J. Nonlinear Var. Anal. 9 (2025), No. 3, pp. 309-327
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.9.2025.3.01

A CLASS OF MIXED VARIATIONAL-HEMIVARIATIONAL INEQUALITIES WITH HISTORY-DEPENDENT OPERATORS AND TIME-DEPENDENT CONSTRAINT SETS

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Abstract. The paper is devoted to investigating a new class of mixed variational-hemivariational inequalities (MVHVI) with history-dependent operators and time-dependent constraint sets. By employing a solvability result for static mixed variational-hemivariaitional inequalities in literature and a fixed-point theorem for history-dependent operators, we prove an existence result for solutions to the MVHVI based on which a convergence result of solutions to the MVHVI is established by considering its perturbation. Moreover, we apply the obtained abstract results to a quasistatic nonsmooth frictional contact problem with long memory, where the existence and convergence of weak solutions to the contact problem are derived.

Keywords. Mixed variational-hemivariational inequalities; Solvability; Convergence; Contact problem.

1. Introduction

Mixed (hemi-)variational inequalities, which are characterized by a new variable (Lagrange multiplier) compared with (hemi-)variational inequalities, have received massive attention in recent years due to their wide applications in nonlinear boundary value problems with or without unilateral constraints, contact mechanical problems as well as engineering problems. Therefore, a large number of research results including solvability, stability analysis, and applications for mixed (hemi-)variational inequalities have been obtained with the help of tools in nonlinear analysis, nonsmooth analysis, and convex analysis etc; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8] and the references therein.

Among most studies on mixed (hemi-)variational inequalities in literature, time-dependent or evolutionary problems with mixed variational inequalities are significant research issues due to their important applications in dynamic contact mechanical problems and engineering problems. Representative results on solvability and applications were obtained in [9, 10, 11] for mixed time-dependent variational inequality problems, in [12, 13, 14, 15] for mixed historical variational inequality problems, and in [16] for mixed evolutionary variational inequalities.

Compared with the abundant research results for mixed (static or time-dependent) variational inequalities and mixed static variational-hemivariational inequalities (see, e.g., [17, 18, 19, 20,

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Received 29 May 2024; Accepted 14 October 2024; Published online 1 March 2025.

21]), there are few studies in literature on mixed time-dependent or evolutionary variational-hemivariational inequalities. At the same time, most of the studies on mixed (hemi-)variational inequalities concern the constant constraint set or the whole space, which makes it convenient for us to use different test functions. While the constraint set of the considered mixed (hemi-)variational inequality varies with parameters (state variable or time variable), it is difficult for us to see the solvability and sensitivity analysis results by using the traditional ways since the test functions used may not belong to the varying constraint set. Because of the difficulty, up to our knowledge, a few researches tried to study some mixed (hemi-)variational inequalities with nonconstant constraint set and their applications in contact mechanics. For example, in 2013, Matei in [22] studied the existence of solutions for a class of static mixed variational problems with solution-dependent constraint set of Lagrange multiplier by using a fixed-point theorem. Furthermore, in [16], Matei employed the saddle point theorem and a fixed-point theorem to study the solvability of an evolutionary mixed variational problem where the constraint set depends on a given element. Both of the results obtained in the above-mentioned references were applied into a contact mechanic problem, respectively.

Let X,Y, and Z be real reflexive Banach spaces with dual spaces X^*,Y^*, and Z^*, where the norm in E and the duality bracket for the pair (E,E^*) are denoted by $\|\cdot\|_E$ and $\langle\cdot,\cdot\rangle_{E^*\times E}$ with E being X,Y, and Z. Motivated and inspired by the results mentioned above, we study the following mixed variational-hemivariational inequality (**MVHVI**) with history-dependent operator and time-dependent constraint set:

Problem 1.1. *Given $f\in C(I;X^*)$ and $g\in C(I;X)$, find $(u,\lambda)\in C(I;X)\times\mathcal{B}(I;Y)$ such that, for all $t\in I$, $(u(t),\lambda(t))\in X\times\Lambda(g(t))$ and*

$$\begin{aligned}&\langle Au(t),v-u(t)\rangle_{X^*\times X}+b(v-u(t),\lambda(t))+\langle \mathcal{S}u(t),v-u(t)\rangle_{X^*\times X}\\&\qquad+j^\circ(\gamma u(t);\gamma v-\gamma u(t))\geq\langle f(t),v-u(t)\rangle_{X^*\times X}\quad \text{for all}\quad v\in X,\qquad(1.1)\\&b(u(t),\mu-\lambda(t))\leq 0\quad\text{for all}\quad \mu\in\Lambda(g(t)).\qquad(1.2)\end{aligned}$$

where I represents an interval $[0,T]$ with $0\leq T<\infty$, $A:X\to X^*$ is an operator, $\Lambda:X\to 2^Y$ is a multivalued mapping, $\mathcal{S}:C(I;X)\to C(I;X^*)$ is a history-dependent operator, $j^\circ(\gamma u(t);\gamma v-\gamma u(t))$ represents the generalized directional derivative of the locally Lipschitz function $j:Z\to\mathbb{R}$ with $\gamma:X\to Z$ being a linear compact operator, $b:X\times Y\to\mathbb{R}$ is a bilinear function, and $C(I;X),\mathcal{B}(I;Y)$ denote the space of continuous functions and the set of bounded functions on I.

The aims of this paper are threefold. First, we study the existence of solutions for **MVHVI** by employing a solvability result in literature for static mixed variational-hemivariational inequalities and a fixed-point theorem. Second, by considering the perturbation of **MVHVI**, we investigate the convergence behavior of its solutions. Under appropriate hypotheses, it is proved that the solution sequence of the perturbed problem converges point-wisely to the solution of the original **MVHVI**. Finally, we employ the obtained abstract results to study a quasistatic nonsmooth frictional contact problem with long memory.

The rest of this paper is structured as follows. In Section 2, we provide some notation and necessary preliminary results which are useful in the sequel. The main existence result of solutions to **MVHVI** is obtained in Section 3. In Section 4, we devote to proving the convergence behavior of solutions to **MVHVI** by introducing its perturbation. Finally, in Section 5, we introduce a mathematical model for a quasistatic nonsmooth frictional contact problem with long memory and establish the existence and convergence of its weak solutions.

2. PRELIMINARIES

In this section, we review some notation and preliminaries, give assumptions for our **MVHVI**, and present an important solvability result for static mixed variational-hemivariational inequalities, which are used in the sequel. Throughout this paper, unless stated otherwise, X, Y, Z are real reflexive Banach spaces with dual spaces X^*, Y^*, Z^*, respectively. As usual, we assume that the weak and strong convergence in different spaces are denoted by " $\rightharpoonup$ " and " $\to$ ", respectively.

We first review some definitions and preliminary results in nonlinear analysis and nonsmooth analysis.

Definition 2.1. [23] A single-valued operator $A : X \to X^*$ is said to be

(a) strongly monotone if there exists $M_A > 0$ such that

$$\langle Au - Av, u - v\rangle_{X^*\times X} \geq M_A \|u - v\|_X^2 \quad \text{for all} \quad u, v \in X;$$

(b) bounded if A maps bounded sets of X into bounded sets of X^*;

(c) hemicontinuous if, for all $u, v, w \in X$, the functional $t \to \langle A(u+tv), w\rangle_{X^*\times X}$ is continuous on $[0,1]$;

(d) pseudomonotone if it is bounded and $u_n \rightharpoonup u$ in X with $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle_{X^*\times X} \leq 0$ implies

$$\langle Au, u - v\rangle_{X^*\times X} \leq \liminf_{n\to\infty}\langle Au_n, u_n - v\rangle_{X^*\times X} \quad \text{for all} \quad v \in X.$$

Definition 2.2. [23] Assume that X is a normed space. A function $j : X \mapsto \mathbb{R}$ is said to be locally Lipschitz continuous at $u \in X$ if there exist a neighborhood U of u and a constant $k = k(U) > 0$ such that

$$|j(w) - j(v)| \leq k\|w - v\|_X \quad \text{for all} \quad w, v \in U.$$

The generalized (Clarke) directional derivative of j at the point $u \in X$ in the direction $v \in X$ is defined by

$$j^\circ(u; v) = \limsup_{w\to u, \lambda\downarrow 0} \frac{j(w+\lambda v) - j(w)}{\lambda}.$$

The subdifferential (the generalized gradient) in the sense of Clarke of j at u is a subset of the dual space X^* given by

$$\partial j(u) = \{\xi \in X^* | j^\circ(u; v) \geq \langle \xi, v\rangle_{X^*\times X}, \forall v \in X\}.$$

Proposition 2.1. [23] *Assume that X is a normed space and $j : X \mapsto \mathbb{R}$ is a locally Lipschitz continuous function. Then*

(a) *The function $X \ni v \mapsto j^\circ(u; v)$ is positively homogeneous and subadditive, i.e., $j^\circ(u; \lambda v) = \lambda j^\circ(u; v)$ for all $\lambda \geq 0, v \in X$, and $j^\circ(u; v_1 + v_2) \leq j^\circ(u; v_1) + j^\circ(u; v_2)$ for all $v_1, v_2 \in X$;*

(b) *For every $v \in X$, $j^\circ(u; v) = \max\{\langle \xi, v\rangle_{X^*\times X}, \forall \xi \in \partial j(u)\}$;*

(c) *The function $X \times X \ni (u, v) \mapsto j^\circ(u; v)$ is upper semicontinuous, i.e., for all $u, v \in X$, $\{u_n\}, \{v_n\} \subset X$ such that $u_n \to u$ and $v_n \to v$ in X, $\limsup j^\circ(u_n; v_n) \leq j^\circ(u; v)$.*

Definition 2.3. [24] Given a normed space Y and a sequence $\{K_n\}$ of nonempty sets in Y, we say that K_n converges to a nonempty set $K \subset Y$ in the Mosco sense, donated by $K_n \xrightarrow{M} K$, if the following two conditions hold:
(a) for each $x \in K$, there exists a sequence $\{x_n\} \subset Y$ with $x_n \in K_n$ such that $x_n \to x$ in Y;
(b) for any sequence $\{x_n\} \subset Y$ with $x_n \in K_n$ and $x_n \rightharpoonup x$ in Y, $x \in K$.

Definition 2.4. [25] Let X, Y be normed spaces. An operator $\mathcal{S} : C(\tilde{I}; X) \to C(\tilde{I}; Y)$ is a history-dependent operator if, for any compact set $J \subset \tilde{I}$, there exists $L_J > 0$ such that

$$\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_Y \leq L_J \int_0^t \|u_1(s) - u_2(s)\|_X ds \quad \text{for all} \quad u_1, u_2 \in C(\tilde{I}; X), t \in J,$$

where $\tilde{I}$ is bounded or unbounded interval.

Remark 2.1. [25] Let X, Y vw normed spaces. An operator $\mathcal{S} : C([0,T]; X) \to C([0,T]; Y)$ is called a history-dependent operator if and only if there exists $L > 0$ such that

$$\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_Y \leq L \int_0^t \|u_1(s) - u_2(s)\|_X ds \quad \text{for all} \quad u_1, u_2 \in C([0,T]; X), t \in [0,T].$$

Theorem 2.1. [25] *Let X be a Banach space, and let $\mathcal{S} : C([0,T]; X) \to C([0,T]; X)$ be a history-dependent operator. Then the operator $\mathcal{S}$ has a unique fixed-point.*

We now present some hypotheses on the data for Problem 1.1.
$\underline{H_A}$: The operator $A : X \to X^*$ is such that
(i) A is hemicontinuous;
(ii) A is strongly monotone with constant $M_A > 0$;
(iii) A is bounded.

$\underline{H_b}$: $b : X \times Y \to \mathbb{R}$ is a bilinear form and
(i) there exists $M_b > 0$ such that $|b(u,v)| \leq M_b \|u\|_X \|v\|_Y$;
(ii) (inf-sup condition) there exists $\alpha > 0$ such that $\inf_{\substack{\mu \in Y \\ \mu \neq 0_Y}} \sup_{\substack{v \in X \\ v \neq 0_X}} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha$.

$\underline{H_j}$: The function $j : Z \to \mathbb{R}$ is such that
(i) j is locally Lipschitz continuous;
(ii) there exists $m_j \geq 0$ such that, for all $u_1, u_2 \in Z$,

$$j^\circ(u_1; u_2 - u_1) + j^\circ(u_2; u_1 - u_2) \leq m_j \|u_1 - u_2\|_Z^2;$$

(iii) there exist $\alpha_j \geq 0$ and $\beta_j \geq 0$ such that, for all $u \in Z$, $\|\partial j(u)\|_{Z^*} \leq \alpha_j + \beta_j \|u\|_Z$.

$\underline{H_\Lambda}$: $\Lambda : X \to 2^Y$ satisfies the following conditions:
(i) Λ has nonempty closed, convex values and $0_Y \in \Lambda(v)$ for all $v \in X$;
(ii) Λ is Mosco continuous, i.e., for every $\{v_n\} \subset X$ with $v_n \to v$ in X, we have $\Lambda(v_n) \xrightarrow{M} \Lambda(v)$.

$\underline{H_\mathcal{S}}$: $\mathcal{S} : C(I; X) \to C(I; X^*)$ is a history-dependent operator with constant $L > 0$.

$\underline{H_\gamma}$: $\gamma : X \to Z$ is a linear compact operator, $\|\gamma\|$ represents the norm of the operator γ in the space $\mathcal{L}(X, Z)$.

$\underline{H_f}$: $f \in C(I; X^*)$.

$\underline{H_0}$: $M_A > m_j \|\gamma\|^2$.

We end this section by introducing the following static mixed variational-hemivariational inequality problem and presenting a solvability theorem for it, which is crucial for the study of solvability for Problem 1.1.

Problem 2.1. *Given $\tilde{f} \in X^*$ and a nonempty subset $K \subset Y$, find $u \in X$ and $\lambda \in K$ such that*

$$\langle Au, v-u\rangle_{X^*\times X} + b(v-u,\lambda) + j^\circ(\gamma u;\gamma v-\gamma u) \geq \langle \tilde{f}, v-u\rangle_{X^*\times X} \quad \textit{for all} \quad v \in X,$$
$$b(u,\mu-\lambda) \leq 0 \quad \textit{for all} \quad \mu \in K.$$

Theorem 2.2. *Let $K \subset Y$ be a nonempty, closed, and convex subset with $0_Y \in K$. Let H_A, H_b, H_j, and H_γ, H_0 hold. Then Problem 2.1 has at least one solution pair (u,λ). Moreover, the first component of the solution, u, is unique.*

Proof. Theorem 2.2 is a direct consequence of [1, Theorem 15] since it is easy to verify that all the hypotheses of [1, Theorem 15] hold under the assumptions of Theorem 2.2. □

3. A SOLVABILITY RESULT

We devote this section to the existence for Problem 1.1 by employing Theorem 2.1 and Theorem 2.2. To this end, we first introduce an auxiliary problem as follows, which is formulated as a mixed time-dependent variational-hemivariational inequality problem.

Problem 3.1. *Given $w \in C(I;X)$, $f \in C(I;X^*)$ and $g \in C(I;X)$, find $(u_w,\lambda_w) \in C(I;X)\times \mathcal{B}(I;Y)$ such that, for all $t \in I$, $(u_w(t),\lambda_w(t)) \in X \times \Lambda(g(t))$ and*

$$\langle Au_w(t), v-u_w(t)\rangle_{X^*\times X} + b(v-u_w(t),\lambda_w(t)) + \langle \mathcal{S}w(t), v-u_w(t)\rangle_{X^*\times X}$$
$$+ j^\circ(\gamma u_w(t);\gamma v-\gamma u_w(t)) \geq \langle f(t), v-u_w(t)\rangle_{X^*\times X} \quad \textit{for all} \quad v \in X, \tag{3.1}$$
$$b(u_w(t),\mu-\lambda_w(t)) \leq 0 \quad \textit{for all} \quad \mu \in \Lambda(g(t)). \tag{3.2}$$

With the help of Theorem 2.2, we can obtain for Problem 3.1 the following theorem which reveals the existence of its solution pair (u,λ) in $C(I;X)\times\mathcal{B}(I;Y)$ and the uniqueness of u, i.e., the first component of solution pair.

Theorem 3.1. *Let $H_A, H_b, H_j, H_\gamma, H_\Lambda, H_f, H_\mathcal{S}, H_0$ hold. Then Problem 3.1 has at least one solution pair $(u_w,\lambda_w) \in C(I;X)\times\mathcal{B}(I;Y)$, which is unique for its first component u_w in $C(I;X)$.*

Proof. Letting $\tilde{f} = f - \mathcal{S}w$, which belongs to $C(I;X^*)$, we see that (3.1) can be reformulated equivalently as

$$\langle Au_w(t), v-u_w(t)\rangle_{X^*\times X} + b(v-u_w(t),\lambda_w(t))$$
$$+ j^\circ(\gamma u_w(t);\gamma v-\gamma u_w(t)) \geq \langle \widetilde{f}(t), v-u_w(t)\rangle_{X^*\times X} \quad \text{for all} \quad v \in X. \tag{3.3}$$

Obviously, for any $t \in I$, we can deduce from Theorem 2.2 that Problem 3.1 has a solution pair in $X\times Y$, which allows us to define a function pair $(u_w,\lambda_w): I \to X\times Y$ such that $(u_w(t),\lambda_w(t))$ is a solution pair to the system of (3.1) and (3.2) for all $t \in I$. Therefore, it is sufficient to obtain the solvability of Problem 3.1 by demonstrating that u_w is continuous and λ_w is bounded on I. We divide the detailed proof into the following two steps.
Step 1. $(u_w(t),\lambda_w(t))$ is bounded on I.

From the definition, $(u_w(t), \lambda_w(t))$ is a solution pair to the system of (3.3) and (3.2) for all $t \in I$. Setting $v = 0_X$ in (3.3) and $\mu = 0_Y$ in (3.2) and adding the obtained inequalities yield that

$$\begin{aligned} M_A \|u_w(t)\|_X^2 &\leq \langle Au_w(t) - A0_X, u_w(t)\rangle_{X^*\times X} \\ &\leq \langle \widetilde{f}(t), u_w(t)\rangle_{X^*\times X} + j^\circ(\gamma u_w(t); -\gamma u_w(t)) + \|A0\|_{X^*}\|u_w(t)\|_X, \end{aligned}$$

where the hypothesis of strong monotonicity for the operator A is used. Moreover, it is obvious to see from the definition of $\widetilde{f}$ that $\|\widetilde{f}(t)\|_{X^*} \leq \|f\|_{C(I;X^*)} + \|\mathcal{S}w\|_{C(I;X^*)}$ for all $t \in I$. Therefore, it follows from the hypotheses $H_0, H_j(ii)(iii)$ that, for all $t \in I$,

$$\|u_w(t)\|_X \leq \frac{\|A0_X\|_{X^*} + \alpha_j\|\gamma\| + \|f\|_{C(I,X^*)} + \|\mathcal{S}w\|_{C(I;X^*)}}{M_A - m_j\|\gamma\|^2},$$

which indicates that $u_w(t)$ is bounded on I.

Next, we turn to the boundedness of $\lambda_w(t)$ on I. For any $z \in X$, we substitute $u_w(t) - \frac{z}{\|z\|_X}$ $(z \neq 0_X)$ for v in (3.3) to get that

$$\langle Au_w(t), -\frac{z}{\|z\|_X}\rangle_{X^*\times X} + b(-\frac{z}{\|z\|_X}, \lambda_w(t)) + j^\circ(\gamma u_w(t); -\frac{\gamma z}{\|z\|_X}) \geq \langle \widetilde{f}(t), -\frac{z}{\|z\|_X}\rangle_{X^*\times X},$$

which together with hypotheses $H_b(ii), H_j(ii)$ yields that

$$\alpha\|\lambda_w(t)\|_Y \leq \|\widetilde{f}\|_{C(I;X^*)} + \|Au_w(t)\|_{X^*} + \alpha_j\|\gamma\| + \beta_j\|\gamma\|^2\|u_w(t)\|_X.$$

Therefore, by applying the boundedness of $u_w(t)$ obtained previously and hypothesis $H_A(iii)$, we can easily obtain that $\lambda_w(t)$ is bounded on I.

Step 2. $(u_w, \lambda_w) \in C(I;X) \times \mathcal{B}(I;Y)$.

From Step 1, $\lambda_w \in \mathcal{B}(I;Y)$. Thus it suffices to show that $u_w \in C(I;X)$. To this end, for each point $t \in I$ and any sequence $\{t_n\}$ satisfying $t_n \to t$, we show that $u_w(t_n) \to u_w(t)$ in X. On the one hand, since $(u_w(t_n), \lambda_w(t_n))$ is the solution to system (3.2)-(3.3) corresponding to t_n, it follows that

$$\begin{aligned} &\langle Au_w(t_n), v - u_w(t_n)\rangle_{X^*\times X} + b(v - u_w(t_n), \lambda_w(t_n)) \\ &\qquad + j^\circ(\gamma u_w(t_n); \gamma v - \gamma u_w(t_n)) \geq \langle \widetilde{f}(t_n), v - u_w(t_n)\rangle_{X^*\times X} \quad \text{for all} \quad v \in X, \end{aligned} \tag{3.4}$$

$$b(u_w(t_n), \mu - \lambda_w(t_n)) \leq 0 \quad \text{for all} \quad \mu \in \Lambda(g(t_n)), \tag{3.5}$$

and $\{u_w(t_n)\}$ and $\{\lambda_w(t_n)\}$ are bounded on I by Step 1. This indicates by the reflexivity of spaces X and Y that there exist a subsequence, still denoted by $(u_w(t_n), \lambda_w(t_n))$, and a pair $(\widetilde{u_w}, \widetilde{\lambda_w}) \in X \times Y$ such that $(u_w(t_n), \lambda_w(t_n)) \rightharpoonup (\widetilde{u_w}, \widetilde{\lambda_w})$ in $X \times Y$ as $n \to \infty$. Since $g \in C(I;X)$, we know that $g(t_n) \to g(t)$ in X, which together with the fact that $\lambda_w(t_n) \in \Lambda(g(t_n))$, $\lambda_w(t_n) \rightharpoonup \widetilde{\lambda_w}$ and the assumption $H_\Lambda(ii)$ implies that $\widetilde{\lambda_w} \in \Lambda(g(t))$. Moreover, using the assumption $H_\Lambda(ii)$ again, we can find a sequence $\{\widetilde{\lambda}_n\}$ in Y such that $\widetilde{\lambda}_n \in \Lambda(g(t_n))$ and $\widetilde{\lambda}_n \to \widetilde{\lambda_w}$. For all $v \in X$, we can obtain by letting $\mu = \widetilde{\lambda}_n$ in (3.5) that

$$0 \geq b(u_w(t_n), \widetilde{\lambda}_n - \lambda_w(t_n)) = b(u_w(t_n) - v, \widetilde{\lambda}_n) - b(u_w(t_n) - v, \lambda_w(t_n)) + b(v, \widetilde{\lambda}_n - \lambda_w(t_n)),$$

which implies that, for all $v \in X$,

$$\begin{aligned} b(v - u_w(t_n), \lambda_w(t_n)) &\leq b(v - u_w(t_n), \widetilde{\lambda}_n) + b(v, \lambda_w(t_n) - \widetilde{\lambda}_n) \\ &= b(v - u_w(t_n), \widetilde{\lambda_w}) + b(v - u_w(t_n), \widetilde{\lambda}_n - \widetilde{\lambda_w}) + b(v, \lambda_w(t_n) - \widetilde{\lambda}_n). \end{aligned}$$

By taking the upper limit on both sides of above inequality, we can obtain that

$$\limsup_{n\to\infty} b(v-u_w(t_n),\lambda_w(t_n)) \le b(v-\widetilde{u_w},\widetilde{\lambda_w}) \quad \text{for all} \quad v\in X, \tag{3.6}$$

which indicates that

$$\limsup_{n\to\infty} b(\widetilde{u_w}-u_w(t_n),\lambda_w(t_n)) \le 0. \tag{3.7}$$

Moreover, setting $v=\widetilde{u_w}$ in (3.4), we have

$$\begin{aligned}&\langle Au_w(t_n),u_w(t_n)-\widetilde{u_w}\rangle_{X^*\times X}\\ &\le \langle \widetilde{f}(t_n),u_w(t_n)-\widetilde{u_w}\rangle_{X^*\times X}+j^\circ(\gamma u_w(t_n);\gamma\widetilde{u_w}-\gamma u_w(t_n))+b(\widetilde{u_w}-u_w(t_n),\lambda_w(t_n)),\end{aligned}$$

which together with (3.7) yields $\limsup_{n\to\infty}\langle Au_w(t_n),u_w(t_n)-\widetilde{u_w}\rangle_{X^*\times X}\le 0$, where we use the assumptions H_f, H_γ, and Proposition 2.1(c). This permits us to use the pseudomonotonicity of A, which can be obtained by the assumption H_A to find

$$\langle A\widetilde{u_w},\widetilde{u_w}-v\rangle_{X^*\times X} \le \liminf_{n\to\infty}\langle Au_w(t_n),u_w(t_n)-v\rangle_{X^*\times X} \quad \text{for all} \quad v\in X, \tag{3.8}$$

and

$$\langle Au_w(t_n),u_w(t_n)-\widetilde{u_w}\rangle_{X^*\times X}\to 0,\ Au_w(t_n)\rightharpoonup A\widetilde{u_w} \quad \text{as} \quad n\to+\infty. \tag{3.9}$$

Therefore, it follows from (3.4), (3.6), and (3.8) that

$$\begin{aligned}\langle A\widetilde{u_w},\widetilde{u_w}-v\rangle_{X^*\times X} &\le \liminf_{n\to\infty}\langle Au_w(t_n),u_w(t_n)-v\rangle_{X^*\times X} \le \limsup_{n\to\infty}\langle Au_w(t_n),u_w(t_n)-v\rangle_{X^*\times X}\\ &\le \limsup_{n\to\infty} b(v-u_w(t_n),\lambda_w(t_n))+\limsup_{n\to\infty}\langle\widetilde{f}(t_n),u_w(t_n)-v\rangle_{X^*\times X}\\ &\quad +\limsup_{n\to\infty} j^\circ(\gamma u_w(t_n);\gamma v-\gamma u_w(t_n))\\ &\le b(v-\widetilde{u_w},\widetilde{\lambda_w})+\langle\widetilde{f}(t),\widetilde{u_w}-v\rangle_{X^*\times X}+j^\circ(\gamma\widetilde{u_w};\gamma v-\gamma\widetilde{u_w}).\end{aligned} \tag{3.10}$$

On the other hand, for any $\mu\in\Lambda(g(t))$, the assumption $H_\Lambda(ii)$ and the fact $g(t_n)\to g(t)$ ensure that there exists a sequence $\mu_n\in\Lambda(g(t_n))$ such that $\mu_n\to\mu$, which together with (3.5) implies that $b(u_w(t_n),\lambda_w(t_n)-\mu_n)\ge 0$. Moreover, by choosing $v=\widetilde{u_w}$ in (3.4), we see that

$$\begin{aligned}&\langle Au_w(t_n),\widetilde{u_w}-u_w(t_n)\rangle_{X^*\times X}+j^\circ(\gamma u_w(t_n);\gamma\widetilde{u_w}-\gamma u_w(t_n))-\langle\widetilde{f}(t_n),\widetilde{u_w}-u_w(t_n)\rangle_{X^*\times X}\\ &\ge b(u_w(t_n)-\widetilde{u_w},\lambda_w(t_n))=b(u_w(t_n),\lambda_w(t_n)-\mu_n)+b(u_w(t_n),\mu_n)-b(\widetilde{u_w},\lambda_w(t_n))\\ &\ge b(u_w(t_n),\mu_n-\mu)+b(u_w(t_n),\mu)-b(\widetilde{u_w},\lambda_w(t_n)).\end{aligned} \tag{3.11}$$

Now, by taking upper limit at the both sides of (3.11), it follows from (3.9) that

$$b(\widetilde{u_w},\mu-\widetilde{\lambda_w})\le 0 \quad \text{for all} \quad \mu\in\Lambda(g(t)). \tag{3.12}$$

In conclusion, we can assert from inequalities (3.10) and (3.12) that $(\widetilde{u_w},\widetilde{\lambda_w})$ solves system (3.1)-(3.2) corresponding to $t\in I$, which together with the uniqueness of $u_w(t)$ to system (3.1)-(3.2) entails that $\widetilde{u_w}=u_w(t)$, i.e., $u_w(t_n)\rightharpoonup u_w(t)$. Then, we further show that $u_w(t_n)\to u_w(t)$ in X. We test (3.10) by $u_w(t_n)$ and test (3.4) by $u_w(t)$ and add the resulting inequalities to find

$$\begin{aligned}&\langle Au_w(t)-Au_w(t_n),u_w(t_n)-u_w(t)\rangle_{X^*\times X}+b(u_w(t_n)-u_w(t),\widetilde{\lambda_w}-\lambda_w(t_n))\\ &\quad +j^\circ(\gamma u_w(t);\gamma u_w(t_n)-\gamma u_w(t))+j^\circ(\gamma u(t_n);\gamma u_w(t)-\gamma u_w(t_n))\\ &\ge\langle\widetilde{f}(t)-\widetilde{f}(t_n),u_w(t_n)-u_w(t)\rangle_{X^*\times X},\end{aligned}$$

which together with the hypotheses $H_A(ii)$ and $H_j(ii)$ implies that

$$(M_A - m_j\|\gamma\|^2)\|u_w(t_n) - u_w(t)\|_X^2 \leq b(u_w(t) - u_w(t_n), \lambda_w(t_n) - \widetilde{\lambda_w}) + c\|\widetilde{f}(t_n) - \widetilde{f}(t)\|_{X^*}. \quad (3.13)$$

Since $\Lambda(g(t_n)) \xrightarrow{M} \Lambda(g(t))$ by $H_\Lambda(ii)$, it follows from $\widetilde{\lambda_w} \in \Lambda(g(t))$ that there exists a sequence $\{\widetilde{\lambda}(t_n)\}$ in Y such that $\widetilde{\lambda}(t_n) \in \Lambda(g(t_n))$ and $\widetilde{\lambda}(t_n) \to \widetilde{\lambda_w}$, which allows us to take $\mu = \widetilde{\lambda}(t_n)$ in (3.5) to see that $b(u_w(t_n), \widetilde{\lambda}(t_n) - \lambda_w(t_n)) \leq 0$. Therefore, combing the hypothesis $H_b(i)$, we deduce that

$$\begin{aligned}
&\limsup_{n\to\infty} b(u_w(t) - u_w(t_n), \lambda_w(t_n) - \widetilde{\lambda_w}) \\
&= \limsup_{n\to\infty} b(u_w(t_n), \widetilde{\lambda_w} - \widetilde{\lambda}(t_n)) + \limsup_{n\to\infty} b(u_w(t_n), \widetilde{\lambda}(t_n) - \lambda_w(t_n)) \\
&\quad + \limsup_{n\to\infty} b(u_w(t), \lambda_w(t_n) - \widetilde{\lambda_w}) \leq 0. \qquad (3.14)
\end{aligned}$$

By taking upper limit at the both sides of (3.13) and using H_0 , we see from (3.14) that

$$\limsup \|u_w(t_n) - u_w(t)\|_{X^*}^2 \leq 0,$$

which implies that $u_w(t_n) \to u_w(t)$ as $n \to \infty$. It is obvious that the above argument shows that every subsequence of the sequence $\{u_w(t_n)\}$ contains a subsequence converging to the unique $u_w(t)$, which implies that the whole sequence $u_w(t_n) \to u_w(t)$ by [23, Proposition 1.14].

Finally, for the uniqueness of u_w in $C(I;X)$ with (u_w, λ_w) being the solution pair to Problem 3.1, it is a direct consequence of the uniqueness of $u_w(t)$ in X with $(u_w(t), \lambda_w(t))$ being the solution pair of (3.1)-(3.2) for all $t \in I$. □

By virtue of Theorem 3.1, we can define a mapping $K : C(I;X) \to C(I;X)$ by

$$Kw = u_w \quad \text{for all} \quad w \in C(I;X), \qquad (3.15)$$

where u_w is the unique first component of solution pair to Problem 3.1 corresponding to w. We now prove that K has a unique fixed point to see the following solvability result for Problem 1.1.

Theorem 3.2. *Let $H_A, H_b, H_j, H_\gamma, H_\mathcal{S}, H_\Lambda, H_f, H_0$ hold. Then Problem 1.1 has at least one solution pair $(u, \lambda) \in C(I;X) \times \mathcal{B}(I;Y)$ which is unique for its first component u in $C(I;X)$.*

Proof. As stated before, it is sufficient to prove that the mapping K defined by (3.15) has a unique fixed point to obtain the existence of solution pair to Problem 1.1 and the uniqueness of its first component. In fact, for any $w_i \in C(I;X)$ with $i = 1,2$, let (u_{w_i}, λ_{w_i}) be the solution pair of Problem 3.1 corresponding to w_i, respectively. To simplify the writing, for $i = 1,2$, (u_{w_i}, λ_{w_i}) is denoted by (u_i, λ_i) in the following. For $i \neq j = 1,2$, we test with functions $u_j(t)$ and $\lambda_j(t)$ in two inequalities of Problem 3.1 corresponding to w_i respectively and add the resulting inequalities to find, for all $t \in I$,

$$\begin{aligned}
&\langle Au_1(t) - Au_2(t), u_1(t) - u_2(t)\rangle_{X^*\times X} \leq b(u_1(t) - u_2(t), \lambda_2(t) - \lambda_1(t)) \\
&+ j^\circ(\gamma u_1(t); \gamma u_2(t) - \gamma u_1(t)) + j^\circ(\gamma u_2(t); \gamma u_1(t) - \gamma u_2(t)) \\
&+ \langle \mathcal{S}w_1(t) - \mathcal{S}w_2(t), u_2(t) - u_1(t)\rangle_{X^*\times X}, \\
&b(u_1(t) - u_2(t), \lambda_2(t) - \lambda_1(t)) \leq 0.
\end{aligned}$$

Now, from the hypotheses $H_A(ii), H_j(ii), H_{\mathcal{S}}, H_0$, it follows from (3.15) that

$$\|Kw_1(t) - Kw_2(t)\|_X \leq \frac{L}{M_A - m_j\|\gamma\|^2} \int_0^t \|w_1(s) - w_2(s)\|_X ds \quad \text{for all} \quad t \in I,$$

which indicates that K is a history-dependent operator. Thus K has a unique fixed point by Theorem 2.1. □

4. A Convergence Result

In this section, based on the solvability result obtained in Section 3, we prove a convergence result which reveals the continuous dependence in the sense of point by point of u, i.e., the first component of the solution pair to Problem 1.1, on the data $\mathcal{S}, f$ and g in Problem 1.1. We denote by $\mathcal{S}_n, f_n$ and g_n the perturbation of data $\mathcal{S}, f$ and g in Problem 1.1, which corresponds the following perturbation of Problem 1.1, i.e., a perturbed mixed variational-hemivariational inequality with history-dependent operator and time-dependent constraint set.

Problem 4.1. *Find $u_n \in C(I;X)$ and $\lambda_n \in \mathcal{B}(I;Y)$ such that, for all $t \in I$, $(u_n(t), \lambda_n(t)) \in X \times \Lambda(g_n(t))$ and*

$$\begin{aligned} &\langle Au_n(t), v - u_n(t)\rangle_{X^*\times X} + b(v - u_n(t), \lambda_n(t)) + \langle \mathcal{S}_n u_n(t), v - u_n(t)\rangle_{X^*\times X} \\ &\qquad + j^\circ(\gamma u_n(t); \gamma v - \gamma u_n(t)) \geq \langle f_n(t), v - u_n(t)\rangle_{X^*\times X} \quad \textit{for all} \quad v \in X, \qquad (4.1) \\ &b(u_n(t), \mu - \lambda_n(t)) \leq 0 \quad \textit{for all} \quad \mu \in \Lambda(g_n(t)). \qquad (4.2) \end{aligned}$$

Before we present the convergence result of this section, we first present some additional hypotheses on the relationship between the original data and the perturbed data in Problems 1.1 and 4.1

- $\underline{H_1}$: For all $n \in \mathbb{N}$, $\mathcal{S}_n : C(I;X) \to C(I;X^*)$ is a history-dependent operator with a positive bounded constant $C_{\mathcal{S}_n}$ and $\mathcal{S}_n w \to \mathcal{S}w$ in $C(I;X^*)$ for all $w \in C(I;X)$.
- $\underline{H_2}$: $f_n \in C(I;X^*)$ is such that $f_n \to f$ in $C(I;X^*)$ and $g_n \in C(I;X)$ is such that $g_n(t) \to g(t)$ in X for all $t \in I$.

The main results in this section are summarized in the following theorem where the solvability of Problem 4.1 and a convergence result are presented.

Theorem 4.1. *Let $H_A, H_b, H_j, H_\gamma, H_{\mathcal{S}}, H_\Lambda, H_f, H_0, H_1$, and H_2 hold. Then*
(i) for each $n \in \mathbb{N}$, Problem 4.1 has a solution pair $(u_n, \lambda_n) \in C(I;X) \times \mathcal{B}(I;Y)$ which is unique and bounded for its first component in $C(I;X)$;
(ii) for all $t \in I$, $u_n(t) \to u(t)$ in X as $n \to \infty$, where $u \in C(I;X)$ is the first component of the solution pair to Problem 1.1.

Proof. (i) Under the hypotheses in Theorem 4.1, the existence of solution pairs to Problems 1.1 and 4.1 are guaranteed by Theorem 3.2 and we assume that (u, λ) and (u_n, λ_n) are solution pairs to Problems 1.1 and 4.1, respectively. We now prove the boundedness of $\{u_n\}$ in $C(I;X)$. Similar to the techniques used in the proof of Theorem 3.1, we choose two special test functions, $v = 0_X$ in (4.1) and $\mu = 0_Y$ in (4.2), and add the obtained inequalities to derive that

$$\langle Au_n(t), u_n(t)\rangle_{X^*\times X} \leq \langle \mathcal{S}_n u_n(t), -u_n(t)\rangle_{X^*\times X} + j^\circ(\gamma u_n(t); -\gamma u_n(t)) + \langle f_n(t), u_n(t)\rangle_{X^*\times X}.$$

Moreover, by taking advantage of the hypotheses $H_A(ii)$, $H_j(ii)$ we can further deduce that

$$(M_A - m_j\|\gamma\|^2)\|u_n(t)\|_X \leq \|\mathcal{S}_n u_n(t)\|_{X^*} + \|f_n\|_{C(I;X^*)} + \alpha_j\|\gamma\| + \|A0\|_{X^*}. \qquad (4.3)$$

By virtue of the assumption H_1, there exists a constant $M>0$ such that $C_{\mathcal{S}_n}<M$. Thus we can infer that

$$\|\mathcal{S}_n u_n(t)\|_{X^*} \le C_{\mathcal{S}_n}\int_0^t \|u_n(s)\|_X ds + \|\mathcal{S}_n 0\|_{C(I;X^*)} \le M\int_0^t \|u_n(s)\|_X ds + \|\mathcal{S}_n 0\|_{C(I;X^*)},$$

which together with (4.3) and H_2 implies that

$$(M_A - m_j\|\gamma\|^2)\|u_n(t)\|_X \le M\int_0^t \|u_n(s)\|_X ds + \|\mathcal{S}_n 0\|_{C(I;X^*)} + \|f_n\|_{C(I;X^*)} + \alpha_j\|\gamma\| + \|A0\|_X.$$

This allows us to use Gronwall inequality and the assumption H_0 to see that

$$\|u_n(t)\|_X \le \frac{\|\mathcal{S}_n 0\|_{C(I;X^*)} + \|f_n\|_{C(I;X^*)} + \alpha_j\|\gamma\| + \|A0\|_{X^*}}{M_A - m_j\|\gamma\|^2} e^{\frac{TM}{M_A - m_j\|\gamma\|^2}}. \tag{4.4}$$

By hypotheses H_1 and H_2, $\{\mathcal{S}_n 0\}$ and $\{f_n\}$ are bounded. Thus there exists a constant $c>0$ such that $\|\mathcal{S}_n 0\|_{C(I;X^*)} + \|f_n\|_{C(I;X^*)} \le c$. Therefore, it follows from (4.4) that

$$\|u_n\|_{C(I;X)} = \max_{t\in I}\|u_n(t)\|_X \le \frac{c+\alpha_j\|\gamma\| + \|A0\|_{X^*}}{M_A - m_j\|\gamma\|^2} e^{\frac{TM}{M_A - m_j\|\gamma\|^2}},$$

i.e., $\{u_n\}$ is bounded in $C(I;X)$.

(ii) In order to prove the convergence result, we introduce the following auxiliary problem with the first component u of solution pair to Problem 1.1.

Problem 4.2. *Find $\tilde{u}\in C(I;X)$ and $\tilde{\lambda}\in \mathcal{B}(I;Y)$ such that, for all $t\in I$, $(\tilde{u}(t),\tilde{\lambda}(t))\in X\times\Lambda(g(t))$ and*

$$\begin{aligned}
&\langle A\tilde{u}(t), v-\tilde{u}(t)\rangle_{X^*\times X} + b(v-\tilde{u}(t),\tilde{\lambda}(t)) + \langle \mathcal{S}u(t), v-\tilde{u}_n(t)\rangle_{X^*\times X}\\
&\qquad + j^\circ(\gamma\tilde{u}(t);\gamma v-\gamma\tilde{u}(t)) \ge \langle f(t), v-\tilde{u}(t)\rangle_{X^*\times X} \quad \text{for all} \quad v\in X, \qquad (4.5)\\
&b(\tilde{u}(t),\mu-\tilde{\lambda}(t)) \le 0 \quad \text{for all} \quad \mu\in\Lambda(g(t)). \qquad (4.6)
\end{aligned}$$

This yields the following perturbed problem as follows:

Problem 4.3. *Find $\tilde{u}_n\in C(I;X)$ and $\tilde{\lambda}_n\in\mathcal{B}(I;Y)$ such that, for all $t\in I$, $(\tilde{u}_n(t),\tilde{\lambda}_n(t))\in X\times \Lambda(g_n(t))$ and*

$$\begin{aligned}
&\langle A\tilde{u}_n(t), v-\tilde{u}_n(t)\rangle_{X^*\times X} + b(v-\tilde{u}_n(t),\tilde{\lambda}_n(t)) + \langle \mathcal{S}_n u(t), v-\tilde{u}_n(t)\rangle_{X^*\times X}\\
&\qquad + j^\circ(\gamma\tilde{u}_n(t);\gamma v-\gamma\tilde{u}_n(t)) \ge \langle f_n(t), v-\tilde{u}_n(t)\rangle_{X^*\times X} \quad \text{for all} \quad v\in X, \qquad (4.7)\\
&b(\tilde{u}_n(t),\mu-\tilde{\lambda}_n(t)) \le 0 \quad \text{for all} \quad \mu\in\Lambda(g_n(t)). \qquad (4.8)
\end{aligned}$$

It is obvious that, under the assumptions of Theorem 4.1, Problem 4.2 and Problem 4.3 admit by Theorem 3.1 a solution pair in $C(I;X)\times\mathcal{B}(I;Y)$, respectively. We denote by $(\tilde{u},\tilde{\lambda})$, and $(\tilde{u}_n,\tilde{\lambda}_n)$ the solution pairs to Problem 4.2 and Problem 4.3. Moreover, similar to the Step 1 of the proof for Theorem 3.1, we can find a positive constant C, which is independent of n and t such that $\|\tilde{u}_n(t)\|_X + \|\tilde{\lambda}_n(t)\|_Y \le C$.

We claim that $\tilde{u}_n(t)\to\tilde{u}(t)$ in X for all $t\in I$. First, for all $t\in I$, because of the boundedness of the sequence $\{(\tilde{u}_n(t),\tilde{\lambda}_n(t))\}$ and the reflexivity of spaces X and Y, there exist a subsequences, still denotes by $\{(\tilde{u}_n(t),\tilde{\lambda}_n(t))\}$, and $(\tilde{u}_t,\tilde{\lambda}_t)\in X\times Y$ such that $(\tilde{u}_n(t),\tilde{\lambda}_n(t))\rightharpoonup(\tilde{u}_t,\tilde{\lambda}_t)$. By hypothesis H_2, we see that $g_n(t)\to g(t)$ in X for all $t\in I$. It follows from the fact $\tilde{\lambda}_n(t))\in$

$\Lambda(g_n(t))$ and the assumption $H_\Lambda(ii)$ that $\tilde{\lambda}_t \in \Lambda(g(t))$. Furthermore we can find a sequence $\{\bar{\lambda}_n\} \subset Y$ with $\bar{\lambda}_n \in \Lambda(g_n(t))$ such that $\bar{\lambda}_n \to \tilde{\lambda}_t$. This allows us to set $\mu = \bar{\lambda}_n$ in (4.8) to see that

$$0 \geq b(\tilde{u}_n(t), \bar{\lambda}_n - \tilde{\lambda}_n(t)) = b(\tilde{u}_n(t) - v, \bar{\lambda}_n) - b(\tilde{u}_n - v, \tilde{\lambda}_n(t)) + b(v, \bar{\lambda}_n - \tilde{\lambda}_n(t)),$$

which implies that

$$\begin{aligned} b(\tilde{u}_n - v, \tilde{\lambda}_n(t)) &\leq b(v - \tilde{u}_n(t), \bar{\lambda}_n) + b(v, \tilde{\lambda}_n(t) - \bar{\lambda}_n) \\ &\leq b(v - \tilde{u}_n(t), \tilde{\lambda}_t) + b(v - \tilde{u}_n(t), \bar{\lambda}_n - \tilde{\lambda}_t) + b(v, \tilde{\lambda}_n(t) - \bar{\lambda}_n), \text{for all } v \in V. \end{aligned}$$

Passing upper limit on both sides of above inequality yields that

$$\limsup_{n\to\infty} b(v - \tilde{u}_n(t), \tilde{\lambda}_n(t)) \leq b(v - \tilde{u}_t, \tilde{\lambda}_t) \quad \text{for all} \quad v \in X, \tag{4.9}$$

which indicates that

$$\limsup_{n\to\infty} b(\tilde{u}_t - \tilde{u}_n(t), \tilde{\lambda}_n(t)) \leq 0. \tag{4.10}$$

Next, by setting $v = \tilde{u}_t$ in (4.7) we can obtain that

$$\begin{aligned} \langle A\tilde{u}_n(t), \tilde{u}_n(t) - \tilde{u}_t \rangle_{X^* \times X} \leq \langle f_n(t) - \mathcal{S}_n u(t), \tilde{u}_n(t) - \tilde{u}_t \rangle_{X^* \times X} + j^\circ(\gamma \tilde{u}_n(t); \gamma \tilde{u}_t - \tilde{u}_n(t)) \\ + b(\tilde{u}_t - \tilde{u}_n(t), \tilde{\lambda}_n(t), \end{aligned}$$

which yields by passing the upper limit that $\limsup_{n\to\infty} \langle A\tilde{u}_n(t), \tilde{u}_n(t) - \tilde{u}_t \rangle_{X^* \times X} \leq 0$, where we use the assumptions H_1, H_2, Proposition 2.1(c), and (4.10). This allows to use the pseudomonotonicity of A to deduce that

$$\langle A\tilde{u}_t, \tilde{u}_t - v \rangle_{X^* \times X} \leq \liminf_{n\to\infty} \langle A\tilde{u}_n(t), \tilde{u}_n(t) - v \rangle_{X^* \times X} \quad \text{for all} \quad v \in X \tag{4.11}$$

or equivalently

$$\langle A\tilde{u}_n(t), \tilde{u}_n(t) - \tilde{u}_t \rangle_{X^* \times X} \to 0 \quad \text{and} \quad A\tilde{u}_n(t) \rightharpoonup A\tilde{u}_t \quad \text{as} \quad n \to \infty.$$

Therefore, by virtue of the assumptions H_1, H_2, and Proposition 2.1(c), it follows from (4.7), (4.9), and (4.11) that, for all $v \in X$,

$$\begin{aligned} \langle A\tilde{u}_t, \tilde{u}_t - v \rangle_{X^* \times X} &\leq \liminf_{n\to\infty} \langle A\tilde{u}_n(t), \tilde{u}_n(t) - v \rangle_{X^* \times X} \leq \limsup_{n\to\infty} \langle A\tilde{u}_n(t), \tilde{u}_n(t) - v \rangle_{X^* \times X} \\ &\leq \limsup_{n\to\infty} b(v - \tilde{u}_n(t), \tilde{\lambda}_n(t)) + \limsup_{n\to\infty} \langle f_n(t) - \mathcal{S}_n u(t), \tilde{u}_n(t) - v \rangle_{X^* \times X} \\ &\quad + \limsup_{n\to\infty} j^\circ(\gamma \tilde{u}_n(t); \gamma v - \gamma \tilde{u}_n(t)) \\ &\leq b(v - \tilde{u}_t, \tilde{\lambda}_t) + \langle f(t) - \mathcal{S}u(t), \tilde{u}_t - v \rangle_{X^* \times X} + j^\circ(\gamma \tilde{u}_t; \gamma v - \gamma \tilde{u}_t). \end{aligned} \tag{4.12}$$

Furthermore, for any $\mu \in \Lambda(g(t))$, with similar argument as before, hypotheses $H_\Lambda(ii)$ and H_2 permit us to find a sequence $\mu_n \in \Lambda(g_n(t))$ such that $\mu_n \to \mu$, which together with (4.8) implies that $b(\tilde{u}_n(t), \mu_n - \tilde{\lambda}_n(t)) \leq 0$. By choosing $v = \tilde{u}_t$ in (4.7), we can find that

$$\begin{aligned} &\langle A\tilde{u}_n(t), \tilde{u}_t - \tilde{u}_n(t) \rangle_{X^* \times X} + j^\circ(\gamma \tilde{u}_n(t); \gamma \tilde{u}_t - \gamma \tilde{u}_n(t)) - \langle f_n(t) - \mathcal{S}_n(t), \tilde{u}_t - \tilde{u}_n(t) \rangle_{X^* \times X} \\ &\geq b(\tilde{u}_n(t) - \tilde{u}_t, \tilde{\lambda}_n(t)) \geq b(\tilde{u}_n(t), \tilde{\lambda}_n(t) - \mu_n) + b(\tilde{u}_n(t), \mu_n) - b(\tilde{u}_t, \tilde{\lambda}_n(t)) \\ &\geq b(\tilde{u}_n(t), \mu_n - \mu) + b(\tilde{u}_n(t), \mu) - b(\tilde{u}_t, \tilde{\lambda}_n(t)). \end{aligned} \tag{4.13}$$

Taking upper limit at the both sides of (4.13) yields that

$$b(\tilde{u}_t, \mu - \tilde{\lambda}_t) \leq 0 \quad \text{for all} \quad \mu \in \Lambda(g(t)), \tag{4.14}$$

which together with (4.12) implies that $(\tilde{u}_t, \tilde{\lambda}_t)$ is a solution pair of (4.5) and (4.6). By the uniqueness of $\tilde{u}$ of solution pair to Problem 4.2, it is obvious that $\tilde{u}(t) = \tilde{u}_t$ for all $t \in I$, i.e., $\tilde{u}_n(t) \rightharpoonup \tilde{u}(t)$ in X. Then, we further prove that $\tilde{u}_n(t) \to \tilde{u}(t)$ in X for all $t \in I$. For all $t \in I$, we take $v = \tilde{u}_n(t)$ in (4.12), $v = \tilde{u}(t)$ in (4.7) and add the resulting inequalities to see that

$$\begin{aligned}&\langle A\tilde{u}(t) - A\tilde{u}_n(t), \tilde{u}_n(t) - \tilde{u}(t)\rangle_{X^*\times X} + b(\tilde{u}_n(t) - \tilde{u}(t), \tilde{\lambda}_t - \tilde{\lambda}_n(t))\\ &\quad + j^\circ(\gamma\tilde{u}(t); \gamma\tilde{u}_n(t) - \gamma\tilde{u}(t)) + j^\circ(\gamma\tilde{u}_n(t); \gamma\tilde{u}(t) - \gamma\tilde{u}_n(t))\\ &\geq \langle f(t) - \mathcal{S}u(t) - (f_n(t) - \mathcal{S}_n u(t)), \tilde{u}_n(t) - \tilde{u}(t)\rangle_{X^*\times X},\end{aligned}$$

which together with the hypotheses $H_A(ii)$ and $H_j(ii)$ implies that

$$\begin{aligned}&(M_A - m_j\|\gamma\|^2)\|\tilde{u}_n(t) - \tilde{u}(t)\|_{X^*}^2\\ &\leq b(\tilde{u}(t) - \tilde{u}_n(t), \tilde{\lambda}_n(t) - \tilde{\lambda}_t) + c\|f(t) - \mathcal{S}u(t) - (f_n(t) - \mathcal{S}_n u(t))\|_{X^*}.\end{aligned} \tag{4.15}$$

Due to the hypothesis $H_b(i)$, it follows that

$$\begin{aligned}\limsup_{n\to\infty} b(\tilde{u}(t) - \tilde{u}_n(t), \tilde{\lambda}_n(t) - \tilde{\lambda}_t) &= \limsup_{n\to\infty} b(\tilde{u}_n(t), \tilde{\lambda}_t - \bar{\lambda}_n) + \limsup_{n\to\infty} b(\tilde{u}_n(t), \bar{\lambda}_n - \tilde{\lambda}_n(t))\\ &\quad + \limsup_{n\to\infty} b(\tilde{u}(t), \tilde{\lambda}_n(t) - \tilde{\lambda}_t) \leq 0,\end{aligned} \tag{4.16}$$

where $\{\bar{\lambda}_n\}$ is a sequence we find before such that $\bar{\lambda}_n \to \tilde{\lambda}_t$. Therefore, by taking upper limit on the both sides of (4.15), we obtan from H_1, H_2, and (4.16) that $\limsup_{n\to\infty}\|\tilde{u}_n(t) - \tilde{u}(t)\|_X^2 \leq 0$, which implies that, for all $t \in I$, $\tilde{u}_n(t) \to \tilde{u}(t)$ as $n \to \infty$. Obviously, the above argument indicates that, for all $t \in I$, every subsequence of the sequence $\{\tilde{u}_n(t)\}$ contains a subsequence converging to the unique $\tilde{u}(t)$, which implies that, for all $t \in I$, the whole sequence $\tilde{u}_n(t) \to \tilde{u}(t)$ by [23, Proposition 1.14].

Finally, we end the proof of this theorem by showing that $u_n(t) \to \tilde{u}(t)$ and $\tilde{u}(t) = u(t)$ to obtain that $u_n(t) \to u(t)$ for all $t \in I$. By letting $v = \tilde{u}(t)$ in (1.1), $\mu = \tilde{\lambda}_t$ in (1.2), $v = u(t)$ in (4.12), $\mu = \lambda(t)$ in (4.14), respectively, we can deduce from the obtained inequalities that

$$\langle Au(t) - A\tilde{u}(t), u(t) - \tilde{u}(t)\rangle_{X^*\times X} \leq j^\circ(\gamma u(t); \gamma\tilde{u}(t) - \gamma u(t)) + j^\circ(\gamma\tilde{u}(t); \gamma u(t) - \gamma\tilde{u}(t)),$$

which indicates by the hypotheses $H_A(ii)$ and $H_j(ii)$ that

$$(M_A - m_j\|\gamma\|^2)\|u(t) - \tilde{u}(t)\|_X^2 \leq 0.$$

This together with the assumption H_0 implies that $\tilde{u}(t) = u(t)$ for all $t \in I$. Moreover, by similar technique, we let $v = \tilde{u}_n(t)$ in (4.1), $\mu = \tilde{\lambda}_n(t)$ in (4.2), $v = u_n(t)$ in (4.7) and $\mu = \lambda_n(t)$ in (4.8) to get from the obtained inequalities that

$$\begin{aligned}&\langle Au_n(t) - A\tilde{u}_n(t), u_n(t) - \tilde{u}_n(t)\rangle_{X^*\times X}\\ &\leq j^\circ(\gamma u_n(t); \gamma\tilde{u}_n(t) - \gamma u_n(t)) + j^\circ(\gamma\tilde{u}_n(t); \gamma u_n(t) - \gamma\tilde{u}_n(t))\\ &\quad + \langle \mathcal{S}_n u_n(t) - \mathcal{S}_n u(t), \tilde{u}_n(t) - u_n(t)\rangle_{X^*\times X}.\end{aligned}$$

for all $t \in I$. The assumptions $H_A(ii), H_j(ii), H_1$ indicate that, for all $t \in I$,

$$(M_A - m_j\|\gamma\|^2)\|\tilde{u}_n(t) - u_n(t)\|_X \leq \|\mathcal{S}_n u_n(t) - \mathcal{S}_n u(t)\|_{X^*} \leq M\int_0^t \|u_n(s) - u(s)\|_X ds,$$

which together with the smallness condition H_0 implies that, for all $t \in I$,

$$\|u_n(t)-\tilde{u}(t)\|_X \leq \|\tilde{u}_n(t)-\tilde{u}(t)\|_X + \frac{M}{M_A - m_j\|\gamma\|^2}\int_0^t \|u_n(s)-\tilde{u}(s)\|_X ds,$$

where we use the result $\tilde{u}(t) = u(t)$ obtained before. Now, by the Gronwall inequality, we can get that, for all $t \in I$,

$$\|u_n(t)-\tilde{u}(t)\|_X \leq \|\tilde{u}_n(t)-\tilde{u}(t)\|_X + \frac{M}{M_A - m_j\|\gamma\|^2}\int_0^t \|\tilde{u}_n(s)-\tilde{u}(s)\|_X ds,$$

which implies by the Lebesgue dominated convergence theorem that, for all $t \in I$, $u_n(t) \to \tilde{u}(t)$ as $n \to \infty$. □

5. APPLICATIONS

In this section, we apply the abstract results obtained in Sections 3 and 4 to a quasistatic viscoelastic contact model with long memory, which involves a multivalued version of the normal damped response contact condition and a classic version of Coulomb's friction law. Before presenting its mathematical model, we recall in the following some notation used in the contact mechanics.

We suppose that a viscoelastic body with long memory in its reference configuration occupies a bounded domain in $\mathbb{R}^d$ with $d = 2,3$. The boundary $\Gamma := \partial\Omega$ is assumed to be Lipschitz continuous and partitioned into three mutually non-overlapping measurable parts $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_C}$ with $meas(\Gamma_D) > 0$. Let $I = [0,T]$ be a time interval with $T > 0$. Here and below, we use boldface letters for vectors and tensors. The outward unit normal on Γ is denoted by $\boldsymbol{\nu}$ and the normal and tangential components of a vector $\boldsymbol{u}$ on Γ are given by $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$. Also the normal and tangential components of a tensor $\boldsymbol{\sigma}$ on Γ are given by $\sigma_\nu = \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$. The indices i, j run between 1 and d and a index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}$. Throughout this section, we adopt the summation convention over a repeated index and we skip, for simplicity, the dependence of variable $\boldsymbol{x}$ sometimes.

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \qquad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}} \qquad \forall \boldsymbol{u} = (u_i), \boldsymbol{v} = (v_i) \in \mathbb{R}^d$$

and

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} : \boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

The deformation operator $\boldsymbol{\varepsilon}$ and divergence operators Div are defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \qquad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad Div\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The mathematical model of the contact problem we consider in this section is as follows.

Problem 5.1. *Find a displacement field $\boldsymbol{u}\colon \Omega\times I\to\mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}\colon \Omega\times I\to\mathbb{S}^d$ such that, for all $t\in I$,*

$$\boldsymbol{\sigma}(t)=\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t))+\mathscr{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\int_0^t\mathscr{C}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}'(s))ds \qquad \text{in} \quad \Omega, \tag{5.1}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t)+\boldsymbol{f}_0(t)=0 \qquad \text{in} \quad \Omega, \tag{5.2}$$

$$\boldsymbol{u}(t)=\mathbf{0} \qquad \text{on} \quad \Gamma_D, \tag{5.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu}=\boldsymbol{f}_N(t) \qquad \text{on} \quad \Gamma_N, \tag{5.4}$$

$$-\sigma_\nu(t)\in\partial j_\nu(u'_\nu(t)) \qquad \text{on} \quad \Gamma_C, \tag{5.5}$$

$$\begin{aligned}&\|\boldsymbol{\sigma}_\tau(t)\|\le F_b(\|\boldsymbol{g}(t)\|),\\ &-\boldsymbol{\sigma}_\tau(t)=F_b(\|\boldsymbol{g}(t)\|)\frac{\boldsymbol{u}'_\tau(t)}{\|\boldsymbol{u}'_\tau(t)\|} \quad \text{if} \quad \boldsymbol{u}'_\tau(t)\neq\mathbf{0}\end{aligned} \qquad \text{on} \quad \Gamma_C, \tag{5.6}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0 \qquad \text{in} \quad \Omega. \tag{5.7}$$

In this model, equation (5.1) represents the viscoelastic constitutive law in which $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ represent the viscosity operator, the elasticity operator, and the relaxation tensor, respectively. Equation (5.2) represents the equilibrium equation in which $\boldsymbol{f}_0$ denotes the density of body forces. Conditions (5.3) and (5.4) are displacement and traction boundary conditions which mean that the body is fixed on Γ_D and the surface tractions $\boldsymbol{f}_N$ act on Γ_N. Conditions (5.5) and (5.6) are contact conditions. Equation (5.5) represents a multivalued extension of the normal damped response condition, where ∂j_ν is the generalized subdifferential of locally Lipschitz functional j. The normal damped response condition describes the contact between the body and the foundation covered with a thin lubricant layer, such as oil. Equation (5.6) represents a frictional condition describing the classic version of Coulomb's law, where F_b is the friction bound which depends on a given function $\boldsymbol{g}$.

In order to present the variational formulation of Problem 5.1, we introduce some spaces that we will use and impose some hypotheses on its data. Let $H^1(\Omega;\mathbb{R}^d)$, $L^2(\Omega;\mathbb{S}^d)$ be Soblev spaces and we define

$$X=\{\boldsymbol{v}\in H^1(\Omega;\mathbb{R}^d)\mid \boldsymbol{v}=\mathbf{0}\ \text{ a.e. on }\Gamma_D\},\quad \mathscr{H}=L^2(\Omega;\mathbb{S}^d). \tag{5.8}$$

The space X is a Hilbert space with the canonical inner product

$$\langle\boldsymbol{u},\boldsymbol{v}\rangle_X=\int_\Omega\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{v})\,d\boldsymbol{x} \quad \forall\boldsymbol{u},\boldsymbol{v}\in X,$$

and the associated norm $\|\cdot\|_X$. The space $\mathscr{H}$ is a Hilbert space equipped with the inner product

$$\langle\boldsymbol{\sigma},\boldsymbol{\tau}\rangle_{\mathscr{H}}=\int_\Omega\boldsymbol{\sigma}:\boldsymbol{\tau}\,d\boldsymbol{x} \quad \forall\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathscr{H},$$

and the associated norm $\|\cdot\|_{\mathscr{H}}$. Moreover, since *meas*$(\Gamma_D>0)$, it is known, by the Korn inequality, that the norm $\|\cdot\|_X$ is equivalent to the usual norm $\|\cdot\|_{H^1(\Omega;\mathbb{R}^d)}$.
The space of fourth order symmetric tensor field is defined by

$$Q_\infty=\{E=(\mathscr{E}_{ijkl})\mid \mathscr{E}_{ijkl}=\mathscr{E}_{jikl}=\mathscr{E}_{klij}\in L^\infty(\Omega),1\le i,j,k,l\le d\}$$

which is a real Banach space with the norm $\|E\|_{Q_\infty}=\sum_{1\le i,j,k,l\le d}\|\mathscr{E}_{ijkl}\|_{L^\infty(\Omega)}$ for all $E\in Q_\infty$. Let $Z=L^2(\Gamma;\mathbb{R}^d)$. The trace operator $\gamma: H^1(\Omega;\mathbb{R}^d)\to Z=L^2(\Gamma;\mathbb{R}^d)$ is linear continuous and compact while the trace operator γ: $H^1(\Omega;\mathbb{R}^d) \to H^{\frac{1}{2}}(\Gamma;\mathbb{R}^d)$ is a linear continuous and surjective operator. It follows from the converse theorem (see, e.g., [26, Theorem 5.7]) that there exists a linear and continuous operator l such that $l: H^{\frac{1}{2}}(\Gamma;\mathbb{R}^d) \to H^1(\Omega;\mathbb{R}^d)$, $\gamma(l(\boldsymbol{\xi})) = \boldsymbol{\xi}$ for all $\boldsymbol{\xi} \in H^{\frac{1}{2}}(\Gamma;\mathbb{R}^d)$. The operator l is called the right inverse of the operator γ: $H^1(\Omega;\mathbb{R}^d) \to H^{\frac{1}{2}}(\Gamma;\mathbb{R}^d)$. Note that $\gamma(l(\gamma\boldsymbol{w})) = \gamma\boldsymbol{w}$ for all $\boldsymbol{w} \in X$, which indicates $l(\gamma\boldsymbol{w})$ has the same trace as $\boldsymbol{w}$. Therefore, for each $\boldsymbol{w} \in X$, $l(\gamma\boldsymbol{w}) \in X$. Also recall that $\gamma(X)$ is a closed subspace of $\gamma(H^1(\Omega;\mathbb{R}^d))$, and thus it is a Hilbert space. We denote by Y the dual space of $\gamma(X)$ and $\langle\cdot,\cdot\rangle_Y$the duality pairing between Y and $\gamma(X)$.

$\underline{H(\mathscr{A})}$: The operator $\mathscr{A}: \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies
(a) $\mathscr{A}(\cdot,\boldsymbol{\varepsilon})$ is measurable on Ω for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$;
(b) there exists $L_{\mathscr{A}} > 0$ such that $\|\mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \le L_{\mathscr{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|$ for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$;
(c) there exists $m_{\mathscr{A}} > 0$ such that $(\mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathscr{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2$ for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$;
(d) $\mathscr{A}(\boldsymbol{x},\boldsymbol{0}) = \boldsymbol{0}$ for a.e. $\boldsymbol{x} \in \Omega$.

$\underline{H(\mathscr{B})}$: The operator $\mathscr{B}: \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies
(a) $\mathscr{B}(\cdot,\boldsymbol{\varepsilon})$ is measurable on Ω for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$;
(b) there exists $L_{\mathscr{B}} > 0$ such that $\|\mathscr{B}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathscr{B}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \le L_{\mathscr{B}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|$ for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$;
(c) $\mathscr{B}(\boldsymbol{x},\boldsymbol{0}) = \boldsymbol{0}$ for a.e. $\boldsymbol{x} \in \Omega$.

$\underline{H(\mathscr{C})}$: $\mathscr{C} \in C(I;Q_\infty)$

$\underline{H(F_b)}$:The function $F_b: \Gamma_C \times \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ = [0,+\infty)$) satisfies
(a) $F_b(\cdot,r)$ is measurable on Γ_C for all $r \in \mathbb{R}$;
(b) there exists $L_{F_b} > 0$ such that $|F_b(\boldsymbol{x},r_1) - F_b(\boldsymbol{x},r_2)| \le L_{F_b}|r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_C$;
(c) $F_b(\cdot,0) \in L^2(\Gamma_C)$.

$\underline{H(j_\nu)}$: The function $j_\nu: \Gamma_C \times \mathbb{R} \to \mathbb{R}$ is such that
(a) $j_\nu(\cdot,r)$ is measurable on Γ_C for all $r \in \mathbb{R}$ and there exists $\bar{e} \in L^2(\Gamma_C)$ such that $j_\nu(\cdot,\bar{e}(\cdot)) \in L^1(\Gamma_C)$;
(b) $j_\nu(\boldsymbol{x},\cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $\boldsymbol{x} \in \Gamma_C$;
(c) there exists $m_{j_\nu} \ge 0$ such that $j_\nu^0(\boldsymbol{x},r_1;r_2 - r_1) + j_\nu^0(\boldsymbol{x},r_2;r_1 - r_2) \le m_{j_\nu}|r_1 - r_2|^2$ for a.e. $\boldsymbol{x} \in \Gamma_C$, all r_1, $r_2 \in \mathbb{R}$;
(d) there exist $\alpha_{j_\nu} > 0$ and $\beta_{j_\nu} > 0$ such that $|\partial j_\nu(\boldsymbol{x},r)| \le \alpha_{j_\nu} + \beta_{j_\nu}|r|$ for a.e. $\boldsymbol{x} \in \Gamma_C$, all $r \in \mathbb{R}$;
(e) either $j_\nu(\boldsymbol{x},\cdot)$ or $-j_\nu(\boldsymbol{x},\cdot)$ is regular for a.e. $\boldsymbol{x} \in \Gamma_C$.

$\underline{H_3}$: $\boldsymbol{u}_0 \in X$, $\boldsymbol{f}_0 \in W^{1,2}(I;H^1(\Omega;\mathbb{R}^d)), \boldsymbol{f}_N \in W^{1,2}(I;H^1(\Gamma_N;\mathbb{R}^d))$, $\boldsymbol{g} \in C(I;X)$.

$\underline{H_4}$: $m_{\mathscr{A}} > m_{j_\nu}\|\gamma\|^2$.

We now introduce the following operators. We can use H_3 and Riesz's representation theorem to define $\boldsymbol{f} \in C(I;X^*)$ by $\langle\boldsymbol{f}(t),\boldsymbol{v}\rangle_X = \langle\boldsymbol{f}_0(t),\boldsymbol{v}\rangle_{L^2(\Omega;\mathbb{R}^d)} + \langle\boldsymbol{f}_N(t),\boldsymbol{v}\rangle_{L^2(\Gamma_N;\mathbb{R}^d)}$ for all $\boldsymbol{v} \in X$, a.e. $t \in I$. And we define operators $A: X \to X^*$, $b: X \times Y \to \mathbb{R}$, $S: C(I;X) \to C(I;X^*)$, functional

$j : Z \to \mathbb{R}$, set-valued mapping $\Lambda : X \to 2^Y$, as follows.

$$\langle A\boldsymbol{u},\boldsymbol{v}\rangle_X = \int_\Omega \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,d\boldsymbol{x}, \quad \forall \boldsymbol{u},\boldsymbol{v}\in X, \tag{5.9}$$

$$\begin{aligned}\langle S\boldsymbol{u}(t),\boldsymbol{v}\rangle_X &= \langle \mathscr{B}\boldsymbol{\varepsilon}(R\boldsymbol{u}(t)) + \int_0^t \mathscr{C}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathscr{H}},\\ &\qquad \forall t\in[0,T], \boldsymbol{v}\in X, \boldsymbol{u}\in C(I;X),\end{aligned} \tag{5.10}$$

$$j(\boldsymbol{v}) = \int_{\Gamma_C} j_\nu(v_\nu)\,d\Gamma, \qquad \forall \boldsymbol{v}\in Z, \tag{5.11}$$

$$b(\boldsymbol{v},\boldsymbol{\mu}) = \langle \boldsymbol{\mu},\gamma\boldsymbol{v}\rangle_Y, \qquad \forall \boldsymbol{v}\in X,\ \boldsymbol{\mu}\in Y, \tag{5.12}$$

$$\Lambda(\boldsymbol{\xi}) = \{\boldsymbol{\mu}\in Y : \langle \boldsymbol{\mu},\gamma\boldsymbol{v}\rangle_Y \le \int_{\Gamma_C} F_b(\|\boldsymbol{\xi}\|)\|\boldsymbol{v}_\tau\|\,d\Gamma \quad \forall \boldsymbol{v}\in X\},\ \forall \boldsymbol{\xi}\in X, \tag{5.13}$$

where $R : C(I;X)\to C(I;X)$ is defined by

$$R\boldsymbol{w}(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{w}(s)ds, \quad \text{for all } \boldsymbol{w}\in C(I;X),\ t\in I.$$

Now, by exploiting a standard process, we can derive the following variational formulation of Problem 5.1 with the help of Green formula.

Problem 5.2. *Find $\boldsymbol{w}\in C(I;X)$ and $\lambda\in\mathcal{B}(I;Y)$ with $(\boldsymbol{w}(t),\lambda(t))\in X\times\Lambda(\boldsymbol{g}(t))$ such that*

$$\begin{cases}\langle A\boldsymbol{w}(t),\boldsymbol{v}-\boldsymbol{w}(t)\rangle + b(\boldsymbol{v}-\boldsymbol{w}(t),\lambda(t)) + j^\circ(\gamma\boldsymbol{w}(t);\gamma\boldsymbol{v}-\gamma\boldsymbol{w}(t)) + \\ \qquad\qquad \langle S\boldsymbol{w}(t),\boldsymbol{v}-\boldsymbol{w}(t)\rangle \ge \langle \boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{w}(t)\rangle \quad \text{for all} \quad \boldsymbol{v}\in X, t\in I,\\ b(\boldsymbol{w}(t),\boldsymbol{\mu}-\lambda(t)) \le 0 \quad \text{for all} \quad \boldsymbol{\mu}\in\Lambda(\boldsymbol{g}(t)), t\in I.\end{cases}$$

It should be noted that (Rw,λ) is called a weak solution to Problem 5.1 if (w,λ) is a solution to Problem 5.2.

Then, we are in a position to apply the abstract results obtained in Sections 3 and 4 to get the following results on the existence, regularity, and convergence of weak solutions to Problem 5.1.

Theorem 5.1. *Let $H(\mathscr{A})$, $H(\mathscr{B})$, $H(\mathscr{C})$, $H(j_\nu)$, $H(F_b)$, H_3, and H_4 hold. Then Problem 5.1 has a weak solution $(\boldsymbol{u},\lambda)\in C^1(I;X)\times\mathcal{B}(I;Y)$, which is unique for its first component. Further, if $(\boldsymbol{u}_n,\lambda_n)$ is the weak solution to Problem 5.1 with the perturbed data $(\boldsymbol{f}_0^n,\boldsymbol{f}_N^n,\boldsymbol{g}_n,\boldsymbol{u}_0^n)$ of the original data $(\boldsymbol{f}_0,\boldsymbol{f}_N,\boldsymbol{g},\boldsymbol{u}_0)$, then $\boldsymbol{u}_n(t)\to\boldsymbol{u}(t)$ inX for all $t\in I$ under the following assumptions*

$$\{\boldsymbol{f}_0^n\}\subset W^{1,2}(I;H^1(\Omega;\mathbb{R}^d)), \quad \boldsymbol{f}_0^n \rightharpoonup \boldsymbol{f}_0 \quad in \quad W^{1,2}(I;H^1(\Omega;\mathbb{R}^d)); \tag{5.14}$$

$$\{\boldsymbol{f}_N^n\}\subset W^{1,2}(I;H^1(\Gamma_N;\mathbb{R}^d)), \quad \boldsymbol{f}_N^n \rightharpoonup \boldsymbol{f}_N \quad in \quad W^{1,2}(I;H^1(\Gamma_N;\mathbb{R}^d)); \tag{5.15}$$

$$\{\boldsymbol{g}_n\}\subset C(I;X), \quad \boldsymbol{g}_n(t)\to\boldsymbol{g}(t) \quad in \quad X \quad for\ all\ \ t\in I; \tag{5.16}$$

$$\{\boldsymbol{u}_0^n\}\subset V, \quad \boldsymbol{u}_0^n\to\boldsymbol{u}_0 \quad in \quad V. \tag{5.17}$$

Proof. We apply Theorem 3.2 and Theorem 4.1 to prove Theorem 5.1.

First, we focus on the application of Theorem 3.2 by verifying its hypotheses. For the operator A defined by (5.9) and the functional j defined by (5.11), we can deduce from the assumptions $H(\mathscr{A})$ and $H(j_\nu)$ that the hypotheses H_A, H_j in Theorem 3.2 are satisfied with coefficients $M_A = m_{\mathscr{A}}, m_j = m_{j_\nu}, \alpha_j = \sqrt{2meas(\Gamma_C)}\alpha_{j_\nu}, \beta_j = \sqrt{2}\beta_{j_\nu}$. The details can be found in, for example, [25, 27, 28, 29, 30]. Meanwhile, taking into account the definition (5.12) of $b(\cdot,\cdot)$, the properties of trace operator γ, and its right inverse l, we can refer to [21, Theorem 14] to see that H_b holds. Moreover, it is obvious that the trace operator $\gamma: H^1(\Omega;\mathbb{R}^d) \to L^2(\Gamma;\mathbb{R}^d)$ is linear continuous and compact, i.e., H_γ holds, and H_f, H_0 are direct consequences of H_3, H_4, respectively. It remains for us to prove the hypotheses $H_{\mathbb{S}}$ for operator S and H_Λ for the set-valued mapping Λ.

Let $t \in I$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2 \in C(I;X)$ and $\boldsymbol{v} \in X$. By using definition (5.10), assumptions $H(\mathscr{B})$, and $H(\mathscr{C})$, we see that

$$\begin{aligned}\langle S\boldsymbol{u}_1(t) - S\boldsymbol{u}_2(t), \boldsymbol{v}\rangle_X &\le (\|\mathscr{B}(t,\boldsymbol{\varepsilon}(R(\boldsymbol{u}_1(t)))) - \mathscr{B}(t,\boldsymbol{\varepsilon}(R(\boldsymbol{u}_2(t))))\|_{\mathscr{H}} \\ &\quad + \|\int_0^t \mathscr{C}(t-s)(\boldsymbol{u}_1(s) - u_2(s))ds\|_{\mathscr{H}})\|\boldsymbol{v}\|_X \\ &\le (L_{\mathscr{B}} + \|\mathscr{C}\|_{C(I;Q_\infty)})\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|ds\|\boldsymbol{v}\|_X,\end{aligned}$$

which implies $H_{\mathbb{S}}$ holds with $L = L_{\mathscr{B}} + \|\mathscr{C}\|_{C(I;Q_\infty)}$. From (5.13), it is clear that Λ has nonempty closed and convex values and $0 \in \Lambda(\boldsymbol{g}(t))$ for all $t \in I$. We further confirm that $H_\Lambda(ii)$ holds, i.e, Λ is Mosco continuous. On the one hand, let $\boldsymbol{z}_n \to \boldsymbol{z}$ in X and $\boldsymbol{\mu} \in \Lambda(\boldsymbol{v})$.We define $\boldsymbol{\mu}_n \in Y$ by $\boldsymbol{\mu}_n = \alpha_n\boldsymbol{\mu}$, where

$$\alpha_n = \frac{\int_{\Gamma_C} \boldsymbol{F}_b(\boldsymbol{x}, \|\boldsymbol{z}_n\|)\|\boldsymbol{w}_{n\tau}\|d\Gamma}{\int_{\Gamma_C} \boldsymbol{F}_b(\boldsymbol{x}, \|\boldsymbol{z}\|)\|\boldsymbol{w}_\tau\|d\Gamma}.$$

For any $\boldsymbol{w} \in V$, we have

$$\langle \boldsymbol{\mu}_n, \gamma\boldsymbol{w}\rangle = \alpha_n\langle \boldsymbol{\mu}, \gamma\boldsymbol{w}\rangle \le \int_{\Gamma_C} \boldsymbol{F}_b(\boldsymbol{x}, \|\boldsymbol{z}_n\|)\|\boldsymbol{w}_{n\tau}\|d\Gamma,$$

which indicates that $\boldsymbol{\mu}_n \in \Lambda(\boldsymbol{z}_n)$. We infer that $\alpha_n \to 1$. In fact, it follows from the hypothesis $H(F_b)$ and Hölder inequality that

$$\begin{aligned}&|\int_{\Gamma_C} \boldsymbol{F}_b(\boldsymbol{x}, \|\boldsymbol{z}_n\|)\|\boldsymbol{w}_{n\tau}\|d\Gamma - \int_{\Gamma_C} \boldsymbol{F}_b(\boldsymbol{x}, \|\boldsymbol{z}\|)\|\boldsymbol{w}_\tau\|d\Gamma| \\ &\le M\|\boldsymbol{w}_{n\tau} - \boldsymbol{w}_\tau\|_{\gamma(X)} + L_{\boldsymbol{F}_b}\|\boldsymbol{z}_n - \boldsymbol{z}\|_X\|\boldsymbol{w}_\tau\|_{\gamma(X)}\end{aligned}$$

which indicates that $\int_{\Gamma_C} \boldsymbol{F}_b(\boldsymbol{x}, \|\boldsymbol{z}_n\|)\|\boldsymbol{w}_{n\tau}\|d\Gamma \to \int_{\Gamma_C} \boldsymbol{F}_b(\boldsymbol{x}, \|\boldsymbol{z}\|)\|\boldsymbol{w}_\tau\|d\Gamma$ in $\mathbb{R}$ since $\boldsymbol{z}_n \to \boldsymbol{z}$ in X and $\boldsymbol{w}_{n\tau} \to \boldsymbol{w}_\tau$ in $\gamma(X)$. Thus $\alpha_n \to 1$ which implies that $\boldsymbol{\mu}_n \to \boldsymbol{\mu}$. On the other hand, let $\{\boldsymbol{\mu}_n\} \in \Lambda(\boldsymbol{z}_n)$ such that $\boldsymbol{\mu}_n \rightharpoonup \boldsymbol{\mu}$ in Y. Thus,

$$\langle \boldsymbol{\mu}_n, \gamma\boldsymbol{v}\rangle \le \int_{\Gamma_C} F_b(\boldsymbol{x}, \|\boldsymbol{z}_n\|)\|\boldsymbol{v}_\tau\|d\Gamma.$$

By passing to the upper limit on both sides of above formulation and applying Fatou Lemma, we are able to derive that, for all $\boldsymbol{v} \in X, \langle \boldsymbol{\mu}, \boldsymbol{v}\rangle \le \int_{\Gamma_C} F_b(\boldsymbol{x}, \|\boldsymbol{z}\|)\|\boldsymbol{v}_\tau\|d\Gamma$, which indicates that $\boldsymbol{\mu} \in \Lambda(\boldsymbol{z})$.

Now, we are in a position to use Theorem 3.2 to see the existence of solution pair $(\boldsymbol{w}, \lambda) \in C(I;X) \times \mathcal{B}(I;Y)$ of Problem 5.2 with $\boldsymbol{w}$ being unique, which indicates that $(\boldsymbol{u} = R\boldsymbol{w}, \lambda) \in C^1(I;X) \times \mathcal{B}(I;Y)$ is a weak solution to Problem 5.1 and $\boldsymbol{u}$ is unique.

Then, let us turn to the application of Theorem 4.1 for the convergence result of Problem 5.1. Let $(\boldsymbol{u}_n, \lambda_n)$ be the weak solution to Problem 5.1 with the perturbed data $(\boldsymbol{f}_0^n, \boldsymbol{f}_N^n, \boldsymbol{g}_n, \boldsymbol{u}_0^n)$, which is guaranteed by Theorem 4.1(i). It follows that $(\boldsymbol{u}_n, \lambda_n)$ solves the following inequalities with the perturbed data:

$$\begin{cases} \langle A\boldsymbol{u}_n'(t), \boldsymbol{v} - \boldsymbol{u}_n'(t)\rangle + b(\boldsymbol{v} - \boldsymbol{u}_n'(t), \lambda_n(t)) + j^\circ(\gamma\boldsymbol{u}_n'(t); \gamma\boldsymbol{v} - \gamma\boldsymbol{u}_n'(t)) + \\ \qquad \langle S_n\boldsymbol{u}_n'(t), \boldsymbol{v} - \boldsymbol{u}_n'(t)\rangle \geq \langle \boldsymbol{f}_n(t), \boldsymbol{v} - \boldsymbol{u}_n'(t)\rangle \quad \text{for all} \quad \boldsymbol{v} \in X, t \in I, \\ b(\boldsymbol{u}_n'(t), \boldsymbol{\mu} - \lambda_n(t)) \leq 0 \quad \text{for all} \quad \boldsymbol{\mu} \in \Lambda(\boldsymbol{g}_n(t)), t \in I, \end{cases} \tag{5.18}$$

where $\boldsymbol{f}_n$ is defined by

$$\langle \boldsymbol{f}_n(t), \boldsymbol{v}\rangle_X = \langle \boldsymbol{f}_0^n(t), \boldsymbol{v}\rangle_{L^2(\Omega;\mathbb{R}^d)} + \langle \boldsymbol{f}_N^n(t), \boldsymbol{v}\rangle_{L^2(\Gamma_N;\mathbb{R}^d)} \tag{5.19}$$

for $\boldsymbol{v} \in V$, $t \in I$, and S_n is similarly defined by (5.10) with R^n rather than R. Here $R^n : C(I;X) \to C(I;X)$ is defined by $R^n\boldsymbol{w}(t) = \boldsymbol{u}_0^n + \int_0^t \boldsymbol{w}(s)ds$ for $\boldsymbol{w} \in C(I;X)$, $t \in I$. By the regularity and convergence hypotheses (5.14) and (5.15), we use the compactness theorem of the Aubin-Simon type [31, Throrem 2.4.1] to deduce that $\boldsymbol{f}_n \in C(I;X^*)$ and $\boldsymbol{f}_n \to \boldsymbol{f}$ in $C(I;X^*)$, which together with hypothesis (5.16) implies the hypothesis H_2 holds. Moreover, by similar argument for the history-dependent operator S, we can prove that S_n is also a history-dependent operator, and the definitions of S and S_n further infer that $\|S_n\boldsymbol{w} - S\boldsymbol{w}\|_{C(I;X^*)} \leq L_{\mathscr{B}}\|\boldsymbol{u}_0^n - \boldsymbol{u}_0\|_X$ for all $\boldsymbol{w} \in C(I;X)$, which together with (5.17) yields that H_1 holds. Therefore, invoking Theorem 4.1(ii), we can deduce that

$$\boldsymbol{u}_n'(t) \to \boldsymbol{u}'(t) \quad \text{in} \quad X \quad \text{for all} \quad t \in I. \tag{5.20}$$

Therefore,

$$\|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\|_X = \|R^n\boldsymbol{u}'(t) - R\boldsymbol{u}'\|_{X^*} \leq \|\boldsymbol{u}_0^n - \boldsymbol{u}_0\|_X + \int_0^T \|\boldsymbol{u}_n'(s) - \boldsymbol{u}'(s)\|_X ds. \tag{5.21}$$

By applying Lebesgue dominated convergence theorem, it follows from the hypothesis (5.17), (5.20) and (5.21) that $\boldsymbol{u}_n(t) \to \boldsymbol{u}(t)$ in X for all $t \in I$. □

Acknowledgements
This research was supported by the National Natural Science Foundation of China (12171070) and the Open Fund of National Center for Applied Mathematics in Sichuan (2023-KFJJ-02-001).

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