\\ x^2-u^2, & \text{if } x>0\end{cases}$$

and

$$g_t(x,v_t):=\begin{cases}-v_t x^2-1, & \text{if } x\leq 0,\\ \frac{t}{v_t}x-1, & \text{if } x>0,\end{cases}$$

where $u\in U$ and $v_t\in V_t$, $t\in T$. Clearly, $A=[-1,1]$, $\bar{x}=0$ is a local optimal solution of (RP). By simple calculations, $\Lambda(\bar{x})=\{0\}$,

$$f'(\cdot,u)(\bar{x};d)=(\max_{u\in U}(\cdot,u)+\delta_A)'(\bar{x};d)=\begin{cases}-2d, & \text{if } d\leq 0,\\ 0, & \text{if } d>0,\end{cases}$$

and

$$g_t'(\cdot,v_t)(\bar{x},d)=\begin{cases}0, & \text{if } d\leq 0,\\ \frac{t}{v_t}d, & \text{if } d>0.\end{cases}$$

Then,

$$\partial^T f(\cdot,u)(\bar{x}) = \partial^T(\max_{u\in U}(\cdot,u)+\delta_A)(\bar{x}) = [-2,0]$$

and

$$\partial^T g_t(\cdot,v_t)(\bar{x}) = \left[0,\frac{t}{v_t}\right] \quad \text{for each} \quad t\in T.$$

Note that

$$N_C(\bar{x}) = \bigcup_{\lambda\in\Lambda(\bar{x}),v\in V}\sum_{t\in T}\lambda_t\partial^T g_t(\cdot,v_t)(\bar{x}) = \{0\}.$$

It follows that condition $(RTCQ)$ is satisfied at $\bar{x}$. Then, by Corollary 3.1, there exists $(\bar{\lambda},\bar{u},\bar{v}) \in \mathbb{R}_+^{(T)}\times U\times V$ such that (3.12)- (3.14) hold. In fact, let $\bar{u}=\bar{\lambda}=0$. It is easy to see that $f(\bar{x},\bar{u}) = \max_{u\in U} f(\bar{x},u)$, $\bar{\lambda}_t g_t(\bar{x},\bar{v}_t)=0$ for all $\bar{v}\in V$ and

$$0\in\partial^T f(\cdot,\bar{u})(\bar{x})+\sum_{t\in T}\bar{\lambda}_t\partial^T g_t(\cdot,\bar{v}_t)(\bar{x})+N_C(\bar{x}).$$

Thus, Corollary 3.1 is applicable. Below we illustrate that Theorem 4.1 holds. To do this, we see that

$$\bigcup_{(\lambda,u,v)\in\Lambda(\bar{x})\times U(\bar{x})\times V}\partial^T\left(f(\cdot,u)+\sum_{t\in T}\lambda_t g_t(\cdot,v_t)+\delta_C\right)(\bar{x}) = [-2,0].$$

It follow that the condition $(WRTCQ)$ is satisfied at $\bar{x}$. Note that

$$L(x,\lambda,u,v) = \begin{cases} -x^2-2x-u^2+\sum\limits_{t\in T}\lambda_t(-v_t x^2-1), & \text{if } x\le 0,\\ x^2-u^2+\sum\limits_{t\in T}\lambda_t(\frac{t}{v_t}x-1), & \text{if } x>0.\end{cases}$$

Let $\varepsilon=\frac{1}{2}$, $\bar{u}=0$, $\bar{\lambda}=0$, and $\bar{v}\in V$. Then, for each $(x,\lambda,u,v)\in\mathbb{B}(\bar{x},\varepsilon)\times\mathbb{R}_+^{(T)}\times U\times V$, $L(\bar{x},\lambda,u,v) = -u^2+\sum_{t\in T}-\lambda_t\le 0$, $L(\bar{x},\bar{\lambda},\bar{u},\bar{v})=0$, and $L(x,\bar{\lambda},\bar{u},\bar{v})\ge 0$. This implies that $(\bar{x},\bar{\lambda},\bar{u},\bar{v})$ is a local saddle point of L. Furthermore,

$$\begin{aligned}\max_{u\in U} f(\bar{x},u)-\langle p,\bar{x}\rangle &= \inf_{x\in\mathbb{B}(\bar{x},\varepsilon)}\{\max_{u\in U} f(x,u)-\langle p,x\rangle\}\\ &= \inf_{x\in\mathbb{B}(\bar{x},\varepsilon)} L_p(x,\bar{\lambda},\bar{u},\bar{v}) = 0.\end{aligned}$$

Thus, Theorem 4.1 is applicable.

5. Applications to Robust Conic Programming

Throughout this section, let X,Y,Z, and $\overline{Z}$ be real locally convex Hausdorff topological vector spaces. For each $p\in X^*$, we consider the following uncertain conic programming problem:

$$(UCP)_p \qquad \begin{array}{l}\inf\left\{f(x,u)-\langle p,x\rangle\right\}\\ \text{s.t. } x\in C, g(x,v)\in -S,\end{array}$$

where f is a function from $X\times Z$ to $\overline{\mathbb{R}}$, g is a function from $X\times\overline{Z}$ to Y, $C\subseteq X$ is a nonempty convex set, $S\subseteq Y$ is a nonempty closed convex cone, u and v are uncertain parameters belong to the corresponding convex and compact sets $U \subseteq Z$ and $V \subseteq \overline{Z}$, respectively. As usual, we consider the minmax type robust (worst-case) problem:

$$(RCP)_p \qquad \begin{array}{l} \inf \left\{ \max_{u\in U} f(x,u) - \langle p,x\rangle \right\} \\ \text{s.t. } g(x,v) \in -S, \forall v \in V, \\ \quad x \in C, \end{array}$$

where the uncertain objective and constraint are enforced for every possible value of the parameters within the corresponding uncertainty sets U and V. In this section, we always assume that $f(\cdot,u)$ and $g(\cdot,v)$ are tangentially convex at each $x \in X$.

For each $\lambda \in S^{\oplus}$, we define

$$(\lambda g)(x) := \begin{cases} \langle \lambda, g\rangle, & \text{if} \quad x \in \operatorname{dom} g, \\ +\infty, & \text{otherwise.} \end{cases}$$

It is easy to see that, for each $x \in C$ and $v \in V$, $g(x,v) \in -S$ if and only if $(\lambda g)(x,v) \leq 0$ for each $\lambda \in S^{\oplus}$. Moreover, problem $(UCP)_p$ can be reformulated as an example of $(UP)_p$ by setting

$$T = S^{\oplus}, \quad g_\lambda = (\lambda g) \quad \text{for each} \quad \lambda \in T = S^{\oplus}.$$

As before, we use A to denote the solution set:

$$A := \{x \in C : g(x,v) \in -S, \forall v \in V\} = \{x \in C : (\lambda g)(x,v) \leq 0, \forall v \in V, \lambda \in S^{\oplus}\}.$$

The corresponding Lagrangian function L_p and the dual problem $(RCD)_p$ can be expressed respectively as

$$L_p(x,\lambda,u,v) = f(x,u) + (\lambda g)(x,v) - \langle p,x\rangle \quad \text{for each} \quad (x,\lambda,u,v) \in C \times S^{\oplus} \times U \times V$$

and

$$(RCD)_p \qquad \max_{(\lambda,u,v)\in S^{\oplus}\times U\times V} \inf_{x\in C} L_p(x,\lambda,u,v).$$

Generalizing the corresponding notions in Section 3 to suit the conic programming, we introduce the following constraint qualifications.

$$(C\text{-}RTCQ) \qquad \begin{aligned} &\partial^T(\max_{u\in U} f(\cdot,u) + \delta_A)(x) \\ &= \bigcup_{u\in U(x)} \partial^T f(\cdot,u)(x) + \bigcup_{\lambda\in\Lambda(x),v\in V} \partial^T(\lambda g)(\cdot,v)(x) + N_C(x), \end{aligned}$$

$$(C\text{-}WRTCQ) \qquad \begin{aligned} &\partial^T(\max_{u\in U} f(\cdot,u) + \delta_A)(x) \\ &= \bigcup_{(u,\lambda,v)\in U(x)\times\Lambda(x)\times V} \partial^T(f(\cdot,u) + (\lambda g)(\cdot,v) + \delta_C)(x), \end{aligned}$$

where $x \in A$, $\Lambda(x) := \{\lambda \in S^{\oplus} : (\lambda g)(x,v) = 0, \forall v \in V\}$, and

$$U(x) := \{\overline{u} \in U : f(x,\overline{u}) = \max_{u\in U} f(x,u)\}.$$

Thus, by Theorems 3.1 and 4.1, we have the following theorems straightforwardly.

Theorem 5.1. *Let $\bar{x} \in \mathrm{int}A$. Then the following statements are equivalent.*

(i) *The condition (C-RTCQ) holds at $\bar{x}$.*

(ii) *For each $p \in X^*$, $\bar{x}$ is a local optimal solution of $(RCP)_p$ if and only if there exists $(\bar{\lambda}, \bar{u}, \bar{v}) \in S^{\oplus} \times U \times V$ such that $\bar{\lambda} g(\bar{x}, \bar{v}) = 0$, $f(\bar{x}, \bar{u}) = \max_{u \in U} f(\bar{x}, u)$ and*

$$p \in \partial^T f(\cdot, \bar{u})(\bar{x}) + \partial^T (\bar{\lambda} g)(\cdot, \bar{v})(\bar{x}) + N_C(\bar{x}).$$

Theorem 5.2. *Let $\bar{x} \in \mathrm{int}A$. Then the following statements are equivalent.*

(i) *The condition (C-WRTCQ) holds at $\bar{x}$.*

(ii) *For each $p \in X^*$, if $\bar{x}$ is a local optimal solution of $(RCP)_p$, then there exists $(\bar{\lambda}, \bar{u}, \bar{v}) \in S^{\oplus} \times U \times V$ such that $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v})$ is a local saddle point of L_p.*

(iii) *For each $p \in X^*$, if $\bar{x}$ is a local optimal solution of $(RCP)_p$, then there exists $\mathbb{B}(\bar{x}, \varepsilon) \subseteq A$ such that*

$$\begin{aligned} \max_{u \in U} f(\bar{x}, u) - \langle p, \bar{x} \rangle &= \inf_{x \in \mathbb{B}(\bar{x}, \varepsilon)} \left\{ \max_{u \in U} f(x, u) - \langle p, x \rangle \right\} \\ &= \max_{(\lambda, u, v) \in S^{\oplus} \times U \times V} \inf_{x \in \mathbb{B}(\bar{x}, \varepsilon)} L(x, \lambda, u, v). \end{aligned}$$

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