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THE THEORY OF HARDY-SOBOLEV SPACES OVER THE UPPER HALF-PLANE

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Abstract. We mainly investigate some results of Hardy-Sobolev spaces on the upper half-plane. In this paper, the Paley-Wiener theorem of Hardy-Sobolev spaces is established. The theorem asserts that the reproducing kernel of the Hardy-Sobolev spaces can be found. On account of this, the estimation of the reproducing kernel is produced. We also consider the multiplication operators on the spaces, and the spectrum of multiplication operators is characterized. In addition, the condition for the boundedness of weighted composition operators can be founded by applying the property of self-maps and the reproducing kernel.

Keywords. Holomorphic Fourier transform; Hardy-Sobolev spaces; Paley-Wiener theorem; Reproducing kernel; Weighted composition operator.

1. INTRODUCTION

Function spaces, as a branch of functional analysis, has been widely developed. Classical Hardy spaces have formed a rich theoretical system [1, 20]. It consists of functions in the Lebesgue spaces L^p that are smooth inside the domain and have finite integral growth on the boundary. Sobolev developed a novel kind of function spaces during his investigation of the elastic wave problem in the 1930s. This kind of spaces later came to be known as Sobolev spaces, which are Banach space constructed by weakly differentiable functions. In harmonic analysis and partial differential equations theory, Hardy spaces are the best alternative to Lebesgue spaces $L^p(0 < p \leq 1)$ and the Sobolev space requires that the weak derivative of the function belongs to $L^p(p > 1)$ [2]. As a result, it makes sense to further research Hardy-Sobolev spaces, which demand that a function's derivative also belongs to Hardy spaces. Hardy-Sobolev spaces include a number of well-known classical function spaces, such as Hardy spaces, Bergmann spaces, Dirichlet spaces, and so on. Since Hardy-Sobolev spaces have more sophisticated spatial structure, they have more deep outcomes than generic classical spaces.

In recent years, scholars studied the Hardy-Sobolev spaces defined on the unit disk or unit ball. He-Cao-Zhu [3] obtained some growth estimates for Hardy-Sobolev functions of the unit

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ball and gave the representation of the spectra of composition operators and described the Fredholmness of composition operators equivalently. Ortega-Fàbrega [4] gave characterizations of pointwise multipliers of the holomorphic Hardy-Sobolev spaces on the unit ball of $\mathbb{C}^n$. In [5–7], the multipler algebra on Hardy-Sobolev spaces over unit disks and unit balls was studied with computing the spectrum and essential spectrum of multiplication operators. There are fewer tools available on the half-planes, However. The theory of Hardy-Sobolev spaces over the half-plane or tubes still remains to be studied. Espallargas [8] and Galé et al. [9] established the extended form of the Paley-Wiener theorem for Hardy-Sobolev spaces on the right-hand half-plane and derived a series of results. Cho-Kim [21] considered Hardy–Sobolev spaces for high dimension case and established an atomic decomposition theorem, analogous to the atomic decomposition characterization of Hardy spaces. Rotkevich [22,23] used the method of pseudoanalytic continuation to obtain the characterization of Hardy–Sobolev spaces on strongly convex domains in terms of polynomial approximations, and then promoted the related conclusions to strictly pseudoconvex domains. Observe that Hardy-Sobolev spaces have wide applications in the areas, such as PDE, wavelet analysis, signal analysis, and so on. For example, Ward-Partington [10] established rational wavelet decompositions of the Hardy–Sobolev classes on the half-plane by using Daubechies wavelet theory. Dang-Qian-Yang [11] extended some formulas to signals in Sobolev spaces by introducing Hardy-Sobolev derivatives.

In this paper, we focus on the function and operator theories of Hardy-Sobolev spaces. More precisely, this paper mainly considers the problems such as the Paley-Wiener theorem, reproducing kernels, multiplication operators, and weighted composition operators. These problems could supplement each other and are important for the development of Hardy-Sobolev spaces. It is well known that the classical Paley–Wiener theorem has important applications in the function theory of complex variables and approximation of real variables. Due to the property of isometric isomorphism, Paley-Wiener type theorem provides effective tools for studying these two related function spaces. Functions with compact and convex spectrum on $\mathbb{R}^n$ were described by the classical Paley-Wiener theorem through the properties of their analytic extensions in the complex domain; see [24]. In addition, the good reproducibility of the reproducing kernel has brought great convenience to many fitting and reconstruction problems in mathematics and engineering, and has also given rise to the great development of many new interdisciplinary fields. Elements in the reproducing kernel spaces can be generated by the inner product with the reproducing kernel, which determines projections, which are the basic method for discussing optimal approximation problems in inner product spaces, which determines the great convenience of reproducing kernels in various numerical analysis, data processing, and optimization problems.

Weighted composition operators, which are a generalized composition operators, are the combination of composition operators and multiplication operators. Investigating weighted composition operators is of great significance for studying harmonic analysis, functional analysis, operator algebra, and topological dynamical systems. It is important to develop the Paley-Wiener type theorem, reproducing kernel and operator theories in Hardy-Sobolev spaces. Our main results are as follows.

A. Let $\mathscr{F}$ be the holomorphic Fourier transform $\mathscr{F}: L^2(\mathbb{R}^+) \to H_2(\mathbb{C}^+)$. There exists $\mathscr{F}(\mathscr{T}_2^{(n)}) = H_2^{(n)}$, and then $H_2^{(n)}$ is a Hilbert space endowed with inner product

$$\langle F, G\rangle_{H_2^{(n)}} = \int_{-\infty}^{+\infty} x^{2n} (F^*)^{(n)}(x) \overline{(G^*)^{(n)}(x)} dx, \qquad F, G \in H_2^{(n)},$$

and its associated norm $\|F\|_{H_2^{(n)}} = \|z^n F^{(n)}\|_{H_2}$, $F \in H_2^{(n)}$. Moreover, $\mathscr{F}$ is an isometric isomorphism from $\mathscr{T}_2^{(n)}$ onto $H_2^{(n)}$, i.e., $F, G \in H_2^{(n)}$ if and only if there exists $f, g \in \mathscr{T}_2^{(n)}$ and $\langle f, g\rangle_{\mathscr{T}_2^{(n)}} = \langle \mathscr{F}(f), \mathscr{F}(g)\rangle_{H_2^{(n)}}$.

B. Let n be a positive integer. Then the function K_n defined on $\mathbb{C}^+ \times \mathbb{C}^+$ by

$$K_n(z,w) = \frac{1}{((n-1)!)^2}\int_0^1\int_0^1 (1-y)^{n-1}(1-x)^{n-1}\frac{i}{yz - x\overline{w}}dxdy, \qquad z, w \in \mathbb{C}^+,$$

is the reproducing kernel of $H_2^{(n)}$, i.e., if $K_{n,w}(z) = K_n(z,w)$, then $K_{n,w} \in H_2^{(n)}$ and $F(w) = \langle F, K_{n,w}\rangle_{H_2^{(n)}}$ for all $F \in H_2^{(n)}$.

C. Let n be the positive integer. Then, for $z \in \mathbb{C}^+$,

$$\frac{1}{(n-1)!\sqrt{2n-1}}\frac{1}{\sqrt{|z|}} \leq \|K_{n,z}\|_{H_2^{(n)}} \leq \frac{\sqrt{\pi}}{(n-1)!\sqrt{n}}\frac{1}{\sqrt{|z|}}.$$

D. For any $\varphi \in \mathscr{M}_n$, the spectrum of the multiplication operator $M_\varphi : H_2^{(n)} \to H_2^{(n)}$ is given by $\sigma(M_\varphi) = \overline{\varphi(\mathbb{C}^+)}$, which is the closure of the range of φ in the complex plane.

E. If $\psi \in \mathscr{M}_n$ satisfying that there exists a positive consant δ such that $|\psi(z)| \geq \delta$ for all $z \in \mathbb{C}^+$ and $C_{\psi,\varphi}$ is a bounded composition operator on $H_2^{(n)}$ for $n \geq 1$, then the self-map φ satisfies $\varphi(\infty) = \infty$ and $\varphi'(\infty) < \infty$.

F. Let $\psi \in H_\infty^{(n)}$ and φ be an analytic self-map on $\mathbb{C}^+$ satisfying that $\varphi'(\infty) < \infty$ and $\sup_{z\in\mathbb{C}^+} |z^k \frac{\varphi^{(k)}(z)}{\varphi(z)}| < \infty$, $k = 1, 2, \ldots, n$. Then the weighted composition operator $C_{\psi,\varphi} : H_2^{(n)} \to H_2^{(n)}$ is bounded.

The present paper is organized as follows. In Section 2, we introduce some notations and definitions, which are necessary in this paper. In Section 3, we establish the Paley-Wiener type theorem of Hardy-Sobolev spaces. In Section 4, we compute the reproducing kernel on the spaces and give the estimation of the reproducing kernel additionally. In Section 5, the multiplication operators and weighted composition operators on the spaces is discussed. We give some properties about the multiplication operators and establish the spectrum of the operator. In addition, we consider the condition for the boundedness of weighted composition operators, which are related to the reproducing kernel, multiplication operators and self-map.

2. Preliminaries

Let $\mathbb{R}^+$ be the interval (0,∞ 0) and $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ denote the upper half-plane. Let $L^p(\mathbb{R}^+)$ be the usual Lebesgue space on (0,∞) with its norm

$$\|f\|_p = \left(\int_0^\infty |f(t)|^p dt\right)^{\frac{1}{p}} < \infty, \qquad f \in L^p(\mathbb{R}^+).$$

In particular, $L^2(\mathbb{R}^+)$ is a Hilbert space with its inner product given by $\langle f, g\rangle_2 = \int_0^\infty f(t)\overline{g(t)}dt$ for all $f, g \in L^2(\mathbb{R}^+)$. One usually writes $\langle\cdot,\cdot\rangle$ to denote the inner product of Hilbert spaces. Recall that the classical Hardy Space $H_2(\mathbb{C}^+)$ is the space of holomorphic functions on the upper half-plane $\mathbb{C}^+$ such that

$$\|f\|_{H_2} = \sup_{y>0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} |f(x+iy)|^2 dx < \infty.$$

In fact, $H_2(\mathbb{C}^+)$ is a Hilbert space with inner product $\langle f,g\rangle_{H_2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f^*(x)\overline{g^*(x)}dx$ for all $f,g \in H_2(\mathbb{C}^+)$, where $f^*(x) = \lim_{y\to 0} f(x+iy)$ a.e. $x \in \mathbb{R}$. In the case of unit disks, Hardy space $H_2(\mathbb{D})$ is endowed with the norm

$$\|F\|_{H_2(\mathbb{D})} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^2 d\theta \right)^{\frac{1}{2}}, \qquad F \in H_2(\mathbb{D}).$$

$\mathscr{T}_p^{(n)} (1 \le p < \infty)$ consist of all the measurable functions on $\mathbb{R}^+$ which belongs to the closure of $C_c^\infty(\mathbb{R}^+)$ corresponding to the norm

$$\|f\|_{\mathscr{T}_p^{(n)}} = \left(\int_0^\infty |f^{(n)}(t)t^n|^p dt \right)^{\frac{1}{p}} < \infty, \qquad f \in C_c^\infty(\mathbb{R}^+).$$

In particular, $\mathscr{T}_1^{(n)}$, a subalgebra of $L^1(\mathbb{R}^+)$, is a convolution Banach algebra with norm $\|\cdot\|_{\mathscr{T}_1^{(n)}}$. It was first introduced to approach Cauchy problems in [12,13] and it was further studied in [14]. For $p = 2$, it was proved that the range $\mathscr{L}(\mathscr{T}_2^{(n)})$, where $\mathscr{L}$ denotes to the Laplace transform, is an isometric isomorphism onto the Hardy-Sobolev space on the right half-plane [8].

In this paper, we consider the holomorphic Fourier transform $\mathscr{F}$ acting on the space $\mathscr{T}_2^{(n)}$, where $\mathscr{F}$ is given by $\mathscr{F}(f)(z) = \int_0^\infty f(t)e^{itz}dt$, $f \in L^2(\mathbb{R}^+), z \in \mathbb{C}^+$. Let f be any function in $L^2(\mathbb{R}^+)$ and $\mathscr{F}(f)(z) = F(z) = \int_0^\infty f(t)e^{itz}dt$, $z \in \mathbb{C}^+$. If $z = x+iy \in \mathbb{C}^+$, then $|e^{itz}| = e^{-ty}$, which shows that the integral above exists as a Lebesgue integral. By applying a series of theorems about real and complex analysis, it is easy to see that F is a holomorphic function on $\mathbb{C}^+$. The classical Paley-Wiener theorem [17] shows that the holomorphic Fourier transform $\mathscr{F} : L^2(\mathbb{R}^+) \to H_2(\mathbb{C}^+)$ is an isometric isomorphism, i.e., $F \in H_2(\mathbb{C}^+)$ if and only if there exists an $f \in L^2(\mathbb{R}^+)$ such that $F = \mathscr{F}(f)$ and $\|F\|_{H_2} = \|f\|_2$. Sometimes, one writes H_2 for short to denote the Hardy space on the upper half-plane.

Let X be a set. Recall that $\mathscr{H}$ is a reproducing kernel Hilbert space, or, more briefly, a RKHS on X if $\mathscr{H}$ is a Hilbert space of all complex functions on X and satisfies the condition that the evaluation functional, $E_x : \mathscr{H} \to \mathbb{C}$, defined by $E_x(f) = f(x)$, is bounded for every $x \in X$. If $\mathscr{H}$ is an RKHS on X, then an application of the Riesz representation theorem shows that the linear evaluation functional is given by the inner product with a unique vector in $\mathscr{H}$. Thus, for each $x \in X$, there exists a unique vector $k_x \in \mathscr{H}$ such that, for every $f \in \mathscr{H}, f(x) = E_x(f) = \langle f, k_x\rangle$. The function k_x is called a kernel-function and the span generated by all of kernel-functions is dense in $\mathscr{H}$. The function $K : X \times X \to \mathbb{C}$ defined by $K(x,y) = k_y(x) = \langle k_y, k_x\rangle, x,y \in X$ is said to be the reproducing kernel of $\mathscr{H}$ [15]. In fact, H_2 is a RKHS with its reproducing kernel given by $K(z,w) = K_w(z) = \frac{i}{z-\overline{w}}$, $z,w \in \mathbb{C}^+$. In addition, a kernel function K is said to be nonnegative, positive, definite or sometimes positive semidefinite meaning that $\sum_{i,j=1}^n \alpha_i\overline{\alpha_j}K(x_i,x_j) \ge 0$, $n \ge 1$, $\alpha_i, \alpha_j \in \mathbb{C}$, $x_i, x_j \in X$, $i, j = 1,2,\ldots,n$. Let $\mathscr{S}$ be the class of all analytic self-maps of $\mathbb{C}^+$. The definition of the angular derivative on the unit disc $\mathbb{D}$ and $\mathbb{C}^+$ was introduced in [18]. In this paper, we mainly consider the self-map φ on the upper half-plane which fixes the point at infinity and has a finite angular derivative at ∞ in the sense of Carathéodory. It means that $\varphi(\infty)=\infty$ and $\varphi'(\infty)<\infty$, where $\varphi(\infty)=\lim_{|z|\to\infty}\varphi(z)$ and $\varphi'(\infty)=\lim_{z\to\infty}\frac{z}{\varphi(z)}$, and the second limit above represent the non-tangential limit.

Here we use some notations and terminologies that are appeared in [8, 9]. We use W^{-1} to denote the integration operator on the space $C_c^\infty(\mathbb{R}^+)$, which is given by

$$W^{-1}\varphi(t)=\int_t^\infty \varphi(s)ds, \qquad \varphi\in C_c^\infty(\mathbb{R}^+), \quad t\geq 0.$$

For any function $\varphi\in C_c^\infty(\mathbb{R}^+)$, it is easy to show that $W^{-1}\varphi\in C_c^\infty(\mathbb{R}^+)$ so that we can define the integration operator $W^{-n}=W^{-1}(W^{-(n-1)})$ for every positive integer n. By applying Fubini's theorem, we have

$$W^{-n}\varphi(t)=\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\varphi(s)ds, \qquad \varphi\in C_c^\infty(\mathbb{R}^+), \quad t\geq 0.$$

In fact, the index n can be extended to any positive real number α. For more details about $W^{-\alpha}$, we refer to [8, 25].

Next, we consider the operator W^{-n} acting on the Hilbert space $L^2(t^n)$. For $n\in\mathbb{N}$, let $L^2(t^n)$ be the space of all complex measurable functions on $\mathbb{R}^+$ whose norm satisfies

$$\|\varphi\|_{L^2(t^n)}=\left(\int_0^\infty |\varphi(t)t^n|^2dt\right)^{\frac{1}{2}}<\infty.$$

Taking $m\in\mathbb{N}$ and Borel function φ, we have $W^{-n}\varphi\in L^2(\mathbb{R}^+)$ for all $\varphi\in L^2(t^n)$ by applying Hardy's inequality [16]

$$\int_0^\infty \left(W^{-m}\varphi(t)\right)^2 dt\leq\left(\frac{\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}\right)^2\int_0^\infty (t^m\varphi(t))^2dt.$$

As a matter of fact, the integration operator $W^{-n}:L^2(t^n)\to L^2(\mathbb{R}^+)$ is injective, which enables us to define Hardy-Lebesgue space $\mathscr{T}_2^{(n)}$. For $n\in\mathbb{N}$, let Hardy-Lebesgue spaces $\mathscr{T}_2^{(n)}$ be the range of operator W^{-n} on $L^2(t^n)$, i.e., $\mathscr{T}_2^{(n)}=W^{-n}(L^2(t^n))$, which is a subspace of $L^2(\mathbb{R}^+)$. It is clear that $\mathscr{T}_2^{(n)}$ is a Hilbert space with inner product $\langle f,g\rangle_{\mathscr{T}_2^{(n)}}=\int_0^\infty f^{(n)}(t)\overline{g^{(n)}(t)}t^{2n}dt$, $f,g\in\mathscr{T}_2^{(n)}$. For $n\geq k\geq 0$, let $\varphi(t)=|f^{(n)}(t)|t^k$ and $m=n-k$. By using Hardy's inequality, we have that the size of the space $\mathscr{T}_2^{(n)}$ decreases as n increases, i.e., $\mathscr{T}_2^{(n)}\subseteq\mathscr{T}_2^{(k)}\subseteq L^2(\mathbb{R}^+)$, $n\geq k\geq 0$. For $n\in\mathbb{N}$, let Hardy-Sobolev spaces $H_2^{(n)}$ be the linear space of all holomorphic functions F on $\mathbb{C}^+$ such that $z^kF^{(k)}\in H_2$, $k=0,1,\ldots,n$. It is clear that $H_2^{(n)}\subseteq H_2^{(0)}=H_2$. Also, we write $H_\infty^{(n)}$ to denote the linear space of all holomorphic functions F on $\mathbb{C}^+$ satisfying $z^kF^{(k)}\in H^\infty, \quad k=0,1,\ldots,n$, where H^∞ denotes the bounded analytic function space on the upper half-plane. Since $H_2^{(n)}$ can be embedded into H_2, it follows that the evaluation functional on $H_2^{(n)}$ is also bounded, so it is a RKHS as well [9]. The multiplier algebra of $H_2^{(n)}$ is the set of all functions φ analytic on $\mathbb{C}^+$ such that $\varphi f\in H_2^{(n)}$ whenever $f\in H_2^{(n)}$. We denote it by $\mathscr{M}_n$. Given $\varphi\in\mathscr{M}_n$, we write $M_\varphi:H_2^{(n)}\to H_2^{(n)}$ for the multiplication operator $M_\varphi(f)=\varphi f \quad (f\in H_2^{(n)})$. The weighted composition operator with the symbol ψ,φ acting on some function spaces is given by $C_{\psi,\varphi}f=\psi f\circ\varphi$.

3. Paley-Wiener Theorem

Paley-Wiener type theorems play an important role in the classical function theory. Now, we obtain the Paley-Wiener theorem of Hardy-Sobolev spaces as follows.

Theorem 3.1. *(Paley-Wiener) Let $\mathscr{F}$ be the holomorphic Fourier transform $\mathscr{F}: L^2(\mathbb{R}^+) \to H_2(\mathbb{C}^+)$. Then there exists $\mathscr{F}(\mathscr{T}_2^{(n)}) = H_2^{(n)}$, and $H_2^{(n)}$ is a Hilbert space endowed with inner product $\langle F,G\rangle_{H_2^{(n)}} = \int_{-\infty}^{+\infty} x^{2n}(F^*)^{(n)}(x)\overline{(G^*)^{(n)}(x)}dx$, $F,G \in H_2^{(n)}$, corresponding to the norm $\|F\|_{H_2^{(n)}} = \|z^n F^{(n)}\|_{H_2}$, $F \in H_2^{(n)}$. Moreover, $\mathscr{F}$ is an isometric isomorphism from $\mathscr{T}_2^{(n)}$ onto $H_2^{(n)}$, i.e., $F, G \in H_2^{(n)}$ if and only if there exists $f,g \in \mathscr{T}_2^{(n)}$ and $\langle f,g\rangle_{\mathscr{T}_2^{(n)}} = \langle \mathscr{F}(f), \mathscr{F}(g)\rangle_{H_2^{(n)}}$.*

In order to prove Theorem 3.1, we need the help of following lemmas.

Lemma 3.1. *[9] For $n \in \mathbb{N}$ and $\varphi \in L^2(t^n)$, we have, for $1 \le k \le n$,*

$$W^{-k}\varphi(t) = \frac{1}{(k-1)!}\int_t^{\infty}(s-t)^{n-1}\varphi(s)ds, \qquad t > 0.$$

Moreover, $W^{-k}\varphi$ is $(k-1)$-times differentiable with $(W^{-k}\varphi)^{(l)} = (-1)^l W^{-(k-l)}\varphi$ for every $1 \le l \le k-1$, and $(-1)^k(W^{-k}\varphi)^{(k)} = \varphi$.

Using Holder's inequality and taking $\varphi = (-1)^n f^{(n)}$ in Lemma 3.1, we obtain

$$|f^{(k)}(t)| \le C_{n,k} t^{-k-\frac{1}{2}}\|f\|_{\mathscr{T}_2^{(n)}}, \qquad f \in \mathscr{T}_2^{(n)}, \quad 0 \le k \le n-1, \quad t > 0,$$

where $C_{n,k}$ is a constant related to n and k.

The next lemma characterizes the equivalent inner-product of spaces $\mathscr{T}_2^{(n)}$.

Lemma 3.2. *[9] Let $f,g \in \mathscr{T}_2^{(n)}$. Then $\langle f,g\rangle_{\mathscr{T}_2^{(n)}} = \langle (t^n f)^{(n)}, (t^n g)^{(n)}\rangle_2$, where $\langle f,g\rangle_2 = \int_0^{\infty} f(t)\overline{g(t)}dt$.*

Lemma 3.3. *[9] For $n \ge 0$, the $(n+1)$-square matrix $C_n = (c_{i,j})_{0\le i,j\le n}$ defined by*

$$c_{i,j} = \begin{cases} 0, & i < j; \\ \binom{i}{j}\frac{i!}{j!}, & i \ge j, \end{cases}$$

satisfies that

(i) C_n is invertible and $C_n^{-1} = ((-1)^{i+j}c_{i,j})_{0\le i,j\le n}$;

(ii) for all $f \in \mathscr{T}_2^{(n)}$, $t > 0$,

$$(t^n f)^{(n)}(t) = \sum_{k=0}^{n} c_{n,k} t^k f^{(k)}(t),\ t^n f^{(n)}(t) = \sum_{k=0}^{n}(-1)^{k+n}c_{n,k}(t^k f)^{(k)}(t).$$

Next, we give some identities about holomorphic Fourier transform $\mathscr{F}$ and higher derivative, which are used in the proof of Theorem 3.1.

Lemma 3.4. *For all $n \in \mathbb{N}$, $z \in \overline{C}^{+}\backslash\{0\}$, and $f \in \mathscr{T}_2^{(n)}$,*

$$(-1)^k z^k [\mathscr{F}(f)]^{(k)}(z) = \sum_{j=0}^{k} c_{k,j} \mathscr{F}(t^j f^{(j)})(z), \qquad k = 0,1,\dots,n;$$

$$(-1)^k \mathscr{F}(t^k f^{(k)})(z) = \sum_{j=0}^{k} c_{k,j} z^j [\mathscr{F}(f)]^{(j)}(z), \qquad k = 0,1,\dots,n,$$

where $c_{k,j}$ is a constant defined by Lemma 3.3.

Proof. Let j,k be nonnegative integers such that $0 \le j \le k \le n$. Taking $h = t^k f$ and integrating by parts k times, we have $\mathscr{F}(h^{(k)})(z) = (-iz)^k \mathscr{F}(h)(z) - \sum_{j=0}^{k-1}(-iz)^{k-1-j} h^{(j)}(0)$. By applying Leibniz's rule of derivation in $(t^k f)^{(j)}$, we have $(t^k f)^{(j)}(0) = \lim_{t \to 0^+} (t^k f)^{(j)}(t) = 0$, $j = 0,1,\dots,k-1$. Thus

$$\mathscr{F}((t^k f)^{(k)})(z) = (-iz)^k \mathscr{F}(t^k f)(z) = (-1)^k z^k [\mathscr{F}(f)]^{(k)}(z).$$

and

$$(-1)^k z^k [\mathscr{F}(f)]^{(k)}(z) = \mathscr{F}((t^k f)^{(k)})(z) = \sum_{j=0}^{k} c_{k,j} \mathscr{F}(t^j f^{(j)})(z),$$

where the second equality is based on Lemma 3.3. By applying the invertibility of the matrix defined by Lemma 3.3, we have $\mathscr{F}(t^j f^{(j)})(z) = \sum_{k=0}^{j} (-1)^j c_{j,k} z^k [\mathscr{F}(f)]^{(k)}(z)$. Interchange the position of j and k, we derive $(-1)^k \mathscr{F}(t^k f^{(k)})(z) = \sum_{j=0}^{k} c_{k,j} z^j [\mathscr{F}(f)]^{(j)}(z)$. $\square$

Using the above lemma, now we can finish the proof of Theorem 3.1.

Proof. If $f \in \mathscr{T}_2^{(n)}$, then, for $0 \le k \le n$, it is easy to see that $t^k f^{(k)} \in L^2(\mathbb{R}^+)$. By applying Lemma 3.4 and the classical Paley-Wiener theorem, we derive that $z^k [\mathscr{F}(f)]^{(k)} \in H_2$. Thus $\mathscr{F}(f) \in H_2^{(n)}$.

Conversely, if $F \in H_2^{(n)} \subseteq H_2$, then, according to the classical Paley-Wiener theorem, there exists a function $f \in L^2(\mathbb{R}^+)$ such that $F = \mathscr{F}(f)$. All we need is to prove that $f \in \mathscr{T}_2^{(n)}$. To this end, we consider the function $G(z) = (-1)^n \sum_{j=0}^{n} c_{n,j} z^j F^{(j)}(z)$, $z \in \mathbb{C}^+$. Since $F \in H_2^{(n)}$, we have $G \in H_2$. Therefore, there exists a function $g \in L^2(\mathbb{R}^+)$ such that $G = \mathscr{F}(g)$. In addition, we define another function

$$h(s) = -\frac{1}{(n-1)!} \int_s^{\infty} (s-t)^{n-1} \frac{g(t)}{t^n} dt.$$

Note that $\frac{g(t)}{t^n} \in L^2(t^n)$. It is clear that it belongs to $\mathscr{T}_2^{(n)}$. By applying Lemma 3.1, we derive $g = t^n h^{(n)}$. According to Lemma 3.4, we have

$$G(z) = \mathscr{F}(g)(z) = \mathscr{F}(t^n h^{(n)})(z) = (-1)^n \sum_{j=0}^{n} c_{n,j} z^j [\mathscr{F}(h)]^{(j)}(z), \qquad z \in \mathbb{C}^+.$$

It follows from $F = \mathscr{F}(f)$ that $(-1)^n [z^n \mathscr{F}(h-f)]^{(n)}(z) = (-1)^n \sum_{j=0}^{n} c_{n,j} z^j [\mathscr{F}(h-f)]^{(j)}(z) = 0$, which implies that $z^n \mathscr{F}(h-f) = P_n(z)$, where $P_n(z)$ denotes the polynomial with its degree less than or equal to $n-1$. In view of $P_n(z) z^{-n} = \mathscr{F}(h-f) \in H_2$, we have $\mathscr{F}(h-f) = 0$. By the injectivity of $\mathscr{F}$, we derive $f = h \in \mathscr{T}_2^{(n)}$. Therefore, $\mathscr{F}(\mathscr{T}_2^{(n)}) = H_2^{(n)}$. Taking $f, g \in \mathscr{T}_2^{(n)}$, we have $F = \mathscr{F}(f)$, $G = \mathscr{F}(g)$ in $H_2^{(n)}$. By Lemma 3.2, we have

$$\begin{aligned}\langle f, g\rangle_{\mathscr{T}_2^{(n)}} = \langle (t^n f)^{(n)}, (t^n g)^{(n)}\rangle_2 = \langle \mathscr{F}[(t^n f)^{(n)}], \mathscr{F}[(t^n g)^{(n)}]\rangle_{H_2} \\ = \langle (-1)^n z^n [\mathscr{F}(f)]^{(n)}, (-1)^n z^n [\mathscr{F}(g)]^{(n)}\rangle_{H_2} \\ = \langle F, G\rangle_{H_2^{(n)}}.\end{aligned}$$

□

Note that the Cayley transform $\Phi(\lambda) = i\frac{1-\lambda}{1+\lambda}$, $\lambda \in \mathbb{D}$ maps the unit disc $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{C}^+$ so that we could deal with Hardy-Sobolev spaces on $\mathbb{D}$ instead on the upper half-plane according to the following proposition.

Proposition 3.1. *Let F be an holomorphic function on $\mathbb{C}^+$ and $F_{\mathbb{D}}$ be the function on the unit disc given by $F_{\mathbb{D}}(\lambda) = F\left(i\frac{1-\lambda}{1+\lambda}\right)$, $\lambda \in \mathbb{D}$. Then $F \in H_2^{(1)}$ if and only if $F_{\mathbb{D}} \in (1+\lambda)H_2(\mathbb{D})$ and $F'_{\mathbb{D}} \in (1-\lambda)^{-1}H_2(\mathbb{D})$.*

Proof. The condition $F \in H_2^{(1)}$ is equivalent to $F \in H_2(\mathbb{C}^+)$ and $zF' \in H_2(\mathbb{C}^+)$ by the definition of Hardy-Sobolev spaces. Note that $F \in H_2(\mathbb{C}^+)$ if and only if $F_{\mathbb{D}} \in (1+\lambda)H_2(\mathbb{D})$ [20, Theorem 13.17], it follows that $zF' \in H_2(\mathbb{C}^+)$ is equivalent to

$$i\frac{1-\lambda}{1+\lambda}F'\left(i\frac{1-\lambda}{1+\lambda}\right) \in (1+\lambda)H_2(\mathbb{D}). \tag{3.1}$$

In addition, we have

$$F'_{\mathbb{D}}(\lambda) = \frac{-2i}{(1+\lambda)^2}F'\left(i\frac{1-\lambda}{1+\lambda}\right). \tag{3.2}$$

Combining formula (3.1) and (3.2), we obtain $F'_{\mathbb{D}}(\lambda) \in (1-\lambda)^{-1}H_2(\mathbb{D})$. □

Proposition 3.1 can be extended for $n > 1$ in the same way so that it offers a tool to investigate the function in Hardy-Sobolev spaces. We give another proposition of functions about Hardy-Sobolev spaces to end this section.

Proposition 3.2. *If $F \in H_2^{(n)}$, then $z^{k-1}F^{(k-1)}$, $k = 1, 2, \ldots, n$ can be extended continuously at all nonzero point on the real axis and the point at infinity is included.*

Proof. Let $F \in H_2^{(1)}$. By Proposition 3.1, we have $F_{\mathbb{D}} \in (1+\lambda)H_2(\mathbb{D}) \subseteq H_2(\mathbb{D})$ and $(1-\lambda)F'_{\mathbb{D}} \in H_2(\mathbb{D})$. Note that

$$[(1-\lambda)F_{\mathbb{D}}(\lambda)]' = -F_{\mathbb{D}}(\lambda) + (1-\lambda)F'_{\mathbb{D}}(\lambda) \subseteq H_2(\mathbb{D}) \subseteq H_1(\mathbb{D}).$$

Then, function $(1-\lambda)F_{\mathbb{D}}(\lambda)$ can be extended continuously on the unit circle, that is, $F_{\mathbb{D}}$ extends by continuity to the unit circle except possibly at 1. Since the Cayley transform maps 1 to 0 and -1 to ∞, the conclusion on the extensibility of F by continuity follows. Now, we take $F \in H_2^{(n)}$, $n > 1$. In view of

$$z\frac{d}{dz}\left(z^{k-1}F^{(k-1)}(z)\right) = (k-1)z^{k-1}F^{(k-1)}(z) + z^k F^k(z), \qquad z \in \mathbb{C}^+,$$

we have $z^{k-1}F^{(k-1)}(z)$, which belongs to $H_2^{(1)}$ for $1 \le k \le n$. Thus the continuity on the real axis of functions $z^{k-1}F^{(k-1)}$, $k = 1, 2, \ldots, n$ follows by the previous discussion. □

4. Reproducing Kernel of Hardy-Sobolev spaces

By applying Theorem 3.1, we can determine the integral form of reproducing kernels about Hardy-Sobolev spaces.

Theorem 4.1. *Let n be a positive integer. Then the function K_n defined on $\mathbb{C}^+ \times \mathbb{C}^+$ by*

$$K_n(z,w) = \frac{1}{((n-1)!)^2} \int_0^1 \int_0^1 (1-y)^{n-1}(1-x)^{n-1} \frac{i}{yz - x\overline{w}} dxdy, \qquad z,w \in \mathbb{C}^+,$$

is the reproducing kernel of $H_2^{(n)}$, i.e., if $K_{n,w}(z) = K_n(z,w)$, then $K_{n,w} \in H_2^{(n)}$ and $F(w) = \langle F, K_{n,w}\rangle_{H_2^{(n)}}$ for all $F \in H_2^{(n)}$.

Proof. Take $w \in \mathbb{C}^+$ and $n \in \mathbb{N}^*$, and let

$$\varphi_{w,n}(t) = \frac{1}{t^n} \int_0^1 \frac{(1-x)^{n-1}}{(n-1)!} e^{-i\overline{w}tx} dx, \qquad t > 0.$$

Observe that $\varphi_{w,n} \in L^2(t^n)$. In fact,

$$\left(\int_0^\infty |\varphi_{w,n}(t)|^2 t^{2n} dt\right)^{\frac{1}{2}} \leq \left(\int_0^\infty \left(\int_0^1 \frac{(1-x)^{n-1}}{(n-1)!} e^{-(\Im w)tx} dx\right)^2 dt\right)^{\frac{1}{2}}$$

$$\leq \int_0^1 \frac{(1-x)^{n-1}}{(n-1)!} \left(\int_0^\infty e^{-2(\Im w)tx}\right)^{\frac{1}{2}} dx = \frac{\sqrt{\frac{\pi}{2}}}{\Gamma(n+\frac{1}{2})\sqrt{\Im w}}.$$

Hence, $g_{w,n}(t) = W^{-n}\varphi_{w,n}(t) \in \mathscr{T}_2^{(n)}$ and

$$g_{w,n}(t) = \frac{1}{((n-1)!)^2} \int_t^\infty \frac{(s-t)^{n-1}}{s^n} \int_0^1 (1-x)^{n-1} e^{-i\overline{w}sx} dxds.$$

By Lemma 3.1, we have

$$g_{w,n}^{(n)}(t) = (-1)^n \varphi_{w,n}(t) = \frac{(-1)^n}{t^n} \int_0^1 \frac{(1-x)^{n-1}}{(n-1)!} e^{-i\overline{w}tx} dx = \frac{(-1)^n}{t^{2n}} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} e^{-i\overline{w}s} ds.$$

Taking $F \in H_2^{(n)}$, we see by Theorem 3.1 that there exists a function $f \in \mathscr{T}_2^{(n)}$ such that $F = \mathscr{F}(f)$. By applying Fubini's theorem, we have, for all $f \in \mathscr{T}_2^{(n)}$,

$$\begin{aligned} F(w) = \mathscr{F}(f)(w) = \int_0^\infty f(t)e^{iwt} dt &= (-1)^n \int_0^\infty \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f^{(n)}(s) ds\, e^{iwt} dt \\ &= \int_0^\infty f^{(n)}(t) \int_0^t (-1)^n \frac{(t-s)^{n-1}}{(n-1)!} e^{iws} ds \\ &= \int_0^\infty f^{(n)}(t) t^{2n} \overline{g_{w,n}^{(n)}(t)} dt \\ &= \langle f, g_{w,n}\rangle_{\mathscr{T}_2^{(n)}}. \end{aligned}$$

On the other hand, using the change of variables $s = \frac{t}{y}$, we have

$$g_{w,n}(t) = \int_0^1 \int_0^1 \frac{(1-y)^{n-1}(1-x)^{n-1}}{y[(n-1)!]^2} e^{\frac{-i\overline{w}tx}{y}} dxdy.$$

For $z \in \mathbb{C}^+$, we see that there exists

$$\begin{aligned}\mathscr{F}(g_{w,n})(z) &= \int_0^1 \int_0^1 \frac{(1-y)^{n-1}(1-x)^{n-1}}{y[(n-1)!]^2} \mathscr{F}(e^{\frac{-i\overline{w}tx}{y}})(z)dxdy \\ &= \int_0^1 \int_0^1 \frac{(1-y)^{n-1}}{(n-1)!} \frac{(1-x)^{n-1}}{(n-1)!} \frac{i}{yz - x\overline{w}} dxdy \\ &= K_{n,w}(z).\end{aligned}$$

It follows from Theorem 3.1 that $F(w) = \langle f, g_{w,n} \rangle_{\mathscr{T}_2^{(n)}} = \langle F, K_{n,w} \rangle_{H_2^{(n)}}$, $w \in \mathbb{C}^+$. $\square$

In the rest of this section, we need to estimate the norm of the reproducing kernel, which is important for the boundedness of weighted composition operators in Section 5. Before we give the estimation, we first prove the following key lemma.

Lemma 4.1. *For $\theta \in (0, \pi)$, let $I(\theta) = \int_0^1 \frac{sin\theta}{t^2+1-2tcos2\theta} dt$. Then*

$$I(\theta) = \begin{cases} \frac{1}{2}, & \theta = \frac{\pi}{2}; \\ \frac{\pi - 2\theta}{4cos\theta}, & \theta \neq \frac{\pi}{2}. \end{cases}$$

and $\frac{1}{2} \le I(\theta) \le \frac{\pi}{4}$.

Proof. A straightforward calculation shows that $I(\frac{\pi}{2}) = \int_0^1 \frac{dt}{(t+1)^2} = \frac{1}{2}$. For $0 < \theta < \frac{\pi}{2}$, we have

$$\begin{aligned}I(\theta) &= \int_0^1 \frac{sin\theta}{(t - cos2\theta)^2 + sin^2 2\theta} dt \\ &= \frac{1}{2cos\theta} \int_0^{\frac{1}{sin2\theta}} \frac{dx}{\left(x - \frac{cos2\theta}{sin2\theta}\right)^2 + 1} \\ &= \frac{1}{2cos\theta} \int_{-\frac{cos2\theta}{sin2\theta}}^{\frac{sin\theta}{cos\theta}} \frac{dy}{y^2+1} = \frac{\pi - 2\theta}{4cos\theta}.\end{aligned}$$

Note that if $t \in (0, \frac{\pi}{2}), 1 \le \frac{t}{sint} \le \frac{\pi}{2}$, we have $I(t) = \frac{t}{2sint} \in \left[\frac{1}{2}, \frac{\pi}{4}\right]$ with changing the variables $t = \frac{\pi}{2} - \theta$. Since the case that $\frac{\pi}{2} < \theta < \pi$ is similar to the case $0 < \theta < \frac{\pi}{2}$, we omit the proof here. $\square$

The following Theorem 4.2 gives the estimation of the reproducing kernel.

Theorem 4.2. *Let n be the positive integer. Then, for $z \in \mathbb{C}^+$,*

$$\frac{1}{(n-1)!\sqrt{2n-1}} \frac{1}{\sqrt{|z|}} \le \|K_{n,z}\|_{H_2^{(n)}} \le \frac{\sqrt{\pi}}{(n-1)!\sqrt{n}} \frac{1}{\sqrt{|z|}}.$$

Proof. For $z = |z|(cos\theta + isin\theta) \in \mathbb{C}^+$, we have

$$\begin{aligned}\frac{i}{yz - x\overline{z}} &= \frac{i}{|z|} \frac{1}{y(cos\theta + isin\theta) - x(cos\theta - isin\theta)} \\ &= \frac{i}{|z|} \frac{1}{(y-x)cos\theta + i(y+x)sin\theta} \\ &= \frac{1}{|z|} \frac{(y+x)sin\theta - i(y-x)cos\theta}{x^2 + y^2 - 2xycos2\theta}.\end{aligned}$$

Since the norm is always real, we derive by using the symmetry of the domain of integration that

$$\begin{aligned}
\|K_{n,z}\|^2_{H_2^{(n)}} &= \langle K_{n,z}, K_{n,z}\rangle_{H_2^{(n)}} = K_{n,z}(z)\\
&= \frac{1}{[(n-1)!]^2}\int_0^1\int_0^1 (1-y)^{n-1}(1-x)^{n-1}\frac{i}{yz-x\bar{z}}dxdy\\
&= \frac{1}{[(n-1)!]^2}\frac{1}{|z|}\int_0^1\int_0^1 (1-y)^{n-1}(1-x)^{n-1}\frac{(x+y)sin\theta}{x^2+y^2-2xycos2\theta}dxdy\\
&= \frac{2}{[(n-1)!]^2|z|}\int_0^1\int_0^1 \frac{(1+t)(1-yt)^{n-1}(1-y)^{n-1}sin\theta}{t^2+1-2tcos2\theta}dtdy.
\end{aligned}$$

A straightforward calculation shows that $(1-y)^{n-1} \le (1+t)(1-yt)^{n-1} \le 2$ for $t, y \in (0,1)$. Hence

$$\|K_{n,z}\|^2_{H_2^{(n)}} \le \frac{4}{[(n-1)!]^2|z|}\int_0^1\int_0^1 \frac{(1-y)^{n-1}sin\theta}{t^2+1-2tcos2\theta}dtdy \le \frac{\pi}{[(n-1)!]^2 n}\frac{1}{|z|}.$$

On the other hand, we have

$$\begin{aligned}
\|K_{n,z}\|^2_{H_2^{(n)}} &\ge \frac{2}{[(n-1)!]^2|z|}\int_0^1\int_0^1 \frac{(1-y)^{2n-2}sin\theta}{t^2+1-2tcos2\theta}dtdy\\
&= \frac{2I(\theta)}{[(n-1)!]^2(2n-1)|z|} \ge \frac{1}{[(n-1)!]^2(2n-1)}\frac{1}{|z|}.
\end{aligned}$$

This completes the proof. □

5. Operator Theory

In this section, we are in a position to consider the multiplication operator and the weighted composition operator on $H_2^{(n)}$, Hardy-Sobolev spaces . We first study the spectrum of multiplication operators and list some facts.

Lemma 5.1. *If $\varphi \in \mathscr{M}_n$, then M_φ is a bounded linear operator from $H_2^{(n)}$ to itself.*

Proof. Assume that $f_m \to 0$ in $H_2^{(n)}$ and $M_\varphi(f_m) \to g$ in $H_2^{(n)}$. Note that $H_2^{(n)}$ is a RKHS so that, for each $z \in \mathbb{C}^+$, point evaluation functional at z is bounded, which means that there exists a constant $M > 0$ such that, for all functions $f \in H_2^{(n)}$, $|f(z)| \le M\|f\|$. For each $z \in \mathbb{C}^+$, $g(z) = \lim_{m\to\infty} M_\varphi(f_m)(z) = \varphi(z)\lim_{m\to\infty} f_m(z) = 0$. Thus $g = 0$, and we complete the proof by applying the close graph theorem. □

From Lemma 5.1, we can define the norm on $\mathscr{M}_n$ by $\|\varphi\|_{\mathscr{M}_n} = \|M_\varphi\| = \sup_{\|f\|_{H_2^{(n)}}\le 1}\|\varphi f\|_{H_2^{(n)}}$.

Lemma 5.2. *If $\varphi \in \mathscr{M}_n$, then $M^*_\varphi(K_{n,w}) = \overline{\varphi(w)}K_{n,w}$, $w \in \mathbb{C}^+$.*

Proof. For all $w \in \mathbb{C}^+$, if $f \in H_2^n$, then

$$\langle f, M^*_\varphi(K_{n,w})\rangle = \langle M_\varphi(f), K_{n,w}\rangle = \langle \varphi f, K_{n,w}\rangle = \varphi(w)f(w) = \langle f, \overline{\varphi(w)}K_{n,w}\rangle.$$

Hence $M^*_\varphi(K_{n,w}) = \overline{\varphi(w)}K_{n,w}$. □

Lemma 5.3. *If $\varphi \in \mathscr{M}_n$, then $\varphi \in H^\infty$ and $\|\varphi\|_{H^\infty} \leq \|\varphi\|_{\mathscr{M}_n}$.*

Proof. Let $\varphi \in \mathscr{M}_n$. It follows from Lemma 5.1 that M_φ^* is a bounded linear operator on $H_2^{(n)}$. Therefore, the eigenvalues of M_φ^* are bounded and their modulus is less than or equal to the spectral radius of M_φ^*, and also no more than $\|M_\varphi^*\|$. Note that $\|M_\varphi\| = \|M_\varphi^*\|$. According to the Lemma 5.2, we derive that the values of φ are bounded and of modulus no more than $\|\varphi\|_{\mathscr{M}_n}$. □

Lemma 5.4. *If $\varphi \in H_\infty^{(n)}$, then $\varphi \in \mathscr{M}_n$.*

Proof. Note that if $\varphi \in H_\infty^{(n)}$, then there exists a constant M such that, for all $k \in \mathbb{N}$ and $0 \leq k \leq n$, $\sup_{z,k} |z^k \varphi^{(k)}| \leq M$. Hence, if $f \in H_2^{(n)}$, then

$$\|\varphi f\|_{H_2^{(n)}} = \|z^n(\varphi f)^{(n)}\|_{H_2} = \|z^n \sum_{k=0}^{n} \binom{n}{k} \varphi^{(k)} f^{(n-k)}\|_{H_2} \leq M \sum_{k=0}^{n} \binom{n}{k} \|z^{n-k} f^{(n-k)}\|_{H_2}.$$

The last formula is finite due to the definition of the spaces $H_2^{(n)}$. This completes the proof. □

Next, we characterize the spectrum of multiplication operator M_φ.

Theorem 5.1. *For any $\varphi \in \mathscr{M}_n$, the spectrum of the multiplication operator $M_\varphi : H_2^{(n)} \to H_2^{(n)}$ is given by $\sigma(M_\varphi) = \overline{\varphi(\mathbb{C}^+)}$, which is the closure of the range of φ in the complex plane.*

Before we prove this theorem, we need the following proposition about $\mathscr{M}_n$.

Proposition 5.1. *If $\varphi \in \mathscr{M}_n$ and there exists a positive consant δ such that $|\varphi(z)| \geq \delta$ for all $z \in \mathbb{C}^+$, then $\frac{1}{\varphi} \in \mathscr{M}_n$.*

We use the following Lemma 5.5 to prove Proposition 5.1.

Lemma 5.5. *[7] Let N be a positive integer, f and g be N times differentiable on the open interval I, and g be non-vanishing on I. Then $D^N(\frac{f}{g}) = \frac{(-1)^N}{g^{N+1}} \sum_{k=0}^{N} (-1)^k \binom{N+1}{k} g^k D^N(g^{N-k} f)$. where $D^n = \frac{d^n}{dx^n}$, $n \geq 0$ is the n-th order differential operator.*

First, we prove Proposition 5.1.

Proof. We need to show that $\|z^n \left(\frac{f}{\varphi}\right)^{(n)}\|_{H_2} < \infty$ for all $f \in H_2^{(n)}$. If $\varphi \in \mathscr{M}_n$, then each function $\varphi^{(n-k)} f$ belongs to $H_2^{(n)}$, which means that $z^n D^n(\varphi^{(n-k)} f) \in H_2$ for all integer $0 \leq k \leq n$. By Lemma 5.5, we derive $\|\frac{f}{\varphi}\|_{H_2^{(n)}} = \|z^n \left(\frac{f}{\varphi}\right)^{(n)}\|_{H_2} = \|\frac{(-1)^n}{\varphi^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n+1}{k} \varphi^k z^n D^n(\varphi^{(n-k)} f)\|_{H_2}$. The last equation is finite since φ and $\frac{1}{\varphi}$ are both bounded holomorphic functions on $\mathbb{C}^+$. Thus $\frac{f}{\varphi} \in H_2^{(n)}$ whenever $f \in H_2^{(n)}$. □

Now we are in a position to prove Theorem 5.1.

Proof. For any $w \in \mathbb{C}^+$, it is easy to see that $\overline{\varphi(w)}$ is an eigenvalue of the operator M_φ^* according to the Lemma 5.2. Thus $\overline{\varphi(w)} \in \sigma(M_\varphi^*)$, which implies that $\varphi(w) \in \sigma(M_\varphi)$. Then we derive $\varphi(\mathbb{C}^+) \subseteq \sigma(M_\varphi)$. If $\lambda \in \mathbb{C} - \varphi(\mathbb{C}^+)$, then there exists a positive constant δ such that $|\lambda - \varphi(w)| \geq \delta$ for all $w \in \mathbb{C}^+$. Since $\lambda - \varphi \in \mathscr{M}_n$ due to the linear property of $\mathscr{M}_n$, we see that

$\psi = \frac{1}{\lambda-\varphi}$ is also the multiplier of $H_2^{(n)}$ by Proposition 5.1. It is easy to check that $M_\psi(\lambda I - M_\varphi) = (\lambda I - M_\varphi)M_\psi = I$. Hence, $\lambda I - M_\varphi$ is invertible, so $\lambda \notin \sigma(M_\varphi)$, which implies that $\sigma(M_\varphi) \subseteq \varphi(\mathbb{C}^+)$. Therefore, we conclude $\sigma(M_\varphi) = \varphi(\mathbb{C}^+)$. Taking closure on both sides, we complete the proof immediately. □

Corollary 5.1. *If $\varphi \in \mathscr{M}_n$, then $r(M_\varphi) = \|\varphi\|_\infty$, where $r(\cdot)$ denotes the spectral radius of operator.*

In the rest of this section, we discuss the boundedness of weighted composition operators on Hardy-Sobolev spaces. Recall that the notation $\mathscr{S}$ denotes the class of all analytic self-maps on $\mathbb{C}^+$. The following Julia-Carathéodory theorem for the upper half-plane was established in [18].

Lemma 5.6. *(Julia-Carathéodory) For $\varphi \in \mathscr{S}$, the following statement are equivalent:*
(a) $\varphi(\infty) = \infty$ and $\varphi'(\infty) < \infty$;
(b) $\sup_{z\in\mathbb{C}^+} \frac{\mathrm{Im} z}{\mathrm{Im}\varphi(z)} < \infty$;
(c) $\limsup_{z\to\infty} \frac{\mathrm{Im} z}{\mathrm{Im}\varphi(z)} < \infty$.
Moreover, the quantities in (b) and (c) are both equal to $\varphi'(\infty)$.

Next, we present a property of analytic self-map $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$.

Lemma 5.7. *If $\varphi \in \mathscr{S}$ and satisfies $\sup_{z\in\mathbb{C}^+} \frac{|z|}{|\varphi(z)|} < \infty$, then $\varphi(\infty) = \infty$ and $\varphi'(\infty) < \infty$.*

Proof. Let γ be the Caley transformation $\gamma(\lambda) = i\frac{1+\lambda}{1-\lambda}$, $\lambda \in \mathbb{D}$, which conformally maps $\mathbb{D}$ onto $\mathbb{C}^+$. Given $\varphi \in \mathscr{S}$, let $\phi = \gamma^{-1} \circ \varphi \circ \gamma$. Then, for $\lambda \in \mathbb{D}$ and $z = \gamma(\lambda) \in \mathbb{C}^+$, we have

$$\frac{z}{\varphi(z)} = \frac{1-\phi(\lambda)}{1+\phi(\lambda)}\frac{1+\lambda}{1-\lambda}.$$

Let $M = \sup_{z\in\mathbb{C}^+} \frac{|z|}{|\varphi(z)|}$. Since $M < \infty$, we have $\varphi(\infty) = \infty$ and $\lim\limits_{\lambda\to 1} \phi(\lambda) = 1$. In view of

$$|\frac{1-\phi(\lambda)}{1-\lambda}| \le M|\frac{1+\phi(\lambda)}{1+\lambda}|,$$

we derive $\limsup_{\lambda\to 1} |\frac{1-\phi(\lambda)}{1-\lambda}| < \infty$, so $\phi'(1) < \infty$. By the formula (2.18) in [18, p. 3178], we know that $\varphi'(\infty) < \infty$. Therefore, the proof of Lemma 5.7 follows. □

Lemma 5.8. *If $\psi \in \mathscr{M}_n$, then $C_{\psi,\varphi}$ is a bounded composition operator on $H_2^{(n)}$ if and only if $C_{\psi,\varphi} H_2^{(n)} \subseteq H_2^{(n)}$.*

Proof. If $C_{\psi,\varphi}$ is a bounded composition operator on $H_2^{(n)}$, then $\|C_{\psi,\varphi} f\|_{H_2^{(n)}} \le \|C_{\psi,\varphi}\|\|f\|_{H_2^{(n)}}$. If $f \in H_2^{(n)}$, we have $C_{\psi,\varphi} f \in H_2^{(n)}$. On the other hand, we assume that $f_m \to 0$ and $C_{\psi,\varphi} f_m \to g$. Then, for each $z \in \mathbb{C}^+$, $g(z) = \lim\limits_{m\to\infty} C_{\psi,\varphi} f_m(z) = \psi(z) \lim\limits_{m\to\infty} f(\varphi(z))$. Since $H_2^{(n)}$ is a RKHS, we have $g(z) = 0$ for all $z \in \mathbb{C}^+$ and then $g = 0$. From the closed graph principle, we obtain that $C_{\psi,\varphi}$ is bounded. □

Next, we investigate some properties of the weighted composition operator $C_{\psi,\varphi}$ on $H_2^{(n)}$ as follows.

Theorem 5.2. *If ψ is in $\mathscr{M}_n$ satisfying that there exists a positive consant δ such that $|\psi(z)| \geq \delta$ for all $z \in \mathbb{C}^+$ and $C_{\psi,\varphi}$ is a bounded composition operator on $H_2^{(n)}$ for $n \geq 1$, then the self-map φ satisfies $\varphi(\infty) = \infty$ and $\varphi'(\infty) < \infty$.*

Proof. Using the property of reproducing kernel, we can write

$$\langle C^*_{\psi,\varphi} K_{n,z}, f\rangle = \langle K_{n,z}, \psi f \circ \varphi\rangle = \overline{\psi(z) f \circ \varphi(z)} = \langle \overline{\psi(z)} K_{n,\varphi(z)}, f\rangle, \quad f \in H_2^{(n)},$$

and then $C^*_{\psi,\varphi} K_{n,z} = \overline{\psi(z)} K_{n,\varphi(z)}$, $z \in \mathbb{C}^+$. Hence,$\sup_{z\in\mathbb{C}^+} |\psi(z)| \frac{\|K_{n,\varphi(z)}\|_{H_2^{(n)}}}{\|K_{n,z}\|_{H_2^{(n)}}} \leq \|C^*_{\psi,\varphi}\| = \|C_{\psi,\varphi}\|$. By Theorem 4.2, we derive

$$B := |\psi(z)|^2 \frac{\|K_{n,\varphi(z)}\|^2_{H_2^{(n)}}}{\|K_{n,z}\|^2_{H_2^{(n)}}} \geq \frac{|\psi(z)|^2 \frac{1}{[(n-1)!]^2(2n-1)|\varphi(z)|}}{\frac{\pi}{[(n-1)!]^2 n|z|}} = \frac{n|\psi(z)|^2}{(2n-1)\pi} \frac{|z|}{|\varphi(z)|}.$$

In view of $\psi^{-1} \in H^\infty$ and $B < \infty$ for all $z \in \mathbb{C}^+$, we have $\sup_{z\in\mathbb{C}^+} \frac{|z|}{|\varphi(z)|} < \infty$. Thus, by Lemma 5.7, $\varphi(\infty) = \infty$ and $\varphi'(\infty) < \infty$. $\square$

We also give a sufficient condition for boundedness of weighted composition operators.

Theorem 5.3. *If $\psi \in H_\infty^{(n)}$, $\varphi \in \mathscr{S}$ satisfy $\varphi'(\infty) < \infty$ and $\sup_{z\in\mathbb{C}^+} |z^k \frac{\varphi^{(k)}(z)}{\varphi(z)}| < \infty$, $k = 1,2,\ldots,n$, then the weighted composition operator $C_{\psi,\varphi} : H_2^{(n)} \to H_2^{(n)}$ is bounded.*

Proof. By Lemma 5.6, we need to show that if f is an arbitrary function in $H_2^{(n)}$, then $\psi f \circ \varphi \in H_2^{(n)}$. Faà di Bruno shows that $(f\circ\varphi)^{(n)} = \sum_{k=1}^n c_k (f^{(k)}\circ\varphi)(\varphi')^{m_{k_1}}(\varphi'')^{m_{k_2}}\ldots(\varphi^{(n)})^{m_{k_n}}$, where c_k, $k = 1,2,\ldots,n$ and m_{i_j}, $i,j = 1,2\ldots,n$ are nonnegative integers and satisfy that, for $k = 1,2,\ldots,n$, $\sum_{p=1}^n p m_{k_p} = n$ and $\sum_{p=1}^n m_{k_p} = k$. It is easy to see that there exists a $M \geq 0$ such that, for $k = 0,1,\ldots,n$, $\sup_{z\in\mathbb{C}^+} |z^k \psi^{(k)}| \leq M$. Thus, we have the following formula:

$$\begin{aligned}
&\|z^n(\psi f\circ\varphi)^{(n)}\|_{H_2}\\
=&\|z^n \sum_{k=0}^n \binom{n}{k} \psi^{(n-k)} \sum_{l=1}^k c_l (f^{(l)}\circ\varphi)(\varphi')^{m_{l_1}}\ldots(\varphi^{(k)})^{m_{l_k}}\|_{H_2}\\
\leq&\sum_{k=0}^n\sum_{l=1}^k \binom{n}{k} c_l \|z^{n-k}\psi^{(n-k)} z^k (f^{(l)}\circ\varphi)(z)(\varphi'(z))^{m_{l_1}}\ldots(\varphi^{(k)}(z))^{m_{l_k}}\|_{H_2}\\
=&\sum_{k=0}^n\sum_{l=1}^k \binom{n}{k} c_l \|z^{n-k}\psi^{(n-k)}(f^{(l)}\circ\varphi)(z)\left(\frac{z\varphi'(z)}{\varphi(z)}\right)^{m_{l_1}}\cdots\left(\frac{z^k\varphi^{(k)}(z)}{\varphi(z)}\right)^{m_{l_k}}(\varphi(z))^l\|_{H_2}\\
\leq&\sum_{k=0}^n\sum_{l=1}^k \binom{n}{k} c_l M \prod_{j=1}^k \left\|\frac{z^j\varphi^{(j)}(z)}{\varphi(z)}\right\|_\infty^{m_{l_j}} \|(\varphi(z))^l(f^{(l)}\circ\varphi)(z)\|_{H_2}.
\end{aligned}$$

Not all analytic self-maps φ on $\mathbb{C}^+$ induce bounded composition operators on H_2, the necessary and sufficient condition for boundedness being $\varphi'(\infty) < \infty$ ([19]). Given that $z^l f^{(l)}(z) \in H_2$ for all $l = 1,2,\ldots,k$ and $\varphi'(\infty) < \infty$, it follows that C_φ is a bounded operator on H_2, that is,

$$\|C_\varphi(z^l f^{(l)}(z))\|_{H_2} = \|(z^l f^{(l)})\circ\varphi\|_{H_2} = \|(\varphi(z))^l(f^{(l)}\circ\varphi)(z)\|_{H_2} < \infty, \quad l = 1,2,\ldots,k,$$

so we show that $\|z^n(\psi f\circ\varphi)^{(n)}\|_{H_2}$ is finite and $\psi f\circ\varphi\in H_2^{(n)}$. The proof is completed. □

Moreover, for the richness of the properties of operators in Hardy-Sobolev spaces, we have a related conclusion based on [9, Theorem 7.1], which is also a main result in Hardy-Sobolev spaces. Applying the theorem above in the case of $H_2^{(n)}$, we have the following corollary.

Corollary 5.2. *Let $\varphi\in\mathscr{S}$ and $n\in\mathbb{N}$. Then the weighted composition operator is a bounded operator on $H_2^{(n)}$ if and only if there exists a $M\geq 0$ such that the kernel given by*

$$L_n(z,w)=\int_0^1\int_0^1(1-y)^{n-1}(1-x)^{n-1}i\left(\frac{M^2}{yz-x\overline{w}}-\frac{\overline{\psi(z)}\psi(w)}{y\varphi(z)-x\overline{\varphi(w)}}\right)dxdy,$$

is nonnegative.

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References

[1] P.L. Duren. Theory of H^p Spaces, Academic Press, 1970.
[2] R.A. Adams, J.J.F Fournier, Sobolev Spaces, Elsevier, 2003.
[3] L. He, G.F. Cao, Z.H. He, Composition operators on Hardy-Sobolev spaces, Indian J. Pure Appl. Math. 46 (2015), 255-267.
[4] J.M. Ortega. J. Fàbrega, Multipliers in Hardy-Sobolev spaces, Integral Equations Operator Theory 55 (2006), 535-560.
[5] G.F. Cao, L. He, Hardy-Sobolev spaces and their multipliers, Sci. Chin. Math. 57(2014), 2361-2368.
[6] G.F. Cao, L. He, Fredholmness of multipliers on Hardy–Sobolev spaces, J. Math. Anal. Appl. 418 (2014), 1-10.
[7] G. Cao, L. He, K. Zhu, Spectral theory of multiplication operators on Hardy–Sobolev spaces, J. Funct. Anal. 275 (2018),1259-1279.
[8] J.J.R. Espallargas, Álgebras y módulos de convolución sobre $\mathbb{R}^+$ definidos mediante derivación fraccionaria, Phd. thesis, Universidad de Zaragoza, 2008.
[9] J.E. Galé, V. Matache, P.J. Miana, L. Sánchez-Lajusticia, Hilbertian Hardy-Sobolev spaces on a half-plane, J. Math. Anal. Appl. 489 (2020), 124131.
[10] N.F.D. Ward, J.R. Partington, Construction of rational wavelets and frames in Hardy–Sobolev spaces with applications to system modeling, SIAM J. Control Optim. 36 (1998), 654-679.
[11] P. Dang, T. Qian, Y. Yang, Hardy-Sobolev derivatives of phase and amplitude, and their appliacations. Math. Meth. Appl. Sci. 35 (2012), 2017-2030.
[12] W. Arendt, Vector-valued Laplace transforms and Cauchy problems, Isr. J. Math. 59 (1987), 327-352.
[13] J.E. Galé, L. Sanchez-Lajusticia, A Sobolev algebra of Volterra type, J. Aust. Math. Soc. 92 (2012), 313-334.
[14] J.E. Galé, P.J. Miana, J.J. Royo. Estimates of the Laplace transform on convolution Sobolev algebras, J. Approx. Theory. 164 (2012), 162-178.
[15] V.I Paulsen, M. Raghupathi, An Introduction to the Theory of Reproducing Kernel Hilbert Spaces, vol. 152, Cambridge University Press, Cambridge, 2016.
[16] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, Cambridge, 1952.
[17] W. Rudin, Real and Complex Analysis, MGH, 1987.

[18] B.R. Choe, H. Koo, W. Smith, Difference of composition operators over the half-plane, Trans. Am. Math. Soc. 369 (2017), 3173-3205.

[19] S. Elliott, M.T. Jury, Composition operators on Hardy spaces of a half plane, Bull. London Math. Soc. 44 (2012), 489-495.

[20] J. Mashreghi, Representation Theorems in Hardy Spaces, vol 74, Cambridge University Press, Cambridge, 2009.

[21] Y.K. Cho, J. Kim, Atomic decomposition on Hardy–Sobolev spaces, Studia Math. 177(2006), 25-42.

[22] A. Rotkevich, Constructive description of Hardy–Sobolev spaces on strictly pseudoconvex domains, J. Geom. Anal. 32 (2022), 41.

[23] A. Rotkevich, Constructive description of Hardy–Sobolev spaces on strongly convex domains in $\mathbb{C}^n$, J. Math. Anal. Appl. 465 (2018), 1025-1038.

[24] E.M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, 1971.

[25] S.G. Samko, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach Science Publishers, 1993.