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VARIATIONAL SETS AND APPLICATIONS TO SENSITIVITY ANALYSIS

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Abstract. In this paper, we first develop the chain and sum rules of the first-order variational sets of type 2. Then, by virtue of the variational sets of type 2, we investigate the sensitivity of variational inequalities. Finally, in terms of these sets, we establish sensitivity results for parametric set-valued equilibrium problems under the weak efficiency. Several examples are provided to illustrate our main results.

Keywords. Parametric equilibrium problems; Sensitivity analysis; Variational sets; Weak perturbation maps.

1. Introduction

Sensitivity analysis, which is of great importance in optimization and some related fields in applied mathematics, mainly concerns derivatives of solution/optimal-valued maps to perturbed optimization problems, which provides quantitative information as regards the perturbation map of a parameterized vector optimization problem when it is perturbed. Therefore, the concept of generalized derivatives plays an essential role in this topic. In set-valued optimization problems, since the classical derivatives do not exist in some cases, it is particularly important to consider the generalized derivatives to replace the classical derivatives. Thus some new generalized derivatives were used to discuss sensitivity. In [1, 2, 3, 4], the sensitivity results for parametric multi-objective optimization problems were discussed via radial derivatives or contingent derivatives. By virtue of higher-order adjacent derivatives, Wang and Li [5] obtained some results on higher-order sensitivity analysis in nonconvex vector optimization. By using variational sets, Anh and Khanh [6] obtained some results on sensitivity analysis for nonsmooth vector optimization. With the aid of the higher-order contingent derivatives and a separation theorem for convex sets, Xu and Peng [7] obtained some results on higher-order sensitivity analysis in set-valued optimization. Xue et al. [8] investigated sensitivity analysis for a parametric vector variational inequality problem by using generalized differentiation. Anh and Thinh [9] established sensitivity analysis for the solution map of the parametric inclusion with the help of higher-order generalized (weak) tangent epiderivatives.

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Variational sets were first introduced and applied in [10]. Since the image of the variational sets is larger than that of some known generalized derivatives, it is difficult to obtain the necessary optimality conditions through separation technology and more strict than the existing results; see [10, 11]. Vector equilibrium theory is an important part of nonlinear analysis. Vector variational inequality, vector optimization, and vector complementarity are special cases of vector equilibrium. Recently, the vector equilibrium problem has been studied extensively. However, some important results are mainly about the solution existence (see, e.g., [12, 13, 14, 15]), stability analysis (see, e.g., [16, 17, 18]), optimality conditions (see, e.g., [19, 20, 21, 22, 23]). To the best of our knowledge, up to now, few sensitivity analysis results were obtained for vector equilibrium problems. Anh [24] proposed a parameterized vector equilibrium problem via the sum of two given set-valued maps and studied sensitivity analysis for this problem in terms of the second-order contingent derivatives. By using the S-derivative of a set-valued map, Deng and Zhao [25] investigated the sensitivity analysis in vector equilibrium problems. In view of $r_n > 0$ in the definition for variational sets of type 2 and $r_n \to 0^+$ in the definition for variational sets of type 1, the existence condition for variational sets of type 2 is weaker than that of type 1. The chain and sum rules for variational sets of type 1 were given in [26]. Observe that the chain and sum rules for variational sets of type 2 are not perfect, we in this paper establish the chain and sum rules for variational sets of type 2 from a different perspective (from that in [27]). This also motivates us to apply these rules to the sensitivity analysis of variational inequalities. Inspired by [26, 27], we investigate the sensitivity analysis in parametric set-valued equilibrium problems under the weak efficiency by using variational sets of type 2. We also mention here that there is a significant difference between this paper and [26, 27] in terms of derivatives, which further facilitates our research.

The organization of this paper is as follows. Section 2 is devoted to some definitions and concepts needed in the sequel. In Section 3, we obtain the chain and sum rules for variational sets of type 2. In Section 4, we establish the first-order sensitivity results of variational inequalities. In Section 5, we study the sensitivity of these sets to parameter set-valued equilibrium problems in the sense of weak efficient solutions. Some conclusions are discussed in Section 6, the last section.

2. Preliminaries

Throughout this paper, let X, Y, and Z be three normed spaces, B_Y be used to denote the closed unit ball in Y, and 0_X, 0_Y, and 0_Z be used to denote the original points of X, Y, and Z, respectively. We denote by $\mathscr{U}(x_0)$ the set of all neighborhoods of x_0. For $B \subseteq Y$, $\operatorname{int}B$ and $\operatorname{cl}B$ stand for the interior and closure of B, respectively. Let $C \subseteq Y$ be a closed convex cone, $\hat{C} \subseteq \operatorname{int}C \cup \{0_Y\}$ be a closed convex cone, $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}$ be the set of real numbers, and $\mathbb{R}_+$ be the set of nonnegative real numbers. A nonempty convex subset $E \subseteq C$ is called a base of C if $0_Y \notin \operatorname{cl}E$ and $C = \operatorname{cone}E := \{re \mid r \geq 0, e \in E\}$. If E is compact, then C is said to have a compact base E. Clearly, the cone C has a compact base if and only if $C \cap \partial B_Y$ is compact. If Y is a finite dimensional space, then C has a compact base. If C has a convex base, then C is convex and pointed. Moreover, we use the following cones,

$$\operatorname{cone}_+ B := \{rb \mid r > 0, b \in B\},\ C^* := \{y^* \in Y^* \mid \langle y^*, c\rangle \geq 0, \forall c \in C\} \text{ (dual cone).}$$

When C is solid (i.e., $\text{int}C \neq \emptyset$), $b_0 \in B$ is said to be a weak efficient point of B, denoted by $b_0 \in \text{WMin}_C B$, if $(B - \{b_0\}) \cap (-\text{int}C) = \emptyset$. Let F and $S : X \to 2^Y$ be two set-valued maps, where X is the space of perturbation parameters and Y be an objective space ordered partially by a closed convex cone C. A nonempty subset B is said to have the weak domination property (see [6]) if $B \subseteq \text{WMin}_C B + \text{int}C \cup \{0_Y\}$. For a set-valued map $F : X \to 2^Y$, the domain and graph of F are defined by $\text{dom}F := \{x \in X \mid F(x) \neq \emptyset\}$ and $\text{gr}F := \{(x,y) \in X \times Y \mid y \in F(x)\}$. The so-called profile map of F is the map $(F + C)(x) := F(x) + C$. The so-called closure map of F is the map $\text{cl}F$ defined by $\text{gr}(\text{cl}F) := \text{cl}(\text{gr}F)$. B has the weak domination property around $x_0 \in X$ with respect to $\hat{C}$ iff there exists a neighborhood V of $x_0 \in X$ such that $F(x) \subseteq \text{WMin}_C F(x) + \hat{C}$. The Painlevé-Kuratowski upper limit is defined by

$$\operatorname*{Limsup}_{x \xrightarrow{F} x_0} F(x) := \{y \in Y \mid \exists x_n \in \text{dom}F,\ y_n \in F(x_n)\ s.t.\ x_n \to x_0,\ y_n \to y,\ \forall n \in \mathbb{N}\},$$

where $x \xrightarrow{F} x_0$ means that $x \in \text{dom}F$ and $x \to x_0$. The Painlevé-Kuratowski lower limit is

$$\operatorname*{Liminf}_{x \xrightarrow{F} x_0} F(x) := \{y \in Y \mid \forall x_n \in \text{dom}F,\ y_n \in F(x_n)\ s.t.\ x_n \to x_0,\ y_n \to y,\ \forall n \in \mathbb{N}\}.$$

If $\operatorname*{Limsup}_{x \xrightarrow{F} x_0} F(x) = \operatorname*{Liminf}_{x \xrightarrow{F} x_0} F(x)$, then this value is called the Painlevé-Kuratowski limit of F at x_0 and denoted by $\operatorname*{Lim}_{x \xrightarrow{F} x_0} F(x)$.

Definition 2.1. [28] Let H be a nonempty subset of X, $x \in H$, and $u \in X$. The contingent cone of H at x is defined by $T(H,x) := \{u \in X \mid \exists r_n \to 0^+,\ \exists u_n \to u,\ s.t.\ x + r_n u_n \in H,\ \forall n \in \mathbb{N}\}$.

Definition 2.2. [29] Let H be a nonempty subset of X, $x \in H$, and $u \in X$. The radial cone of H at x is defined by $R(H,x) := \{u \in X \mid \exists r_n > 0,\ \exists u_n \to u,\ s.t.\ x + r_n u_n \in H,\ \forall n \in \mathbb{N}\}$.

Definition 2.3. [4] Let $F:\ X \to 2^Y$ be a set-valued map and $(x_0, y_0) \in \text{gr}F$. The first-order radial derivative of F at (x_0, y_0) is a set-valued map $D_R F(x_0, y_0) : X \to 2^Y$ defined by

$$D_R F(x_0,y_0)(u) := \{v \in Y \mid \exists r_n > 0,\ \exists (u_n, v_n) \to (u,v), s.t.\ y_0 + r_n v_n \in F(x_0 + r_n u_n),\ \forall n \in \mathbb{N}\}.$$

According to Definition 2.3, one has $0_Y \in D_R F(x_0,y_0)(0_X)$.

Definition 2.4. [10] Let $F:\ X \to 2^Y$ be a set-valued map and $(x_0, y_0) \in \text{gr}F$.

(i) The first-order variational set of type 2 of F at (x_0, y_0) is defined by

$$\overline{W}(F,(x_0,y_0)) := \operatorname*{Limsup}_{x \xrightarrow{F} x_0} (\text{cone}_+(F(x) - y_0)).$$

(ii) The first-order lower variational set of type 2 of F at (x_0, y_0) is defined by

$$\underline{W}(F,(x_0,y_0)) := \operatorname*{Liminf}_{x \xrightarrow{F} x_0} (\text{cone}_+(F(x) - y_0)).$$

F is said to have the first-order proto-variational set of type 2 at (x_0, y_0) iff

$$\overline{W}(F,(x_0,y_0)) = \underline{W}(F,(x_0,y_0)).$$

(iii) The first-order variational set of type 1 of F at (x_0,y_0) is defined by

$$V(F,(x_0,y_0)) := \operatorname*{Limsup}_{\substack{x \xrightarrow{F} x_0, \\ r\to 0^+}} \frac{1}{r}(F(x)-y_0),$$

where $x \xrightarrow{F} x_0$ means $x \in \operatorname{dom} F$ and $x \to x_0$.

According to Definition 2.4, one has $0_Y \in W(F,(x_0,y_0))$.

Remark 2.1. Definition 2.4 can be also expressed equivalently as follows:

(i) $\overline{W}(F,(x_0,y_0)) = \{v \in Y \mid \exists r_n > 0,\ \exists (x_n,v_n) \to (x_0,v), s.t.\ y_0 + r_n v_n \in F(x_n),\ \forall n \in \mathbb{N}\}$;

(ii) $\underline{W}(F,(x_0,y_0)) = \{v \in Y \mid \forall r_n > 0,\ \exists (x_n,v_n) \to (x_0,v), s.t.\ y_0 + r_n v_n \in F(x_n),\ \forall n \in \mathbb{N}\}$;

(iii) $V(F,(x_0,y_0)) = \{v \in Y \mid \exists r_n \to 0^+,\ \exists (x_n,v_n) \to (x_0,v), s.t.\ y_0 + r_n v_n \in F(x_n),\ \forall n \in \mathbb{N}\}$.

It is clear that $\underline{W}(F,(x_0,y_0)) \subseteq V(F,(x_0,y_0)) \subseteq \overline{W}(F,(x_0,y_0))$.
However, the reverse conclusion not necessarily hold; see the following example.

Example 2.1. Let $X = Y = \mathbb{R}, C = \mathbb{R}_+$, and $F : X \to 2^Y$ be defined by $F(x) = \{x^2\}$. Taking $(x_0,y_0) = (0,0)$, one has $\underline{W}(F,(x_0,y_0)) = \{0\}$ and $V(F,(x_0,y_0)) = \mathbb{R}$, which imply that $V(F,(x_0,y_0)) \not\subseteq \underline{W}(F,(x_0,y_0))$. Taking $(x_0,y_0) = (1,0)$, one sees that $V(F,(x_0,y_0)) = \emptyset$ and $\overline{W}(F,(x_0,y_0)) = \mathbb{R}_+$, which imply that $\overline{W}(F,(x_0,y_0)) \not\subseteq V(F,(x_0,y_0))$.

Definition 2.5. [30] Let $M \subseteq X$ be a convex subset and C be a closed convex cone. The map $P : M \to 2^Y$ is said to be C-convex on M if, for all $x_1,x_2 \in M$ and $\lambda \in (0,1)$,

$$\lambda P(x_1) + (1-\lambda)P(x_2) \subseteq P(\lambda x_1 + (1-\lambda)x_2) + C.$$

Inspired by [24], we propose the following definition.

Definition 2.6. Let F and $S : X \to 2^Y$ be two set-valued maps. F is said to be C-dominated by S near x_0 iff $F(x) \subseteq S(x) + C$ for all $x \in \overline{X}$ in some $\overline{X} \in \mathscr{U}(x_0)$.

Let $y_0 \in S(x_0)$ and F be C-dominated by S near x_0. Since $S(x) \subseteq F(x)$ for all $x \in U$, then $S(x) + C = F(x) + C$ for all $x \in \overline{U}$. Thus $\overline{W}(F+C,(x_0,y_0)) = \overline{W}(S+C,(x_0,y_0))$.

Definition 2.7. [6] Let $F :\ X \to 2^Y$ be a set-valued map and $(x_0,y_0) \in \operatorname{gr} F$. The first-order singular variational set of type 2 of F at (x_0,y_0) is defined by

$$\begin{aligned} W^{\infty(1)}(F,(x_0,y_0)) :&= \operatorname*{Limsupcone_+}^{\infty(1)}_{x \xrightarrow{F} x_0}(F(x)-y_0) \\ &= \{y \in Y \mid \exists x_n \xrightarrow{F} x_0,\ \exists \lambda_n \to 0^+, \\ &\quad \exists y_n \in \operatorname{cone}_+(F(x_n)-y_0),\ s.t.\ \lambda_n y_n \to y,\ \forall n \in \mathbb{N}\}. \end{aligned}$$

Lemma 2.1. [6] *Let $x_0 \in S$ and $y_0 \in S(x_0)$. Let $\hat{C} \subseteq \operatorname{int} C \cup \{0_Y\}$ be a closed convex cone with a compact convex base, and F fulfill the weak domination property around x_0 with respect to $\hat{C}$. Then $\overline{W}\left(S+\hat{C},(x_0,y_0)\right) = \overline{W}\left(F+\hat{C},(x_0,y_0)\right)$.*

Lemma 2.2. [6] *Let $\hat{C} \subseteq \operatorname{int} C \cup \{0_Y\}$ be a closed convex cone with a compact convex base. Then*

$$\operatorname{WMin}_C \overline{W}\left(F+\hat{C},(x_0,y_0)\right) \subseteq \overline{W}(F,(x_0,y_0)). \tag{2.1}$$

Lemma 2.3. [6] *Let* $\hat{C} \subseteq \operatorname{int} C \cup \{0_Y\}$ *be a closed convex cone with a compact convex base. Suppose that either of the following conditions holds:*

(i) $\overline{W}\left(F+\hat{C},(x_0,y_0)\right)$ *has the weak domination property;*

(ii) $W^{\infty(1)}\left(F,(x_0,y_0)\right) \cap (-\hat{C}) = \{0_Y\}$.

Then $\operatorname{WMin}_C \overline{W}\left(F+\hat{C},(x_0,y_0)\right) = \operatorname{WMin}_C \overline{W}\left(F,(x_0,y_0)\right)$.

Now we illustrate the relation between the variational set of type 2 of F and that of S.

Proposition 2.1. *Let* $(x_0,y_0) \in \operatorname{gr} S$, $\hat{C} \subseteq \operatorname{int} C \cup \{0_Y\}$ *be a closed convex cone with a compact base, and* F *have the weak domination property around* x_0 *with respect to* $\hat{C}$. *Suppose that either of the following two conditions holds:*

(i) $\overline{W}(F+\hat{C},(x_0,y_0))$ *has the weak domination property;*

(ii) $W^{\infty(1)}(F,(x_0,y_0)) \cap (-\hat{C}) = \{0_Y\}$.

Then

$$\operatorname{WMin}_C \overline{W}\left(F,(x_0,y_0)\right) \subseteq \overline{W}\left(S,(x_0,y_0)\right). \tag{2.2}$$

Proof. From Lemma 2.3, we have $\operatorname{WMin}_C \overline{W}\left(F,(x_0,y_0)\right) = \operatorname{WMin}_C \overline{W}\left(F+\hat{C},(x_0,y_0)\right)$, which together with Lemma 2.1 yields $\operatorname{WMin}_C \overline{W}\left(F,(x_0,y_0)\right) = \operatorname{WMin}_C \overline{W}\left(S+\hat{C},(x_0,y_0)\right)$. In view of (2.1), we find from Lemma 2.2 that $\operatorname{WMin}_C \overline{W}\left(F,(x_0,y_0)\right) \subseteq \overline{W}\left(S,(x_0,y_0)\right)$. This completes the proof. □

Under the conditions of Proposition 2.1, the inverse inclusion relation of (2.2) may not necessarily hold, which can be seen via the following example.

Example 2.2. Let $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $C=\mathbb{R}^2_+$, and $F:\ X \to 2^Y$ be defined by

$$F(x)=\begin{cases}(\{0\}\times\mathbb{R})\cup(\mathbb{R}\times\{0\})\cup\left\{(x_0,y_0)\in Y \mid x_0^2+y_0^2=1\right\}, & \text{if } x=0,\\ \emptyset, & \text{if } x\neq 0.\end{cases}$$

Then

$$S(x)=\begin{cases}((-\infty,-1)\times\{0\})\cup(\{0\}\times(-\infty,-1))\cup \\ \{(x_0,y_0)\in Y \mid x_0^2+y_0^2=1, x\le 0, y\le 0\}, & \text{if } x=0,\\ \emptyset, & \text{if } x\neq 0.\end{cases}$$

Since $\overline{W}(F,(0,(-1,0))) = (\mathbb{R}_+\times\mathbb{R})\cup(\mathbb{R}_-\times\{0\})$,

$$\operatorname{WMin}_C \overline{W}(F,(0,(-1,0))) = (\{0\}\times\mathbb{R}_-)\cup(\mathbb{R}_-\times\{0\}).$$

However, we have

$$\overline{W}(S,(0,(-1,0))) = (\mathbb{R}_-\times\{0\})\cup\{(x_0,y_0)\in Y \mid y_0\le -x_0, x_0\ge 0\}.$$

Hence, $\overline{W}(S,(0,(-1,0))) \not\subseteq \operatorname{WMin}_C \overline{W}(F,(0,(-1,0)))$.

To see the inverse inclusion relation of (2.2) in Proposition 2.1, we impose three conditions as follows.

Proposition 2.2. *Let* $(x_0,y_0) \in \operatorname{gr} F$, *let* C *have a compact convex base. Assume that the following conditions are satisfied:*

(i) F *is* C*-dominated by* S *near* x_0*;*

(ii) F *has the first-order proto-variational set of type 2 at* (x_0,y_0)*;*

(iii) for all $x \in \overline{X}$ in some $\overline{X} \in \mathscr{U}(x_0)$, $m-e \in \mathrm{int}C \cup (-\mathrm{int}C)$, $\forall m, e \in F(x)$, $m \neq e$.

Then

$$\overline{W}(S,(x_0,y_0)) \subseteq \mathrm{WMin}_C \overline{W}(F,(x_0,y_0)). \tag{2.3}$$

If, additionally, the conditions in Proposition 2.1 are fulfilled, then (2.3) becomes an equality.

Proof. First, we prove that $S(x)$ is a single point set for all $x \in \overline{X}$ in some $\overline{X} \in \mathscr{U}(x_0)$. Indeed, if $m, e \in S(x)$, then $m, e \in F(x)$. Suppose that $S(x)$ is not a single point set. Then, for any $m, e \in S(x) \subseteq F(x)$ with $e \neq m$, according to assumption (iii), one has $m-e \in \mathrm{int}C \cup (-\mathrm{int}C)$, that is, $m-e \in \mathrm{int}C$ or $m-e \in -\mathrm{int}C$. It is clear that $m-e \neq 0_Y$.

To prove that $S(x)$ is a single point set, we divide $m-e$ into two cases.

Case I. If $m-e \in \mathrm{int}C$, then $(S(x)-\{m\}) \cap (-\mathrm{int}C) \neq \emptyset$, which contradicts $m \in S(x)$.

Case II. If $m-e \in -\mathrm{int}C$, then $(S(x)-\{e\}) \cap (-\mathrm{int}C) \neq \emptyset$, which contradicts $e \in S(x)$. So, it is obvious that $S(x)$ is a single point set for all $x \in \overline{X}$ in some $\overline{X} \in \mathscr{U}(x_0)$.

Next, we prove inclusion relation (2.3). Let $v \in \overline{W}(S,(x_0,y_0))$. Then, there exist sequences $\{r_n\}$ with $r_n > 0$ and $\{(x_n, v_n)\}$ with $(x_n, v_n) \to (x_0, v)$ such that

$$y_0 + r_n v_n \in S(x_n) \subseteq F(x_n), \ \forall n \in \mathbb{N}. \tag{2.4}$$

Suppose to the contrary that $v \notin \mathrm{WMin}_C \overline{W}(F,(x_0,y_0))$. Then, $(\overline{W}(F,(x_0,y_0)) - \{v\}) \cap (-\mathrm{int}C) \neq \emptyset$. It follows that there exists $\overline{v} \in \overline{W}(F,(x_0,y_0))$ such that $v - \overline{v} \in \mathrm{int}C$. Since S has the first-order proto-variational set of type 2 at (x_0,y_0), for the preceding sequences $\{r_n\}$ and $\{x_n\}$, there exists a sequence $\{\overline{v}_n\}$ with $\overline{v}_n \to \overline{v}$ such that

$$y_0 + r_n \overline{v}_n \in F(x_n), \ \forall n \in \mathbb{N}. \tag{2.5}$$

Since F is C-dominated by S near x_0, one sees that there exists $X' \subseteq \mathscr{U}(x_0)$ such that

$$F(x) \subseteq S(x) + C, \ \forall x \in X'. \tag{2.6}$$

It is easily seen that there exists a natural number $N > 0$ such that $x_n \in \overline{X} \cap X'$, $\forall n > N$. Thus it follows from (2.5) and (2.6) that $y_0 + r_n \overline{v}_n \in S(x_n) + C$ for all $n > N$. Since $S(x)$ is a single point set, it follows from (2.4) that $y_0 + r_n \overline{v}_n - (y_0 + r_n v_n) = r_n(\overline{v}_n - v_n) \in C$ for all $n > N$. Since $\overline{v}_n - v_n \to \overline{v} - v$ and C has a convex base, one obtains $\overline{v} - v \in C$, which contradicts $v - \overline{v} \in \mathrm{int}C$. Thus $v \in \mathrm{WMin}_C \overline{W}(F,(x_0,y_0))$ and (2.3) holds. The rest of the proof follows from Proposition 2.1. Hence, this completes the proof. □

To explain Proposition 2.2, we provide the following example.

Example 2.3. Let $X = Y = \mathbb{R}$ and $C = \hat{C} = \mathbb{R}_+$. Let $F : X \to 2^Y$ be defined by $F(x) = \{x^2\}$ if $x \geq 0$; $F(x) = \emptyset$ if $x < 0$. Then $S(x) = \{0\}$ if $x \geq 0$; $S(x) = \emptyset$ if $x < 0$. Let $(x_0,y_0) = (0,0) \in \mathrm{gr}\,F$. It is easy to see that $\overline{W}(F,(0,0)) = \mathbb{R}_+$. Thus $\mathrm{WMin}_C \overline{W}(F,(0,0)) = \{0\}$. We also have

$$\begin{aligned}
&\overline{W}(S,(0,0)) = \{0\}, \ \overline{W}(F+\hat{C},(0,0)) = \mathbb{R}_+, \\
&W^{\infty(1)}(F,(0,0)) = \mathbb{R}_+, \ \mathrm{WMin}_C \overline{W}(F+\hat{C},(0,0)) = \{0\}, \\
&W^{\infty(1)}(F,(0,0)) \cap (-\hat{C}) = \{0\}.
\end{aligned}$$

Thus $\overline{W}(S,(0,0)) = \mathrm{WMin}_C \overline{W}(F,(0,0))$, and Proposition 2.2 holds.

3. Chain and Sum Rules

In this section, we establish the chain and sum rules of variational sets of type 2. We first recall some definitions from [19, 27].

Definition 3.1. [19] Let $I : X \to 2^Y$ be a set-valued map and $x \in \operatorname{dom} I$. I is said to be closed at x if $I(x) = (\operatorname{cl} I)(x)$. I is said to be compact with respect to its domain if any sequence $\{(x_n, y_n)\} \subseteq \operatorname{gr} I$ has a convergent subsequence as soon as $\{x_n\}$ is a convergent sequence.

Definition 3.2. [27] Let $L : X \times Z \to 2^Y$ be a set-valued map, $((x_0, z_0), y_0) \in \operatorname{gr} L$, and $w \in Z$. The first-order quasi-variational set of the set-valued map L of type 1 at (x_0, z_0) with respect to w is the set

$$V_q(L, (x_0, z_0[w]), y_0) := \{v \in Y \mid \exists h_n \to 0^+,\ \exists (x_n, v_n, w_n) \to (x_0, v, w),$$
$$s.t.\ y_0 + h_n v_n \in L(x_n, z_0 + h_n w_n),\ \forall n \in \mathbb{N}\}.$$

Inspired by the radial cone in [29], we propose the following definition.

Definition 3.3. Let $L : X \times Z \to 2^Y$ be a set-valued map, $((x_0, z_0), y_0) \in \operatorname{gr} L$, and $w \in Z$.

(i) The first-order radial-variational set of the set-valued map L at (x_0, z_0) with respect to w is the set

$$W(L, (x_0, z_0[w]), y_0) := \{v \in Y \mid \exists r_n > 0,\ \exists (x_n, w_n, v_n) \to (x_0, w, v),$$
$$s.t.\ y_0 + r_n v_n \in L(x_n, z_0 + r_n w_n),\ \forall n \in \mathbb{N}\}.$$

(ii) The map L is said to have the first-order semi-radial-variational set with respect to w if

$$W_{srv}(L, (x_0, z_0[w]), y_0) := \{v \in Y \mid \forall r_n > 0,\ \forall (x_n, w_n) \to (x_0, w),\ \exists v_n \to v,$$
$$s.t.\ y_0 + r_n v_n \in L(x_n, z_0 + r_n w_n),\ \forall n \in \mathbb{N}\}.$$

Let $F : X \to 2^Y$ and $G^{-1} : Z \to 2^Y$ be two set-valued maps. For a chain rule, we define a resultant set-valued map $C : X \times Z \to 2^Y$ as follows:

$$C(x, z) := F(x) \cap G^{-1}(z). \tag{3.1}$$

Then, $\operatorname{dom} C = \operatorname{gr} G \circ F$.

Proposition 3.1. *Let $(x_0, z_0) \in \operatorname{gr} G \circ F$, $w \in Z$, C defined by (3.1) be closed at (x_0, z_0), and $D_R G(x, y)[T]$ be defined by $\bigcup_{t \in T} D_R G(x, y)(t)$ for all $x \in X$, $y \in Y$, and $T \subseteq X$. Suppose that the following condition is satisfied, for all $y_0 \in C(x_0, z_0)$,*

$$\overline{W}\left(F, (x_0, y_0)\right) \cap D_R G^{-1}\left(z_0, y_0\right)(w) \subseteq W\left(C, (x_0, z_0[w]), y_0\right). \tag{3.2}$$

Then

$$\bigcup_{y_0 \in C(x_0, z_0)} D_R G\left(y_0, z_0\right)\left[W\left(F, (x_0, y_0)\right)\right] \subseteq \overline{W}\left(G \circ F, (x_0, z_0)\right). \tag{3.3}$$

If, additionally, Y is finite dimensional, G and F are compact with respect to their domains, and the following assumption is satisfied, for every $y_0 \in C\left(x_0, z_0\right)$,

$$V_q(C, (x_0, z_0[0_Z]), y_0) = \{0_Y\}, \tag{3.4}$$

then

$$\bigcup_{y_0 \in C(x_0, z_0)} D_R G\left(y_0, z_0\right)\left[\overline{W}\left(F, (x_0, y_0)\right)\right] = \overline{W}\left(G \circ F, (x_0, z_0)\right). \tag{3.5}$$

Proof. First, we prove (3.3). Let $w \in \bigcup_{y_0 \in C(x_0,z_0)} D_R G(y_0,z_0)\left[\overline{W}(F,(x_0,y_0))\right]$. Then there exists $y_0 \in C(x_0,z_0)$ such that $w \in D_R G(y_0,z_0)\left[\overline{W}(F,(x_0,y_0))\right]$. It follows that $v \in \overline{W}(F,(x_0,y_0))$ with $w \in D_R G(y_0,z_0)(v)$. We see that there exist sequences $\{r_n\}$ with $r_n > 0$ and $\{(v_n,w_n)\}$ with $(v_n,w_n) \to (v,w)$ such that $z_0 + r_n w_n \in G(y_0 + r_n v_n)$, which implies that $y_0 + r_n v_n \in G^{-1}(z_0 + r_n w_n)$, i.e., $v \in D_R G^{-1}(z_0,y_0)(w)$. It follows from (3.2) that $v \in W(C,(x_0,z_0[w]),y_0)$. In view of Definition 3.3, one sees that there exist sequences $\{(\bar{x}_n,\bar{v}_n,\bar{w}_n)\}$ with $(\bar{x}_n,\bar{v}_n,\bar{w}_n) \to (x_0,v,w)$ and $\{t_n\}$ with $t_n > 0$ such that $y_0 + t_n\bar{v}_n \in C(\bar{x}_n, z_0 + t_n\bar{w}_n)$, which implies that

$$z_0 + t_n\bar{w}_n \in G(t_n\bar{v}_n + y_0) \subseteq (G \circ F)(\bar{x}_n),$$

i.e., $w \in \overline{W}(G \circ F,(x_0,z_0))$. So (3.3) holds.

Next we prove the inverse inclusion relation of (3.3). Let $w \in \overline{W}(G \circ F,(x_0,z_0))$. To prove $w \in \bigcup_{y_0 \in C(x_0,z_0)} D_R G(y_0,z_0)\left[\overline{W}(F,(x_0,y_0))\right]$, we divide w into two cases.

(i) If $w = 0_Z$, then $0_Z \in \bigcup_{y_0 \in C(x_0,z_0)} D_R G(y_0,z_0)\left[\overline{W}(F,(x_0,y_0))\right]$.

(ii) If $w \neq 0_Z$, then there exist sequences $\{r_n\}$ with $r_n > 0$ and $\{(x_n,w_n)\}$ with $(x_n,w_n) \to (x_0,w)$ such that $z_0 + r_n w_n \in (G \circ F)(x_n)$,, that is, there exists a sequence $\{y_n\}$ with $y_n \in F(x_n)$ such that $z_0 + r_n w_n \in G(y_n)$. Since F and G are compact with respect to their domains and $x_n \to x_0$, $\{y_n\}$ has a subsequence (denoted by the same notion y_n) converging to some $\bar{y}$, which implies that $\{z_n\}$, $z_n = z_0 + r_n w_n$, also has a convergent subsequence (denoted by z_n) with the limit point $\bar{z}$. Since $r_n > 0$, $z_n \to \bar{z}$, and $w_n \to w \neq 0_Z$, we see that r_n converges to some $k \geq 0$.

To prove $w \in \bigcup_{y_0 \in C(x_0,z_0)} D_R G(y_0,z_0)\left[\overline{W}(F,(x_0,y_0))\right]$, we divide k into two cases.

Case I. Let $y_0 \in C(x_0,z_0)$ and set $v_n = \frac{y_n - y_0}{r_n}$ with $v_n \to \frac{\bar{y}-y_0}{k}$. If $k > 0$, then $y_0 + r_n v_n \in F(x_n)$ and $z_0 + r_n w_n \in G(y_0 + r_n v_n)$, which means that $\frac{\bar{y}-y_0}{k} \in \overline{W}(F,(x_0,y_0))$ and $w \in D_R G(y_0,z_0)(\frac{\bar{y}-y_0}{k})$. Thus, $w \in \bigcup_{y_0 \in C(x_0,z_0)} D_R G(y_0,z_0)\left[\overline{W}(F,(x_0,y_0))\right]$.

Case II. If $k = 0$, we see that $y_n \in C(x_n, z_0 + r_n w_n)$, i.e., $(x_n, z_0 + r_n w_n, y_n) \in \mathrm{gr}C$, which implies that $(x_0,z_0,\bar{y}) \in \mathrm{cl}(\mathrm{gr}C) = \mathrm{gr}(\mathrm{cl}C)$. Hence, $\bar{y} \in \mathrm{cl}C(x_0,z_0) = C(x_0,z_0)$, since C is closed at (x_0,z_0). Suppose that $y_n = \bar{y}$ for all $n \in \mathbb{N}$. Then $0_Y \in \overline{W}(F,(x_0,\bar{y}))$ and $w \in D_R G(\bar{y},z_0)(0_Y)$. This demonstrates $w \in \bigcup_{y_0 \in C(x_0,z_0)} D_R G(y_0,z_0)\left[\overline{W}(F,(x_0,y_0))\right]$. Suppose that $y_n \neq \bar{y}$ for all $n \in \mathbb{N}$. Setting $s_n = \|y_n - \bar{y}\|$ and $v_n = \frac{y_n - \bar{y}}{s_n}$, one obtains that $s_n \to 0^+$ and the sequence $\{v_n\}$ or some subsequence has a limit v with $\|v\| = 1$, since Y is finite dimensional and $\|v_n\| = 1$. We can conclude that $\{\frac{r_n}{s_n}\}$ does not converge to 0. In fact, if $\{\frac{r_n}{s_n}\}$ converges to 0, then $\bar{y} + s_n v_n = y_n \in C(x_n, z_0 + s_n(\frac{r_n}{s_n} w_n))$, which yields that $v \in V_q(C,(x_0,z_0[0_Z]),\bar{y})$, a contradiction with (3.4). Thus $\{\frac{s_n}{r_n}\}$ is bounded and then $\{\frac{s_n}{r_n}\}$ (taking a subsequence if necessary) has a limit $q \geq 0$. For all $n \in \mathbb{N}$, one has

$$\bar{y} + r_n\left(\frac{s_n}{r_n} v_n\right) = y_n \in C(x_n, z_0 + r_n w_n).$$

Therefore, $qv \in W(C,(x_0,z_0[w]),\bar{y})$. It follows from the definition of $W(C,(x_0,z_0[w]),\bar{y})$ that $qv \in \overline{W}(F,(x_0,\bar{y}))$ and $w \in D_R G(\bar{y},z_0)(qv)$. Then

$$w \in \bigcup_{y_0 \in C(x_0,z_0)} D_R G(y_0,z_0)\left[\overline{W}(F,(x_0,y_0))\right].$$

From (i) and (ii), we know that the inverse inclusion relation of (3.3) holds. Hence, (3.5) holds, and the proof is complete. □

We remark here that [27, Proposition 3.8] demonstrates the chain rule with inclusion relations, while Proposition 3.1 demonstrates the chain rule with equality relations. Furthermore, the conditions of Proposition 3.1 are different from those in [27, Proposition 3.8]. To explain Proposition 3.1, we provide the following example.

Example 3.1. Let $X=Y=Z=\mathbb{R}$, $w\in Y$, $F:X\to 2^Y$, and $G:Y\to 2^Z$ be two set-valued maps defined by

$$F(x)=\begin{cases}\{0\}, & \text{if } x\neq 0,\\ \{1\}, & \text{if } x=0,\end{cases}\qquad G(y)\ =\begin{cases}[0,1], & \text{if } y\neq 0,\\ \{0\}, & \text{if } y=0.\end{cases}$$

For the chain rule $G\circ F$, we obtain

$$(G\circ F)(x)=\begin{cases}\{0\}, & \text{if } x\neq 0,\\ [0,1], & \text{if } x=0;\end{cases}\qquad G^{-1}(z)=\begin{cases}\mathbb{R}\backslash\{0\}, & \text{if } z\in(0,1],\\ \mathbb{R}, & \text{if } z=0,\\ \emptyset, & \text{otherwise;}\end{cases}\quad \textit{and}$$

$$C(x,z)=\begin{cases}\{0\}, & \text{if } x\neq 0 \text{ and } z=0,\\ \{1\}, & \text{if } x=0 \text{ and } z\in[0,1],\\ \emptyset, & \text{otherwise.}\end{cases}$$

Let $(x_0,z_0)=(0,0)\in \operatorname{gr}(G\circ F)$. Then $C(x_0,z_0)=\{1\}$. For any $y_0\in C(x_0,z_0)$, we have

$$\begin{aligned}&\overline{W}(F,(x_0,y_0))=\mathbb{R}_-,\ D_RG^{-1}(z_0,y_0)(w)=\mathbb{R},\\ &W(C,(x_0,z_0[w]),y_0)=\mathbb{R}_-,\ V_q(C,(x_0,z_0[0]),y_0)=\{0\},\\ &\overline{W}(G\circ F,(x_0,z_0))=\mathbb{R}_+,\ D_RG(y_0,z_0)\left[\overline{W}(F,(x_0,y_0))\right]=\mathbb{R}_+.\end{aligned}$$

It is obvious that conditions (3.2) and (3.4) hold in Proposition 3.1, and

$$\bigcup_{y_0\in C(x_0,z_0)} D_RG(y_0,z_0)\left[\overline{W}(F,(x_0,y_0))\right]=\overline{W}(G\circ F,(x_0,z_0)).$$

Thus Proposition 3.1 holds.

We now discuss the sum rule of two set-valued maps $M,\ N:X\to 2^Y$. For $(x,z)\in X\times Y$, setting $S(x,z):=M(x)\cap(z-N(x))$, one has $\operatorname{dom} S=\operatorname{gr}(M+N)$.

Proposition 3.2. *Let $(x_0,z_0)\in\operatorname{gr}(M+N)$, $v\in Y$ and S be closed at (x_0,z_0). Suppose that the following condition is satisfied, for all $y_0\in S(x_0,z_0)$,*

$$\overline{W}(M,(x_0,y_0))\cap\left[v-\overline{W}(N,(x_0,z_0-y_0))\right]\subseteq W(S,(x_0,z_0[v]),y_0). \tag{3.6}$$

Then

$$\bigcup_{y_0\in S(x_0,z_0)}\left(\overline{W}(M,(x_0,y_0))+\overline{W}(N,(x_0,z_0-y_0))\right)\subseteq\overline{W}(M+N,(x_0,z_0)). \tag{3.7}$$

If, additionally, Y is finite dimensional, M and N are compact with respect to their domains, and the following assumption is satisfied, for every $y_0\in S(x_0,z_0)$,

$$V_q(S,(x_0,z_0[0_Y]),y_0)=\{0_Y\}, \tag{3.8}$$

then

$$\bigcup_{y_0\in S(x_0,z_0)}\left(\overline{W}(M,(x_0,y_0))+\overline{W}(N,(x_0,z_0-y_0))\right)=\overline{W}(M+N,(x_0,z_0)). \tag{3.9}$$

Proof. We first prove (3.7). Let $v \in \bigcup_{y_0 \in S(x_0,z_0)} \left(\overline{W}(M,(x_0,y_0)) + \overline{W}(N,(x_0,z_0-y_0))\right)$. Then there exists $y_0 \in S(x_0,z_0)$ such that $v \in (\overline{W}(M,(x_0,y_0)) + \overline{W}(N,(x_0,z_0-y_0)))$. It follows from (3.6) that there exists some $w \in \overline{W}(M,(x_0,y_0))$ such that $w \in W(S,(x_0,z_0[v]),y_0)$. Then, there are sequences $\{r_n\}$ with $r_n > 0$ and $\{(x_n,v_n,w_n)\}$ with $(x_n,v_n,w_n) \to (x_0,v,w)$ such that $y_0 + r_n w_n \in S(x_n, z_0 + r_n v_n)$, which implies that $z_0 + r_n v_n \in (M+N)(x_n)$, i.e., $v \in \overline{W}(M+N,(x_0,z_0))$. So (3.7) holds.

Next we prove the inverse inclusion relation of (3.7). Let $v \in W(M+N,(x_0,z_0))$. To prove $v \in \bigcup_{y_0 \in S(x_0,z_0)} (\overline{W}(M,(x_0,y_0)) + \overline{W}(N,(x_0,z_0-y_0)))$, we divide v into two cases.

(i) If $v = 0_Y$, one has $0_Y \in \bigcup_{y_0 \in S(x_0,z_0)} \left(\overline{W}(M,(x_0,y_0)) + \overline{W}(N,(x_0,z_0-y_0))\right)$.

(ii) If $v \neq 0_Y$, then there exist sequences $\{(x_n,v_n)\}$ with $(x_n,v_n) \to (x_0,v)$ and $\{r_n\}$ with $r_n > 0$ such that $z_0 + r_n v_n \in (M+N)(x_n)$. Therefore, there exist $y_{1n} \in M(x_n)$ and $y_{2n} \in N(x_n)$ such that

$$z_0 + r_n v_n = y_{1n} + y_{2n}. \tag{3.10}$$

Since M and N are compact with respect to their domains and $x_n \to x_0$, $\{y_{in}\}(i=1,2)$ have subsequences denoted by the same notions y_{in} converging to $y_i(i=1,2)$. Since $r_n > 0$, $y_{in} \to y_i(i=1,2)$ and $v_n \to v \neq 0_Y$, we see from (3.10) that r_n converges to some $g \geq 0$. To prove

$$v \in \bigcup_{y_0 \in S(x_0,z_0)} \left(\overline{W}(M,(x_0,y_0)) + \overline{W}(N,(x_0,z_0-y_0))\right), \tag{3.11}$$

we divide g into two cases.

Case I. $g > 0$. Let $y_0 \in S(x_0,z_0)$. Setting $v_{1n} = \frac{y_{1n}-y_0}{r_n}$ with $v_{1n} \to \frac{y_1-y_0}{g}$ and $v_{2n} = \frac{y_{2n}-(z_0-y_0)}{r_n}$ with $v_{2n} \to \frac{y_2-(z_0-y_0)}{g}$, one has $y_0 + r_n v_{1n} \in M(x_n)$ and $z_0 - y_0 + r_n v_{2n} \in N(x_n)$, which together with (3.10) yields $v_n = v_{1n} + v_{2n}$. Thus

$$\frac{y_1-y_0}{g} \in \overline{W}(M,(x_0,y_0)),\ \frac{y_2-(z_0-y_0)}{g} \in \overline{W}(N,(x_0,z_0-y_0))$$

and

$$v = \frac{y_1 - y_0 + y_2 - (z_0-y_0)}{g}.$$

Thus (3.11) holds when $g > 0$.

Case II. If $g = 0$, then $y_{1n} \in S(x_n, z_0 + r_n v_n)$, i.e., $(x_n, z_0 + r_n v_n, y_{1n}) \in \operatorname{gr} S$, which implies that $(x_0,z_0,y_1) \in \operatorname{cl}(\operatorname{gr} S) = \operatorname{gr}(\operatorname{cl} S)$. Hence, $y_1 \in \operatorname{cl} S(x_0,z_0) = S(x_0,z_0)$, since S is closed at (x_0,z_0). If $y_{1n} = y_1$ for all $n \in \mathbb{N}$, then $0_Y \in \overline{W}(M,(x_0,y_1))$ and $v \in \overline{W}(N,(x_0,z_0-y_1))$. Thus (3.11) holds. Suppose $y_{1n} \neq y_1$ for all $n \in \mathbb{N}$. Setting $s_n = \|y_{1n} - y_1\|$ and $k_n = \frac{y_{1n}-y_1}{s_n}$, one sees that $s_n \to 0^+$ and the sequence k_n or some subsequence has a limit k with $\|k\| = 1$, since Y is finite dimensional and $\|k_n\| = 1$. We can conclude that $\{\frac{r_n}{s_n}\}$ does not converge to 0. In fact, if $\{\frac{r_n}{s_n}\}$ converges to 0, then $y_1 + s_n k_n = y_{1n} \in S(x_n, z_0 + s_n(\frac{r_n}{s_n} v_n))$. Since $\frac{r_n}{s_n} v_n \to 0_Y$, one obtains $k \in V_q(S,(x_0,z_0[0_Y]),y_1)$, which contradicts (3.8). Thus $\{\frac{s_n}{r_n}\}$ is bounded, and then $\{\frac{s_n}{r_n}\}$ (taking a subsequence if necessary) has a limit $q \geq 0$. It follows that, for all $n \in \mathbb{N}$,

$$y_1 + r_n \left(\frac{s_n}{r_n} k_n\right) = y_{1n} \in S(x_n, z_0 + r_n v_n).$$

Hence, $qk \in W(S,(x_0,z_0[v]),y_1)$. It follows from the definition of $W(S,(x_0,z_0[v]),y_1)$ that $qk \in \overline{W}(M,(x_0,y_1))$ and $v - qk \in \overline{W}(N,(x_0,z_0-y_1))$. Then, (3.11) holds. Thus (3.11) holds when

$g=0$. From (i) and (ii), it is easy to see that the inverse inclusion relation of (3.7) holds. Thus (3.9) holds, and the proof is complete. □

Remark 3.1. Since the result of Proposition 3.2 is that the sum of the variational set is in the variational set of the sum, while the result in [27, Proposition 3.4] is that the variational set of the sum is in the sum of the variational set. In addition, Proposition 3.2 presentes the sum rule with an equality relation.

Remark 3.2. Condition (3.8) is essential to Proposition 3.2.

The following example explains Remark 3.2.

Example 3.2. Let $X=Y=Z=\mathbb{R}$ and $M,N: X\to 2^Y$ be two set-valued maps defined by

$$M(x)=\begin{cases}\{0\}, & \text{if } x=0,\\ [-1,0], & \text{if } x\neq 0.\end{cases}\quad N(x)=\begin{cases}\{-1\}, & \text{if } x=0,\\ \{0,-1\}, & \text{if } x\neq 0.\end{cases}$$

Then

$$(M+N)(x)=\begin{cases}\{-1\}, & \text{if } x=0,\\ [-2,0], & \text{if } x\neq 0;\end{cases}$$

$$S(x,z)=M(x)\cap(z-N(x))=\begin{cases}\{0\}, & \text{if } x=0 \text{ and } z=-1,\\ \{z\}, & \text{if } x\in\mathbb{R}\backslash\{0\} \text{ and } z\in(-1,0],\\ \{z+1\}, & \text{if } x\in\mathbb{R}\backslash\{0\} \text{ and } z\in[-2,-1),\\ \{0,1\}, & \text{if } x\in\mathbb{R}\backslash\{0\} \text{ and } z=-1,\\ \emptyset, & \text{otherwise .}\end{cases}$$

Let $(x_0,z_0)=(1,-1)\in \text{gr}(M+N)$. Then $S(x_0,z_0)=\{-1,0\}$ and, for any $y_0\in S(x_0,z_0)$, (3.8) does not hold. In fact, taking $x_n=1\to 1$, $w_n=\frac{1}{n}\to 0$, $v_n=-1\to -1$, and $r_n=\frac{1}{n}\to 0$, one has $y_0+r_nv_n\in S(x_n,z_0+r_nw_n)$, i.e., $-1\in V_q(S,(x_0,z_0[0]),y_0)$. Observe

$$\overline{W}(M,(x_0,y_0))=\mathbb{R}_+,\ \overline{W}(N,(x_0,z_0-y_0))=\{0\},$$
$$\overline{W}(M+N,(x_0,z_0))=\mathbb{R},\ W(S,(x_0,z_0[v]),y_0)=\mathbb{R}.$$

Theus (3.6) and (3.7) hold, but (3.9) does not hold. This demonstrates that condition (3.8) is essential to Proposition 3.2.

4. Sensitivity Analysis of the Variational Inequality

Let X, W, and Z be three normed spaces, $F: W\times X\to 2^Z$ and $N: X\to 2^Z$ be two set-valued maps, and K be a subset of X. Consider the set-valued map defined by

$$M_K(w,z) := \{x\in K \mid z\in F(w,x)+N(x)\}. \tag{4.1}$$

When K is convex, $N(x)$ is the normal cone to K at x, and w is a parameter, M is the solution map of a parameterized variational inequality. The map M is equivalently expressed by

$$M_K(w,z) := \{x\in X \mid z\in Q_K(w,x)\}.$$

Then M is related to the sum map $Q_K(w,x) := F(w,x) + N_K(w,x)$ and

$$N_K(w,x) := \begin{cases} N(x), & \text{if } (w,x) \in W \times X, \\ \emptyset, & \text{if } (w,x) \in W \times (X \backslash K). \end{cases}$$

Inspired by [26], we propose the following definition.

Definition 4.1. The first-order Studniarski-variational set of the set-valued map $Q : W \times X \to Z$ of type 1 at (w_0, x_0) with $u \in X$ is the set defined by

$$V_S(Q,(w_0,x_0[u]),z_0) := \{z \in Z \mid \exists b_n \to 0^+,\ \exists (w_n,z_n,u_n) \to (w_0,z,u), \\ s.t.\ z_0 + b_n z_n \in Q(w_n, x_0 + b_n u_n),\ \forall n \in \mathbb{N}\}.$$

Now, we recall some concepts and a property from [27] which are needed for calculating the variational set of type 2 of M.

Definition 4.2. [27] Let A and B be linear spaces, and let $E \subseteq A$.

(i) E is said to be star-shaped at $a_0 \in E$, if for all $a \in E$ and $\beta \in [0,1]$, $(1-\beta)a_0 + \beta a \in E$.

(ii) Let E be a star-shaped set at $a_0 \in E$. The set-valued map $T : A \to 2^B$ is said to be star-shaped at a_0 on E if, for all $a \in E$ and $\beta \in [0,1]$, $(1-\beta)T(a_0) + \beta T(a) \subseteq T((1-\beta)a_0 + \beta a)$.

Lemma 4.1. [27] *Let E be a star-shaped set at $x_0 \in E$, and let F be star-shaped at x_0 on E. Then $\overline{W}(F,(x_0,y_0)) = V(F,(x_0,y_0))$.*

Proposition 4.1. *Let Z be a finite dimensional space, $((w_0,z_0),x_0) \in \mathrm{gr} M$, and $u \in X$. (i) If*

$$V_s(Q,(w_0,x_0[0_X]),z_0) = \{0_Z\}, \tag{4.2}$$

then

$$\overline{W}(M,((w_0,z_0),x_0)) \subseteq \{u \in X \mid W(Q,(w_0,x_0[u]),z_0) \neq \emptyset\}. \tag{4.3}$$

(ii) If M is star-shaped at (w_0,z_0) and convex, then

$$\overline{W}(M,((w_0,z_0),x_0)) = \{u \in X \mid W(Q,(w_0,x_0[u]),z_0) \neq \emptyset\}. \tag{4.4}$$

Proof. We first prove (4.3). Let $u \in \overline{W}(M,((w_0,z_0),x_0))$. Then there exist $\{(w_n,z_n,u_n)\}$ with $(w_n,z_n,u_n) \to (w_0,z_0,u)$ and $\{r_n\}$ with $r_n > 0$ such that $x_0 + r_n u_n \in M(w_n,z_n)$, which implies that $z_n \in Q(w_n, x_0 + r_n u_n)$. Setting $s_n = \|z_n - z_0\|$ $(s_n \to 0^+)$ and $d_n = \frac{z_n - z_0}{s_n}$, we see that $\{d_n\}$ has a convergent subsequence with a limit d satisfying $\|d\| = 1$, since Z is finite dimensional and $\|d_n\| = 1$. We conclude that $\{\frac{r_n}{s_n}\}$ does not converge to 0. In fact, if $\{\frac{r_n}{s_n}\}$ converges to 0, then $z_0 + s_n d_n = z_n \in Q(w_n, x_0 + s_n(\frac{r_n}{s_n} u_n))$. Thus $d \in V_s(Q,(w_0,x_0[0_X]),z_0)$, which contradicts (4.2), so $\{\frac{s_n}{r_n}\}$ is bounded and $\{\frac{s_n}{r_n}\}$ (taking a subsequence if necessary) has a limit $q \geq 0$. Observe that $z_0 + r_n\left(\frac{s_n}{r_n} d_n\right) = z_n \in Q(w_n, x_0 + r_n u_n)$, so $qd \in W(Q,(w_0,x_0[u]),z_0)$ and then (4.3) holds. Next we prove the inverse inclusion relation of (4.3) for equation (4.4). Let $d \in W(Q,(w_0,x_0[u]),z_0)$. Then there exist sequences $\{r_n\}$ with $r_n > 0$ and $\{(w_n,u_n,d_n)\}$ with $(w_n,u_n,d_n) \to (w_0,u,d)$ such that $z_0 + r_n d_n \in Q(w_n, x_0 + r_n u_n)$, which implies that $x_0 + r_n u_n \in M(w_n, z_0 + r_n d_n)$. It is clear that one can choose a sequence $\{\lambda_n\}$ with $\lambda_n \to 0^+$ such that

$\frac{\lambda_n}{r_n} \to 0^+$. Then, for n large enough, we have $\frac{\lambda_n}{r_n} \leq 1$ and

$$\begin{aligned} x_0 + \lambda_n u_n &\in M(w_n, z_0) + \frac{\lambda_n}{r_n}(M(w_n, z_0 + r_n d_n) - M(w_n, z_0)) \\ &\subseteq M(w_n, z_0 + \lambda_n d_n) := M(w_n, \overline{z_n}). \end{aligned}$$

Thus $u \in V(M, ((w_0, z_0), x_0))$. Since

$$V(M, ((w_0, z_0), x_0)) \subseteq \overline{W}(M, ((w_0, z_0), x_0)),$$

$u \in \overline{W}(M, ((w_0, z_0), x_0))$, then

$$\{u \in X \mid W(Q, (w_0, x_0[u]), z_0) \neq \emptyset\} \subseteq \overline{W}(M, ((w_0, z_0)), x_0).$$

Combining this with (4.3), we see that (4.4) holds and the proof is complete. □

Now we give an example to demonstrate Proposition 4.1.

Example 4.1. Let $W = X = Z = \mathbb{R}$, $u \in X$, $Q : W \times X \to 2^Z$, $M : W \times Z \to 2^X$ be two set-valued maps, and Q be defined by $Q(w, x) = [0, |2wx|]$ for all $w \in \mathbb{R}$ and $x \in \mathbb{R}$. Then

$$M(w, z) = \begin{cases} \mathbb{R}_+, & \text{if } w \in \mathbb{R}_+ \text{ and } z \in \mathbb{R}_-, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let $((w_0, z_0), x_0) = ((1, 0), 0) \in \operatorname{gr} M$. Thus $V_s(Q, (w_0, x_0[0]), z_0) = \{0\}$. Observe that

$$\begin{aligned} &W(Q, (w_0, x_0[u]), z_0) = \mathbb{R}_+, \quad \overline{W}(M, ((w_0, z_0), x_0)) = \mathbb{R}_+, \\ &\{u \in X \mid W(Q, (w_0, x_0[u]), z_0) \neq \emptyset\} = \mathbb{R}_+. \end{aligned}$$

Thus $\{u \in X \mid W(Q, (w_0, x_0[u]), z_0) \neq \emptyset\} = \overline{W}(M, ((w_0, z_0)), x_0)$, and Proposition 4.1 holds.

In the following proposition, we show the sum rule of the first-order radial-variational sets, which is needed for Theorem 4.1.

Proposition 4.2. *Let $((w_0, x_0), z_0) \in \operatorname{gr}(F + N_K)$ and $v \in Y$. Let S be closed at (x_0, z_0). If $W(F, (w_0, x_0[u]), y_0) \cap [v - W(N_K, (w_0, x_0[u]), z_0 - y_0)] \subseteq W(S, ((w_0, x_0[u]), z_0[v]), y_0)$, for all $y_0 \in S((w_0, x_0), z_0)$, where $S((w, x), z) = F(w, x) \cap (z - N_K(w, x))$, then*

$$\bigcup_{y_0 \in S((w_0, x_0), z_0)} (W(F, (w_0, x_0[u]), y_0) + W(N_K, (w_0, x_0[u]), z_0 - y_0)) \subseteq W(F + N_K, (w_0, x_0[u]), z_0).$$

If, additionally, Z is finite dimensional, F and N_K are compact with respect to their domains and, for every $S((w_0, x_0), z_0)$, $V_q(S, ((w_0, x_0[0_X]), z_0[0_Z]), y_0) = \{0_Z\}$, then

$$\bigcup_{y_0 \in S((w_0, x_0), z_0)} (W(F, (w_0, x_0[u]), y_0) + W(N_K, (w_0, x_0[u]), z_0 - y_0)) = W(F + N_K, (w_0, x_0[u]), z_0).$$

Since the proof is similar to Proposition 4.2, we omit the proof here. Since the solution map (4.1) was studied in terms of the variational sets of type 1 and contingent derivatives in [31, 32], we now apply the sum rule of the first-order radial-variational sets to have the first-order sensitivity analysis in terms of the variational sets of type 2 for (4.1).

Theorem 4.1. *Let $u \in X$ and $((w_0, x_0), z_0) \in \operatorname{gr} M$. Let all the conditions in Propositions 4.1 and 4.2 are satisfied. Then*

$$\overline{W}(M, ((w_0, z_0)), x_0) = \{u \in X \mid \bigcup_{y_0 \in S((w_0, x_0), z_0)} (W(F, (w_0, x_0[u]), y_0) + W(N_K, (w_0, x_0[u]), z_0 - y_0)) \neq \emptyset\}. \tag{4.5}$$

Proof. By using Proposition 4.1, we have

$$\overline{W}(M, ((w_0, z_0)), x_0) = \{u \in X \mid W(Q, (w_0, x_0[u]), z_0) \neq \emptyset\}. \tag{4.6}$$

Since the sum map $Q(w,x) = F(w,x) + N_K(w,x)$, one has

$$\{u \in X \mid W(Q, (w_0, x_0[u]), z_0) \neq \emptyset\} = \{u \in X \mid W(F + N_K, (w_0, x_0[u]), z_0) \neq \emptyset\}. \tag{4.7}$$

It follows from Proposition 4.2 that

$$W(F + N_K, (w_0, x_0[u]), z_0) = \bigcup_{y_0 \in S((w_0, x_0), z_0)} (W(F, (w_0, x_0[u]), y_0) + W(N_K, (w_0, x_0[u]), z_0 - y_0)). \tag{4.8}$$

In view of (4.6), (4.7), and (4.8), we see that (4.5) holds, and this completes the proof. $\square$

The following example illustrates Theorem 4.1.

Example 4.2. Consider the set-valued map (4.1) with $X = W = Z = \mathbb{R}$. Let K be a subset of X, $F : W \times X \to 2^Z$, $M : W \times Z \to 2^X$, $N_K : X \to 2^Z$ be three set-valued maps, F and N_K be defined by

$$F(w,x) = \begin{cases} [0, wx], & \text{if } wx \geq 0, \\ \emptyset, & \text{if } wx < 0 \end{cases} \quad \text{and} \quad N_K(w,x) = \begin{cases} [-wx, 0], & \text{if } wx < 0, \\ \{0\}, & \text{if } wx = 0, \\ \emptyset, & \text{if } wx > 0. \end{cases}$$

Then

$$Q(w,x) = (F + N_K)(w,x) = \begin{cases} [0, wx], & \text{if } wx = 0, \\ \emptyset, & \text{if } wx \neq 0, \end{cases} \textit{and}$$

$$S((w,x),z) = F(w,x) \cap (z - N_K(w,x)) = \begin{cases} \{z\}, & \text{if } wx = 0 \text{ and } z \in [0, wx], \\ \emptyset, & \text{otherwise.} \end{cases}$$

Since $M(w,z) = \{x \in X \mid z \in Q(w,x)\}$,

$$M(w,z) = \begin{cases} \mathbb{R}, & \text{if } z = 0 \text{ and } w = 0, \\ \{0\}, & \text{if } z = 0 \text{ and } w \in \mathbb{R} \backslash \{0\}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

If $((w_0, x_0), z_0) = ((0,0), 0) \in \operatorname{gr}(F + N_K)$, then $S((w_0, x_0), z_0) = \{0\}$. It follows that, for any $y_0 \in S((w_0, x_0), z_0) = \{0\}$,

$$W(F, (w_0, x_0[u]), y_0) = \mathbb{R}_+, \quad W(N_K, (w_0, x_0[u]), z_0 - y_0) = \mathbb{R}_-,$$
$$\overline{W}(M, ((w_0, z_0), x_0)) = \mathbb{R}$$

and

$$\{u \in X \mid \bigcup_{y_0 \in S((w_0,x_0),z_0)} (W(F,(w_0,x_0[u]),y_0) + W(N_K,(w_0,x_0[u]),z_0 - y_0)) \neq \emptyset\} = \mathbb{R}.$$

Thus

$$\begin{aligned}\overline{W}(M,((w_0,z_0)),x_0) = \{u \in X \mid & \bigcup_{y_0 \in S((w_0,x_0),z_0)} (W(F,(w_0,x_0[u]),y_0) + \\ & W(N_K,(w_0,x_0[u]),z_0 - y_0)) \neq \emptyset\},\end{aligned}$$

and Theorem 4.1 holds.

5. Sensitivity Analysis

Let X, Y, and Z be three normed spaces whose norms are all denoted by $\|\cdot\|$. Let $C \subseteq Y$ be a closed convex cone, $F : X \times X \times Z \to 2^Y$, and $K : Z \to 2^X$ be two set-valued maps. Consider the following parametric weakly set-valued equilibrium problem (PWSEP):

$$\text{find } x \in K(w) \text{ such that } F(x,y,w) \cap (-\operatorname{int} C) = \emptyset, \ \forall y \in K(w). \tag{5.1}$$

For each $w \in Z$, the set

$$S(w) := \{x \in K(w) \mid F(x,y,w) \cap (-\operatorname{int} C) = \emptyset, \forall y \in K(w)\}$$

stands for the solution map of problem (PWSEP). For $x \in X$ and $w \in Z$, the map S can be expressed by

$$S(w) = \{x \in K(w) \mid 0_Y \in J(w,x)\}, \tag{5.2}$$

where $J(w,x) = \operatorname{WMin}_C G(w,x)$, $G(w,x) = \bigcup_{y \in K(w)} F(x,y,w) \cup \{0_Y\}$, and $S(w)$ is defined by the generalized equation $0_Y \in J(w,x)$.

In this section, for a subset $H \subseteq X$, we define the distance from $x \in X$ to H by $d(x,H) := \inf_{h \in H} \|x-h\|$ with the convention that $d(x,\emptyset) = \infty$. The closed ball centered at $w_0 \in Z$ with radius λ and centered at $x_0 \in X$ with radius λ are denoted by $B_Z(w_0,\lambda)$ and $B_X(x_0,\lambda)$, respectively. First, we recall the following concept, which is important for this paper.

Definition 5.1. [33] The map S is said to be Robinson metrically regular around $(w_0,x_0) \in \operatorname{gr} S$ if there exist $\mu > 0, \gamma > 0$, and neighborhoods U of w_0, V of x_0 such that $d(x,S(w)) \leq \mu d(0,J(w,x))$ for all $w \in U, x \in V$ satisfying $d(0,J(w,x)) < \gamma$.

Inspired by [33], we propose the following definition.

Definition 5.2. Let $(w_0,x_0) \in \operatorname{gr} S$. S is said to be directionally Robinson metrically regular of order 1 along K around (w_0,x_0) if there exist $\mu > 0, \gamma > 0$, and $\lambda > 0$ such that $d(x_0+tx', S(w')) \leq \mu d(0, J(w', x_0+tx'))$, for any $t \in (0,\lambda), w' \in B_Z(w_0,\lambda)$, and $x' \in B_X(x_0,\lambda)$ satisfying $x_0 + tx' \in K(w')$ and $d(0, J(w', x_0+tx')) < \gamma$.

Remark 5.1. If S is Robinson metrically regular around (w_0,x_0), then S is directionally Robinson metrically regular of order 1 around (w_0,x_0) in the direction x for all $x \in X$. Since the converse may not hold, one has that Definition 5.2 is a generalization of Definition 5.1. We give the following example to illustrate this remark.

Example 5.1. Let $X = Y = Z = \mathbb{R}$, $C = \mathbb{R}_+$, $(w_0, x_0) \in \operatorname{gr} S$, $K : Z \to 2^X$, and $F : X \times X \times Z \to 2^Y$ be defined by

$$F(x, y, w) = x(y + w) \quad \text{and} \quad K(w) = \begin{cases} [-w, w], & \text{if } w \geqslant 0, \\ [w, -w], & \text{if } w < 0. \end{cases}$$

Then, we can easily see that

$$G(w, x) = \bigcup_{y \in K(w)} \{x(y + w)\} \cup \{0\}, \quad J(w, x) = \begin{cases} \{0\}, & \text{if } wx \geqslant 0, \\ \{2wx\}, & \text{if } wx < 0 \end{cases}$$

and $S(w) = \begin{cases} [0, w], & \text{if } w \geqslant 0, \\ [w, 0], & \text{if } w < 0. \end{cases}$ For $\mu = 1$, $\gamma = 2$, and $\lambda = 5$, we take $w_0 = 0$, $x_0 = -2$, $w' = -\frac{1}{4}$, $x' = \frac{1}{2}$, and $t = 4$. Then, $K(w') = [-\frac{1}{4}, \frac{1}{4}]$, $x_0 + tx' = 0$, $J(w', x_0 + tx') = 0$, $J(w', x') = -\frac{1}{4}$, and $S(w') = [-\frac{1}{4}, 0]$. Thus, when $t \in (0, \lambda), w' \in B_Z(w_0, \lambda), x' \in B_X(x_0, \lambda)$,

$$x_0 + tx' \in K\left(w'\right) \text{ and } d\left(0, J\left(w', x_0 + tx'\right)\right) < \gamma,$$

there exist $\mu > 0$, $\gamma > 0$ and $\lambda > 0$ such that $d\left(x_0 + tx', S\left(w'\right)\right) \leq \mu d\left(0, J\left(w', x_0 + tx'\right)\right)$. However, when $w' \in U(w_0)$, $x' \in V(x_0)$, $d\left(0, J\left(w', x'\right)\right) < \gamma$, there does not exist $\mu > 0, \gamma > 0$ such that $d(x', S(w')) \leq \mu d(0, J(w', x'))$. Thus we verify that S is directionally Robinson metrically regular of order 1 around (w_0, x_0) in the direction x for all $x \in X$, but S is not Robinson metrically regular.

Proposition 5.1. *If* $(w_0, x_0) \in \operatorname{gr} S$, *then*

$$\overline{W}\left(S, (w_0, x_0)\right) \subseteq \left\{x \in \overline{W}\left(K, (w_0, x_0)\right) \mid 0_Y \in W\left(J, (w_0, x_0[x]), 0_Y\right)\right\}. \tag{5.3}$$

If, additionally, S is directionally Robinson metrically regular of order 1 along K around (w_0, x_0) *in all directions* $x \in M$, *where* $M := \{x \mid 0_Y \in W(J, (w_0, x_0[x]), 0_Y)\}$, *and J has the first-order semi-radial-variational set at* (w_0, x_0) *with respect to x, then*

$$\overline{W}\left(S, (w_0, x_0)\right) = \left\{x \in \overline{W}\left(K, (w_0, x_0)\right) \mid 0_Y \in W\left(J, (w_0, x_0[x]), 0_Y\right)\right\}. \tag{5.4}$$

Proof. We first prove (5.3). Let $x \in \overline{W}\left(S, (w_0, x_0)\right)$. Then there exist sequences $\{r_n\}$ with $r_n > 0$ and $\{(w_n, x_n)\}$ with $(w_n, x_n) \to (w_0, x)$ such that $x_0 + r_n x_n \in S(w_n)$. From the definition of S, we have $x_0 + r_n x_n \in K(w_n)$ and $0_Y \in J(w_n, x_0 + r_n x_n)$, which implies that $x \in \overline{W}\left(K, (w_0, x_0)\right)$ and $0_Y \in W\left(J, (w_0, x_0[x]), 0_Y\right)$. Thus (5.3) holds.

Next we prove the inverse inclusion relation of (5.3) for equation (5.4). Let $x \in \overline{W}\left(K, (w_0, x_0)\right)$ with $0_Y \in W\left(J, (w_0, x_0[x]), 0_Y\right)$. Then, for x, there exist sequences $\{r_n\}$ with $r_n > 0$ and $\{(w_n, x_n)\}$ with $(w_n, x_n) \to (w_0, x)$ such that $x_0 + r_n x_n \in K(w_n)$. It follows from the first-order semi-radial-variation property of J that, with sequences $\{r_n\}$ and $\{(w_n, x_n)\}$ above, there exists a sequence $\{z_n\}$ with $z_n \to 0_Y$ such that $r_n z_n \in J(w_n, x_0 + r_n x_n)$. Since S is directionally Robinson metrically regular of order 1 along K around (w_0, x_0), there exist $\lambda > 0$, $\mu > 0$, and $\gamma > 0$ such that, for n large enough, $x_n \in B_X(x, \lambda), w_n \in B_Z(w_0, \lambda)$, $d\left(0, J\left(w_n, x_0 + r_n x_n\right)\right) \leq r_n \|z_n\| < \gamma$, and

$$d\left(x_0 + r_n x_n, S(w_n)\right) \leq \mu d\left(0, J\left(w_n, x_0 + r_n x_n\right)\right) \leq \mu r_n \|z_n\|.$$

Thus, for n large enough, there exists $\{y_n\}$ with $y_n \in S(w_n)$ such that $\|x_0 + r_n x_n - y_n\| < \mu r_n \|z_n\|$, which implies that $\left\| \frac{y_n - x_0}{r_n} - x_n \right\| < \mu \|z_n\|$. Setting $\hat{x}_n = \frac{y_n - x_0}{r_n}$, it is obvious that $\hat{x}_n \to x$. Hence, $x \in \overline{W}(S,(w_0,x_0))$. It follows that

$$\left\{x \in \overline{W}(K,(w_0,x_0)) \mid 0_Y \in W(J,(w_0,x_0[x]),0_Y)\right\} \subseteq \overline{W}(S,(w_0,x_0)),$$

which together with (5.3) yields that (5.4) holds and the proof is complete. □

Next, we give an example to illustrate Proposition 5.1.

Example 5.2. From Example 5.1, one obtains $\overline{W}(S,(w_0,x_0)) = \mathbb{R}_+$, $\overline{W}(K,(w_0,x_0)) = \mathbb{R}_+$, and $W(J,(w_0,x_0[x]),0) = \mathbb{R}_-$. Then $\overline{W}(S,(w_0,x_0)) = \left\{x \in \overline{W}(K,(w_0,x_0)) \mid 0 \in W(J,(w_0,x_0[x]),0)\right\}$. It is easy to check that the conditions of Proposition 5.1 hold. Thus Proposition 5.1 holds.

To see Proposition 5.2, we propose the following definition.

Definition 5.3. Let $F : X \times X \times Z \to 2^Y$ be a set-valued map, $((x_0,y_0,w_0),k_0) \in \operatorname{gr} F$, and $(x,y,k) \in X \times X \times Y$. The first-order proto-radial-variational set at (x_0,y_0,w_0) is the set

$$W(F,(x_0[x],y_0[y],w_0),k_0) := \{v \in Y \mid \exists r_n > 0,\ \exists (x_n,y_n,w_n,v_n) \to (x,y,w_0,v),$$
$$s.t.\ k_0 + r_n v_n \in F(x_0 + r_n x_n, y_0 + r_n y_n, w_n),\ \forall n \in \mathbb{N}\}.$$

Example 5.3. Let $X = Y = Z = \mathbb{R}_+$, and let $F : X \times X \times Z \to 2^Y$ be defined by $F(x,y,w) := xw$. Taking $k_0 = w_0 = x_0 = y_0 = 0$, we obtain $W(F,(x_0[x],y_0[y],w_0),k_0) = \{0\}$.

Proposition 5.2. *Let X be finite dimensional and $(w_0,x_0) \in \operatorname{gr} S$. Let K be compact (i.e., $\operatorname{gr} K$ is a compact set) and, for each $y_0 \in K(w_0)$,*

$$V_S(H,(w_0,x_0[0_X]),y_0) = \{0_X\}, \tag{5.5}$$

where $H(w,x) = \{y \in X \mid y \in K(w), F(x,y,w) \neq \emptyset\}$. Then

$$W(G,(w_0,x_0[x]),0_Y) \subseteq \bigcup_{y_0 \in K(w_0)} \bigcup_{y \in \overline{W}(K,(w_0,y_0))} W(F,(x_0[x],y_0[y],w_0),0_Y) \cup \{0_Y\}. \tag{5.6}$$

If, additionally, F has the first-order semi-radial-variational set at $((x_0,y_0,w_0),0)$ with respect to (x,y), then

$$W(G,(w_0,x_0[x]),0_Y) = \bigcup_{y_0 \in K(w_0)} \bigcup_{y \in \overline{W}(K,(w_0,y_0))} W(F,(x_0[x],y_0[y],w_0),0_Y) \cup \{0_Y\}. \tag{5.7}$$

Proof. We first prove (5.6). Let $v \in W(G,(w_0,x_0[x]),0_Y)$. There are only two cases for v as follows:

Case I. If $v = 0_Y$, then it is trivial.

Case II. If $v \neq 0_Y$, then there exist sequences $\{r_n\}$ with $r_n > 0$ and $\{(w_n,x_n,v_n)\}$ with $(w_n,x_n,v_n) \to (w_0,x,v)$ such that $r_n v_n \in G(w_n, x_0 + r_n x_n)$. From the definition of G, one sees that there exists a sequence $\{y_n\}$ with $y_n \in K(w_n)$ such that $r_n v_n \in F(x_0 + r_n x_n, y_n, w_n)$. It follows from the compactness of K that $\{y_n\}$ has a subsequence converging to $y_0 \in K(w_0)$.

We now prove that $\{\frac{y_n - y_0}{r_n}\}$ is bounded. Without loss of generality, suppose to the contrary that $\frac{\|y_n - y_0\|}{r_n} \to +\infty$. It follows that $y_n \in H(w_n, x_0 + r_n x_n)$. Then,

$$y_0 + \|y_n - y_0\| \left(\frac{y_n - y_0}{\|y_n - y_0\|}\right) \in H\left(w_n, x_0 + \|y_n - y_0\|^{1/2} \left(\frac{r_n}{\|y_n - y_0\|^{1/2}} x_n\right)\right).$$

Since X is finite dimensional, then $\frac{y_n-y_0}{\|y_n-y_0\|}$ has a subsequence converging to $\hat{y}$ with $\|\hat{y}\|=1$. It is easy to see that $\frac{r_n}{\|y_n-y_0\|^{1/2}}x_n \to 0_X$. Since $s_n=\sqrt{\|y_n-y_0\|}\to 0^+$, we obtain $\hat{y}\in V_S(H,(w_0,x_0[0_X]),y_0)$, which contradicts (5.5). Thus $\{\frac{y_n-y_0}{r_n}\}$ is bounded. Let $\bar{y}_n=\frac{y_n-y_0}{r_n}$. Without loss of generality, we assume that $\bar{y}_n\to\bar{y}$. Thus $y_0+r_n\bar{y}_n\in K(w_n)$ and

$$r_nv_n\in F(x_0+r_nx_n,y_0+r_n\bar{y}_n,w_n),$$

which implies that $\bar{y}\in\overline{W}(K,(w_0,y_0))$ and $v\in W(F,(x_0[x],y_0[\bar{y}],w_0),0_Y)$. Hence, (5.6) holds.

Next, we prove the inverse inclusion relation of (5.6). Let

$$v\in\bigcup_{y_0\in K(w_0)}\bigcup_{y\in\overline{W}(K,(w_0,y_0))}W(F,(x_0[x],y_0[y],w_0),0_Y)\cup\{0_Y\}.$$

To prove

$$v\in W(G,(w_0,x_0[x]),0_Y), \tag{5.8}$$

we divide v into two cases.

Case I. If $v=0_Y$, then, for any sequences $\{r_n\}$ with $r_n>0$ and $\{(w_n,x_n)\}$ with $(w_n,x_n)\to(w_0,x)$, we have $0_Y\in G(w_n,x_0+r_nx_n)$, which implies that (5.8) holds.

Case II. If $v\neq 0_Y$, then one sees that there exist $y_0\in K(w_0)$ and $y\in\overline{W}(K,(w_0,y_0))$ such that $v\in W(F,(x_0[x],y_0[y],w_0),0_Y)$. For y, there exist sequences $\{r_n\}$ with $r_n>0$ and $\{(w_n,y_n)\}$ with $(w_n,y_n)\to(w_0,y)$ such that $y_0+r_ny_n\in K(w_n)$. Since F has the first-order semi-radial-variational set at $((x_0,y_0,w_0),0)$ with respect to (x,y), with sequences $\{r_n\}$, $\{w_n\}$ and $\{y_n\}$ above, for v, one has a sequence $\{(w_n,v_n)\}$ with $r_nv_n\in F(x_0+r_nx_n,y_0+r_ny_n,w_n)$, which implies that $r_nv_n\in G(w_n,x_0+r_nx_n)$. Then, (5.8) holds,

$$\bigcup_{y_0\in K(w_0)}\bigcup_{y\in\overline{W}(K,(w_0,y_0))}W(F,(x_0[x],y_0[y],w_0),0_Y)\cup\{0_Y\}\subseteq W(G,(w_0,x_0[x]),0_Y),$$

which together with (5.6) yields that (5.7) holds. Hence, the proof is complete. □

We now give an example to illustrate Proposition 5.2.

Example 5.4. Let $X=Y=Z=\mathbb{R}$, $C=\mathbb{R}_+$, $K:Z\to 2^X$, and $F:X\times X\times Z\to 2^Y$ be defined by

$$F(x,y,w)=\begin{cases}\frac{xw}{2}, & \text{if } x\geqslant 0,\ y\geqslant 0 \text{ and } w\geqslant 0,\\ \emptyset, & \text{otherwise}\end{cases}$$

and

$$K(w)=\begin{cases}[-w,w], & \text{if } w\geqslant 0,\\ [w,-w], & \text{if } w<0.\end{cases}$$

Then,

$$G(w,x)=\begin{cases}\{\frac{xw}{2},0\}, & \text{if } x\geqslant 0 \text{ and } w\geqslant 0,\\ \emptyset, & \text{otherwise}\end{cases}$$

and

$$H(w,x)=[0,w],\ \forall w\geqslant 0,\ x\geqslant 0.$$

Let $(w_0,x_0)=(0,0)\in \operatorname{gr} S$. Then $K(w_0)=\{0\}$. In fact, for any $y_0=0$, we have

$$V_S(H,(w_0,x_0[0]),y_0)=\{0\},\quad W(G,(w_0,x_0[x]),0)=\{0\},$$

$$W(K,(w_0,y_0))=\mathbb{R},\quad W(F,(x_0[x],y_0[y],w_0),0)=\{0\}$$

and

$$\bigcup_{y_0\in K(w_0)}\bigcup_{y\in\overline{W}(K,(w_0,y_0))} W(F,(x_0[x],y_0[y],w_0),0)\cup\{0\}=\{0\}.$$

We see that the conditions in Proposition 5.2 hold. Thus

$$W\left(G,(w_0,x_0[x]),0\right)=\bigcup_{y_0\in K(w_0)}\bigcup_{y\in\overline{W}(K,(w_0,y_0))} W(F,(x_0[x],y_0[y],w_0),0)\cup\{0\},$$

and Proposition 5.2 holds.

Theorem 5.1. *Let $F: X\times X\times Z\to 2^Y$ and $G,J: X\times Z\to 2^Y$. Let the parametric weakly set-valued equilibrium problem (5.1) with $(w_0,x_0)\in \operatorname{gr} S$. Assume that all the conditions of Propositions 5.1 and 5.2 are satisfied. Assume further that the following conditions are satisfied:*

(i) F is C-dominated by S near x_0;
(ii) F has the first-order proto-variational set of type 2 at (x_0,y_0);
(iii) for all $x\in\overline{X}$ in some $\overline{X}\in\mathscr{U}(x_0)$, $m-e\in \operatorname{int}C\cup(-\operatorname{int}C)$, $\forall m,e\in F(x)$, $m\neq e$.

Then

$$\begin{aligned}\overline{W}(S,(w_0,x_0))=\big\{x\in\overline{W}(K,(w_0,x_0))\mid W\left(F,(x_0[x],y_0[y],w_0),0_Y\right)\cap\\ \left(-\operatorname{int}C\right)=\emptyset,\ \forall y_0\in K(w_0),\ \forall y\in\overline{W}\left(K,(w_0,y_0)\right)\big\}.\end{aligned}\tag{5.9}$$

Proof. Using Proposition (2.2), we obtain $\overline{W}\left(J,(w_0,x_0)\right)=\operatorname{WMin}_C\overline{W}\left(G,(w_0,x_0)\right)$. Proposition (5.1) yields

$$\overline{W}\left(S,(w_0,x_0)\right)=\left\{x\in\overline{W}\left(K,(w_0,x_0)\right)\mid 0_Y\in W\left(J,(w_0,x_0[x]),0_Y\right)\right\}.$$

Moreover, it follows from Proposition 5.2 that

$$W\left(G,(w_0,x_0[x]),0_Y\right)=\bigcup_{y_0\in K(w_0)}\bigcup_{y\in\overline{W}(K,(w_0,y_0))} W(F,(x_0[x],y_0[y],w_0),0_Y)\cup\{0_Y\},$$

and then

$$\begin{aligned}\overline{W}(S,(w_0,x_0))=\big\{x\in\overline{W}(K,(w_0,x_0))\mid W\left(F,(x_0[x],y_0[y],w_0),0_Y\right)\cap\\ \left(-\operatorname{int}C\right)=\emptyset,\ \forall y_0\in K(w_0),\ \forall y\in\overline{W}\left(K,(w_0,y_0)\right)\big\}.\end{aligned}$$

Thus the proof is complete. □

Now we consider the following example to illustrate Theorem 5.1.

Example 5.5. Let $X=Y=Z=\mathbb{R}$, $C=\mathbb{R}_+$, $K: Z\to 2^X$, and $F: X\times X\times Z\to 2^Y$ be defined by $F(x,y,w)=\frac{xw}{9}$ and $K(w)=[\frac{-|w|}{3},\frac{|w|}{3}]$. Then $S(w)=\begin{cases}[0,\frac{w}{3}], & \text{if } w\geq 0,\\ [\frac{-|w|}{3},0), & \text{if } w<0.\end{cases}$ Let $(w_0,x_0)=(0,0)\in\operatorname{gr}S$. Then $K(w_0)=\{0\}$. Note that $\overline{W}\left(S,(w_0,x_0)\right)=\mathbb{R}$ and $\overline{W}\left(K,(w_0,x_0)\right)=\mathbb{R}$. For all $y_0\in K(w_0)$, $\overline{W}\left(K,(w_0,y_0)\right)=\mathbb{R}$ and $W\left(F,(x_0[x],y_0[y],w_0),0\right)=\{0\}$. The assumptions in Theorem 5.1 are satisfied, so (5.9) is fulfilled, and Theorem 5.1 holds.

6. Concluding Remarks

In this paper, we first proposed the chain and sum rules of variational sets of type 2 from a different perspective, which is different from [27]. Next, we investigated the sensitivity analysis of variational inequalities. Finally, we discussed the first-order sensitivity analysis of parametric weakly set-valued equilibrium problems in terms of variational sets of type 2. The results presented in this paper mainly improve or generalize the corresponding ones in [26, 27].

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