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KIRCHHOFF TYPE $p(x)$-BIHARMONIC EQUATIONS INVOLVING $(h,r(x))$-HARDY SINGULAR COEFFICIENTS WITH NO-FLUX BOUNDARY CONDITIONS

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Abstract. In this paper, we investigate fourth order variable exponent Kirchhoff type $p(x)$-biharmonic equations involving constant exponent and variable exponent Hardy potentials, in which the $r(x)$-Hardy potentials are seldom mentioned in previous results. We show the existence of at least one or two distinct generalized solutions via variational methods and the theory of variable exponent Sobolev spaces when the Kirchhoff function and the nonlinear term satisfy appropriate assumptions.

Keywords. $p(x)$-biharmonic equations; $(h,r(x))$-Hardy potentials; Variable exponent spaces.

1. Introduction

Elliptic equations with double Hardy terms were investigated as a mathematical model for various problems in electrically conducting materials, singular minimal surfaces, and the examination of non-Newtonian fluids. For the existence of generalized solutions to this type of elliptic equation, we refer to [16–20] and the references therein. The Kirchhoff type equations arise in the models of physical fields, and the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0$$

was presented by Kirchhoff in 1883 as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings; see [15]. Khodabakhshi et al. [14] studied a class of singular elliptic problems with p-Laplace operators, subject to Dirichlet boundary conditions in a bounded domain with smooth boundary

$$\begin{cases}-\Delta_p u+\dfrac{|u|^{p-2}u}{|x|^p}=\lambda f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

and the existence of at least two distinct generalized solutions for this type of singular elliptic equations was obtained.

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In [21], the existence of weak solutions to the following elliptic equations with mixed boundary conditions was demonstrated

$$\begin{cases} -\operatorname{div}\mathbf{A}(x,\nabla u)+\dfrac{a(x)}{|x|^{p}}|u|^{p-2}u=\lambda f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \Gamma_1,\\ \mathbf{A}(x,\nabla u)\cdot\nu=\mu g(x,\gamma(u)) & \text{on } \Gamma_2. \end{cases}$$

In this paper, we focus on the existence of generalized solutions to the following Kirchhoff type $p(x)$-biharmonic equations involving $(h,r(x))$-Hardy singular coefficients with no-flux boundary conditions

$$\begin{cases} M(J(u))(\Delta^2_{p(x)}u+a(x)|u|^{p(x)-2}u)+\dfrac{b(x)|u|^{h-2}u}{|x|^{2h}}+\dfrac{c(x)|u|^{q(x)-2}u}{|x|^{r(x)}}=\lambda f(x,u) & \text{in } \Omega,\\ u=\text{constant},\ \ \Delta u=0 & \text{on } \partial\Omega,\\ \int_{\partial\Omega}(|\Delta u|^{p(x)-2}\Delta u)\cdot\nu ds=0 & \end{cases} \tag{1.1}$$

where

$$J(u)=\int_\Omega\frac{1}{p(x)}|\Delta u|^{p(x)}dx+\int_\Omega\frac{a(x)}{p(x)}|u|^{p(x)}dx,\ \Delta^2_{p(x)}u=\Delta(|\Delta u|^{p(x)-2}\Delta u),$$

Ω is an open bounded subset in $\mathbb{R}^N(N\geq 2)$ with smooth boundary $\partial\Omega$, ν is the outward normal vector field on $\partial\Omega$, $\lambda>0$ is a parameter, $0<a(x),b(x),c(x)\in L^\infty(\Omega)$, $1<h<\min\{p(x),\frac{N}{2}\}$, $p(x),q(x)\in C_+(\overline{\Omega})=\{g|\ g\in C(\overline{\Omega}),g(x)>1 \text{ for all } x\in\overline{\Omega}\}$; further, p is log-Hölder continuous, i.e., there exists $l>0$ such that

$$|p(x)-p(y)|\leq\frac{l}{-\log|x-y|},\forall x,y\in\Omega,0<|x-y|\leq\frac{1}{2},$$

$q(x)<\frac{Np_2^*(x)}{N+r(x)p_2^*(x)}$, and $0\leq r(x)\in C(\overline{\Omega})$, where

$$p_2^*(x)=\begin{cases}\frac{Np(x)}{N-2p(x)} & p(x)<\frac{N}{2},\\ +\infty & p(x)\geq\frac{N}{2},\end{cases}$$

$f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying

$$(f_1)\quad |f(x,u)|\leq M_1(x)+M_2|u|^{s(x)-1},\ \ a.e.\ (x,u)\in\Omega\times\mathbb{R},$$

where $0<M_1(x)\in L^{\frac{s(x)}{s(x)-1}}(\Omega),\ M_2>0,\ s(x)\in C_+(\overline{\Omega})$ with $s(x)<p_2^*(x)$, M is continuous, and there exist constants $m_1,m_2>0$, and $\alpha>\max\{1,\frac{s^+}{p^-}\}$ such that

$$(M)\quad m_1t^{\alpha-1}\leq M(t)\leq m_2t^{\alpha-1},\forall\, t\in\mathbb{R}^+:=[0,+\infty).$$

Recent research addressing operators with variable exponents was documented in [22] and [23] and the problem involving the $p(x)$-biharmonic operator with no-flux boundary conditions can be found in [5].

In recent years, numerous problems involving Hardy potentials were extensively investigated. These studies mostly focus on nonlinear eigenvalue problems where the presence of Hardy-type terms introduces significant challenges and interesting phenomena. In particular, the main objective of the work by Khalil, Alaoui, and Touzani [12] is the following nonlinear fourth-order eigenvalue problem

$$\Delta\left(|\Delta u|^{p-2}\Delta u\right)=\lambda\frac{|u|^{p-2}u}{\delta(x)^{2p}}\quad\text{in }\Omega,\quad u\in W_0^{2,p}(\Omega),$$

where λ is the eigenvalue parameter, $1<p<N/2$, and $\delta(x)=d(x,\partial\Omega)$ represents the distance to the boundary of the domain Ω, and has at least one increasing sequence $(\lambda_k)_{k\geq 1}$ of positive eigenvalues. To achieve this and to overcome the difficulty that the sequence of Palais-Smale type do not ensure the compactness property due to the presence of the Hardy potential on the right hand side of their problem, the authors employed a variational technique based on the Ljusternik-Schnirelmann theory applied to C^1 manifolds. This approach facilitates the construction of an increasing sequence of positive eigenvalues, thereby establishing the existence of an infinite sequence of eigenvalues for the problem. It is worth noting that a similar argument was employed by the authors in [13] for the case of variable exponents. It should be noted that, in our case, the presence of the Hardy potential in the term on the left-hand side of our problem is crucial. Indeed, thanks to the Lemma 2.14 in Section 2, it can be demonstrated that any Palais-Smale sequence admits a strongly convergent subsequence.

This paper delves into a pivotal and original problem in partial differential equations by exploring Kirchhoff-type $p(x)$-biharmonic equations with Hardy-type singular coefficients and no-flux boundary conditions. By incorporating variable exponents and singular terms, the study extends traditional theories and addresses a crucial gap in the literature. The investigation of Kirchhoff problems with Hardy potentials is particularly significant due to its applications in mathematical physics, including the analysis of elasticity in heterogeneous materials, fluid dynamics with spatially varying viscosity, and nonlinear wave propagation in complex media. The Hardy-type potential introduces additional complexity and relevance, as it models singular behavior in physical systems. This work not only advances the theoretical understanding of nonlinear elliptic equations but also has practical implications for various scientific and engineering challenges, highlighting its broad interdisciplinary impact and paving the way for future research.

2. Preliminaries and the Variational Structure

Let $S(\Omega)$ be the set of all measurable real functions defined on Ω, $p^+=\text{ess inf}_{x\in\Omega}\ p(x)$, $p^-=\text{ess sup}_{x\in\Omega}\ p(x)$, and

$$L^{p(x)}(\Omega)=\left\{u\in S(\Omega):\int_\Omega |u(x)|^{p(x)}\,dx<\infty\right\}$$

with a Luxemburg-type norm defined by

$$\|u\|_{L^{p(x)}(\Omega)}=\|u\|_{p(x)}:=\inf\left\{\eta>0:\int_\Omega\Big|\frac{u(x)}{\eta}\Big|^{p(x)}\,dx\leq 1\right\}.$$

Lemma 2.1 ([1] Hölder's Inequality)**.** *Let $p,q,t\geq 1$ be measurable functions defined on Ω and $\frac{1}{t(x)}=\frac{1}{p(x)}+\frac{1}{q(x)}$ for a.e. $x\in\Omega$. If $f\in L^{p(x)}(\Omega)$ and $g\in L^{q(x)}(\Omega)$, then $fg\in L^{t(x)}(\Omega)$ and $\|fg\|_{t(x)}\leq 2\|f\|_{p(x)}\|g\|_{q(x)}$.*

Define the variable exponent Sobolev space $W^{m,p(\cdot)}(\Omega)$ by

$$W^{m,p(\cdot)}(\Omega)=\{u\in L^{p(\cdot)}(\Omega)|\ D^\alpha u\in L^{p(\cdot)}(\Omega),|\alpha|\leq m\},$$

where

$$D^{\alpha}u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_N{}^{\alpha_N}}u$$

with $\alpha = (\alpha_1, \ldots, \alpha_N)$ being a multi-index and $|\alpha| = \sum_{i=1}^{N}\alpha_i$. The space $W^{m,p(\cdot)}(\Omega)$, equipped with the norm $\|u\|_{m,p(\cdot)} := \sum_{|\alpha|\le m}|D^{\alpha}u|_{p(\cdot)}$, is a separable, reflexive, and uniformly convex Banach space.

Let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$. Since p is log-Hölder continuous, one sees that $W_0^{1,p(x)}(\Omega)$ coincides with $\{u \in W^{1,p(x)}(\Omega) : u = 0 \text{ on } \partial\Omega\}$. We consider the subspace of $W^{2,p(x)}(\Omega)$, $X = \{u \in W^{2,p(x)}(\Omega) : u|_{\partial\Omega} \equiv \text{constant}\}$, which can also be viewed as

$$X = \{u + c : u \in W_0^{1,p(x)}(\Omega)\cap W^{2,p(x)}(\Omega), c \in \mathbb{R}\},$$

Then X is a closed subspace of $W^{2,p(x)}(\Omega)$. Thus it is a reflexive and separable Banach space. The norm

$$\|u\| = \|u\|_{W_0^{1,p(x)}(\Omega)} + \|u\|_{W^{2,p(x)}(\Omega)} = \sum_{|\alpha|=2}|D^{\alpha}u|_{p(x)} + \||\nabla u|\|_{p(x)} + \|u\|_{p(x)}$$

is equivalent to the norm

$$\|\Delta u\|_{p(x)} = \inf\Big\{\eta > 0 : \int_{\Omega}\Big|\frac{\Delta u(x)}{\eta}\Big|^{p(x)}dx \le 1\Big\}$$

via [25, Theorem 4.4].

In view of the term $a(x)|u|^{p(x)-2}u$ in equation (1.1), we use the following norm:

$$\|u\|_a = \inf\Big\{\eta > 0 : \int_{\Omega}\Big(\Big|\frac{\Delta u(x)}{\eta}\Big|^{p(x)} + a(x)\Big|\frac{u(x)}{\eta}\Big|^{p(x)}\Big)dx \le 1\Big\},$$

which is equivalent to the usual norm $\|u\|$ (see [2, Remark 2.1]), and it is useful to deal with the modular

$$\rho(u) = \int_{\Omega}\big(\big|\Delta u(x)\big|^{p(x)} + a(x)|u(x)\big|^{p(x)}\big)dx.$$

Proposition 2.1 [10]. *For $p \in C_+(\overline{\Omega}), u, u_n \in L^{p(x)}(\Omega)$,*

$$\min\big\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\big\} \le \int_{\Omega}|u(x)|^{p(x)}dx \le \max\big\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\big\}$$

and $\|u_n\|_{p(x)} \to 0$ ($\to \infty$, respectively) $\Leftrightarrow \rho(u_n) \to 0$ ($\to \infty$, respectively) as $n \to \infty$.

Similar to Proposition 2.1, one has the following lemma.

Lemma 2.2. *For $p \in C_+(\overline{\Omega}), u, u_n \in L^{p(x)}(\Omega)$,*

$$\min\big\{\|u\|_a^{p^-}, \|u\|_a^{p^+}\big\} \le \rho(u) \le \max\big\{\|u\|_a^{p^-}, \|u\|_a^{p^+}\big\},$$

and $\|u_n\|_a \to 0$ ($\to \infty$, respectively) $\Leftrightarrow \rho(u_n) \to 0$ ($\to \infty$, respectively) as $n \to \infty$.

Let $d(x) \in S(\Omega)$ and $d(x) > 0$ for all $x \in \Omega$. Define

$$L_{d(x)}^{p(x)}(\Omega) = \Big\{u \in S(\Omega) : \int_{\Omega}d(x)|u(x)|^{p(x)}dx < \infty\Big\}$$

with a Luxemburg-type norm defined by

$$\|u\|_{L_{d(x)}^{p(x)}(\Omega)} = \|u\|_{(p(x),d(x))} := \inf\Big\{\mu > 0 : \int_{\Omega}d(x)\Big|\frac{u(x)}{\mu}\Big|^{p(x)}dx \le 1\Big\}.$$

Lemma 2.3 ([11]). *If $p \in C_+(\overline{\Omega})$, then*

$$\min\left\{\|u\|^{p^-}_{(p(x),d(x))}, \|u\|^{p^+}_{(p(x),d(x))}\right\} \leq \int_\Omega d(x)|u(x)|^{p(x)}dx \leq \max\left\{\|u\|^{p^-}_{(p(x),d(x))}, \|u\|^{p^+}_{(p(x),d(x))}\right\}$$

for any $u \in L^{p(x)}_{d(x)}(\Omega)$ and for a.e. $x \in \Omega$.

Lemma 2.4 ([9]). *If $p_1, p_2 \in C_+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$ a.e. $x \in \Omega$, then, for $k \in \mathbb{N}$, there exists the continuous embedding $W^{k,p_2(x)}(\Omega) \hookrightarrow W^{k,p_1(x)}(\Omega)$.*

Lemma 2.5 ([10]). *Assume that the boundary of Ω possesses the cone property and $p \in C_+(\overline{\Omega})$. If $s \in C_+(\overline{\Omega})$ and $s(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$, then $X \hookrightarrow L^{s(x)}(\Omega)$ is continuous and compact.*

Lemma 2.6 ([10]). *Let G be a measurable subset in $\mathbb{R}^N$ and $0 < meas(G) < +\infty$. If $f : G \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $|f(x,u(x))| \leq a(x) + b|u(x)|^{\frac{p_1(x)}{p_2(x)}}$ a.e. $(x,u) \in G \times \mathbb{R}$, where $p_1(x), p_2(x) \in C_+(\overline{\Omega})$, $0 < a(x) \in L^{p_2(x)}(G)$, $b > 0$. Then the Nemytskii operator defined by $N_f(u)(x) = f(x,u(x))$ maps $L_{p_1(x)}(G)$ into $L_{p_2(x)}(G)$, and it is continuous and bounded.*

Define the functional $I_\lambda : X \to \mathbb{R}$ by $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$, where $\Phi(u) = \mathscr{M}(J(u)) + \Phi_1(u)$,

$$\mathscr{M}(u) = \int_0^u M(\tau)d\tau, \;\; \Phi_1(u) = \int_\Omega \frac{b(x)|u|^h}{h|x|^{2h}}dx + \int_\Omega \frac{c(x)|u|^{q(x)}}{q(x)|x|^{r(x)}}dx,$$

and

$$\Psi(u) = \int_\Omega F(x,u)dx, \;\; F(x,u) = \int_0^u f(x,\tau)d\tau.$$

By (f_1), the continuous embeddings, the Hölder inequality mentioned in Lemma 2.1 and the compactness result in Lemma 2.14, and by similar arguments as in [26] and [27], we can show that Φ and Ψ are well-defined on X, and Φ and Ψ are continuously Gâteaux differentiable with

$$\Phi'(u)(v) = M(J(u))\Big(\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v dx + \int_\Omega a(x)|u|^{p(x)-2}uvdx\Big)$$
$$+ \int_\Omega \frac{b(x)|u|^{h-2}}{|x|^{2h}}uvdx + \int_\Omega \frac{c(x)|u|^{q(x)-2}}{|x|^{r(x)}}uvdx, \quad \forall u,v \in X,$$

and $\Psi'(u)(v) = \int_\Omega f(x,u)vdx$ for all $u,v \in X$. We say that $u \in X$ is a generalized solution to (1.1) if

$$I'_\lambda(u)(v) = \Phi'(u)(v) - \lambda\Psi'(u)(v) = 0, \quad \forall v \in X.$$

i.e., a critical point of the functional $I_\lambda(u)$ in the space X is a generalized solution to (1.1).

Lemma 2.7 ([6], Example B). *If $\Phi : X \to R$ is a convex lower semicontinuous functional on a reflexive Banach space X, then Φ is sequentially weakly lower semicontinuous on X, that is, for any $u \in X$ and any sequence $u_n \in X$ satisfying $u_n \rightharpoonup u$ in X, there holds $\Phi(u) \leq \underline{\lim}_{n\to\infty}\Phi(u_n)$.*

Lemma 2.8 ([6], Theorem 6.2.1). *Let X be a reflexive Banach space and let $\Phi : X \to R$ be Gâteaux differentiable on X. Then the following conditions are equivalent:*

(i) Φ is convex on X.
(ii) $\Phi(u) - \Phi(v) \geq \langle\Phi'(v), u - v\rangle$, $\forall u,v \in X$, where $\langle .,. \rangle$ is the usual inner product.
(iii) Φ' is monotone on X.

Lemma 2.9. *The operator Φ_1' is strictly monotone in X, and the operator Φ_1 is sequentially weakly lower semicontinuous.*

Proof. According to [24, (2.2)], we sees that there exists a constant C_p such that

$$\langle |x|^{p-2}x-|y|^{p-2}y, x-y\rangle \geq C_p|x-y|^p, \text{ if } p\geq 2,$$

and

$$\langle |x|^{p-2}x-|y|^{p-2}y, x-y\rangle \geq \frac{C_p|x-y|^2}{(|x|+|y|)^{2-p}}, \text{ if } 1<p<2,$$

where $\langle .,.\rangle$ is the usual inner product. Thus, for any $u,v\in X$ satisfying $u\neq v$, the above two inequalities lead to

$$\begin{aligned}\langle \Phi_1'(u)-\Phi_1'(v), u-v\rangle &= \int_\Omega \frac{b(x)}{|x|^{2h}}(|u|^{h-2}u-|v|^{h-2}v)(u-v)dx\\ &\quad+\int_\Omega \frac{c(x)}{|x|^{r(x)}}(|u|^{q(x)-2}u-|v|^{q(x)-2}v)(u-v)dx>0,\end{aligned}$$

which implies that Φ_1 is strictly monotone on X. Hence, by Lemma 2.8, Φ_1' is convex. For any sequence $u_n\in X$ satisfying $u_n\to u$ in X, one has $\Phi_1(u_n)-\Phi_1(u)\geq\langle\Phi_1'(u),u_n-u\rangle$, which leads to

$$\Phi_1(u)\leq\Phi_1(u_n)-\langle\Phi_1'(u),u_n-u\rangle\leq\Phi_1(u_n)+\|\Phi_1'(u)\|\|u_n-u\|_a,$$

and the above inequality implies that $\Phi_1(u)\leq\underline{\lim}_{n\to\infty}\Phi_1(u_n)$. Thus Φ_1 is lower semicontinuous, which together with Lemma 2.7 yields that Φ_1 is sequentially weakly lower semicontinuous on X. □

Lemma 2.10 ([2], Proposition 2.5)**.** *The operator J' is a mapping of (S_+)-type, i.e., if $u_n\rightharpoonup u$ in X, and $\overline{\lim}_{n\to\infty}\langle J'(u_n),u_n-u)\rangle\leq 0$, then $u_n\to u$ in X.*

Lemma 2.11 ([2], Proposition 2.5)**.** *The operator J is sequentially weakly lower semicontinuous, that is, for any $u\in X$ and any sequence $u_n\in X$ satisfying $u_n\rightharpoonup u$ in X, there holds $J(u)\leq\underline{\lim}_{n\to\infty}J(u_n)$.*

Lemma 2.12. *The operator $\Psi': X\to X^*$ is compact, where X^* is the dual of X.*

Proof. Condition (f_1) and the compact embedding $X\hookrightarrow L^{s(x)}(\Omega), 1<s(x)<p_2^*(x)$ imply the compactness of $\Psi'(u)$. In fact, let $\{u_n\}\in X$ be a sequence such that $u_n\rightharpoonup u$. Since $X\hookrightarrow L^{s(x)}(\Omega), 1<s(x)<p_2^*(x)$ is compact, we see that there exists a subsequence, still denoted by $\{u_n\}$, such that $u_n\to u$, strongly in $L^{s(x)}(\Omega)$. We demonstrate that the Nemytskii operator $N_f(u)(x)=f(x,u(x))$ is continuous since $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying (f_1). Thus $N_f(u_n)\to N_f(u)$ in $L^{\frac{s(x)}{s(x)-1}}(\Omega)$. In view of the Hölder's inequality mentioned in Lemma 2.6 and the compact embedding $X\hookrightarrow L^{s(x)}(\Omega), 1<s(x)<p_2^*(x)$, for all $v\in X$, one has

$$\begin{aligned}|\Psi'(u_n)(v)-\Psi'(u)(v)| = |\int_\Omega(f(x,u_n)-f(x,u))vdx| &\leq 2\|N_f(u_n)-N_f(u)\|_{\frac{s(x)}{s(x)-1}}\|v\|_{s(x)}\\ &\leq 2c_s\|N_f(u_n)-N_f(u)\|_{\frac{s(x)}{s(x)-1}}\|v\|_a,\end{aligned}$$

where c_s is the embedding constant of the embedding $X\hookrightarrow L^{s(x)}(\Omega)$. Thus $\Psi'(u_n)\to\Psi'(u)$ in X^*, i.e., Ψ' is completely continuous, and then Ψ' is compact. □

Lemma 2.13. *There exists a constant* $\bar{c}$ *such that* $\int_\Omega \frac{|u(x)|^h}{|x|^{2h}}dx \le \bar{c}\|u\|_a^h$ *for all* $u \in X$.

Proof. Recall the Hardy-Rellich inequality (see [7, Lemma 2.] for more details). If $1 < h < \frac{N}{2}$, then

$$\int_\Omega \frac{|u(x)|^h}{|x|^{2h}}dx \le \frac{1}{H}\int_\Omega |\Delta u|^h dx,\ \forall u \in W_0^{1,h}(\Omega)\cap W^{2,h}(\Omega),$$

where $H = (\frac{N(h-1)(N-2p)}{h^2})^h$. From Lemma 2.3 and $h < p(x)$, one has $X \hookrightarrow W_0^{1,h}(\Omega)\cap W^{2,h}(\Omega)$. Thus, for $\forall u \in X$, there exists a constant $\kappa > 0$ such that $\int_\Omega (|\Delta u|^h + a(x)|u|^h)dx \le \kappa^h\|u\|_a^h$. Hence

$$\int_\Omega \frac{|u(x)|^h}{|x|^{2h}}dx \le \frac{\kappa^h}{H}\|u\|_a^h,\ \forall u \in X.$$

□

From the condition $q(x) < \frac{Np_2^*(x)}{N+r(x)p_2^*(x)}$, we have

$$q(x) < \frac{N-r(x)q(x)}{N}p_2^*(x) \le \frac{N-r(x)}{N}p_2^*(x).$$

By similar arguments as [11, Corollary 2.1], we have the following lemma.

Lemma 2.14. *Assume that* $0 \in \bar{\Omega}$ *and the boundary of* Ω *possesses the cone property and* $p \in C_+(\overline{\Omega}), s, q \in C(\overline{\Omega}), 0 \le r(x) < N, \forall x \in \bar{\Omega}$. *If* q *satisfies* $1 \le q(x) < \frac{N-r(x)}{N}p_2^*(x)$ *for all* $x \in \Omega$, *then* $W^{2,p(x)}(\Omega) \hookrightarrow L^{q(x)}_{|x|^{-r(x)}}(\Omega)$ *is compact.*

The lemma is crucial to our paper as it guarantees the compactness of the embedding

$$W^{2,p(x)}(\Omega) \hookrightarrow L^{q(x)}_{|x|^{-r(x)}}(\Omega),$$

which is essential for ensuring the Palais-Smale condition. This condition is a key requirement for applying the critical point theory to prove the existence of solutions to our Kirchhoff-type biharmonic equations with Hardy-type singular coefficients. By establishing this compactness, the lemma supports the validity of our variational approach and underpins the theoretical framework needed to handle the complexities introduced by variable exponents and singularities in our problem.

Lemma 2.15. *There exists a constant* $\tilde{c}$ *such that* $\int_\Omega \frac{|u|^{q(x)}}{|x|^{r(x)}}dx \le \tilde{c}(\|u\|_a^{q^+} + \|u\|_a^{q^-})$.

Proof. Combining Lemma 2.2 with Lemma 2.14, we see that there exists a positive constant $\tilde{c_1}$ such that

$$\int_\Omega \frac{|u|^{q(x)}}{|x|^{r(x)}}dx \le |u|^{q^+}_{(q(x),|x|^{-r(x)}} + |u|^{q^-}_{(q(x),|x|^{-r(x)}} \le \tilde{c_1}(\|u\|^{q^+}_{W^{2,p(x)}} + \|u\|^{q^-}_{W^{2,p(x)}}).$$

Since $X \hookrightarrow W^{2,p(x)}(\Omega)$, whence there exists a positive constant $\tilde{c_2}$ such that

$$\|u\|^{q^+}_{W^{2,p(x)}} + \|u\|^{q^-}_{W^{2,p(x)}} \le \tilde{c_2}(\|u\|_a^{q^+} + \|u\|_a^{q^-}),$$

we see that there exists $\tilde{c} = \tilde{c_1}\cdot\tilde{c_2}$ such that $\int_\Omega \frac{|u|^{q(x)}}{|x|^{r(x)}}dx \le \tilde{c}(\|u\|_a^{q^+} + \|u\|_a^{q^-})$. □

Lemma 2.16 ([8]). *Let p,q be measurable functions such that $p \in L^{\infty}(\Omega)$ and $1 < p(x)q(x) < \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$ such that $u \neq 0$. Then*

$$\min\{\|u\|_{p(x)q(x)}^{p^-}, \|u\|_{p(x)q(x)}^{p^+}\} \leq \||u|^{p(x)}\|_{q(x)} \leq \max\{\|u\|_{p(x)q(x)}^{p^-}, \|u\|_{p(x)q(x)}^{p^+}\}.$$

We rely on the following critical point theorems to find generalized solutions.

Theorem 2.1 ([6]). *Assume that X is a reflexive Banach space and the functional $I : X \to \mathbb{R}$ is*
(i) coercive on X;
(ii) (sequentially) weakly lower semicontinuous on X.
Then I is bounded from below on X and attains its infimum in X.

Theorem 2.2 ([3]). *Let X be a real Banach space, and let $\Phi, \Psi : X \to R$ be two continuous Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{\Phi(u)\leq r} \Psi(u) < +\infty$ and assume that, for each*

$$\lambda \in \left(0, \frac{r}{\sup_{\Phi(u)\leq r} \Psi(u)}\right)$$

the functional $\Phi - \lambda\Psi$ satisfies the Palais-Smale condition and it is unbounded from below. Then, for each $\lambda \in (0, \frac{r}{\sup_{\Phi(u)\leq r} \Psi(u)})$, the functional $\Phi - \lambda\Psi$ admits at least two distinct critical points.

Theorem 2.3 ([4]). *Let X be a real Banach space, and let $\Phi, \Psi : X \to R$ be two continuous Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r \in \mathbb{R}$ and $\bar{u} \in X$ with $0 < \Phi(\bar{u}) < r$ such that*

$$\frac{1}{r} \sup_{\Phi(u)\leq r} \Psi(u) < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$$

and for each $\lambda \in \left(\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\leq r} \Psi(u)}\right)$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the Palais-Smale condition and it is unbounded from below. Then, for each $\lambda \in \left(\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\leq r} \Psi(u)}\right)$, the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

3. Main Results

Three theorems on the existence of at least one or least one two nontrivial generalized solutions to equation (1.1) are obtained in this section. For $k > 0$, and $r(x) \in C_+(\overline{\Omega})$, we use the following notations $k^{\hat{r}} = \max\{k^{r^-}, k^{r^+}\}$ and $k^{\check{r}} = \min\{k^{r^-}, k^{r^+}\}$. Condition (f_1) implies that there exists a constant $C_1 > 0$ such that

$$|F(x,u)| \leq C_1(M_1(x)|u| + M_2|u|^{s(x)}), \quad a.e. \ (x,u) \in \Omega \times \mathbb{R}. \tag{3.1}$$

Theorem 3.1. Let (f_1) hold. Then (1.1) has at least one generalized solution when $\lambda > 0$.

Firstly, we prove that $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ is coercive on X for any $\lambda > 0$. For $\|u\|_a > 1$, by the compact embedding $X \hookrightarrow L^{s(x)}(\Omega), 1 < s(x) < p_2^*(x)$, the Hölder's inequality, and (3.1), we have

$$\begin{aligned}
&\Phi(u)-\lambda\Psi(u)\\
&=\mathscr{M}\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx+\int_\Omega \frac{a(x)}{p(x)}|u|^{p(x)}\,dx\Big)+\int_\Omega \frac{b(x)|u|^h}{h|x|^{2h}}dx+\int_\Omega \frac{c(x)|u|^{q(x)}}{q(x)|x|^{r(x)}}dx\\
&\quad-\lambda\int_\Omega F(x,u)dx\\
&\geq \frac{m_1}{\alpha p^{+\alpha}}\|u\|_a^{\alpha p^-}-\lambda C_1\int_\Omega (M_1|u|+M_2|u|^s dx)\\
&\geq \frac{m_1}{\alpha p^{+\alpha}}\|u\|_a^{\alpha p^-}-\lambda C_1(2c_s\|M_1\|_{\frac{s(x)}{s(x)-1}}\|u\|_a+M_2c_s^{\hat{s}}\|u\|_a^{s^+}),
\end{aligned}$$

where c_s is the embedding constant of the embedding $X\hookrightarrow L^{s(x)}(\Omega)$. Thus $1<s^+<\alpha p^-$ leads to the coercivity of $\Phi(u)-\lambda\Psi(u)$.

Secondly, we prove that $\Psi(u)$ is weakly semicontinuous. If $u_n\rightharpoonup u$ in X, then $u_n\to u$ in $L^{s(x)}(\Omega)$ due to Lemma 2.4. Hence, by the mean value theorem, the Hölder's inequality, and Lemma 2.16, one sees that there exists $v_0\in X$ such that

$$\begin{aligned}
|\Psi(u_n)-\Psi(u)|&\leq \int_\Omega |u_n-u|(M_1(x)+M_2|v_0|^{s(x)-1})dx\\
&\leq 2\|u_n-u\|_{s(x)}\|M_1\|_{\frac{s(x)}{s(x)-1}}+2M_2\|u_n-u\|_{s(x)}\||v_0|^{s(x)-1}\|_{\frac{s(x)}{s(x)-1}}\\
&\leq 2\|u_n-u\|_{s(x)}\|M_1\|_{\frac{s(x)}{s(x)-1}}+2M_2\|u_n-u\|_{s(x)}(\||v_0|\|_{s(x)}^{s^--1}+\||v_0|\|_{s(x)}^{s^+-1})\\
&\to 0\ (u_n\rightharpoonup u).
\end{aligned}$$

Thus $\Psi(u)$ is weakly continuous. On the other hand, since $J(u)$ is sequentially weakly lower semicontinuous via Lemma 2.11, the continuity and the growth of $\mathscr{M}(J(u))$ allow us to deduce that $\mathscr{M}(J(u))$ is sequentially weakly lower semicontinuous. Furthermore, we know that Φ_1 is sequentially weakly lower semicontinuous in Lemma 2.9. Thus $I_\lambda(u)=\mathscr{M}(J(u))+\Phi_1(u)-\lambda\Psi(u)$ is sequentially weakly lower semicontinuous. In view of Theorem 2.1, we see that I_λ attains its infimum in X. Therefore, (1.1) has at least one generalized solutions, possibly being trivial solution.

We now focus on the existence of nontrivial generalized solutions.

Theorem 3.2. Let (f_1) hold and there exist $\mu>\max\{m_2\alpha(p^+)^\alpha/m_1(p^-)^{\alpha-1},q^+\}$ and $R>0$ such that (f_2) $0<\mu F(x,u)\leq f(x,u)u$ for all $x\in\Omega, |u|\geq R$. Then (1.1) has at least two generalized solutions when $\lambda\in\Big(0,\frac{1}{2c_sC_1\gamma_0\|M_1\|_{\frac{s(x)}{s(x)-1}}+C_1M_2c_s^{\hat{s}}\gamma_0^{\hat{s}}}\Big)$, where $\gamma_0=\max\{(\frac{\alpha}{m_1}p^{+\alpha})^{\frac{1}{\alpha p^-}},(\frac{\alpha}{m_1}p^{+\alpha})^{\frac{1}{\alpha p^+}}\}$, and c_s is the embedding constant of the embedding $X\hookrightarrow L^{s(x)}(\Omega)$.

Proof. In view of

$$\begin{aligned}
\Phi(u)&=\mathscr{M}\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx+\int_\Omega \frac{a(x)}{p(x)}|u|^{p(x)}\,dx\Big)+\int_\Omega \frac{b(x)|u|^h}{h|x|^{2h}}dx+\int_\Omega \frac{c(x)|u|^{q(x)}}{q(x)|x|^{r(x)}}dx\\
&\geq \frac{m_1}{\alpha p^{+\alpha}}\|u\|_a^{\alpha\check{p}},
\end{aligned}$$

one has that Φ is bounded from below. Let $\{u_n\} \subset X$ such that $\{I_\lambda(u_n)\}$ is bounded, and $I'_\lambda(u_n) \to 0$ as $n \to +\infty$, i.e., there exists a constant $L > 0$ independent from n such that $|I_\lambda(u_n)| \leq L$. For n large enough, one has

$$|I'_\lambda(u_n)u_n| \leq \|I'_\lambda(u_n)\| \|u_n\|_a \leq \|u_n\|_a, \tag{3.2}$$

If $|u_n| \geq R$, in view of condition (f_2) with $\mu > \max\{m_2\alpha(p^+)^\alpha/m_1(p^-)^{\alpha-1}, q^+\} > \alpha p^+ > p^+ > h$, one has

$$\begin{aligned}
&\mu I_\lambda(u_n) - I'_\lambda(u_n)u_n \\
&= \mu \mathscr{M}\Big(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx + \int_\Omega \frac{a(x)}{p(x)}|u_n|^{p(x)}\,dx\Big) \\
&\quad + \mu\int_\Omega \frac{b(x)|u_n|^h}{h|x|^{2h}}dx + \mu\int_\Omega \frac{c(x)|u_n|^{q(x)}}{q(x)|x|^{r(x)}}dx - \lambda\mu\int_\Omega F(x,u_n)dx \\
&\quad - M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx + \int_\Omega \frac{a(x)}{p(x)}|u_n|^{p(x)}\,dx\Big)\Big(\int_\Omega |\Delta u_n|^{p(x)}\,dx + \int_\Omega a(x)|u_n|^{p(x)}\,dx\Big) \\
&\quad - \int_\Omega \frac{b(x)|u_n|^h}{|x|^{2h}}dx - \int_\Omega \frac{c(x)|u_n|^{q(x)}}{|x|^{r(x)}}dx + \lambda\int_\Omega f(x,u_n)u_n dx \\
&\geq \Big(\frac{\mu m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{(p^-)^{\alpha-1}}\Big) \times \Big(\int_\Omega |\Delta u_n|^{p(x)}\,dx + \int_\Omega a(x)|u_n|^{p(x)}\,dx\Big)^\alpha.
\end{aligned} \tag{3.3}$$

If $\|u_n\|_a \leq 1$, then $\{u_n\}$ is bounded. If $\|u_n\|_a > 1$, combining (3.2) with (3.3), we obtain

$$\mu L + \|u_n\| \geq \mu I_\lambda(u_n)_a - I'_\lambda(u_n)u_n \geq \Big(\frac{\mu m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{(p^-)^{\alpha-1}}\Big)\|u_n\|_a^{\alpha p^-},$$

which implies that $\{u_n\}$ is bounded. Thus there exists $u \in X$ such that, passing to a subsequence and still denoted by $\{u_n\}$, it converges weakly to u in X and converges strongly to u in $L^{s(x)}(\Omega)$. Moreover,

$$\begin{aligned}
\Big|\int_\Omega f(x,u_n)(u_n-u)\,dx\Big| &\leq \int_\Omega |f(x,u_n)||u_n-u|\,dx \\
&\leq \int_\Omega (M_1(x) + M_2|u_n|^{s(x)-1})|u_n-u|\,dx \\
&\leq 2\|M_1\|_{\frac{s(x)}{s(x)-1}}\|u_n-u\|_{s(x)} + 2M_2\|u_n-u\|_{s(x)}\||u_n|^{s(x)-1}\|_{\frac{s(x)}{s(x)-1}} \\
&\leq 2\|u_n-u\|_{s(x)}\|M_1\|_{\frac{s(x)}{s(x)-1}} + 2M_2\|u_n-u\|_{s(x)}(\||u_n|\|_{s(x)}^{s^--1} + \||u_n|\|_{s(x)}^{s^+-1}) \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which yields

$$\lim_{m\to\infty}\int_\Omega f(x,u_n)(u_n-u)\,dx = 0. \tag{3.4}$$

Since $\{u_n\}$ converges weakly to u in X, we obtain

$$I'_\lambda(u_n)(u_n-u) = \Phi'(u_n)(u_n-u) - \lambda\Psi'(u_n)(u_n-u) \to 0.$$

Furthermore, by (3.4), we have $\Phi'(u_n)(u_n-u)\to 0$, that is,

$$\Phi'(u_n)(u_n-u)=M(J(u_n))\Big(\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n\Delta(u_n-u)dx+\int_\Omega a(x)|u_n|^{p(x)-2}u_n(u_n-u)dx\Big)$$
$$+\int_\Omega \frac{b(x)|u|^{h-2}}{|x|^{2h}}uvdx+\int_\Omega \frac{c(x)|u|^{q(x)-2}}{|x|^{r(x)}}uvdx\to 0,$$

Now, we prove that

$$\Phi'_{11}(u)(v)=\int_\Omega \frac{b(x)|u|^{h-2}}{|x|^{3h}}uvdx$$

and

$$\Phi'_{12}(u)(v)=\int_\Omega \frac{c(x)|u|^{q(x)-2}}{|x|^{r(x)}}uvdx$$

are completely continuous. In fact, by $u_n\rightharpoonup u$ in X, the compact embedding $X\hookrightarrow L^{q(x)}(\Omega)$, it follows that $u_n\to u$ in $L^{q(x)}(\Omega)$ and $|u_n|^{q(x)-2}u_n\to |u|^{q(x)-2}u$ in $L^{\frac{q(x)}{q(x)-1}}$. For all $v\in X$, by Hölder's inequality and Lemma 2.15, there exists a constant $\dot{c}$ such that

$$\begin{aligned}
|\Phi'_{12}(u_n)(v)-\Phi'_{12}(u)(v)| &= \Big|\int_\Omega \frac{c(x)|u_n|^{q(x)-2}}{|x|^{r(x)}}u_nvdx-\int_\Omega \frac{c(x)|u|^{q(x)-2}}{|x|^{r(x)}}uvdx\Big|\\
&\le \int_\Omega |(|u_n|^{q(x)-2}u_n-|u|^{q(x)-2}u)|c(x)\frac{|v|}{|x|^{r(x)}}dx\\
&\le \|c\|_\infty \||u_n|^{q(x)-2}u_n-|u|^{q(x)-2}u\|_{\frac{q(x)}{q(x)-1}}\Big\|\frac{v}{|x|^{r(x)}}\Big\|_{q(x)}\\
&\le \|c\|_\infty \||u_n|^{q(x)-2}u_n-|u|^{q(x)-2}u\|_{\frac{q(x)}{q(x)-1}}\\
&\quad\times \dot{c}\big((\|v\|_a^{q^+}+\|v\|_a^{q^-})^{\frac{1}{q^-}}+(\|v\|_a^{q^+}+\|v\|_a^{q^-})^{\frac{1}{q^+}}\big).
\end{aligned}$$

Thus Φ'_{12} is completely continuous. Similarly, we can obtain Φ'_{11} is completely continuous. Then $\Phi'_{12}(u_n)\to\Phi'_{12}(u)$ and $\Phi'_{12}(u_n)(u_n)\to\Phi'_{12}(u)(u)$. It follows that

$$|\Phi'_{12}(u_n)(u_n)-\Phi'_{12}(u_n)(u)|\le|\Phi'_{12}(u_n)(u_n)-\Phi'_{12}(u)(u)|+|\Phi'_{12}(u_n)(u)-\Phi'_{12}(u)(u)|$$

and

$$|\Phi'_{12}(u_n)(u_n)-\Phi'_{12}(u_n)(u)|\le|\Phi'_{12}(u_n)(u_n)-\Phi'_{12}(u)(u)|+\|\Phi'_{12}(u_n)-\Phi'_{12}(u)\|_*\|u\|.$$

Hence,

$$\int_\Omega \frac{c(x)|u_n|^{q(x)-2}}{|x|^{r(x)}}u_n(u_n-u)dx\to 0\ \ as\ n\to\infty.$$

Similarly, $\int_\Omega \frac{b(x)|u_n|^{h-2}}{|x|^{3h}}u_n(u_n-u)dx\to 0$ as $n\to\infty$. Thus

$$\lim_{n\to\infty} M(J(u_n))\Big(\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n\Delta(u_n-u)dx+\int_\Omega a(x)|u_n|^{p(x)-2}u_n(u_n-u)dx\Big)=0.$$

If

$$J(u_n)=\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx+\int_\Omega \frac{a(x)}{p(x)}|u_n|^{p(x)}dx\to 0\ \ as\ n\ \to\infty,$$

then

$$\int_\Omega |\Delta u_n|^{p(x)}dx + \int_\Omega a(x)|u_n|^{p(x)}dx \to 0 \ \ as\ n\ \to \infty.$$

Hence, $\|u_n\|_a \to 0$ as $n \to \infty$ by Lemma 2.1. If

$$J(u_n) = \int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx + \int_\Omega \frac{a(x)}{p(x)}|u_n|^{p(x)}dx \to t_0 > 0,$$

then $M(J(u_n)) \geq \frac{1}{2}M(t_0) > 0$ for n large enough. It follows that

$$\lim_{n\to\infty}(\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n \Delta(u_n - u)dx + \int_\Omega a(x)|u_n|^{p(x)-2}u_n(u_n - u)dx) = 0,$$

that is, $\lim_{n\to\infty} J'(u_n)(u_n - u) = 0$. Thus $u_n \to u$ since J' is of (S_+)-type by Lemma 2.10. It follows that I_λ satisfies the Palais-Smale condition.

Next, we prove I_λ is unbounded from below. Since $0 < \mu F(x,u) \leq f(x,u)u$ for all $x \in \Omega, |u| \geq R$, one sees that there exist two constants $\bar{\alpha}, \bar{\beta} > 0$ such that $F(t,u) \geq \bar{\alpha}|u|^\mu - \bar{\beta}$ for all $u \in \mathbb{R} \setminus \{0\}$. Thus, for $m > 1$ and fixed $\tilde{u} \in X \setminus \{0\}$, we find that

$$\begin{aligned}
I_\lambda(m\tilde{u}) &= \mathscr{M}(J(m\tilde{u})) + \int_\Omega \frac{b(x)|m\tilde{u}|^h}{h|x|^{2h}}dx + \int_\Omega \frac{c(x)|m\tilde{u}|^{q(x)}}{q(x)|x|^{r(x)}}dx - \lambda \int_\Omega F(x,m\tilde{u})dx \\
&\leq \frac{m_2}{\alpha}(\frac{m^{p^+}}{p^-})^\alpha(\int_\Omega(|\Delta\tilde{u}|^{p(x)} + a(x)|\tilde{u}|^{p(x)})dx)^\alpha + \frac{m^h\kappa^h\|b\|_\infty}{hH}\|\tilde{u}\|_a^h \\
&\quad + \frac{\tilde{c}\|c\|_\infty m^{q^+}}{q^-}(\|\tilde{u}\|_a^{q^+} + \|\tilde{u}\|_a^{q^-}) - \lambda\bar{\alpha}m^\mu \int_\Omega |\tilde{u}|^\mu dx + \lambda\bar{\beta}|\Omega| \\
&\leq \frac{m_2}{\alpha}(\frac{m^{p^+}}{p^-})^\alpha(\|\tilde{u}\|_a^{p^+} + \|\tilde{u}\|_a^{p^-})^\alpha + \frac{m^h\kappa^h\|b\|_\infty}{hH}\|\tilde{u}\|_a^h \\
&\quad + \frac{\tilde{c}\|c\|_\infty m^{q^+}}{q^-}(\|\tilde{u}\|_a^{q^+} + \|\tilde{u}\|_a^{q^-}) - \lambda\bar{\alpha}m^\mu \int_\Omega |\tilde{u}|^\mu dx + \lambda\bar{\beta}|\Omega|,
\end{aligned}$$

where $|\Omega|$ is the Lebesgue measure of Ω, which leads to $I_\lambda(m\tilde{u}) \to -\infty (m \to +\infty)$ since $\mu > \frac{m_2\alpha(p^+)^\alpha}{m_1(p^-)^{\alpha-1}} > \alpha p^+ > p^+ > h$, and $\mu > q^+$. Thus I_λ is unbounded from below. For every $u \in \Phi^{-1}((-\infty,1])$, one has $\Phi(u) \leq 1$ and

$$\|u\|_a \leq \max\{(\frac{\alpha}{m_1}p^{+\alpha})^{\frac{1}{\alpha p^-}}, (\frac{\alpha}{m_1}p^{+\alpha})^{\frac{1}{\alpha p^+}}\} = \gamma_0,$$

so

$$\begin{aligned}
\sup_{u\in\Phi^{-1}((-\infty,1])} \Psi(u) &\leq C_1 \sup_{\|u\|_a\leq\gamma_0} \int_\Omega (M_1|u| + M_2|u|^s)dx \\
&\leq C_1 \sup_{\|u\|_a\leq\gamma_0} (2c_s\|M_1\|_{\frac{s(x)}{s(x)-1}}\|u\|_a + M_2 c_s^{\hat{s}}\|u\|_a^{\hat{s}}) \\
&\leq 2c_s C_1\gamma_0\|M_1\|_{\frac{s(x)}{s(x)-1}} + C_1 M_2 c_s^{\hat{s}}\gamma_0^{\hat{s}},
\end{aligned}$$

where c_s is the embedding constant of the embedding $X \hookrightarrow L^{s(x)}(\Omega)$. Therefore, the hypotheses of Theorem 2.2 are verified. Thus there are two nontrivial critical points for $I_\lambda(u)$, that is, (1.1) has at least two distinct generalized solutions, possibly one being trivial solution.

In the next part, we focus on the existence of two nontrivial generalized solutions. Putting

$$\delta(x) = \sup\{\delta > 0 : B(x,\delta) \subseteq \Omega\}$$

for all $x \in \Omega$, we can demonstrate that there exists $x_0 \in \Omega$ such that $B(x_0,d) \subseteq \Omega$, where $d = \sup_{x\in\Omega} \delta(x)$.

Theorem 3.3. Let (f_1) and (f_2) hold and $F(x,\xi) \geq 0$ for all $(x,\xi) \in B(x_0,d) \times [0,\delta]$. In addition, assume that there are C_δ and γ with

$$C_\delta < \frac{m_1}{\alpha} \min\{\frac{\gamma^{\alpha p^-}}{p^{+\alpha}}, \frac{\gamma^{\alpha p^+}}{p^{+\alpha}}\}, \tag{3.5}$$

where C_δ is defined in the following proof such that

$$\lambda_1 = \frac{2c_s C_1 \gamma p^+ \|M_1\|_{\frac{s(x)}{s(x)-1}} + C_1 M_2 p^+ (c\gamma)^{\hat{s}}}{\gamma^{\check{p}}} < \frac{\text{ess inf}_{B(x_0,\frac{d}{2})} F(x,\delta) |B(x_0,\frac{d}{2})|}{C_\delta} = \lambda_2. \tag{3.6}$$

Thus (1.1) has at least two nontrivial generalized solutions provided $\lambda \in \Lambda = (\frac{1}{\lambda_2}, \frac{1}{\lambda_1})$.

Proof. It is obvious that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Let

$$\bar{u}(x) = \begin{cases} 0 & x \in \Omega \setminus B(x_0,d), \\ \frac{2\delta}{d}(d - |x - x_0|) & x \in B(x_0,d) \setminus B(x_0,\frac{d}{2}), \\ \delta & x \in B(x_0,\frac{d}{2}). \end{cases}$$

Thus

$$\Psi(\bar{u}) = \int_\Omega F(x,\bar{u})dx \geq \int_{B(x_0,\frac{d}{2})} F(x,\delta)dx \geq |B(x_0,\frac{d}{2})| \text{ ess inf}_{B(x_0,\frac{d}{2})} F(x,\delta). \tag{3.7}$$

Direct calculation shows that

$$\Delta\bar{u}(x) = \begin{cases} 0 & x \in \Omega \setminus B(x_0,d) \cup B(x_0,\frac{d}{2}), \\ \frac{2\delta(1-N)}{d(x-x_0)} & x \in B(x_0,d) \setminus B(x_0,\frac{d}{2}), \end{cases}$$

which implies $\Delta\bar{u}(x) \leq \frac{4\delta(N-1)}{d^2}$. Thus

$$\|\Delta\bar{u}(x)\|_{p(x)} \leq \max\{(\frac{4\delta(N-1)}{d^2})^{\frac{\hat{p}}{p^+}} |B(x_0,d)|^{\frac{1}{p^+}}, (\frac{4\delta(N-1)}{d^2})^{\frac{\hat{p}}{p^-}} |B(x_0,d)|^{\frac{1}{p^-}}\} := \bar{M}.$$

Since the norm $\|u\|_a$ is equivalent to the norm $\|\Delta u\|_{p(x)}$, one see that there exists a constant C such that $\|\bar{u}\|_a \leq C\|\Delta\bar{u}\|_{p(x)}$. By (3.5), Lemma 2.1, Lemma 2.13, and Lemma 2.15, one has

$$\begin{aligned}
\Phi(\bar{u}) &\leq \frac{m_2}{\alpha p^{-\alpha}}\Big(\int_\Omega(|\Delta\bar{u}|^{p(x)}+a(x)|\bar{u}|^{p(x)})dx\Big)^\alpha+\frac{\kappa^h\|b\|_\infty}{hH}\|\bar{u}\|_a^h+\frac{\tilde{c}\|c\|_\infty}{q^-}(\|\bar{u}\|_a^{q^+}+\|\bar{u}\|_a^{q^-})\\
&\leq \frac{m_2}{\alpha p^{-\alpha}}(\|\bar{u}\|_a^{p^+}+\|\bar{u}\|_a^{p^-})^\alpha+\frac{\kappa^h\|b\|_\infty}{hH}\|\bar{u}\|_a^h+\frac{\tilde{c}\|c\|_\infty}{q^-}(\|\bar{u}\|_a^{q^+}+\|\bar{u}\|_a^{q^-})\\
&\leq \frac{m_2}{\alpha p^{-\alpha}}(C^{p^+}\|\Delta\bar{u}\|_{p(x)}^{p^+}+C^{p^-}\|\Delta\bar{u}\|_{p(x)}^{p^-})^\alpha+\frac{C^h\kappa^h\|b\|_\infty}{hH}\|\Delta\bar{u}\|_{p(x)}^h\\
&\quad+\frac{\tilde{c}\|c\|_\infty}{q^-}(C^{q^+}\|\Delta\bar{u}\|_{p(x)}^{q^+}+C^{q^-}\|\Delta\bar{u}\|_{p(x)}^{q^-})\\
&\leq \frac{m_2}{\alpha p^{-\alpha}}(C^{p^+}\bar{M}^{p^+}+C^{p^-}\bar{M}^{p^-})^\alpha+\frac{C^h\kappa^h\|b\|_\infty}{hH}\bar{M}^h+\frac{\tilde{c}\|c\|_\infty}{q^-}(C^{q^+}\bar{M}^{q^+}+C^{q^-}\bar{M}^{q^-})\\
&:= C_\delta < \frac{m_1}{\alpha}\min\{\frac{\gamma^{\alpha p^-}}{p^{+\alpha}},\frac{\gamma^{\alpha p^+}}{p^{+\alpha}}\}.
\end{aligned}\tag{3.8}$$

Let $r=\frac{m_1}{\alpha}\min\{\frac{\gamma^{\alpha p^-}}{p^{+\alpha}},\frac{\gamma^{\alpha p^+}}{p^{+\alpha}}\}$. Then $0<\Phi(\bar{u})<r$. For every $u\in\Phi^{-1}((-\infty,r])$, one has $\Phi(u)\leq r$ and

$$\|u\|_a \leq \max\{(\frac{r\alpha}{m_1}p^{+\alpha})^{\frac{1}{\alpha p^-}},(\frac{r\alpha}{m_1}p^{+\alpha})^{\frac{1}{\alpha p^+}}\}=\gamma.$$

Combining with , we obtain

$$\begin{aligned}
\sup_{u\in\Phi^{-1}((-\infty,r])}\Psi(u) &\leq C_1\sup_{\|u\|_a\leq\gamma}\int_\Omega(M_1|u|+M_2|u|^s dx)\\
&\leq C_1\sup_{\|u\|_a\leq\gamma}(2c_s\|M_1\|_{\frac{s(x)}{s(x)-1}}\|u\|_a+M_2c_s^{\hat{s}}\|u\|_a^{\hat{s}})\\
&\leq 2c_sC_1\gamma\|M_1\|_{\frac{s(x)}{s(x)-1}}+C_1M_2(c_s\gamma)^{\hat{s}},
\end{aligned}\tag{3.9}$$

where c_s is the embedding constant of the embedding $X\hookrightarrow L^{s(x)}(\Omega)$. In view of (3.6), (3.7), (3.8), and (3.9), one has

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)}{r} &\leq \frac{2c_sC_1\gamma\|M_1\|_{\frac{s(x)}{s(x)-1}}+C_1M_2(c_s\gamma)^{\hat{s}}}{\frac{\gamma^{\check{p}}}{p^+}}\\
&< \frac{\operatorname{ess\,inf}_{B(x_0,\frac{d}{2})}F(x,\delta)|B(x_0,\frac{d}{2})|}{C_\delta}\leq\frac{\Psi(\bar{u})}{\Phi(\bar{u})}.
\end{aligned}$$

Therefore, the hypotheses of Theorem 2.3 are all verified. Thus, there are two nontrivial critical points for the functional $I_\lambda(u)$, that is, the equation (1.1) has at least two nontrivial generalized solutions when $\lambda\in\Lambda$.

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