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TIME-DEPENDENT TENSOR VARIATIONAL INEQUALITY FOR AN OLIGOPOLISTIC MARKET EQUILIBRIUM PROBLEM

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Abstract. The general tensor variational inequalities are highly valuable for analyzing equilibrium problems in oligopolistic markets. In this paper, we begin by reviewing current conclusions related to the solutions of tensor variational inequality problems. Then, a new algorithm is proposed, which merges the inertial contraction projection method with the Mann-type method to tackle tensor variational inequalities under mild conditions. Finally, we investigate the equilibrium problem in dynamic (i.e., time-dependent) oligopolistic markets characterized by excess supply and demand. A numerical example is given to verify the effectiveness of the proposed algorithm.

Keywords. Dynamic oligopolistic market equilibrium problem; Inertial contraction projection method; Mann-type method; Tensor variational inequality.

1. Introduction

Variational inequalities (VIs) have received extensive attention in recent decades with established results on solution existence, uniqueness, and regularity. To our knowledge, the VIs serve as a vital tool in optimization theory and find applications in various areas, including partial differential equations, optimal control, and mathematical programming. On the other hand, in the era of big data, tensors have attracted extensive attention as one of an effective forms for high-order multidimensional data. Since polynomials can be expressed easily by high-order tensors, a novel development on VI was introduced by Wang *et al.* [34], named tensor variational inequality (TVI). Then, Barbagallo *et al.* [4, 5, 6] studied TVIs in tensor Hilbert spaces, and presented some results on solution existence and uniqueness. When the feasible set is bounded and closed, the TVI problem has at least one solution if the function is continuous. Moreover, if the function is K-pseudomonotone and lower hemicontinuous along a line segment, the solution exists. When the feasible set is unbounded, a unique solution exists if the function is strongly monotone. However, if the function satisfies monotonicity or strict monotonicity only, the existence of solution is not guaranteed.

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From an application perspective, the VIs have been seen a crucial tool for analyzing general oligopolistic market equilibrium problems. Note that there are fruitful and long history for equilibrium problems. Cournot [14] initially delved into the realm of noncooperative behavior between two producers, namely, the duopoly problem. Nash [25, 26] later expanded upon this by generalizing the duopoly problem into the noncooperative game problem with n players. Numerous scholars investigated the existence and uniqueness of outcomes in noncooperative games with various assumptions [18, 19, 29]. Dafermos and Nagurney focused on equilibrium conditions for static oligopoly market problems by using a finite-dimensional variational approach [15]. Barbagallo *et al.* [2] extended this analysis to dynamic oligopoly market equilibrium problems, and explored the equivalence between Nash equilibrium conditions and the variational inequality. Building upon this foundation, they further generalized the problem to accommodate excess production and excess demand [10, 11]. They also introduced the initial model for oligopolistic markets, allowing firms to produce multiple goods, and employ the tensor variational inequality to find the equilibrium solution [11]. Considering unpredictable events, Barbagallo *et al.* [8, 9] studied the random time-dependent oligopolistic market equilibrium problem with both production and demand excesses from the firms' perspective.

From an algorithm perspective, to our knowledge, the regularization method and the projection method are two fundamental approaches for solving VIs. The authors in [1, 35] introduced the projective gradient method and provided rigorous proofs of the algorithm's convergence. However, the convergence was proved under very strong conditions. In response to the limitation, Korpelevich [21] introduced an alternative approach known as the extragradient method (EGM). Nevertheless, the efficiency of the EGM is compromised in cases where the feasible set is generally closed and convex due to the requirement of two projections in each iteration. To mitigate this challenge, Censor *et al.* [13] proposed the subgradient extragradient method. To expedite convergence of gradient-based algorithms, inertial technique was extensively studied; see, e.g., [17, 20, 28, 31, 32]. Bot and Csetnek [12] developed the inertial hybrid proximal extragradient algorithm. Dong *et al.* [16] introduced the inertial contraction projection method (ICPM) and established the weak convergence under appropriate conditions. Nadezhkina and Takahashi [24] applied two widely recognized hybrid extragradient methods for tackling the variational inequality problem, and they also demonstrated the algorithms' strong convergence.

In this paper, we propose a new algorithm in the optimization point of view, which merges the inertial contraction projection method with the Mann-type method to tackle tensor variational inequalities under mild conditions. As an application of the algorithm, we investigate the equilibrium problem in dynamic (i.e. time-dependent) oligopolistic markets characterized by excess supply and demand. A numerical example is given to verify the effectiveness of the proposed algorithm. The remaining parts of this paper are organized as follows. In Section 2, we review some basic preliminaries of tensors and notations used in the paper. In Section 3, an alternative algorithm is given, which combines the inertial contraction projection method with the Mann-type method to calculate tensor variational inequalities. Furthermore, the strong convergence of this algorithm is proved, subject to standard assumptions imposed on the operator. In Section 4, we study the dynamic oligopolistic market equilibrium problem with production and demand excesses. Finally, a practical numerical example is provided in Section 5, the last section.

2. Preliminaries

In this section, we recall some useful preliminaries. For positive integer n, denote $[n] = \{1,2,\cdots,n\}$. Scalars in $\mathbb{R}$ are denoted by unbold lowercase letters such as $a,b,\cdots$, and vectors in $\mathbb{R}^n$ are denoted by bold lowercase letters such as $\mathbf{x},\mathbf{y}$. Tensors are denoted by capital uppercase letters such as $\mathscr{X},\mathscr{Y}$. Denote $\mathscr{X} = (x_{i_1 i_2\cdots i_M})$, where $(i_1,i_2,\cdots,i_M) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_M}$ is a tensor with order M and dimension $n_1\times n_2\times\cdots\times n_M$. $\mathscr{X}$ is an order M dimension n tensor when $i_1,\cdots,i_M \in [n]$. Let $\mathscr{T}_{n_1 n_2\cdots n_M}$ denote the set including all tensors with order M and dimension $n_1\times n_2\times\cdots\times n_M$. The set of all order M dimension n tensors is denoted by $\mathscr{T}_{M,n}$. $\mathscr{O}$ represents the tensor in which all elements are equal to 0.

To move on, we recall the definition of inner product on $\mathscr{T}_{M,n}$ as follows.

Definition 2.1. [27] For any two tensors $\mathscr{X},\mathscr{Y} \in \mathscr{T}_{M,n}$, the inner product of $\mathscr{X}$ and $\mathscr{Y}$, denoted as $\langle\mathscr{X},\mathscr{Y}\rangle$, is defined as

$$\langle\mathscr{X},\mathscr{Y}\rangle = \sum_{i_1,i_2,\cdots,i_M=1}^{n} x_{i_1 i_2\cdots i_M} y_{i_1 i_2\cdots i_M}.$$

Similar to the matrix case, the induced norm $\|\mathscr{X}\| = \sqrt{\langle\mathscr{X},\mathscr{X}\rangle}$ is called the Frobenius norm. Clearly, $(\mathscr{T}_{M,n},\langle\cdot,\cdot\rangle)$ is a Hilbert space.

Next, we recall the definition of infinite dimensional tensor variational inequalities and evolutionary tensor variational inequalities.

Definition 2.2. [7] In the Hilbert space $L^2([0,T],\mathscr{T}_{M,n})$, $\langle\langle\mathscr{X},\mathscr{Y}\rangle\rangle := \int_0^T\langle\mathscr{X}(t),\mathscr{Y}(t)\rangle\mathrm{dt}$, is its duality mapping, where $\mathscr{X} \in (L^2([0,T],\mathscr{T}_{M,n}))^* = L^2([0,T],\mathbb{R}^k)$ and $\mathscr{Y} \in L^2([0,T],\mathscr{T}_{M,n})$.

Definition 2.3. [7] Let $\mathbb{K} \subset L^2([0,T],\mathscr{T}_{M,n})$ be a nonempty, convex, and closed subset, and let $F : [0,T]\times\mathbb{K} \to L^2([0,T],\mathscr{T}_{M,n})$ be a function. The infinite dimensional tensor variational inequality problems involves determining $\mathscr{X} \in \mathbb{K}$ such that

$$\langle\langle F(\mathscr{X}),\mathscr{Y}-\mathscr{X}\rangle\rangle \geq 0, \qquad \forall\,\mathscr{Y} \in \mathbb{K}. \tag{2.1}$$

According to Definitions 2.2 and 2.3, the evolutionary tensor variational inequality aims to find $\mathscr{X}(t) \in \mathbb{K}(t)$ such that

$$\int_0^T \langle F(t,\mathscr{X}(t)),\mathscr{Y}(t)-\mathscr{X}(t)\rangle\mathrm{dt} \geq 0, \qquad \forall\,\mathscr{Y} \in \mathbb{K}(t),$$

where $\mathbb{K}(t) \subset \mathscr{T}_{M,n}$ is a nonempty, convex, and closed subset and $F : [0,T]\times\mathbb{K}(t) \to \mathscr{T}_{M,n}$.

To proceed, we initially revisit the notion of monotonicity for tensor functions in infinite dimensional spaces.

Definition 2.4. [7] Let $\mathbb{K}$ be a nonempty subset of $L^2([0,T],\mathscr{T}_{M,n})$. A tensor function F : $[0,T]\times\mathbb{K} \to L^2([0,T],\mathscr{T}_{M,n})$ is said to be

- monotone on $\mathbb{K}$ if, for each $\mathscr{X},\mathscr{Y} \in \mathbb{K}$, $\langle\langle F(\mathscr{X})-F(\mathscr{Y}),\mathscr{X}-\mathscr{Y}\rangle\rangle \geq 0$;
- strictly monotone on $\mathbb{K}$ if, for each $\mathscr{X},\mathscr{Y} \in \mathbb{K}$ with $\mathscr{X} \neq \mathscr{Y}$,

$$\langle\langle F(\mathscr{X})-F(\mathscr{Y}),\mathscr{X}-\mathscr{Y}\rangle\rangle > 0;$$

- strongly monotone on $\mathbb{K}$ if, for each $\mathscr{X},\mathscr{Y} \in \mathbb{K}$, there exists a constant $v > 0$ such that

$$\langle\langle F(\mathscr{X})-F(\mathscr{Y}),\mathscr{X}-\mathscr{Y}\rangle\rangle \geq v\|\mathscr{X}-\mathscr{Y}\|^2.$$

- pseudomonotone in the sense of Karamardian (K-pseudomotone) on $\mathbb{K}$ if, for each $\mathscr{X},\mathscr{Y}\in\mathbb{K}$,

$$\langle\langle F(\mathscr{X}),\mathscr{Y}-\mathscr{X}\rangle\rangle\geq 0\Rightarrow\langle\langle F(\mathscr{Y}),\mathscr{Y}-\mathscr{X}\rangle\rangle\geq 0.$$

- strong K-pseudomonotone on $\mathbb{K}$ if, for each $\mathscr{X},\mathscr{Y}\in\mathbb{K}$, with $\mathscr{X}\neq\mathscr{Y}$,

$$\langle\langle F(\mathscr{X}),\mathscr{Y}-\mathscr{X}\rangle\rangle\geq 0\Rightarrow\langle\langle F(\mathscr{Y}),\mathscr{Y}-\mathscr{X}\rangle\rangle> 0.$$

3. ALGORITHM

In this section, we introduce a novel algorithm that integrates the inertial projection contraction method with the Mann-type method to address finite dimensional tensor variational inequalities

$$\langle F(\mathscr{X}),\mathscr{Y}-\mathscr{X}\rangle. \tag{3.1}$$

Suppose that $S(\mathbb{K},F)$ is the solution set of (3.1). To prove the convergence of our algorithm, we first recall some properties of the tensor function in finite dimensional space.

Definition 3.1. Let $\mathbb{K}$ be a nonempty subset of $\mathscr{T}_{M,n}$. A tensor function $F:[0,T]\times\mathbb{K}\to\mathscr{T}_{M,n}$ is said to be

- sequentially weakly continuous if, for each sequence $\{\mathscr{X}^k\}\subset\mathbb{K}$ converging weakly to a point $\mathscr{X}\in\mathbb{K}$, $\{F(\mathscr{X}^k)\}$ converges weakly to $F(\mathscr{X})$;
- [4] pseudomonotone in the sense of Karamardian (K-pseudomotone) on $\mathbb{K}$ if, for each $\mathscr{X},\mathscr{Y}\in\mathbb{K}$,

$$\langle F(\mathscr{X}),\mathscr{Y}-\mathscr{X}\rangle\geq 0\Rightarrow\langle F(\mathscr{Y}),\mathscr{Y}-\mathscr{X}\rangle\geq 0.$$

To move on, the following assumptions are needed:

(i) The feasible set $\mathbb{K}\subset\mathscr{T}_{M,n}$ is nonempty, closed and convex.
(ii) The function $F:\mathscr{T}_{M,n}\to\mathscr{T}_{M,n}$ is Lipschitz continuous with constant L and K-pseudomonotone on $\mathscr{T}_{M,n}$ and sequentially weakly continuous on $\mathbb{K}$.
(iii) The solution set $S(\mathbb{K},F)$ is nonempty.

Remark 3.1. By the definition of $\overline{\alpha}_r$, it holds clearly that $\alpha_r\|\mathscr{X}^r-\mathscr{X}^{r-1}\|\leq\tau_r$ for $r\geq 1$. Combining this with $\lim\limits_{r\to\infty}\frac{\tau_r}{\beta_r}=0$, it follows that

$$\lim_{r\to\infty}\frac{\alpha_r}{\beta_r}\|\mathscr{X}^r-\mathscr{X}^{r-1}\|\leq\lim_{r\to\infty}\frac{\tau_r}{\beta_r}=0,$$

which implies that

$$\lim_{r\to\infty}\frac{\alpha_r}{\beta_r}\|\mathscr{X}^r-\mathscr{X}^{r-1}\|=0.$$

To continue, we recall the following preliminary lemmas.

Lemma 3.1. [6] *If $\mathbb{K}$ is a nonempty, convex, and closed subset of $\mathscr{T}_{M,n}$ for any $\mathscr{X}$, $\mathscr{Y}\in\mathscr{T}_{M,n}$ and any $\mathscr{Z}\in\mathbb{K}$, then*

$$\langle\mathscr{X}-P_{\mathbb{K}}(\mathscr{X}),\mathscr{Z}-P_{\mathbb{K}}(\mathscr{X})\rangle\leq 0, \tag{3.2}$$

$$\|P_{\mathbb{K}}(\mathscr{X})-P_{\mathbb{K}}(\mathscr{Y})\|^2\leq\|\mathscr{X}-\mathscr{Y}\|^2-\|P_{\mathbb{K}}(\mathscr{X})-\mathscr{X}+\mathscr{Y}-P_{\mathbb{K}}(\mathscr{Y})\|^2. \tag{3.3}$$

Lemma 3.2. *Let assumptions (i)-(iii) hold. If $\mathscr{Y}^r=\mathscr{W}^r$ or $d^r=0$ for all $r\geq 1$ in Algorithm 1, then $\mathscr{Y}^r\in S(\mathbb{K},F)$.*

Algorithm 1 The inertial contraction projection method with the Mann-type method

Iterative Steps: Let $\lambda \in (0,1/L)$, $\gamma \in (0,2)$, $\alpha > 0$, and $\mathscr{X}^0, \mathscr{X}^1 \in \mathscr{T}_{M,n}$. Choose three positive sequences $\{\tau_r\} \subset (0,\infty)$, $\{\beta_r\} \subset (0,1)$, $\{\theta_r\} \subset (a, 1-\beta_r)$ for some $a > 0$, satisfying

$$\lim_{r\to\infty} \beta_r = 0, \quad \sum_{r=1}^{\infty} \beta_r = \infty, \quad \tau_r = \circ(\beta_r), \quad \lim_{r\to\infty} \frac{\tau_r}{\beta_r} = 0.$$

Substep 1. For $\mathscr{X}^r$ and $\mathscr{X}^{r-1}$ $(r \geq 1)$, take α_r satifying $0 \leq \alpha_r \leq \overline{\alpha}_r$, where

$$\overline{\alpha}_r := \begin{cases} \min\left\{\alpha, \frac{\tau_r}{\|\mathscr{X}^r - \mathscr{X}^{r-1}\|}\right\}, & \text{if } \mathscr{X}^r \neq \mathscr{X}^{r-1}, \\ \alpha, & \text{otherwise.} \end{cases}$$

Set

$$\mathscr{W}^r = \mathscr{X}^r + \alpha_r(\mathscr{X}^r - \mathscr{X}^{r-1}), \quad \mathscr{Y}^r = \mathrm{P}_{\mathbb{K}}(\mathscr{W}^r - \lambda F(\mathscr{W}^r)).$$

If $\mathscr{Y}^r = \mathscr{W}^r$ or $F(\mathscr{Y}^r) = \mathscr{O}$, then stop (indicating $\mathscr{Y}^r$ as a solution of problem (3.1)). Otherwise, proceed to Substep 2.

Substep 2. Calculate $\mathscr{Z}^r = \mathscr{W}^r - \gamma\eta_r d^r$, where

$$d^r = (\mathscr{W}^r - \mathscr{Y}^r) - \lambda(F(\mathscr{W}^r) - F(\mathscr{Y}^r)), \ \varphi(\mathscr{W}^r, \mathscr{Y}^r) := \langle \mathscr{W}^r - \mathscr{Y}^r, d^r \rangle,$$

and

$$\eta_r := \begin{cases} \frac{\varphi(\mathscr{W}^r, \mathscr{Y}^r)}{\|d^r\|^2}, & \text{if } d^r \neq \mathscr{O}, \\ 0, & \text{if } d^r = \mathscr{O}. \end{cases}$$

Substep 3. Calculate

$$\mathscr{X}^{r+1} = (1 - \theta_r - \beta_r)\mathscr{X}^r + \theta_r \mathscr{Z}^r.$$

Set $r := r+1$ and go to Substep 1.

Proof. To prove the main result, we first prove that $\mathscr{W}^r = \mathscr{Y}^r$ if and only if $d^r = 0$. According to the Lipschitz continuity property of F, for each $r \geq 1$, we know that

$$\begin{aligned} \|d^r\| &\geq \|\mathscr{W}^r - \mathscr{Y}^r\| - \lambda\|F(\mathscr{W}^r) - F(\mathscr{Y}^r)\| \\ &\geq (1 - \lambda L)\|\mathscr{W}^r - \mathscr{Y}^r\|, \end{aligned}$$

which together with the following inequality

$$\|d^r\| \leq \|\mathscr{W}^r - \mathscr{Y}^r\| + \lambda\|F(\mathscr{W}^r) - F(\mathscr{Y}^r)\| \leq (1+\lambda L)\|\mathscr{W}^r - \mathscr{Y}^r\|, \qquad \forall r \geq 1,$$

yields

$$(1 - \lambda L)\|\mathscr{W}^r - \mathscr{Y}^r\| \leq \|d^r\| \leq (1+\lambda L)\|\mathscr{W}^r - \mathscr{Y}^r\|, \qquad \forall r \geq 1.$$

Thus $\mathscr{W}^r = \mathscr{Y}^r$ if and only if $d^r = 0$. Therefore, if $\mathscr{Y}^r = \mathscr{W}^r$ or $d^r = 0$ for each $r \geq 1$, it holds that $\mathscr{Y}^r = \mathrm{P}_{\mathbb{K}}(\mathscr{Y}^r - \lambda F(\mathscr{Y}^r))$. By (3.2), for any $\mathscr{Z} \in \mathbb{K}$, we know that

$$\langle \mathscr{Y}^r - \lambda F(\mathscr{Y}^r) - \mathscr{Y}^r, \mathscr{Z} - \mathscr{Y}^r \rangle = -\lambda\langle F(\mathscr{Y}^r), \mathscr{Z} - \mathscr{Y}^r \rangle \leq 0,$$

which implies $\langle F(\mathscr{Y}^r), \mathscr{Z} - \mathscr{Y}^r \rangle \geq 0$. Hence $\mathscr{Y}^r \in S(\mathbb{K}, F)$, and the desired results hold. □

Lemma 3.3. *Suppose that assumptions (i)-(iii) hold. Then $\mathscr{X} \in \mathbb{K}$ is a solution to*

$$\langle F(\mathscr{X}), \mathscr{Y} - \mathscr{X} \rangle \geq 0, \ \forall\, \mathscr{Y} \in \mathbb{K}, \tag{3.4}$$

if and only if

$$\langle F(\mathscr{Y}), \mathscr{Y} - \mathscr{X}\rangle \geq 0,\ \forall\, \mathscr{Y} \in \mathbb{K}. \tag{3.5}$$

Proof. To prove the sufficiency, we assume that $\mathscr{X} \in \mathbb{K}$ is a solution to (3.5). For any $\mathscr{Y} \in \mathbb{K}$ and $0 < \lambda \leq 1$, let

$$\mathscr{X}_\lambda = \lambda \mathscr{Y} + (1-\lambda)\mathscr{X}.$$

Since $\mathbb{K}$ is a convex set, $\mathscr{X}_\lambda \in \mathbb{K}$. From (3.5), we obtain that

$$\langle F(\mathscr{X}_\lambda), \mathscr{X}_\lambda - \mathscr{X}\rangle = \langle F(\mathscr{X}_\lambda), \lambda\mathscr{Y} + (1-\lambda)\mathscr{X} - \mathscr{X}\rangle = \lambda\langle F(\mathscr{X}_\lambda), \mathscr{Y} - \mathscr{X}\rangle \geq 0.$$

Thus

$$\langle F(\mathscr{X}_\lambda), \mathscr{Y} - \mathscr{X}\rangle \geq 0, \tag{3.6}$$

and $\mathscr{X}_\lambda \to \mathscr{X}$ if $\lambda \to 0$. By the continuity of F, it follows that $F(\mathscr{X}_\lambda) \to F(\mathscr{X})$ as $\lambda \to 0$. Combining this with (3.6), it holds that

$$\langle F(\mathscr{X}), \mathscr{Y} - \mathscr{X}\rangle \geq 0,$$

and $\mathscr{X}$ is a solution to (3.4).

Conversely, let $\mathscr{X} \in \mathbb{K}$ be a solution to (3.4). Since F is K-pseudomonotone, then

$$\langle F(\mathscr{X}), \mathscr{Y} - \mathscr{X}\rangle \geq 0 \Rightarrow \langle F(\mathscr{Y}), \mathscr{Y} - \mathscr{X}\rangle \geq 0,\ \forall\, \mathscr{Y} \in \mathbb{K}.$$

Thus equation (3.5) holds and the desired results hold. □

Note that the following result plays a crucial role to guarantee the strong convergence of the proposed algorithm.

Lemma 3.4. *Suppose $\{\mathscr{W}^r\}$ is a sequence generated by Algorithm 1 and that assumptions $(i)-(iii)$ hold. If there exists a subsequence $\{\mathscr{W}^{r_n}\}$ of $\{\mathscr{W}^r\}$ that converges weakly to a point $\mathscr{X} \in \mathscr{T}_{M,n}$ and $\lim_{n\to\infty} \|\mathscr{W}^{r_n} - \mathscr{Y}^{r_n}\| = 0$, then $\mathscr{X} \in S(\mathbb{K}, F)$.*

Proof. Firstly, we prove that $\liminf_{n\to\infty}\langle F(\mathscr{Y}^{r_n}), \mathscr{Z} - \mathscr{Y}^{r_n}\rangle \geq 0$. Clearly,

$$\begin{aligned}&\langle F(\mathscr{Y}^{r_n}), \mathscr{Z} - \mathscr{Y}^{r_n}\rangle \\ &= \langle F(\mathscr{Y}^{r_n}) - F(\mathscr{W}^{r_n}), \mathscr{Z} - \mathscr{W}^{r_n}\rangle + \langle F(\mathscr{W}^{r_n}), \mathscr{Z} - \mathscr{W}^{r_n}\rangle + \langle F(\mathscr{Y}^{r_n}), \mathscr{W}^{r_n} - \mathscr{Y}^{r_n}\rangle.\end{aligned} \tag{3.7}$$

By Lemma 3.1, for any $Z \in \mathbb{K}$, we obtain that

$$\begin{aligned}&\langle \mathscr{W}^{r_n} - \lambda F(\mathscr{W}^{r_n}) - P_{\mathbb{K}}(\mathscr{W}^{r_n} - \lambda F(\mathscr{W}^{r_n})), \mathscr{Z} - P_{\mathbb{K}}(\mathscr{W}^{r_n} - \lambda F(\mathscr{W}^{r_n}))\rangle \\ =&\langle \mathscr{W}^{r_n} - \lambda F(\mathscr{W}^{r_n}) - \mathscr{Y}^{r_n}, \mathscr{Z} - \mathscr{Y}^{r_n}\rangle \leq 0,\end{aligned} \tag{3.8}$$

which is equivalent to $\frac{1}{\lambda}\langle \mathscr{W}^{r_n} - \mathscr{Y}^{r_n}, \mathscr{Z} - \mathscr{Y}^{r_n}\rangle \leq \langle F(\mathscr{W}^{r_n}), \mathscr{Z} - \mathscr{Y}^{r_n}\rangle$. Thus

$$\frac{1}{\lambda}\langle \mathscr{W}^{r_n} - \mathscr{Y}^{r_n}, \mathscr{Z} - \mathscr{Y}^{r_n}\rangle + \langle F(\mathscr{W}^{r_n}), \mathscr{Y}^{r_n} - \mathscr{W}^{r_n}\rangle \leq \langle F(\mathscr{W}^{r_n}), \mathscr{Z} - \mathscr{W}^{r_n}\rangle. \tag{3.9}$$

From the conditions that $\{\mathscr{W}^{r_n}\}$ of $\{\mathscr{W}^r\}$ weakly converges to the point $\mathscr{X} \in \mathscr{T}_{M,n}$, we know that $\{\mathscr{W}^{r_n}\}$ is bounded. By the Lipschitz continuity of F, it implies the boundedness of $\{F(\mathscr{W}^{r_n})\}$. Furthermore, the boundedness of $\{\mathscr{Y}^{r_n}\}$ is guaranteed since $\|\mathscr{W}^{r_n} - \mathscr{Y}^{r_n}\| \to 0$. As $n \to \infty$ in (3.9), it follows that

$$\liminf_{n\to\infty}\langle F(\mathscr{W}^{r_n}), \mathscr{Z} - \mathscr{W}^{r_n}\rangle \geq 0, \qquad \mathscr{Z} \in \mathbb{K}. \tag{3.10}$$

From the Lipschitz continuity of F again and $\lim\limits_{n\to\infty}\|\mathscr{W}^{r_n}-\mathscr{Y}^{r_n}\|=0$, it holds that

$$\lim_{n\to\infty}\|F(\mathscr{W}^{r_n})-F(\mathscr{Y}^{r_n})\|=0,$$

which, together with (3.7) and (3.10), implies that $\liminf\limits_{n\to\infty}\langle F(\mathscr{Y}^{r_n}),\mathscr{Z}-\mathscr{Y}^{r_n}\rangle\geq 0$.

We now prove that $\mathscr{X}\in S(\mathbb{K},F)$. Suppose that $\{\varepsilon_n\}$ is a decreasing positive sequence satisfying $\lim\limits_{n\to\infty}\varepsilon_n=0$. For each $n\geq 1$, we define R_n as the smallest positive integer such that

$$\langle F(\mathscr{Y}^{r_j}),\mathscr{Z}-\mathscr{Y}^{r_j}\rangle+\varepsilon_n\geq 0,\qquad \forall\, j\geq R_n. \tag{3.11}$$

The decreasing of $\{\varepsilon_n\}$ implies that $\{R_n\}$ is increasing. Furthermore, for each $n\geq 1$, $\{\mathscr{Y}^{R_n}\}\subset\mathbb{K}$, considering the case where $F(\mathscr{Y}^{R_n})\neq 0$ and setting

$$\mathscr{V}_{R_n}=\frac{F(\mathscr{Y}^{R_n})}{\|F(\mathscr{Y}^{R_n})\|^2},$$

it follows that $\langle F(\mathscr{Y}^{R_n}),\mathscr{V}^{R_n}\rangle=1$. Thus, by (3.11), we have

$$\langle F(\mathscr{Y}^{R_n}),\mathscr{Z}+\varepsilon_n\mathscr{V}_{R_n}-\mathscr{Y}^{R_n}\rangle\geq 0,\qquad \forall n\geq 1,$$

Since F is K-pseudomonotone on $\mathbb{K}$, we have

$$\langle F(\mathscr{Z}+\varepsilon_n\mathscr{V}^{R_n}),\mathscr{Z}+\varepsilon_n\mathscr{V}^{R_n}-\mathscr{Y}^{R_n}\rangle\geq 0.$$

It follows that

$$\begin{aligned}&\langle F(\mathscr{Z}+\varepsilon_n\mathscr{V}^{R_n}),\mathscr{Z}+\varepsilon_n\mathscr{V}^{R_n}-\mathscr{Y}^{R_n}\rangle+\langle F(\mathscr{Z}),\mathscr{Z}+\varepsilon_n\mathscr{V}^{R_n}-\mathscr{Y}^{R_n}\rangle\\ &\quad-\langle F(\mathscr{Z}),\mathscr{Z}+\varepsilon_n\mathscr{V}^{R_n}-\mathscr{Y}^{R_n}\rangle\\ =&\langle F(\mathscr{Z}),\mathscr{Z}-\mathscr{Y}^{R_n}\rangle+\langle F(\mathscr{Z}),\varepsilon_n\mathscr{V}^{R_n}\rangle+\langle F(\mathscr{Z}+\varepsilon_n\mathscr{V}^{R_n})-F(\mathscr{Z}),\mathscr{Z}+\varepsilon_n\mathscr{V}^{R_n}-\mathscr{Y}^{R_n}\rangle\geq 0,\end{aligned}$$

and then

$$\langle F(\mathscr{Z}),\mathscr{Z}-\mathscr{Y}^{R_n}\rangle\geq\langle F(\mathscr{Z})-F(\mathscr{Z}+\varepsilon_n\mathscr{V}^{R_n}),\mathscr{Z}+\varepsilon_n\mathscr{V}^{R_n}-\mathscr{Y}^{R_n}\rangle-\varepsilon_n\langle F(\mathscr{Z}),\mathscr{V}^{R_n}\rangle. \tag{3.12}$$

Finally, we show that $\lim\limits_{n\to\infty}\varepsilon_n\mathscr{V}^{R_n}=0$. As $\mathscr{W}^{r_n}\rightharpoonup\mathscr{X}$ and $\lim\limits_{n\to\infty}\|\mathscr{W}^{r_n}-\mathscr{Y}^{r_n}\|=0$, we have $\mathscr{Y}^{r_n}\rightharpoonup\mathscr{X}$ as $n\to\infty$. By the fact that $\{\mathscr{Y}^{r}\}\subset\mathbb{K}$, it follows that $\mathscr{X}\in\mathbb{K}$. The sequentially weakly continuity of F implies $\lim\limits_{n\to\infty}F(\mathscr{Y}^{r_n})\rightharpoonup F(\mathscr{X})$. Suppose $F(\mathscr{X})\neq 0$, otherwise $\mathscr{X}$ would be a solution. Since the norm mapping is sequentially weakly lower semi-continuous, we have

$$0<\|F(\mathscr{X})\|\leq\liminf_{n\to\infty}\|F(\mathscr{Y}^{r_n})\|.$$

Given that $\{\mathscr{Y}^{R_n}\}\subset\{\mathscr{Y}^{r_n}\}$ and $\lim_{n\to\infty}\varepsilon_n=0$, we can derive

$$0\leq\limsup_{n\to\infty}\|\varepsilon_n\mathscr{V}^{R_n}\|=\limsup_{n\to\infty}\left(\frac{\varepsilon_n}{\|F(\mathscr{Y}^{r_n})\|}\right)\leq\frac{\limsup_{n\to\infty}\varepsilon_n}{\liminf_{n\to\infty}\|F(\mathscr{Y}^{r_n})\|}=0,$$

which implies that $\lim_{n\to\infty}\varepsilon_n\mathscr{V}^{R_n}=0$. Due to the uniform continuity of F, $\{\mathscr{V}^{R_n}\}$ is bounded and $\lim_{n\to\infty}\varepsilon_n\mathscr{V}^{R_n}=0$. Consequently, the right-hand side of (3.12) tends to 0 as $n\to\infty$. Thus

$$\liminf_{n\to\infty}\langle F(\mathscr{Z}),\mathscr{Z}-\mathscr{Y}^{R_n}\rangle\geq 0.$$

Then, for all $\mathscr{X}\in\mathbb{K}$, we have

$$\langle F(\mathscr{Z}),\mathscr{Z}-\mathscr{X}\rangle=\lim_{n\to\infty}\langle F(\mathscr{Z}),\mathscr{Z}-\mathscr{Y}^{R_n}\rangle=\liminf_{n\to\infty}\langle F(\mathscr{Z}),\mathscr{Z}-\mathscr{Y}^{R_n}\rangle\geq 0.$$

By Lemma 3.3, it follows that $\langle F(\mathscr{X}),\mathscr{Z}-\mathscr{X}\rangle \geq 0$. Thus $\mathscr{X}\in S(\mathbb{K},F)$. The proof is done. □

Lemma 3.5. [30] *Let $\{s_r\}$ be a sequence of nonnegative real numbers, $\{\alpha_r\}$ be a sequence in $(0,1)$ such that $\sum_{r=1}^{\infty}\alpha_r=\infty$, and $\{t_r\}$ be a sequence of real numbers. Suppose*

$$s_{r+1}\leq(1-\alpha_r)s_r+\alpha_r t_r,\qquad \forall r\geq 1.$$

If, for every subsequence $\{s_{r_n}\}$ of $\{s_r\}$ that satisfies $\liminf_{n\to\infty}(s_{r_n+1}-s_{r_n})\geq 0$, it holds that $\limsup_{n\to\infty} t_{r_n}\leq 0$, then $\lim_{r\to\infty}s_r=0$.

Lemma 3.6. *Assume conditions $(i)-(iii)$ hold. Let $\{\mathscr{Z}^r\}$ be a sequence generated by Algorithm 1. Then, for any $\mathscr{X}^*\in S(\mathbb{K},F)$, the following inequality holds*

$$\|\mathscr{Z}^r-\mathscr{X}^*\|^2\leq\|\mathscr{W}^r-\mathscr{X}^*\|^2-\frac{2-\gamma}{\gamma}\|\mathscr{W}^r-\mathscr{Z}^r\|^2.$$

Proof. To prove the result, it is clear that

$$\langle\mathscr{W}^r-\mathscr{X}^*,d^r\rangle=\langle\mathscr{W}^r-\mathscr{Y}^r,d^r\rangle+\langle\mathscr{Y}^r-\mathscr{X}^*,\mathscr{W}^r-\mathscr{Y}^r-\lambda(F(\mathscr{W}^r)-F(\mathscr{Y}^r))\rangle. \tag{3.13}$$

By the fact that $\mathscr{Y}^r=P_{\mathbb{K}}(\mathscr{W}^r-\lambda F(\mathscr{W}^r))$ and (3.2), we can derive

$$\langle\mathscr{Y}^r-\mathscr{W}^r+\lambda F(\mathscr{W}^r),\mathscr{Y}^r-\mathscr{X}^*\rangle\leq 0. \tag{3.14}$$

By the K-pseudomonotonicity of F and $\mathscr{X}^*\in S(\mathbb{K},F)$, we have that

$$\langle F(\mathscr{X}^*),\mathscr{Y}^r-\mathscr{X}^*\rangle\geq 0\Rightarrow\langle F(\mathscr{Y}^r),\mathscr{Y}^r-\mathscr{X}^*\rangle\geq 0. \tag{3.15}$$

Performing operations on (3.14) and (3.15), it follows that

$$\langle\mathscr{W}^r-\mathscr{Y}^r-\lambda(F(\mathscr{W}^r)-F(\mathscr{Y}^r)),\mathscr{Y}^r-\mathscr{X}^*\rangle\geq 0. \tag{3.16}$$

Combining (3.13) and (3.16), we obtain $\langle\mathscr{W}^r-\mathscr{X}^*,d^r\rangle\geq\langle\mathscr{W}^r-\mathscr{Y}^r,d^r\rangle$. Clearly,

$$\begin{aligned}\|\mathscr{Z}^r-\mathscr{X}^*\|^2&=\|\mathscr{W}^r-\mathscr{X}^*\|^2-2\gamma\eta_r\langle\mathscr{W}^r-\mathscr{X}^*,d^r\rangle+\gamma^2\eta_r^2\|d^r\|^2\\&\leq\|\mathscr{W}^r-\mathscr{X}^*\|^2-2\gamma\eta_r\langle\mathscr{W}^r-\mathscr{Y}^r,d^r\rangle+\gamma^2\eta_r^2\|d^r\|^2.\end{aligned} \tag{3.17}$$

By the definition of η_r, we have

$$\eta_r\|d^r\|^2=\langle\mathscr{W}^r-\mathscr{Y}^r,d^r\rangle. \tag{3.18}$$

Substituting (3.18) into (3.17), we can derive

$$\|\mathscr{Z}^r-\mathscr{X}^*\|^2\leq\|\mathscr{W}^r-\mathscr{X}^*\|^2-\frac{2-\gamma}{\gamma}\|\gamma\eta_r d^r\|^2.$$

According to the definition of the sequence $\{\mathscr{Z}^r\}$, we have $\mathscr{Z}^r=\mathscr{W}^r-\gamma\eta_r d^r$. Thus, $\gamma\eta_r d^r=\mathscr{W}^r-\mathscr{Z}^r$, which implies that

$$\|\mathscr{Z}^r-\mathscr{X}^*\|^2\leq\|\mathscr{W}^r-\mathscr{X}^*\|^2-\frac{2-\gamma}{\gamma}\|\mathscr{W}^r-\mathscr{Z}^r\|^2. \tag{3.19}$$

□

Lemma 3.7. *Let $\{\beta_r\}\subset(0,1)$, $\theta_r\subset(a,1-\beta_r)$ for some $a>0$ and $\mathscr{X}^*\in S(\mathbb{K},F)$. Then the sequence $\{\mathscr{X}^r\}$ generated by Algorithm 1 is bounded.*

Proof. By (3.19), it can be derived that $\|\mathscr{Z}^r - \mathscr{X}^*\| \leq \|\mathscr{W}^r - \mathscr{X}^*\|$. Furthermore, by the notion of $\mathscr{W}^r$, we obtain

$$\begin{aligned}\|\mathscr{W}^r - \mathscr{X}^*\| &\leq \|\mathscr{X}^r - \mathscr{X}^*\| + \alpha_r\|\mathscr{X}^r - \mathscr{X}^{r-1}\| \\ &= \|\mathscr{X}^r - \mathscr{X}^*\| + \beta_r \cdot \frac{\alpha_r}{\beta_r}\|\mathscr{X}^r - \mathscr{X}^{r-1}\|.\end{aligned}$$

From the fact $\lim_{r\to\infty} \frac{\alpha_r}{\beta_r}\|\mathscr{X}^r - \mathscr{X}^{r-1}\| = 0$ in Remark 3.1, it can be inferred that there exists a constant $C_1 > 0$ satisfying $\frac{\alpha_r}{\beta_r}\|\mathscr{X}^r - \mathscr{X}^{r-1}\| \leq C_1$ for all $r \geq 1$. It follows that

$$\|\mathscr{Z}^r - \mathscr{X}^*\| \leq \|\mathscr{W}^r - \mathscr{X}^*\| \leq \|\mathscr{X}^r - \mathscr{X}^*\| + \beta_r C_1. \tag{3.20}$$

In addition, we know that

$$\begin{aligned}\|\mathscr{X}^{r+1} - \mathscr{X}^*\| &= \|(1-\theta_r-\beta_r)(\mathscr{X}^r - \mathscr{X}^*) + \theta_r(\mathscr{Z}^r - \mathscr{X}^*) - \beta_r\mathscr{X}^*\| \\ &\leq \|(1-\theta_r-\beta_r)(\mathscr{X}^r - \mathscr{X}^*) + \theta_r(\mathscr{Z}^r - \mathscr{X}^*)\| + \beta_r\|\mathscr{X}^*\|\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}&\|(1-\theta_r-\beta_r)(\mathscr{X}^r - \mathscr{X}^*) + \theta_r(\mathscr{Z}^r - \mathscr{X}^*)\|^2 \\ &\leq (1-\theta_r-\beta_r)^2\|\mathscr{X}^r - \mathscr{X}^*\|^2 + 2(1-\theta_r-\beta_r)\theta_r\|\mathscr{X}^r - \mathscr{X}^*\|\|\mathscr{Z}^r - \mathscr{X}^*\| \\ &\quad + \theta_r^2\|\mathscr{Z}^r - \mathscr{X}^*\|^2 \\ &\leq (1-\theta_r-\beta_r)(1-\beta_r)\|\mathscr{X}^r - \mathscr{X}^*\|^2 + (1-\beta_r)\theta_r\|\mathscr{Z}^r - \mathscr{X}^*\|^2.\end{aligned} \tag{3.22}$$

Substituting (3.20) to (3.22), we have

$$\begin{aligned}&\|(1-\theta_r-\beta_r)(\mathscr{X}^r - \mathscr{X}^*) + \theta_r(\mathscr{Z}^r - \mathscr{X}^*)\|^2 \\ &\leq (1-\theta_r-\beta_r)(1-\beta_r)\|\mathscr{X}^r - \mathscr{X}^*\|^2 + (1-\beta_r)\theta_r(\|\mathscr{X}^r - \mathscr{X}^*\| + \beta_r C_1)^2 \\ &\leq (1-\theta_r-\beta_r)(1-\beta_r)\|\mathscr{X}^r - \mathscr{X}^*\|^2 + (1-\beta_r)\theta_r\|\mathscr{X}^r - \mathscr{X}^*\|^2 \\ &\quad + 2(1-\beta_r)\theta_r\beta_r\|\mathscr{X}^r - \mathscr{X}^*\|C_1 + \beta_r^2 C_1^2 \\ &\leq [(1-\beta_r)\|\mathscr{X}^r - \mathscr{X}^*\| + \beta_r C_1]^2.\end{aligned}$$

Therefore, it concludes that

$$\|(1-\theta_r-\beta_r)(\mathscr{X}^r - \mathscr{X}^*) + \theta_r(\mathscr{Z}^r - \mathscr{X}^*)\| \leq (1-\beta_r)\|\mathscr{X}^r - \mathscr{X}^*\| + \beta_r C_1. \tag{3.23}$$

Based on (3.21) and (3.23), for each $r \geq r_0$, we have

$$\begin{aligned}\|\mathscr{X}^{r+1} - \mathscr{X}^*\| &\leq (1-\beta_r)\|\mathscr{X}^r - \mathscr{X}^*\| + \beta_r(C_1 + \|\mathscr{X}^*\|) \\ &\leq \max\{\|\mathscr{X}^r - \mathscr{X}^*\|, C_1 + \|\mathscr{X}^*\|\}.\end{aligned}$$

By induction, we find that $\{\mathscr{X}^r\}$ is bounded, so are $\{\mathscr{Z}^r\}$ and $\{\mathscr{W}^r\}$. □

Lemma 3.8. *Let $\{\beta_r\} \subset (0,1)$, $\theta_r \subset (a, 1-\beta_r)$ for some $a > 0$ and $\mathscr{X}^* \in S(\mathbb{K}, F)$. Then the sequence $\{\mathscr{X}^r\}$ generated by Algorithm 1 satisfies the following inequality*

$$(1-\beta_r)\theta_r\frac{2-\gamma}{\gamma}\|\mathscr{W}^r - \mathscr{Z}^r\|^2 \leq |\mathscr{X}^r - \mathscr{X}^*\|^2 - |\mathscr{X}^{r+1} - \mathscr{X}^*\|^2 + \beta_r C_4,$$

for some constants $C_4 > 0$.

Proof. First of all, by setting $-2\langle(1-\theta_r-\beta_r)(\mathscr{X}^r-\mathscr{X}^*)+\theta_r(\mathscr{Z}^r-\mathscr{X}^*),\mathscr{X}^*\rangle+\beta_r\|\mathscr{X}^*\|^2\leq C_2$, for some constants $C_2>0$, one has that

$$\begin{aligned}\|\mathscr{X}^{r+1}-\mathscr{X}^*\|^2 &= \|(1-\theta_r-\beta_r)(\mathscr{X}^r-\mathscr{X}^*)+\theta_r(\mathscr{Z}^r-\mathscr{X}^*)-\beta_r\mathscr{X}^*\|^2\\ &= \|(1-\theta_r-\beta_r)(\mathscr{X}^r-\mathscr{X}^*)+\theta_r(\mathscr{Z}^r-\mathscr{X}^*)\|^2\\ &\quad -2\beta_r\langle(1-\theta_r-\beta_r)(\mathscr{X}^r-\mathscr{X}^*)+\theta_r(\mathscr{Z}^r-\mathscr{X}^*),\mathscr{X}^*\rangle+\beta_r^2\|\mathscr{X}^*\|^2\\ &\leq \|(1-\theta_r-\beta_r)(\mathscr{X}^r-\mathscr{X}^*)+\theta_r(\mathscr{Z}^r-\mathscr{X}^*)\|^2+\beta_rC_2,\end{aligned} \tag{3.24}$$

Substituting (3.22) into (3.24) yields

$$\|\mathscr{X}^{r+1}-\mathscr{X}^*\|^2\leq(1-\theta_r-\beta_r)(1-\beta_r)\|\mathscr{X}^r-\mathscr{X}^*\|^2+(1-\beta_r)\theta_r\|\mathscr{Z}^r-\mathscr{X}^*\|^2+\beta_rC_2. \tag{3.25}$$

Combining (3.19) with (3.25), we arrive at

$$\begin{aligned}\|\mathscr{X}^{r+1}-\mathscr{X}^*\|^2 &\leq(1-\theta_r-\beta_r)(1-\beta_r)\|\mathscr{X}^r-\mathscr{X}^*\|^2+(1-\beta_r)\theta_r\|\mathscr{W}^r-\mathscr{X}^*\|^2\\ &\quad -(1-\beta_r)\theta_r\frac{2-\gamma}{\gamma}\|\mathscr{W}^r-\mathscr{Z}^r\|^2+\beta_rC_2.\end{aligned} \tag{3.26}$$

Due to the boundness of $\{\mathscr{X}^r\}$ and $\|\mathscr{W}^r-\mathscr{X}^*\|\leq\|\mathscr{X}^r-\mathscr{X}^*\|+\beta_rC_1$, we have

$$\|\mathscr{W}^r-\mathscr{X}^*\|^2\leq\|\mathscr{X}^r-\mathscr{X}^*\|^2+\beta_rC_3, \tag{3.27}$$

for some constants $C_3>0$. Combining (3.26) with (3.27), it can be concluded that

$$\begin{aligned}\|\mathscr{X}^{r+1}-\mathscr{X}^*\|^2 &\leq(1-\theta_r-\beta_r)(1-\beta_r)\|\mathscr{X}^r-\mathscr{X}^*\|^2+(1-\beta_r)\theta_r\|\mathscr{X}^r-\mathscr{X}^*\|^2\\ &\quad +(1-\beta_r)\theta_r\beta_rC_3-(1-\beta_r)\theta_r\frac{2-\gamma}{\gamma}\|\mathscr{W}^r-\mathscr{Z}^r\|^2+\beta_rC_2\\ &=(1-\beta_r)^2\|\mathscr{X}^r-\mathscr{X}^*\|^2-(1-\beta_r)\theta_r\frac{2-\gamma}{\gamma}\|\mathscr{W}^r-\mathscr{Z}^r\|^2\\ &\quad +\beta_r[(1-\beta_r)\theta_rC_3+C_2]\\ &\leq\|\mathscr{X}^r-\mathscr{X}^*\|^2-(1-\beta_r)\theta_r\frac{2-\gamma}{\gamma}\|\mathscr{W}^r-\mathscr{Z}^r\|^2+\beta_rC_4,\end{aligned}$$

for some constants $C_4>0$. Thus, the desired results hold due to

$$(1-\beta_r)\theta_r\frac{2-\gamma}{\gamma}\|\mathscr{W}^r-\mathscr{Z}^r\|^2\leq|\mathscr{X}^r-\mathscr{X}^*\|^2-|\mathscr{X}^{r+1}-\mathscr{X}^*\|^2+\beta_rC_4. \tag{3.28}$$

□

Lemma 3.9. *Let $\{\beta_r\}\subset(0,1)$, $\theta_r\subset(a,1-\beta_r)$ for some $a>0$, and $\mathscr{X}^*\in S(\mathbb{K},F)$. Then the sequence $\{\mathscr{X}^r\}$ generated by Algorithm 1 satisfies the following inequality*

$$\begin{aligned}\|\mathscr{X}^{r+1}-\mathscr{X}^*\|^2\leq&(1-\beta_r)\|\mathscr{X}^r-\mathscr{X}^*\|^2+\beta_r\Bigg[\frac{\alpha_r}{\beta_r}\|\mathscr{X}^r-\mathscr{X}^{r-1}\|(1-\beta_r)C_5\\ &+2\theta_r\|\mathscr{X}^r-\mathscr{Z}^r\|\|\mathscr{X}^*-\mathscr{X}^{r+1}\|+2\langle\mathscr{X}^*,\mathscr{X}^*-\mathscr{X}^{r+1}\rangle\Bigg],\end{aligned} \tag{3.29}$$

for some constants $C_5>0$.

Proof. Observe that

$$\mathscr{X}^{r+1} = (1-\theta_r-\beta_r)\mathscr{X}^r + \theta_r\mathscr{Z}^r = (1-\theta_r)\mathscr{X}^r + \theta_r\mathscr{Z}^r - \beta_r\mathscr{X}^r.$$

Setting $\mathscr{J}^r = (1-\theta_r)\mathscr{X}^r + \theta_r\mathscr{Z}^r$, we have

$$\begin{aligned}
&\|\mathscr{J}^r - \mathscr{X}^*\|^2 \\
&= \|(1-\theta_r)(\mathscr{X}^r - \mathscr{X}^*) + \theta_r(\mathscr{Z}^r - \mathscr{X}^*)\|^2 \\
&\leq (1-\theta_r)^2\|\mathscr{X}^r - \mathscr{X}^*\|^2 + \theta_r^2\|\mathscr{Z}^r - \mathscr{X}^*\|^2 + 2(1-\theta_r)\theta_r\|\mathscr{X}^r - \mathscr{X}^*\|\|\mathscr{Z}^r - \mathscr{X}^*\| \\
&\leq (1-\theta_r)\|\mathscr{X}^r - \mathscr{X}^*\|^2 + \theta_r\|\mathscr{Z}^r - \mathscr{X}^*\|^2 \\
&\leq (1-\theta_r)\|\mathscr{X}^r - \mathscr{X}^*\|^2 + \theta_r\|\mathscr{W}^r - \mathscr{X}^*\|^2.
\end{aligned} \tag{3.30}$$

In addition, it can be deduced that

$$\begin{aligned}
\|\mathscr{W}^r - \mathscr{X}^*\|^2 &= \|(\mathscr{X}^r - \mathscr{X}^*) + \alpha_r(\mathscr{X}^r - \mathscr{X}^{r-1})\|^2 \\
&= \|(\mathscr{X}^r - \mathscr{X}^*)\|^2 + 2\alpha_r\langle \mathscr{X}^r - \mathscr{X}^*, \mathscr{X}^r - \mathscr{X}^{r-1}\rangle + \alpha_r^2\|\mathscr{X}^r - \mathscr{X}^{r-1}\|^2 \\
&\leq \|(\mathscr{X}^r - \mathscr{X}^*)\|^2 + \alpha_r\|\mathscr{X}^r - \mathscr{X}^{r-1}\|\left[2\|\mathscr{X}^r - \mathscr{X}^*\| + \alpha_r\|\mathscr{X}^r - \mathscr{X}^{r-1}\|\right] \\
&\leq \|(\mathscr{X}^r - \mathscr{X}^*)\|^2 + \alpha_r\|\mathscr{X}^r - \mathscr{X}^{r-1}\|C_5,
\end{aligned} \tag{3.31}$$

where the last inequality holds by setting $2\|\mathscr{X}^r - \mathscr{X}^*\| + \alpha_r\|\mathscr{X}^r - \mathscr{X}^{r-1}\| \leq C_5$, for some constants $C_5 > 0$. Combining (3.30) with (3.31), we obtain that

$$\begin{aligned}
\|\mathscr{J}^r - \mathscr{X}^*\|^2 &\leq (1-\theta_r)\|\mathscr{X}^r - \mathscr{X}^*\|^2 + \theta_r\|\mathscr{X}^r - \mathscr{X}^*\|^2 + \alpha_r\theta_r\|\mathscr{X}^r - \mathscr{X}^{r-1}\|C_5 \\
&\leq \|\mathscr{X}^r - \mathscr{X}^*\|^2 + \alpha_r\|\mathscr{X}^r - \mathscr{X}^{r-1}\|C_5.
\end{aligned} \tag{3.32}$$

By the definition of $\mathscr{J}^r$, we know that $\mathscr{X}^r - \mathscr{J}^r = \theta_r(\mathscr{X}^r - \mathscr{Z}^r)$. Thus

$$\begin{aligned}
\mathscr{X}^{r+1} = \mathscr{J}^r - \beta_r\mathscr{X}^r &= (1-\beta_r)\mathscr{J}^r - \beta_r(\mathscr{X}^r - \mathscr{J}^r) \\
&= (1-\beta_r)\mathscr{J}^r - \beta_r\theta_r(\mathscr{X}^r - \mathscr{Z}^r),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|\mathscr{X}^{r+1} - \mathscr{X}^*\|^2 \\
&= \|(1-\beta_r)(\mathscr{J}^r - \mathscr{X}^*) - (\beta_r\theta_r(\mathscr{X}^r - \mathscr{Z}^r) + \beta_r\mathscr{X}^*)\|^2 \\
&\leq (1-\beta_r)^2\|\mathscr{J}^r - \mathscr{X}^*\|^2 - 2\langle \beta_r\theta_r(\mathscr{X}^r - \mathscr{Z}^r) + \beta_r\mathscr{X}^*, \mathscr{J}^r - \mathscr{X}^*\rangle \\
&\quad + 2\|\beta_r\theta_r(\mathscr{X}^r - \mathscr{Z}^r) + \beta_r\mathscr{X}^*\|^2 \\
&= (1-\beta_r)^2\|\mathscr{J}^r - \mathscr{X}^*\|^2 - 2\langle \beta_r\theta_r(\mathscr{X}^r - \mathscr{Z}^r) + \beta_r\mathscr{X}^*, \mathscr{X}^{r+1} - \mathscr{X}^*\rangle \\
&\leq (1-\beta_r)\|\mathscr{J}^r - \mathscr{X}^*\|^2 + 2\beta_r\theta_r\|\mathscr{X}^r - \mathscr{Z}^r\|\|\mathscr{X}^* - \mathscr{X}^{r+1}\| \\
&\quad + 2\beta_r\langle \mathscr{X}^*, \mathscr{X}^* - \mathscr{X}^{r+1}\rangle.
\end{aligned} \tag{3.33}$$

Moreover, combining (3.32) with (3.33), we have

$$\begin{aligned}\|\mathscr{X}^{r+1}-\mathscr{X}^*\|^2 \leq\ & (1-\beta_r)\|\mathscr{X}^r-\mathscr{X}^*\|^2+(1-\beta_r)\alpha_r\|\mathscr{X}^r-\mathscr{X}^{r-1}\|C_5\\ &+2\beta_r\theta_r\|\mathscr{X}^r-\mathscr{Z}^r\|\|\mathscr{X}^*-\mathscr{X}^{r+1}\|+2\beta_r\langle\mathscr{X}^*,\mathscr{X}^*-\mathscr{X}^{r+1}\rangle\\ =\ & (1-\beta_r)\|\mathscr{X}^r-\mathscr{X}^*\|^2+\beta_r\Bigg[\frac{\alpha_r}{\beta_r}\|\mathscr{X}^r-\mathscr{X}^{r-1}\|(1-\beta_r)C_5\\ &+2\theta_r\|\mathscr{X}^r-\mathscr{Z}^r\|\|\mathscr{X}^*-\mathscr{X}^{r+1}\|+2\langle\mathscr{X}^*,\mathscr{X}^*-\mathscr{X}^{r+1}\rangle\Bigg].\end{aligned}$$

□

By Lemmas 3.4-3.9, we now show that the proposed algorithm converges strongly to the solution with the least norm.

Theorem 3.1. *Let assumptions $(i)-(iii)$ hold. Then the sequence $\{\mathscr{X}^r\}$ generated by Algorithm 1 converges strongly to $\mathscr{X}^*\in S(\mathbb{K},F)$, where*

$$\|\mathscr{X}^*\|=\min\{\|\mathscr{X}\|:\mathscr{X}\in S(\mathbb{K},F)\}. \tag{3.34}$$

Proof. To prove the desired result, it is sufficient to show that $\{\|\mathscr{X}^r-\mathscr{X}^*\|^2\}$ converges to zero. To do that, we assume that $\{\|\mathscr{X}^{r_n}-\mathscr{X}^*\|\}$ is a subsequence of $\{\|\mathscr{X}^r-\mathscr{X}^*\|\}$ such that

$$\liminf_{n\to\infty}(\|\mathscr{X}^{r_n+1}-\mathscr{X}^*\|-\|\mathscr{X}^{r_n}-\mathscr{X}^*\|)\geq 0.$$

By Lemma 3.5 and Lemma 3.9, it is sufficient to show that

$$\begin{aligned}\limsup_{n\to\infty}\Big(&\frac{\alpha_{r_n}}{\beta_{r_n}}\|\mathscr{X}^{r_n}-\mathscr{X}^{r_n-1}\|(1-\beta_{r_n})C_5+2\theta_{r_n}\|\mathscr{X}^{r_n}-\mathscr{Z}^{r_n}\|\|\mathscr{X}^*-\mathscr{X}^{r_n+1}\|\\ &+2\langle\mathscr{X}^*,\mathscr{X}^*-\mathscr{X}^{r_n+1}\rangle\Big)\leq 0.\end{aligned}$$

Observe that

$$\begin{aligned}&\liminf_{n\to\infty}(\|\mathscr{X}^{r_n+1}-\mathscr{X}^*\|^2-\|\mathscr{X}^{r_n}-\mathscr{X}^*\|^2)\\ &=\liminf_{n\to\infty}\left[(\|\mathscr{X}^{r_n+1}-\mathscr{X}^*\|-\|\mathscr{X}^{r_n}-\mathscr{X}^*\|)(\|\mathscr{X}^{r_n+1}-\mathscr{X}^*\|+\|\mathscr{X}^{r_n}-\mathscr{X}^*\|)\right]\geq 0.\end{aligned}$$

By (3.28), we have

$$\begin{aligned}&\limsup_{n\to\infty}\left[(1-\beta_{r_n})\theta_{r_n}\frac{2-\gamma}{\gamma}\|\mathscr{W}^{r_n}-\mathscr{Z}^{r_n}\|^2\right]\\ &\leq\limsup_{n\to\infty}\left[\|\mathscr{X}^{r_n}-\mathscr{X}^*\|^2-\|\mathscr{X}^{r_n+1}-\mathscr{X}^*\|^2+\beta_{r_n}C_4\right]\\ &\leq\limsup_{n\to\infty}\left[\|\mathscr{X}^{r_n}-\mathscr{X}^*\|^2-\|\mathscr{X}^{r_n+1}-\mathscr{X}^*\|^2\right]+\limsup_{n\to\infty}\beta_{r_n}C_4\\ &=-\liminf_{n\to\infty}\left[\|\mathscr{X}^{r_n+1}-\mathscr{X}^*\|^2-\|\mathscr{X}^{r_n}-\mathscr{X}^*\|^2\right]\leq 0,\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|\mathscr{W}^{r_n}-\mathscr{Z}^{r_n}\|=0. \tag{3.35}$$

On the other hand, we prove that $\lim_{n\to\infty}\|\mathscr{W}^{r_n}-\mathscr{Y}^{r_n}\|=0$. By the definition of d^r and the Lipschitz continuity of F, it can be deduced that

$$\begin{aligned}\|d^r\| &= \|\mathscr{W}^r-\mathscr{Y}^r-\lambda(F(\mathscr{W}^r)-F(\mathscr{Y}^r))\| \\ &\le \|\mathscr{W}^r-\mathscr{Y}^r\|+\lambda\|F(\mathscr{W}^r)-F(\mathscr{Y}^r)\| \\ &\le (1+\lambda L)\|\mathscr{W}^r-\mathscr{Y}^r\|.\end{aligned}$$

Then

$$\frac{1}{\|d^r\|}\ge\frac{1}{(1+\lambda L)\|\mathscr{W}^r-\mathscr{Y}^r\|}. \tag{3.36}$$

Furthermore,

$$\begin{aligned}\langle\mathscr{W}^r-\mathscr{Y}^r,d^r\rangle &= \|\mathscr{W}^r-\mathscr{Y}^r\|^2-\langle\mathscr{W}^r-\mathscr{Y}^r,\lambda(F(\mathscr{W}^r)-F(\mathscr{Y}^r))\rangle \\ &\ge (1-\lambda L)\|\mathscr{W}^r-\mathscr{Y}^r\|^2.\end{aligned} \tag{3.37}$$

Thus, due to the definition of $\mathscr{Z}^r$ and η_r and equations (3.36) and (3.37), we have

$$\|\mathscr{Z}^r-\mathscr{W}^r\|=\gamma\eta_r\|d^r\|=\gamma\frac{\langle\mathscr{W}^r-\mathscr{Y}^r,d_r\rangle}{\|d_r\|}\ge\gamma\frac{1-\lambda L}{1+\lambda L}\|\mathscr{W}^r-\mathscr{Y}^r\|. \tag{3.38}$$

Combining (3.35) with (3.38), we obtain that

$$\lim_{n\to\infty}\|\mathscr{W}^{r_n}-\mathscr{Y}^{r_n}\|=0. \tag{3.39}$$

Next, we aims to prove $\|\mathscr{X}^{r_n+1}-\mathscr{X}^{r_n}\|\to 0$, as $n\to\infty$. By the definition of $\mathscr{W}^r$ and $\lim_{r\to\infty}\frac{\alpha_r}{\beta_r}\|\mathscr{X}^r-\mathscr{X}^{r-1}\|=0$, we have

$$\|\mathscr{X}^{r_n}-\mathscr{W}^{r_n}\|=\alpha_{r_n}\|\mathscr{X}^{r_n}-\mathscr{X}^{r_n-1}\|=\beta_{r_n}\cdot\frac{\alpha_{r_n}}{\beta_{r_n}}\|\mathscr{X}^{r_n}-\mathscr{X}^{r_n-1}\|\to 0, \tag{3.40}$$

as $n\to\infty$. Therefore, by (3.35) and (3.40), we have $\|\mathscr{X}^{r_n}-\mathscr{Z}^{r_n}\|\to 0$. From definitions of $\mathscr{X}^{r+1}$ and $\{\beta_r\}$, one has

$$\|\mathscr{X}^{r_n+1}-\mathscr{X}^{r_n}\|\le\theta_{r_n}\|\mathscr{X}^{r_n}-\mathscr{Z}^{r_n}\|+\beta_{r_n}\|\mathscr{X}^{r_n}\|\to 0.$$

Finally, we prove $\limsup_{n\to\infty}\langle\mathscr{X}^*,\mathscr{X}^*-\mathscr{X}^{r_n+1}\rangle\le 0$. Since $\{\mathscr{X}^{r_n}\}$ is bounded, one sees that there exists a subsequence $\{\mathscr{X}^{r_{n_j}}\}$ of $\{\mathscr{X}^{r_n}\}$, which converges weakly to some $\mathscr{X}'\in\mathscr{T}_{M,n}$, such that

$$\limsup_{n\to\infty}\langle\mathscr{X}^*,\mathscr{X}^*-\mathscr{X}^{r_n}\rangle=\lim_{j\to\infty}\langle\mathscr{X}^*,\mathscr{X}^*-\mathscr{X}^{r_{n_j}}\rangle=\langle\mathscr{X}^*,\mathscr{X}^*-\mathscr{X}'\rangle.$$

According to $\mathscr{X}^{r_{n_j}}\rightharpoonup\mathscr{X}'$ and $\|\mathscr{X}^{r_n}-\mathscr{W}^{r_n}\|\to 0$, we know that $\mathscr{W}^{r_{n_j}}\rightharpoonup\mathscr{X}'$. By Lemma 3.4, it follows that

$$\|\mathscr{W}^{r_n}-\mathscr{Y}^{r_n}\|=\|\mathscr{W}^{r_n}-P_{\mathbb{K}}(\mathscr{W}^{r_n}-\lambda F(\mathscr{W}^{r_n}))\|\to 0,$$

and $\mathscr{X}'\in S(\mathbb{K},F)$. From $\|\mathscr{X}^*\|=\min\{\|\mathscr{X}\|:\mathscr{X}\in S(\mathbb{K},\mathbb{F})\}$ and the properties of projection, we can derive $\mathscr{X}^*=P_{S(\mathbb{K},F)}(0)$. By (3.2), we have

$$\limsup_{n\to\infty}\langle\mathscr{X}^*,\mathscr{X}^*-\mathscr{X}^{r_n}\rangle=\langle\mathscr{X}^*,\mathscr{X}^*-\mathscr{X}'\rangle\le 0.$$

By $\|\mathscr{X}^{r_n+1}-\mathscr{X}^{r_n}\|\to 0$, it holds that

$$\limsup_{n\to\infty}\langle\mathscr{X}^*,\mathscr{X}^*-\mathscr{X}^{r_n+1}\rangle\le 0. \tag{3.41}$$

Thus

$$\limsup_{n\to\infty}\Big(\frac{\alpha_{r_n}}{\beta_{r_n}}\|\mathscr{X}^{r_n}-\mathscr{X}^{r_n-1}\|(1-\beta_{r_n})C_5+2\theta_{r_n}\|\mathscr{X}^{r_n}-\mathscr{Z}^{r_n}\|\|\mathscr{X}^*-\mathscr{X}^{r_n+1}\| \\ +2\langle\mathscr{X}^*,\mathscr{X}^*-\mathscr{X}^{r_n+1}\rangle\Big)\leq 0.$$

Combining this with Lemma 3.5, we obtain that

$$\lim_{r\to\infty}\|\mathscr{X}^r-\mathscr{X}^*\|^2=\lim_{r\to\infty}\|\mathscr{X}^r-\mathscr{X}^*\|=0,$$

which implies that $\{\mathscr{X}^r\}$ converges strongly to $\mathscr{X}^*$. The desired results hold. □

4. Dynamic Oligopolistic Market Equilibrium

In this section, we consider an economic model comprising m firms $S_i (i\in[m])$ producing $k\in[l]$ different commodities, and n demand markets $D_j (j\in[n])$, typically spatially separated. Suppose that, within a time period $[0,T], T>0$, commodities produced by firm S_i are consumed by demand market D_j. For the sake of simplicity, for $i\in[m]$, $j\in[n]$, and $k\in[l]$, we introduce the following notations:

- $x_{ij}^k(t)$ denotes the nonnegative shipment quantity of commodity k from supplier S_i to demand market D_j at time $t\in[0,T]$.
- $\varepsilon_i^k(t)$ represents the surplus production of commodity k by producer S_i at time $t\in[0,T]$.
- $\delta_j^k(t)$ represents the excess demand for commodity k by demand market D_j at time $t\in[0,T]$.
- $p_i^k(t)$ indicates the production quantity of commodity k by producer S_i at time $t\in[0,T]$.
- $q_j^k(t)$ indicates the demand quantity of commodity k by demand market D_j at time $t\in[0,T]$.

We organize the production output into a matrix-function $P:[0,T]\to\mathbb{R}_+^{ml}$, the demand output into a matrix-function $Q:[0,T]\to\mathbb{R}_+^{nl}$, the commodity shipments into a tensor-function $\mathscr{X}:[0,T]\to\mathscr{T}_{mnl}$, the production surplus into a matrix-function $\varepsilon:[0,T]\to\mathbb{R}_+^{ml}$, and the demand excess into a matrix-function $\delta:[0,T]\to\mathbb{R}_+^{nl}$. Moreover, we assume that the following feasibility conditions hold:

$$\begin{aligned} p_i^k(t)&=\sum_{j=1}^n x_{ij}^k(t)+\varepsilon_i^k(t),\forall i\in[m];\ \forall k\in[l],\ \text{a.e. in [0,T]},\\ q_j^k(t)&=\sum_{i=1}^m x_{ij}^k(t)+\delta_j^k(t),\forall j\in[n];\ \forall k\in[l],\ \text{a.e. in [0,T]}.\end{aligned}\tag{4.1}$$

Therefore, at time $t\in[0,T]$, the quantity produced by each firm S_i of type k must equal the commodity shipments of that type from the firm to all demand markets plus the production surplus. Similarly, the quantity demanded by each demand market D_j of type k must equal the shipments of that type from all firms to the demand market plus the demand excess, at time $t\in[0,T]$. Consequently, the total production p_i of firm S_i and the total demand q_j of demand market D_j are given by

$$p_i=\sum_{k=1}^l\left[\sum_{j=1}^n x_{ij}^k(t)+\varepsilon_i^k(t)\right],\forall i\in[m],\ q_j=\sum_{k=1}^l\left[\sum_{i=1}^m x_{ij}^k(t)+\delta_j^k(t)\right],\forall j\in[n],\tag{4.2}$$

respectively.

For analytical convenience, we make the following assumptions

$$\varepsilon \in L^2([0,T],\mathbb{R}^{ml}_+), \quad \delta \in L^2([0,T],\mathbb{R}^{nl}_+), \quad P \in L^2([0,T],\mathbb{R}^{ml}_+), \quad Q \in L^2([0,T],\mathbb{R}^{nl}_+).$$

Therefore, the feasible set is defined as

$$\begin{aligned}\widetilde{\mathbb{K}} = \Big\{ & (\mathscr{X},\varepsilon,\delta) \in L^2([0,T],\mathscr{T}_{mnl} \times \mathbb{R}^{ml} \times \mathbb{R}^{nl}) : \\ & \underline{x}^k_{ij}(t) \leq x^k_{ij}(t) \leq \overline{x}^k_{ij}(t),\ \forall i \in [m],\ \forall j \in [n],\ \forall k \in [l],\ \text{a.e. in } [0,T], \\ & \varepsilon^k_i(t) \geq 0,\ \forall i \in [m],\ \forall k \in [l],\ \text{a.e. in } [0,T], \\ & p^k_i(t) = \sum_{j=1}^{n} x^k_{ij}(t) + \varepsilon^k_i(t),\ \forall i \in [m],\ \forall k \in [l],\ \text{a.e. in } [0,T], \\ & \delta^k_j(t) \geq 0,\ \forall j \in [n],\ \forall k \in [l],\ \text{a.e. in } [0,T], \\ & q^k_j(t) = \sum_{i=1}^{m} x^k_{ij}(t) + \delta^k_j(t),\ \forall j \in [n],\ \forall k \in [l],\ \text{a.e. in } [0,T] \Big\},\end{aligned}$$

where $\underline{x}^k_{ij}(t), \overline{x}^k_{ij}(t)$ are nonnegative bounds. It is evident that $\widetilde{\mathbb{K}}$ is a convex closed and bounded subset of $\mathscr{T}_{mnl}$. Furthermore, we define the following concepts, where $i \in [m], j \in [n], k \in [l]$:

- $\widetilde{f}^k_i$ represents the production cost for firm S_i to produce the k-th commodity. We assume that the production cost for firm S_i depends on the entire production pattern, as follows:
$$\widetilde{f}^k_i = \widetilde{f}^k_i(t,\mathscr{X}(t),\varepsilon(t)); \tag{4.3}$$
- $\widetilde{d}^k_j$ represents the unit demand price for the k-th commodity at demand market D_j. We assume that the demand price for demand market D_j depends on the entire consumption pattern, as follows:
$$\widetilde{d}^k_j = \widetilde{d}^k_j(t,\mathscr{X}(t),\delta(t)); \tag{4.4}$$
- $\widetilde{g}^k_i$ represents the cost for firm S_i to store the k-th commodity. We assumed that this cost depends on the entire production pattern, as follows:
$$\widetilde{g}^k_i = \widetilde{g}^k_i(t,\mathscr{X}(t),\varepsilon(t)); \tag{4.5}$$
- $\widetilde{c}^k_{ij}$ represents the cost to transport the k-th commodity from producer S_i to demander D_j. We assume that the transaction cost depends on the entire shipment pattern, as follows:
$$\widetilde{c}^k_{ij} = \widetilde{c}^k_{ij}(t,\mathscr{X}(t)). \tag{4.6}$$

The profit $\widetilde{v}_i(t,\mathscr{X}(t),\varepsilon(t),\delta(t))$, $i \in [m]$ of the firm S_i at the time $t \in [0,T]$, is given by

$$\begin{aligned}\widetilde{v}_i(t,\mathscr{X}(t),\varepsilon(t),\delta(t)) = \sum_{k=1}^{l} \Bigg[& \sum_{j=1}^{n} \widetilde{d}^k_j(t,\mathscr{X}(t),\delta(t)) x^k_{ij}(t) - \widetilde{f}^k_i(t,\mathscr{X}(t),\varepsilon(t)) \\ & - \widetilde{g}^k_i(t,\mathscr{X}(t),\varepsilon(t)) - \sum_{j=1}^{n} \widetilde{c}^k_{ij}(t,\mathscr{X}(t)) x^k_{ij}(t) \Bigg].\end{aligned}$$

Now, we rewrite $\widetilde{\mathbb{K}}$ equivalently. According to (4.1), we can express $\varepsilon_i^k(t)$ in terms of $p_i^k(t)$ and $x_{ij}^k(t)$, and $\delta_j^k(t)$ in terms of $q_j^k(t)$ and $x_{ij}^k(t)$, namely:

$$\begin{aligned} \varepsilon_i^k(t) &= p_i^k(t) - \sum_{j=1}^n x_{ij}^k(t),\ i \in [m],\ \text{a.e.in } [0,T], \\ \delta_j^k(t) &= q_j^k(t) - \sum_{i=1}^m x_{ij}^k(t),\ j \in [n],\ \text{a.e.in } [0,T]. \end{aligned} \tag{4.7}$$

Then, the equivalent feasible set $\mathbb{K}$ is obtained by substituting these expressions into $\widetilde{\mathbb{K}}$,

$$\begin{aligned} \mathbb{K} = \Big\{ &\mathscr{X} \in L^2([0,T],\mathscr{T}_{mnl}) : \\ &\underline{x}_{ij}^k(t) \le x_{ij}^k(t) \le \overline{x}_{ij}^k(t),\ \forall i \in [m],\ \forall j \in [n],\ \forall k \in [l],\ \text{a.e. in } [0,T], \\ &\sum_{j=1}^n x_{ij}^k(t) \le p_i^k(t),\ \forall i \in [m],\ \forall k \in [l],\ \text{a.e. in } [0,T], \\ &\sum_{i=1}^m x_{ij}^k(t) \le q_j^k(t),\ \forall j \in [n],\ \forall k \in [l],\ \text{a.e. in } [0,T] \Big\}. \end{aligned}$$

It can be observed that $\mathbb{K}$ encompasses the production and demand surpluses outlined by $\widetilde{\mathbb{K}}$. Subsequently, considering (4.7) along with (4.3)-(4.5), the costs of production, demand, and storage are formulated, as follows,

$$f_i^k = f_i^k(t,\mathscr{X}(t)) = \widetilde{f}_i^k(t,\mathscr{X}(t),\varepsilon(t)),\ d_j^k = d_j^k(t,\mathscr{X}(t)) = \widetilde{d}_j^k(t,\mathscr{X}(t),\delta(t)),$$
$$g_i^k = g_i^k(t,\mathscr{X}(t)) = \widetilde{g}_i^k(t,\mathscr{X}(t),\varepsilon(t)).$$

Similarly, the profit function is

$$\begin{aligned} v_i(t,\mathscr{X}(t)) &= \widetilde{v}_i(t,\mathscr{X}(t),\varepsilon(t),\delta(t)) \\ &= \sum_{k=1}^l \left[\sum_{j=1}^n d_j^k(t,\mathscr{X}(t)) x_{ij}^k(t) - f_i^k(t,\mathscr{X}(t)) - g_i^k(t,\mathscr{X}(t)) - \sum_{j=1}^n c_{ij}^k(t,\mathscr{X}(t)) x_{ij}^k(t) \right]. \end{aligned}$$

In the dynamic oligopoly market model, each company supplies goods in a non-cooperative manner at time $t \in [0,T]$, and they all aim to maximize their profit function within the optimal allocation model of other companies. Our objective is to determine a non-negative commodity distribution tensor function $\mathscr{X}$, where m producers and n demand markets will be in an equilibrium state defined by a generalization of the Cournot-Nash equilibrium principle.

Definition 4.1. A feasible tensor function $\mathscr{X}^* \in \mathbb{K}$ is a dynamic oligopolistic market equilibrium if and only if, for each $i = 1,\cdots,m$ and a.e. in [0,T],

$$v_i(t,\mathscr{X}^*(t)) \ge v_i(t,X_i(t),\widehat{\mathscr{X}_i}^*(t)), \tag{4.8}$$

where $\widehat{\mathscr{X}_i}^*(t) = (X_1^*(t),\cdots,X_{i-1}^*(t),X_{i+1}^*(t),\cdots,X_m^*(t))$, $X_i(t)$ is a slice of dimension nl.

To obtain an equivalent formulation of Definition 4.1 characterized by tensor variational inequality, we suppose that the following conditions hold on the profit function v_i and the tensor-function $\nabla_D v = \left(\frac{\partial v_i}{\partial x_{ij}^k}\right)_{ijk}$, where $i = 1 \in [m]$, $j \in [n]$, and $k \in [l]$:

(i) $v_i(t,\mathscr{X}(t))$ is continuously differentiable for each $i\in[m]$, a.e. in [0,T],
(ii) $\nabla_D v$ is a Carathéodory function such that

$$\exists\, h\in L^2([0,T],\mathbb{R}):\|\nabla_D v(t,\mathscr{X}(t))\|\le h(t)\|\mathscr{X}(t)\|,\ \text{a.e. in [0,T]}, \tag{4.9}$$

(iii) $v_i(t,\mathscr{X}(t))$ is pseudoconcave with respect to the variable $\mathscr{X}\in\mathscr{T}_{mnl}$, $i\in[m]$, namely the following condition holds a.e. in [0,T]:

$$\left\langle \frac{\partial v_i}{\partial X_i}(t,X_1,\cdots,X_i,\cdots,X_m),X_i-Y_i\right\rangle\ge 0\Rightarrow v_i(t,X_1,\cdots,X_i,\cdots,X_m)\ge v_i(t,X_1,\cdots,Y_i,\cdots,X_m).$$

Theorem 4.1. *Let the assumptions $(i)-(iii)$ hold. Then $\mathscr{X}^*\in\mathbb{K}$ is a dynamic oligopolistic market equilibrium if and only if it satisfies the evolutionary variational inequality*

$$\langle\langle -\nabla_D v(t,\mathscr{X}^*(t)),\mathscr{X}-\mathscr{X}^*\rangle\rangle\ge 0,\qquad \forall\mathscr{X}\in\mathbb{K}. \tag{4.10}$$

Proof. First of all, we prove the equivalence between (4.10) and

$$\langle -\nabla_D v(t,\mathscr{X}^*(t)),\mathscr{X}(t)-\mathscr{X}^*(t)\rangle=-\sum_{i=1}^m\sum_{j=1}^n\sum_{k=1}^l\frac{\partial v_i(t,\mathscr{X}^*(t))}{\partial x_{ij}^k(t)}(x_{ij}^k(t)-(x_{ij}^k(t))^*)\ge 0, \tag{4.11}$$

$\forall\mathscr{X}(t)\in\mathbb{K}(t)$, a.e. in [0,T], where

$$\begin{aligned}\mathbb{K}(t)=\Big\{&\mathscr{X}(t)\in\mathscr{T}_{mnl}:\underline{x}_{ij}^k(t)\le x_{ij}^k(t)\le\overline{x}_{ij}^k(t),\ \forall i\in[m],\ \forall j\in[n],\ \forall k\in[l],\\ &\sum_{j=1}^n x_{ij}^k(t)\le p_i^k(t),\ \forall i\in[m],\ \forall k\in[l],\sum_{i=1}^m x_{ij}^k(t)\le q_j^k(t),\ \forall j\in[n],\ \forall k\in[l]\Big\}.\end{aligned} \tag{4.12}$$

Indeed, if (4.11) is not true, then, for any $-\nabla_D v(t,\mathscr{X}^*(t))\in\mathscr{T}_{mnl}$, there exist $\mathscr{X}_1(t)\in\mathbb{K}(t)$ and a measurable set $J\subseteq[0,T]$ with Lebesgue measure $\mu(J)>0$ such that

$$\langle -\nabla_D v(t,\mathscr{X}^*(t)),\mathscr{X}_1(t)-\mathscr{X}^*(t)\rangle<0,\ \forall t\in J.$$

Let

$$\widetilde{\mathscr{X}}(t)=\begin{cases}\mathscr{X}^*(t) & \text{in [0,T]}\setminus J,\\ \mathscr{X}_1(t) & \text{in } J.\end{cases}$$

Then

$$\begin{aligned}\langle\langle -\nabla_D v(t,\mathscr{X}^*(t)),\widetilde{\mathscr{X}}(t)-\mathscr{X}^*\rangle\rangle&=\int_{[0,T]\setminus J}\langle -\nabla_D v(t,\mathscr{X}^*(t)),\mathscr{X}^*(t)-\mathscr{X}^*(t)\rangle dt\\ &\quad+\int_J\langle -\nabla_D v(t,\mathscr{X}^*(t)),\mathscr{X}_1(t)-\mathscr{X}^*(t)\rangle dt<0.\end{aligned}$$

Thus we have the equivalence between (4.10) and (4.11). This equivalence is crucial for constructing discretization schemes aimed at computing numerical solutions to the equilibrium problem in dynamic oligopolistic markets. Next, we proceed to prove this theorem further.

To move on, we suppose that $\mathscr{X}^*(t)$ is a solution of (4.8). We now prove that $\mathscr{X}^*(t)$ is a solution to (4.10). By the equivalence between (4.10) and (4.11), we subsequently demonstrate the validity of (4.11). Since $\nabla_D v(t,\mathscr{X}^*(t))$ is a Carathéodory function, (4.9) holds and

$\mathscr{X}, \mathscr{X}^* \in L^2([0,T], \mathscr{T}_{mnl})$, we have that $t \mapsto \langle \nabla_D v(t, \mathscr{X}^*(t)), \mathscr{X} - \mathscr{X}^* \rangle \in L^2([0,T], \mathbb{R})$ and

$$\begin{aligned}\langle -\nabla_D v(t, \mathscr{X}^*(t)), \mathscr{X}(t) - \mathscr{X}^*(t) \rangle &= \sum_{i=1}^{m} \left\langle -\frac{\partial v_i(t, \mathscr{X}^*(t))}{\partial X_i^*(t)}, X_i(t) - X_i^*(t) \right\rangle \\ &= -\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} \frac{\partial v_i(t, \mathscr{X}^*(t))}{\partial x_{ij}^k(t)} (x_{ij}^k(t) - (x_{ij}^k(t))^*) \geq 0,\end{aligned}$$

which yields (4.10).

Conversely, we assume that $\mathscr{X}^*(t)$ is a solution of (4.10) but not a solution to (4.8). This means that there exists $\bar{i}$ such that $v_{\bar{i}}(t, \mathscr{X}^*(t)) < v_{\bar{i}}(t, X_{\bar{i}}(t), \widehat{\mathscr{X}_{\bar{i}}^*}(t))$. Since that $v_i(t, \mathscr{X}(t))$ is pseudoconcave, it follows that

$$\langle -\nabla_D v(t, \mathscr{X}^*(t)), \mathscr{X}(t) - \mathscr{X}^*(t) \rangle < 0, \text{ a.e. in } [0,T].$$

Thus

$$\langle\langle -\nabla_D v(t, \mathscr{X}^*(t)), \mathscr{X} - \mathscr{X}^* \rangle\rangle < 0, \ \forall\ \mathscr{X} \in \mathbb{K},$$

which contradicts the assumed conditions, and the desired results hold. □

In what follows, we present the conditions under which the solution to the dynamic oligopolistic market with both production and demand excess remain continuous over time. Before that, we revisit the classical concept of convergence for subsets of a given metric space (X,d), introduced in [22, 23].

Let $\{K_n\}_{n\in\mathbb{N}}$ denote a sequence of subsets in X, where sequences are indexed by the elements of $\mathbb{N}$, the set of positive integers. Recall that

$$d - \underline{\lim}_n K_n := \left\{ x \in X : \exists \{x_n\}_{n\in\mathbb{N}} \text{ eventually in } K_n \text{ such that } x_n \xrightarrow{d} x \right\},$$

$$d - \overline{\lim}_n K_n := \left\{ x \in X : \exists \{x_n\}_{n\in\mathbb{N}} \text{ frequently in } K_n \text{ such that } x_n \xrightarrow{d} x \right\},$$

where eventually denotes that there exists $\delta \in \mathbb{N}$ such that $x_n \in K_n$ for any $n \geq \delta$, and frequently denotes the existence of an infinite subset $N \subseteq \mathbb{N}$ such that $x_n \in K_n$ for any $n \in N$ (in the latter case, according to the notation provided above, we also state the existence of a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}} \subseteq \{x_n\}_{n\in\mathbb{N}}$ such that $x_{k_n} \in K_{k_n}$). Next, we review the Kuratowski's convergence of sets.

Definition 4.2. [22, 23] The sequence $\{K_n\}_{n\in\mathbb{N}}$ converges to some subset $K \subseteq X$ in Kuratowski's sense, briefly write $K_n \to K$, if and only if $d - \underline{\lim}_n K_n = d - \overline{\lim}_n K_n = K$.

Thus, in order to verify that $K_n \to K$, it suffices to check that

(i) $K \subset d - \underline{\lim}_n K_n$, i.e., for any $x \in K$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ eventually in K_n such that $x_n \xrightarrow{d} x$;

(ii) $d - \overline{\lim}_n K_n \subseteq K$, i.e., for any sequence $\{x_n\}_{n\in\mathbb{N}}$ frequently in K_n such that $x_n \xrightarrow{d} x$ for some $x \in K_n$, then $x \in K$.

Moreover, Definition 4.2 can also be expressed as follows.

Remark 4.1. [11] Let (X,d) be a metric space and K be a nonempty, closed and convex subset of X. A sequence of nonempty, closed and convex sets K_n of X converges to K in Kuratowski's sense, as $n \to +\infty$, that is, $K_n \to K$, if and only if

$(M1)$ for any $x \in K$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to $x \in X$ such that x_n lies in K_n for all $n \in N$;

$(M2)$ for any subsequence $\{x_n\}_{n\in\mathbb{N}}$ converging to $x \in X$ such that x_n lies in K_n for all $n \in \mathbb{N}$, $x \in K$.

The following Lemma demonstrates that the feasible set $\mathbb{K}$ of the dynamic oligopolistic market problem satisfies the property of set convergence in the Kuratowski's sense in the case of excess production and demand.

Lemma 4.1. *Let* $\underline{\mathscr{X}}, \overline{\mathscr{X}} \in C^0([0,T], \mathscr{T}_{mnl})$, $P \in C^0([0,T], \mathbb{R}^{ml}_+)$, $Q \in C^0([0,T], \mathbb{R}^{nl}_+)$ *and let* $\{t_r\}_{r\in\mathbb{N}}$ *be a sequence such that* $t_r \in [0,T]$, *for all* $r \in \mathbb{N}$ *and* $t_r \to t$, *with* $t \in [0,T]$, *as* $r \to +\infty$. *Then the sequence of sets*

$$\mathbb{K}(t_r) = \Bigg\{ \mathscr{X}(t_r) \in \mathscr{T}_{mnl} : \underline{x}^k_{ij}(t_r) \le x^k_{ij}(t_r) \le \overline{x}^k_{ij}(t_r),\ \forall i \in [m],\ \forall j \in [n],\ \forall k \in [l],$$
$$\sum_{j=1}^{n} x^k_{ij}(t_r) \le p^k_i(t_r),\ \forall i \in [m], \sum_{i=1}^{m} x^k_{ij}(t_r) \le q^k_j(t_r),\ \forall j \in [n] \Bigg\},$$

for all $r \in \mathbb{N}$, *converges to* (4.12).

Proof. To show (M_1) holds, we let $\{t_r\}_{r\in\mathbb{N}}$ be a sequence such that $t_r \to t$, with $t \in [0,T]$, as $r \to +\infty$. Due to the continuity of $\underline{\mathscr{X}}$, $\overline{\mathscr{X}}$, P, Q, it follows that $\underline{\mathscr{X}}(t_r) \to \underline{\mathscr{X}}(t)$, $\overline{\mathscr{X}}(t_r) \to \overline{\mathscr{X}}(t)$, $P(t_r) \to P(t)$, and $Q(t_r) \to Q(t)$ as $r \to +\infty$, respectively. Let $\mathscr{X}(t) \in \mathbb{K}(t)$ be fixed. For $i \in [m]$, $j \in [n]$ and $k \in [l]$, if

$$a^k_{ij}(t_r) = x^k_{ij}(t) - \underline{x}^k_{ij}(t_r) + \frac{mp^k_i(t_r) + nq^k_j(t_r)}{mn} - \frac{mp^k_i(t) + nq^k_j(t)}{mn},$$

then

$$\lim_{r\to+\infty} a^k_{ij}(t_r) = x^k_{ij}(t) - \underline{x}^k_{ij}(t) \ge 0. \tag{4.13}$$

Thus, there exists an index w_1 such that, for $r > w_1$,

$$a^k_{ij}(t_r) \ge 0,\ \forall\, i \in [m],\ j \in [n],\ k \in [l]. \tag{4.14}$$

Then, we have

$$\lim_{r\to+\infty} \left[\frac{1}{m} \sum_{j=1}^{n} q^k_j(t_r) - \frac{1}{m} \sum_{j=1}^{n} q^k_j(t) - \varepsilon^k_i(t) \right] = -\varepsilon^k_i(t) \le 0,\ \forall\, i \in [m],\ k \in [l],$$

where ε is the production excess function. Therefore, there exists an index w_2 satisfying $r > w_2$ and

$$\frac{1}{m} \sum_{j=1}^{n} q^k_j(t_r) - \frac{1}{m} \sum_{j=1}^{n} q^k_j(t) - \varepsilon^k_i(t) \le 0. \tag{4.15}$$

Similarly, we have

$$\lim_{r\to+\infty} \left[\frac{1}{n} \sum_{i=1}^{m} p^k_i(t_r) - \frac{1}{n} \sum_{i=1}^{m} p^k_i(t) - \delta^k_j(t) \right] = -\delta^k_j(t) \le 0,\ \forall\, j \in [n],\ k \in [l],$$

where δ is the demand excess function. Therefore, there exists an index w_3 satisfying $r > w_3$ and

$$\frac{1}{n}\sum_{i=1}^{m} p_i^k(t_r) - \frac{1}{n}\sum_{i=1}^{m} p_i^k(t) - \delta_j^k(t) \le 0. \tag{4.16}$$

As a consequence, we consider the sequence $\{\mathscr{X}(t_r)\}_{r\in\mathbb{N}}$ with entries satisfying the following two properties:

(i) for $r > w = \max\{w_1, w_2, w_3\}$, $\forall\, i \in [m],\ j \in [n],\ k \in [l]$,

$$x_{ij}^k(t_r) = \underline{x}_{ij}^k(t_r) + \min\left\{x_{ij}^k(t) - \underline{x}_{ij}^k(t), \overline{x}_{ij}^k(t_r) - \underline{x}_{ij}^k(t_r), a_{ij}^k(t_r)\right\}, \tag{4.17}$$

(ii) for $r \le w$, $\forall\, i \in [m],\ j \in [n],\ k \in [l]$, $x_{ij}^k(t_r) = P_{\mathbb{K}(t_r)}(x_{ij}^k(t))$, where $P_{\mathbb{K}(t_r)}(\cdot)$ denotes the Hilbertian projection on $\mathbb{K}(t_r)$.

Apparently, if $r \le w$, then $\mathscr{X}(t_r) \in \mathbb{K}(t_r)$. If $r > w$, by the definition of $\overline{\mathscr{X}}$, $\underline{\mathscr{X}}$ and (4.14), it is evident that

$$\min\left\{x_{ij}^k(t) - \underline{x}_{ij}^k(t), \overline{x}_{ij}^k(t_r) - \underline{x}_{ij}^k(t_r), a_{ij}^k(t_r)\right\} \ge 0.$$

From (4.17), we have $\underline{x}_{ij}^k(t_r) \le x_{ij}^k(t_r)$ for all $i \in [m],\ j \in [n]$, and $k \in [l]$. In addition, since $\min\left\{x_{ij}^k(t) - \underline{x}_{ij}^k(t), \overline{x}_{ij}^k(t_r) - \underline{x}_{ij}^k(t_r), a_{ij}^k(t_r)\right\} \le \overline{x}_{ij}^k(t_r) - \underline{x}_{ij}^k(t_r)$, $\forall\ i \in [m],\ j \in [n],\ k \in [l]$, we have $x_{ij}^k(t_r) \le \overline{x}_{ij}^k(t_r)$. Moreover, it is clear that

$$\begin{aligned}\min&\left\{x_{ij}^k(t) - \underline{x}_{ij}^k(t), \overline{x}_{ij}^k(t_r) - \underline{x}_{ij}^k(t_r), a_{ij}^k(t_r)\right\} \le a_{ij}^k(t_r)\\ &= x_{ij}^k(t) - \underline{x}_{ij}^k(t_r) + \frac{mp_i^k(t_r) + nq_j^k(t_r)}{mn} - \frac{mp_i^k(t) + nq_j^k(t)}{mn}.\end{aligned}$$

Combining this with (4.17), we obtain that

$$x_{ij}^k(t_r) \le x_{ij}^k(t) + \frac{mp_i^k(t_r) + nq_j^k(t_r)}{mn} - \frac{mp_i^k(t) + nq_j^k(t)}{mn},\ \forall\, i \in [m],\ j \in [n],\ k \in [l].$$

Taking into account (4.15) and the definition of $p_i^k(t)$, it results that

$$\begin{aligned}\sum_{j=1}^{n} x_{ij}^k(t_r) &\le \sum_{j=1}^{n} x_{ij}^k(t) + p_i^k(t_r) + \frac{1}{m}\sum_{j=1}^{n} q_j^k(t_r) - p_i^k(t) - \frac{1}{m}\sum_{j=1}^{n} q_j^k(t)\\ &\le \sum_{j=1}^{n} x_{ij}^k(t) + p_i^k(t_r) - p_i^k(t) + \varepsilon_i^k(t)\\ &= \sum_{j=1}^{n} x_{ij}^k(t) + p_i^k(t_r) - \sum_{j=1}^{n} x_{ij}^k(t) - \varepsilon_i^k(t) + \varepsilon_i^k(t)\\ &= p_i^k(t_r), \quad \forall i = 1, \cdots, m,\ k = 1, \cdots, l.\end{aligned}$$

Similarly, by (4.16) and the definition of $q_j^k(t)$, we obtain that

$$\begin{aligned}\sum_{i=1}^m x_{ij}^k(t_r) &\leq \sum_{i=1}^m x_{ij}^k(t) + \frac{1}{n}\sum_{i=1}^m p_i^k(t_r) + q_j^k(t_r) - \frac{1}{n}\sum_{i=1}^m p_i^k(t) - q_j^k(t) \\ &\leq \sum_{i=1}^m x_{ij}^k(t) + q_j^k(t_r) - q_j^k(t) + \delta_j^k(t) \\ &= \sum_{i=1}^m x_{ij}^k(t) + q_j^k(t_r) - \sum_{i=1}^m x_{ij}^k(t) - \delta_j^k(t) + \delta_j^k(t) \\ &= q_j^k(t_r), \quad \forall j = 1,\cdots,n,\ k = 1,\cdots,l.\end{aligned}$$

Above all, for all $r \in \mathbb{N}$, $\mathscr{X}(t_r) \in \mathbb{K}(t_r)$. In view of (4.13), it follows that

$$\lim_{r\to+\infty} x_{ij}^k(t_r) = \underline{x}_{ij}^k(t) + \min\left\{x_{ij}^k(t) - \underline{x}_{ij}^k(t), \overline{x}_{ij}^k(t) - \underline{x}_{ij}^k(t), x_{ij}^k(t) - \underline{x}_{ij}^k(t)\right\} = x_{ij}^k(t).$$

In conclusion, the proof of condition $(M1)$ is completed.

In the following, we show that $(M2)$ holds. Let $\{t_r\}_{r\in\mathbb{N}}$ be a sequence such that $t_r \to t$ $(r \to +\infty)$, with $t \in [0,T]$. Let $\{\mathscr{X}(t_r)\}_{r\in\mathbb{N}}$ be a sequence such that $\mathscr{X}(t_r) \in \mathbb{K}(t_r)$, and $\mathscr{X}(t_r) \to \mathscr{X}(t)$ $(r \to +\infty)$. We now prove that $\mathscr{X}(t) \in \mathbb{K}(t)$. As $\mathscr{X}(t_r) \in \mathbb{K}(t_r)$, for all $r \in \mathbb{N}$, it results in

$$\underline{x}_{ij}^k(t_r) \leq x_{ij}^k(t_r) \leq \overline{x}_{ij}^k(t_r), \quad \forall i = 1,\cdots,m,\ \forall j = 1,\cdots,n,\ \forall k = 1,\cdots,l,$$

$$\sum_{j=1}^n x_{ij}^k(t_r) \leq p_i^k(t_r), \quad \forall i = 1,\cdots,m,\ \forall k = 1,\cdots,l,$$

$$\sum_{i=1}^m x_{ij}^k(t_r) \leq q_j^k(t_r), \quad \forall j = 1,\cdots,n,\ \forall k = 1,\cdots,l.$$

From the continuity of $\underline{\mathscr{X}}, \overline{\mathscr{X}}, P, Q$, for $r \to +\infty$, it holds that

$$\underline{x}_{ij}^k(t) \leq x_{ij}^k(t) \leq \overline{x}_{ij}^k(t), \quad \forall i = 1,\cdots,m,\ \forall j = 1,\cdots,n,\ \forall k = 1,\cdots,l,$$

$$\sum_{j=1}^n x_{ij}^k(t) \leq p_i^k(t), \quad \forall i = 1,\cdots,m,\ \forall k = 1,\cdots,l,$$

$$\sum_{i=1}^m x_{ij}^k(t) \leq q_j^k(t), \quad \forall j = 1,\cdots,n,\ \forall k = 1,\cdots,l.$$

Therefore, $\mathscr{X}(t) \in \mathbb{K}(t)$ and the condition $(M2)$ is proven. □

Combining Theorem 4.2 in [3] and Lemma 4.1, we have the following result.

Theorem 4.2. *Let the production function P, the demand function Q, and the capacity constraints $\underline{\mathscr{X}}$, $\overline{\mathscr{X}}$ be continuous in $[0,T]$. Furthermore, let $-\nabla_D v$ be strictly pseudomonotone and continuous on $[0,T]$. Then, the unique dynamic market equilibrium distribution in the presence of production and demand excesses $\mathscr{X}^* \in \mathbb{K}$ is continuous on $[0,T]$.*

5. Numerical Example

In order to obtain the solution of the dynamic market equilibrium problem, by discretizing the time interval, we first obtain finite-dimensional tensor variational inequalities (i.e., static tensor variational inequalities). Then, we solve finite-dimensional tensor variational inequalities by Algorithm 1.

The following example was also studied in [2]. The difference between the example in this section and the example in [2] is that, for each firm, we consider the case with more than one productions while only one production is considered in [2]. In detail, we consider the case with both production and demand excesses, involving two firms and two demand markets, as illustrated in Figure 1, over the time interval $[0,1]$.

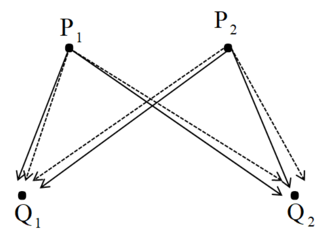


Figure 1. Network structure of the numerical dynamic spatial oligopoly problem.

Let $\overline{\mathscr{X}}, \underline{\mathscr{X}} \in L^2([0,1],\mathscr{T}_{222})$ be the capacity constraints, and $\overline{X}^k, \underline{X}^k \in L^2([0,1],\mathbb{R}_+^{2\times 2})(k=1,2)$ represent the maximum and minimum production of the k-th commodity, respectively:

$$\overline{X}^1(t) = \begin{bmatrix} 10t+1 & 4t \\ 7t+\frac{1}{2} & 10t+\frac{1}{5} \end{bmatrix}, \overline{X}^2(t) = \begin{bmatrix} 4t+2 & 5t+2 \\ 2t+1 & t+\frac{1}{2} \end{bmatrix}, \underline{X}^1(t) = \begin{bmatrix} 0 & \frac{1}{5}t \\ 0 & 0 \end{bmatrix},$$

$$\underline{X}^2(t) = \begin{bmatrix} t & 0 \\ \frac{1}{20}t & 0 \end{bmatrix}.$$

Let $P \in L^2([0,1],\mathbb{R}_+^{2\times 2})$ and $Q \in L^2([0,1],\mathbb{R}_+^{2\times 2})$ be the production and demand function, respectively, a.e. in $[0,1]$

$$P = \begin{bmatrix} 7t+2 & 6t+2 \\ 8t+3 & 11t+5 \end{bmatrix}, \quad Q = \begin{bmatrix} 10t+3 & 8t+2 \\ 9t+1 & 6t+3 \end{bmatrix}.$$

Hence, the feasible set is

$$K = \Bigg\{ \mathscr{X} \in L^2([0,1],\mathscr{T}_{222}) :$$
$$\underline{x}_{ij}^k(t) \le x_{ij}^k(t) \le \overline{x}_{ij}^k(t),\ \forall i=1,2,\ \forall j=1,2,\ \forall k=1,2, \text{ a.e. in } [0,1],$$
$$\sum_{j=1}^{2} x_{ij}^k(t) \le p_i^k(t),\ \forall i=1,2\ \forall k=1,2, \text{ a.e. in } [0,1],$$
$$\sum_{i=1}^{2} x_{ij}^k(t) \le q_j^k(t),\ \forall j=1,2\ \forall k=1,2, \text{ a.e. in } [0,1] \Bigg\}.$$

Consider the profit functions v_i defined as follows,

$$\begin{aligned}v_1(t,\mathscr{X}(t)) =& -(t+4)(x_{11}^1)^2(t)-\frac{5}{2}(x_{12}^1)^2(t)-(3t+2)(x_{11}^2)^2(t)-4(2t+1)(x_{12}^2)^2(t)\\&-\frac{1}{2}x_{11}^1(t)x_{22}^2(t)+(t+2)x_{11}^1(t)+2tx_{12}^1(t)+2x_{11}^2(t)+3tx_{12}^2(t),\\v_2(t,\mathscr{X}(t)) =& -(x_{21}^1)^2(t)-(3t+1)(x_{22}^1)^2(t)-2(x_{21}^2)^2(t)-(t+2)(x_{22}^2)^2(t)\\&+4(\frac{t}{2}+1)x_{21}^1(t)+7tx_{22}^1(t)+(1+\frac{t}{5})x_{21}^2(t))+tx_{22}^2(t).\end{aligned}$$

Then the components of $\nabla_D v$ are given by

$$\begin{aligned}&\frac{\partial v_1}{\partial x_{11}^1}=-2(t+4)x_{11}^1(t)-\frac{1}{2}(t+1)x_{22}^2(t)+t+2, &&\frac{\partial v_1}{\partial x_{12}^1}=-5x_{12}^1(t)+2t,\\&\frac{\partial v_1}{\partial x_{11}^2}=-2(3t+2)x_{11}^2(t)+2, &&\frac{\partial v_1}{\partial x_{12}^2}=-8(2t+1)x_{12}^2(t)+3t,\\&\frac{\partial v_2}{\partial x_{21}^1}=-2x_{21}^1(t)+2t+4, &&\frac{\partial v_2}{\partial x_{22}^1}=-2(3t+1)x_{22}^1(t)+7t,\\&\frac{\partial v_2}{\partial x_{21}^2}=-4x_{21}^2(t)+\frac{t}{5}+1, &&\frac{\partial v_2}{\partial x_{22}^2}=-2(t+2)x_{22}^2(t)+t.\end{aligned}$$

Note that $-\nabla_D v$ is Lipschitz continuous. Furthermore, we now confirm that $-\nabla_D v$ is a strongly monotone operator, a.e. in $[0,1]$

$$\begin{aligned}&\langle -\nabla_D v(t,\mathscr{X}(t))+\nabla_D v(t,\mathscr{Y}(t)),\mathscr{X}(t)-\mathscr{Y}(t)\rangle\\&=(2t+8)[x_{11}^1(t)-y_{11}^1(t)][x_{11}^1(t)-y_{11}^1(t)]+5[x_{12}^1(t)-y_{12}^1(t)][x_{12}^1(t)-y_{12}^1(t)]\\&+(6t+4)[x_{11}^2(t)-y_{11}^2(t)][x_{11}^2(t)-y_{11}^2(t)]+(16t+8)[x_{12}^2(t)-y_{12}^2(t)][x_{12}^2(t)-y_{12}^2(t)]\\&+(6t+2)[x_{22}^1(t)-y_{22}^1(t)][x_{22}^1(t)-y_{22}^1(t)]+2[x_{21}^1(t)-y_{21}^1(t)][x_{21}^1(t)-y_{21}^1(t)]\\&+4[x_{21}^2(t)-y_{21}^2(t)][x_{21}^2(t)-y_{21}^2(t)]+(\frac{5}{2}t+\frac{9}{2})[x_{22}^2(t)-y_{22}^2(t)][x_{22}^2(t)-y_{22}^2(t)]\\&\geq\|\mathscr{X}(t)-\mathscr{Y}(t)\|^2.\end{aligned}$$

By Theorem 4.2, we know that the unique dynamic market equilibrium solution is continuous on $[0,1]$. We consider the point-to-point variational inequality (4.11) for each $t\in[0,1]$ and provide a partition of $[0,1]$ such that $0=t_0<t_1<\ldots<t_u<\ldots<t_{20}=1$. For each point t_u, the finite-dimensional variational inequality can be formulated as follows:

$$\langle -\nabla_D v(t_u,\mathscr{X}^*(t_u)),\mathscr{X}(t_u)-\mathscr{X}^*(t_u)\rangle\geq 0,\qquad\forall\mathscr{X}(t_u)\in\mathbb{K}(t_u),\text{a.e. in }[0,1],\tag{5.1}$$

where

$$\begin{aligned}\mathbb{K}(t_u)=\Bigg\{&\mathscr{X}(t_u)\in\mathscr{T}_{mnl}:\underline{x}_{ij}^k(t_u)\leq x_{ij}^k(t_u)\leq\overline{x}_{ij}^k(t_u),\ \forall i\in[2],\ \forall j\in[2],\ \forall k\in[2],\\&\sum_{j=1}^2 x_{ij}^k(t_u)\leq p_i^k(t_u),\ \forall i\in[2],\ \forall k\in[2],\sum_{i=1}^2 x_{ij}^k(t_u)\leq q_j^k(t_u),\ \forall j\in[2],\ \forall k\in[2]\Bigg\}.\end{aligned}$$

To continue, let $L=30$, $\lambda=0.9/L$, $\gamma=0.5$, $\beta_r=1/200(r+1)$, $\theta_r=0.8-\beta_r$, $\tau_r=1/(r+1)^2$, $\alpha_r=\overline{\alpha}_r$, $\alpha=0.5$ for all $r\geq 1$ and set the stopping criterion to $E(\mathscr{X}^{*r}(t_u))=\|\mathscr{X}^{*r+1}(t_u)-$

$\mathscr{X}^{*r}(t_u)\| \leq 10^{-5}$. Then we choose the suitable initial points $\mathscr{X}^{*0}(t_u)$ and $\mathscr{X}^{*1}(t_u)$:

$$(X^1(t_u))^{*0} = \begin{bmatrix} \frac{1}{3}t_u & \frac{1}{3}t_u \\ \frac{1}{3}t_u & \frac{1}{3}t_u \end{bmatrix}, (X^2(t_u))^{*0} = \begin{bmatrix} \frac{1}{10}t_u & t_u \\ \frac{1}{10}t_u & \frac{1}{10}t_u \end{bmatrix},$$

$$(X^1(t_u))^{*1} = \begin{bmatrix} \frac{1}{5}t_u + 1 & \frac{1}{5}t_u + 5 \\ \frac{1}{5}t_u + 3 & \frac{1}{5}t_u + 2 \end{bmatrix}, (X^2(t_u))^{*1} = \begin{bmatrix} \frac{6}{5}t_u + 10 & \frac{1}{20}t_u + 1 \\ \frac{1}{20}t_u + 6 & \frac{1}{20}t_u + 2 \end{bmatrix}.$$

By Algorithm 1, we obtain the equilibrium curves (depicted in Figure 2), as well as the curves representing production and demand excesses (shown in Figure 3 and Figure 4, respectively). Additionally, we illustrate the convergence behavior of the sequences generated by Algorithm 1 at $t_u = 0, 0.25, 0.5, 0.75, 1$ in Figure 5.

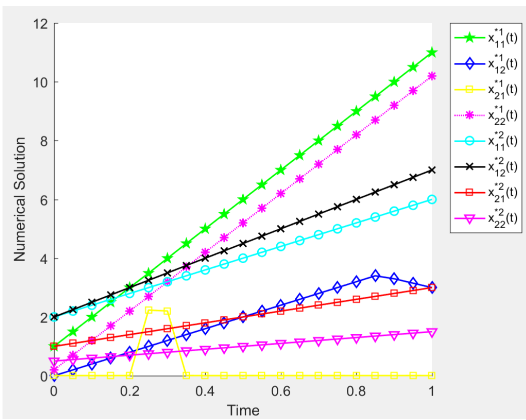


FIGURE 2. Curves of equilibria.

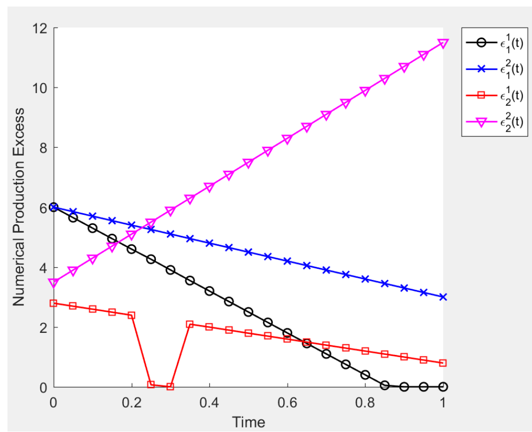


FIGURE 3. Curves of production excess.

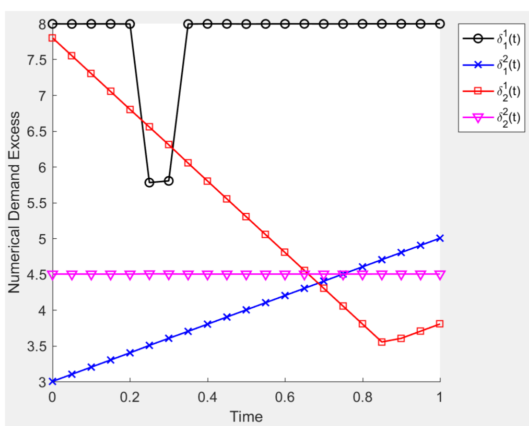
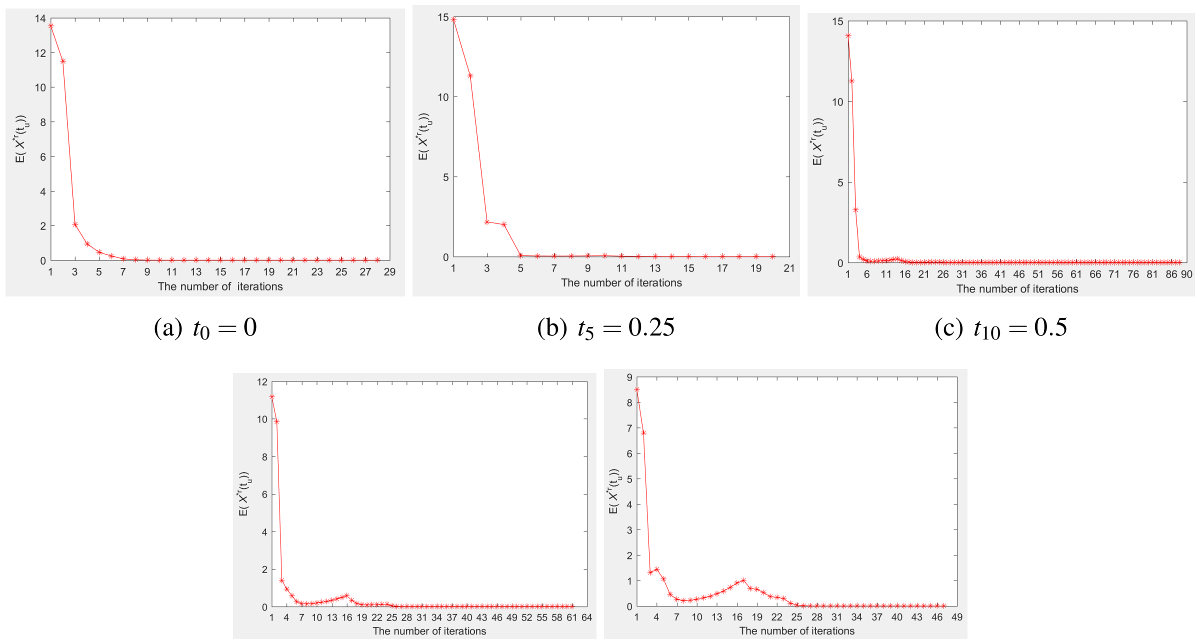


FIGURE 4. Curves of demand excess.



(a) $t_0 = 0$ (b) $t_5 = 0.25$ (c) $t_{10} = 0.5$

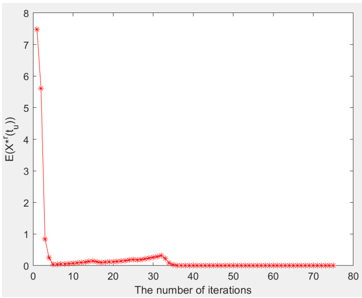
(d) $t_{15} = 0.75$ (e) $t_{20} = 1$

FIGURE 5. The error plotting of Algorithm 1.

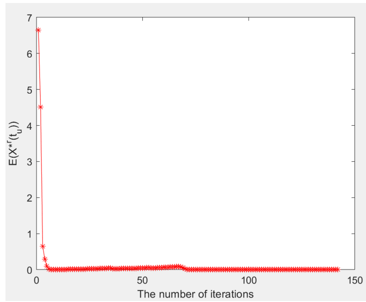
To verify the performance of the algorithm according to different parameters λ, α and γ, we first fix $\alpha = 0.5$, $\gamma = 0.5$, and set $\lambda = 1/50,\ 1/100,\ 1/150$. The corresponding iteration steps of the algorithm are 75, 142, and 209. Then, we fix $\lambda = 1/50$, $\gamma = 0.5$, and set $\alpha = 0.1,\ 1,\ 5$. The corresponding iteration steps of the algorithm are 76, 75, and 84. Finally, we fix $\lambda = 1/50$, $\alpha = 0.5$, and set $\gamma = 0.5,\ 1,\ 1.5$. The corresponding iteration steps of the algorithm are 75, 36, and 24. The iteration diagrams of the algorithm are shown in Figure 6. It can be observed that different values of λ, α, and γ does not affect the convergence of the algorithm, but it affects the number of iterations.

6. CONCLUSIONS

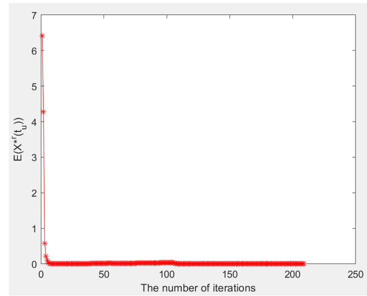
In this paper, we investigated the TVI and its applications. An alternative algorithm was proposed to tackle TVIs with certain monotonicity or continuity conditions. Then the equilibrium problems from dynamic (i.e., time-dependent) oligopolistic markets were investigated, and numerical experiments were given to verify the effectiveness of the proposed algorithm. We also mention here that the study on TVIs only in its early stage, and there are challenging problems, which need to be investigated. For example, as an efficient tool for multi-dimensional data, can we consider more applications of TVIs? Can we consider the dynamic (i.e., time-dependent) oligopolistic markets with stochastic productions and demanding markets? Maybe those problems are hard to answer, however it is really interesting and meaningful for practical applications.



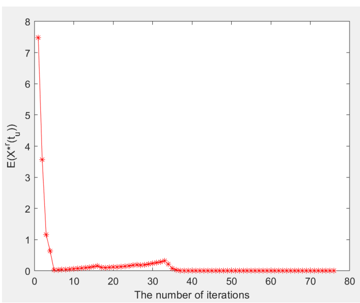
(a) $\alpha = 0.5$, $\gamma = 0.5$, and $\lambda = 1/50$



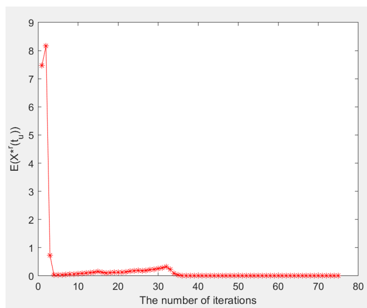
(b) $\alpha = 0.5$, $\gamma = 0.5$, and $\lambda = 1/100$



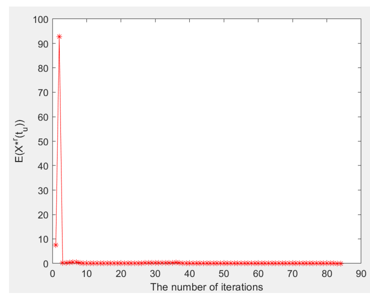
(c) $\alpha = 0.5$, $\gamma = 0.5$, and $\lambda = 1/150$



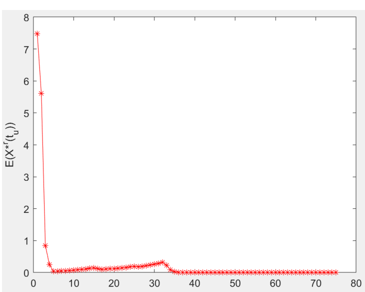
(d) $\lambda = 1/50$, $\gamma = 0.5$, and $\alpha = 0.1$



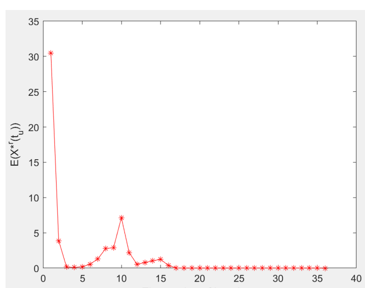
(e) $\lambda = 1/50$, $\gamma = 0.5$, and $\alpha = 1$



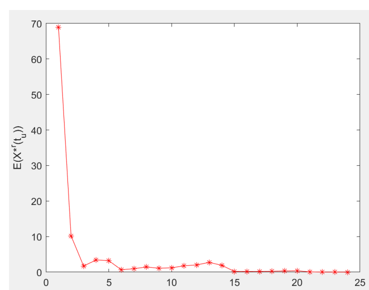
(f) $\lambda = 1/50$, $\gamma = 0.5$, and $\alpha = 5$



(g) $\lambda = 1/50$, $\alpha = 0.5$, and $\gamma = 0.5$



(h) $\lambda = 1/50$, $\alpha = 0.5$, and $\gamma = 1$



(i) $\lambda = 1/50$, $\alpha = 0.5$, and $\gamma = 1.5$

FIGURE 6. The error plotting of Algorithm 1 with varying parameters λ, α, and γ.

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