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CHARACTERIZING ROBUST ε-OPTIMAL SOLUTION SETS FOR UNCERTAIN CONVEX OPTIMIZATION PROBLEMS

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Abstract. This paper is devoted to the investigation of robust ε-optimal solution sets for convex optimization problems with data uncertainty in both the objective and constraints. We first establish some properties of the Lagrangian-type function corresponding to a multiplier associated to the given robust ε-optimal solution. Then, we describe a new approach to characterize the robust ε-optimal solution sets of this uncertain convex optimization problem via the study of the relationships among the set of Lagrange multipliers corresponding to robust ε-optimal solutions, the set of ε-Kuhn-Tucker vectors, and the set of robust ε-optimal solutions of its Lagrangian dual problem.

Keywords. Robust ε-optimal solution sets; Subdifferential; Uncertain convex optimization.

1. Introduction

Consider the following convex optimization problem

$$\begin{cases} \inf\limits_{x\in C} f_0(x) \\ \text{s.t. } g_0(x) \in -K, \end{cases} \tag{1.1}$$

where X and Y are locally convex vector spaces, $C \subseteq X$ is a nonempty closed convex set, $K \subseteq Y$ is a nonempty closed convex cone, $f_0 : X \to \mathbb{R}$ is a convex function, and $g_0 : X \to Y$ is a continuous K-convex function.

One interesting and important area for (1.1) is to characterize the optimal solution set, which is also a fundamental topic for many mathematical optimization problems that have multiple optimal solutions. Over the past few decades, many successful treatments on characterizations of optimal solution sets of different classes of (1.1) were widely given from several different perspectives. More precisely, Mangasarian [1] gave some simple characterizations of optimal solution sets of a convex optimization problem with smooth functions. Burke and Ferris [2] obtained another more specific characterization of the optimal solution set of a convex optimization problem with nonsmooth functions. Jeyakumar and Yang [3] described characterizations of the optimal solution set of a pseudo-linear optimization problem. Characterizations of the

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optimal solution set of quasi-convex and pseudoconvex optimization problems were presented in [4–7]. Long et al. [8, 9] obtained some characterizations of the optimal solution set of non-convex semi-infinite optimization problems. In [10], the authors proposed a new approach to characterize the optimal solution set of a DC optimization problem via its dual problem.

Recently, following the framework of robust optimization [11–14], the investigation of the robust optimal solution set for (1.1) with uncertain data attracted an increasing interest. Various properties and characterizations of the set of all robust optimal solutions of convex optimization problems with uncertain data were established in [15] under a robust Slater-type condition. By using a robust-type subdifferential constraint qualification, some completely characterizations of robust optimal solution set were obtained in [16] for an uncertain convex optimization problem in locally convex vector spaces. Similar results were also obtained in [17] for a fractional optimization problem with uncertain data in both the objective and constraints. By using a robust Farkas-Minkowski constraint qualification, Li and Wang [18] established various properties and characterizations of the set of all robust optimal solutions of convex optimization problems with uncertain data. A new dual approach to characterize robust optimal solution set of uncertain convex optimization problems was proposed in [19] via their duality properties. The characterizations of the sets of all the robust weak sharp solutions were obtained in [20] for an uncertain convex optimization problem in terms of the similar method considered in [16]. By using a new robust-type constraint qualification, some characterizations of robust optimal solution set were presented in [21] for a nonconvex uncertain semi-infinite programming problems in terms of tangential subdifferentials.

We observe that there exists only two papers [22, 23] in the literature devoting to characterizing approximate solution sets for optimization problems although there are many papers concerning approximate solutions of optimization problems; see, e.g. [24–34]. More precisely, the expression of ε-optimal solution set of a convex infinite optimization problem was given in [22] in terms of the relationships among the set of Lagrange multipliers corresponding to its ε-optimal solutions, the set of ε-optimal solutions of its Lagrange dual problem, and the set of ε-Kuhn-Tucker vectors. By using the ε-subdifferential of the convex objective and the structure of semidefinite matrices, the characterization of robust ε-optimal solution set was given in [23] for a convex semidefinite programming problem that involves uncertain data. Obviously, the characterization of ε-optimal solution sets has received less attention compared with others. This is also the main motivation for the study of robust ε-optimal solution set of convex optimization problems with uncertain data in both the objective and constraints. It is worth noting that our results can be regarded as the generalizations of the characterizations of robust *exact* optimal solution sets for various optimization problems published recently in the literature.

The paper is organized as follows. Section 2 recalls some notions and auxiliary results which are used in the following sections. Section 3 gives some new properties of the Lagrange-type function associated with the considered uncertain convex optimization problem. Section 4 is devoted to a representation of the robust ε-optimal solution set of the considered uncertain convex optimization problem. Section 5, the last section, ends this paper with concluding remarks.

2. Preliminaries

In this paper, we used the standard notations of convex analysis and nonsmooth analysis; see, e.g., [35, 36]. Let X be a locally convex vector space with its topological dual space X^*, endowed with the weak* topology $w(X^*,X)$. Let $\langle x^*,x\rangle = x^*(x)$ be the value of the continuous linear functional $x^* \in X^*$ at $x \in X$. Let D be a closed and convex cone in X. The (positive) dual cone of D is defined by $D^* = \{x^* \in X^* \mid \langle x^*,x\rangle \geq 0, \forall x \in D\}$. Furthermore, for a nonempty set $C \subseteq X$, the indicator function $\delta_C : X \to \mathbb{R}\cup\{+\infty\}$ of C is defined by

$$\delta_C(x) := \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let $\phi : X \to \mathbb{R}\cup\{+\infty\}$ be an extended real valued function. The effective domain, the epigraph, and the conjugate function of ϕ are defined by

$$\operatorname{dom} \phi := \{x \in X : f(x) < +\infty\},$$

$$\operatorname{epi}\phi := \{(x,r) \in X \times \mathbb{R} : \phi(x) \leq r\},$$

and

$$\phi^*(x^*) := \sup_{x\in X} \{\langle x^*,x\rangle - \phi(x)\},$$

respectively. We say that ϕ is a proper function iff $\operatorname{dom} \phi \neq \emptyset$. We say that ϕ is a convex function iff epi ϕ is a convex set, or equivalently,

$$\phi(\alpha x + (1-\alpha)y) \leq \alpha\phi(x) + (1-\alpha)\phi(y), \forall x,y \in X \text{ and } \alpha \in [0,1].$$

Moreover, we say that the function ϕ is concave iff $-\phi$ is convex. For any proper convex function $\phi : X \to \mathbb{R}\cup\{+\infty\}$, the ε-subdifferential of ϕ at $\bar{x} \in \operatorname{dom} \phi$ is defined by

$$\partial_\varepsilon \phi(\bar{x}) = \{x^* \in X^* \mid \phi(x) - \phi(\bar{x}) \geq \langle x^*, x-\bar{x}\rangle - \varepsilon, \forall x \in X\}.$$

Obviously, if $\varepsilon = 0$, then $\partial_\varepsilon \phi(\bar{x})$ becomes

$$\partial \phi(\bar{x}) = \{x^* \in X^* \mid \phi(x) - \phi(\bar{x}) \geq \langle x^*, x-\bar{x}\rangle, \forall x \in X\}.$$

Let Y be another locally convex vector space with its topological dual space Y^*, endowed with the weak* topology $w(Y^*,Y)$. For Y, we consider the partial ordering "$\leq_K$" induced by the convex cone $K \subseteq Y$ and defined by $\vartheta_1 \leq_K \vartheta_2 \iff \vartheta_2 - \vartheta_1 \in K$ for all $\vartheta_1, \vartheta_2 \in Y$. We say that $h : X \to Y$ is K-convex iff, for any $x,y \in X$ and $\alpha \in [0,1]$,

$$h(\alpha x + (1-\alpha)y) \leq_K \alpha h(x) + (1-\alpha)h(y).$$

We say that h is K-concave iff $-h$ is K-convex. Moreover, for any $\lambda \in K^*$, the composition of mapping $\lambda \circ h$ is denoted by λh. When $\lambda = 0$, we take $0h := \delta_{\operatorname{dom} h}$.

Let X, Y, $\overline{Z}$, and Z be locally convex vector spaces. Let $C \subseteq X$ be a nonempty, closed, and convex set, and let $K \subseteq Y$ be a nonempty, closed, and convex cone. In this paper, we consider the following uncertain optimization problem

$$\text{(UP)} \qquad \begin{cases} \inf\limits_{x\in C} f(x,u) \\ \text{s.t. } g(x,v) \in -K, \end{cases}$$

where $f : X \times \overline{Z} \to \mathbb{R}$ and $g : X \times Z \to Y$ are given functions, u and v are uncertain parameters that belong to the corresponding convex and compact uncertainty sets $\mathscr{U} \subseteq \overline{Z}$, and $\mathscr{V} \subseteq Z$, respectively. In the sequel, we assume that $f(\cdot,u)$ is a convex function for any $u \in \mathscr{U}$, $f(x,\cdot)$ is a concave function for any $x \in X$, and $g(\cdot,v)$ is a K-convex function for any $v \in \mathscr{V}$.

For (UP), it is usually associated with the robust (worst-case) counterpart given below:

$$\text{(RUP)} \qquad \begin{cases} \inf\limits_{x\in C} \max\limits_{u\in\mathscr{U}} f(x,u) \\ \text{s.t. } g(x,v) \in -K, \forall v \in \mathscr{V}. \end{cases}$$

To provide some characterizations of the robust ε-optimal solution set of (UP), we first recall some important concepts and auxiliary results which are used later in this paper.

Definition 2.1. The robust feasible set of (UP) is defined by

$$\mathscr{F} := \left\{ x \in C \mid g(x,v) \in -K, \forall v \in \mathscr{V} \right\}.$$

Definition 2.2. [31] Let $\varepsilon \geq 0$. We say that $\hat{x} \in \mathscr{F}$ is a robust ε-optimal solution of (UP) iff it is an ε-optimal solution to (RUP), i.e.,

$$\max_{u\in\mathscr{U}} f(x,u) \geq \max_{u\in\mathscr{U}} f(\hat{x},u) - \varepsilon, \forall x \in \mathscr{F}.$$

The set of all robust ε-optimal solutions for (UP) is denoted by

$$\mathbb{S}_\varepsilon(\text{UP}) := \left\{ x \in \mathscr{F} \;\middle|\; \max_{u\in\mathscr{U}} f(y,u) \geq \max_{u\in\mathscr{U}} f(x,u) - \varepsilon, \forall y \in \mathscr{F} \right\}.$$

Remark 2.1. If $\varepsilon = 0$, then the robust ε-optimal solution of (UP) becomes the exact robust optimal solution of (UP). We denote the robust optimal solution set of (UP) by $\mathbb{S}(\text{UP})$ for convenience in the sequel. We refer to [15, 16, 18, 19] for some characterizations of $\mathbb{S}(\text{UP})$.

In order to derive some characterizations of $\mathbb{S}_\varepsilon(\text{UP})$, we assume that $\mathbb{S}_\varepsilon(\text{UP}) \neq \emptyset$. Let $\bar{x} \in \mathbb{S}_\varepsilon(\text{UP})$. Then, under a robust type constraint qualification condition, [31, Theorem 3.1] demonstrated that there exist $\bar{u} \in \mathscr{U}$, $\bar{v} \in \mathscr{V}$, $\bar{\lambda} \in K^*$, and $\varepsilon_i \geq 0$, $i = 1,2,3$, such that

$$0 \in \partial_{\varepsilon_1} f(\cdot,\bar{u})(\bar{x}) + \partial_{\varepsilon_2}\left(\left(\bar{\lambda} g\right)(\cdot,\bar{v})\right)(\bar{x}) + \partial_{\varepsilon_3}\delta_C(\bar{x}), \tag{2.1}$$

and

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon \leq \left(\bar{\lambda} g\right)(\bar{x},\bar{v}). \tag{2.2}$$

In the special case that $\varepsilon = 0$, under a robust type subdifferential constraint qualification condition, [16, Theorem 3.1] demonstrated that $\bar{x} \in \mathbb{S}(\text{UP})$ if and only if there exist $\bar{u} \in \mathscr{U}$, $\bar{v} \in \mathscr{V}$, and $\bar{\lambda} \in K^*$ such that

$$0 \in \partial f(\cdot,\bar{u})(\bar{x}) + \partial\left(\left(\bar{\lambda} g\right)(\cdot,\bar{v})\right)(\bar{x}) + \partial\delta_C(\bar{x}), \; \left(\bar{\lambda} g\right)(\bar{x},\bar{v}) = 0, \tag{2.3}$$

and

$$f(\bar{x},\bar{u}) = \max_{u\in\mathscr{U}} f(\bar{x},u). \tag{2.4}$$

As usual, the Lagrangian-type function associated with (UP) is defined by

$$\mathscr{L}(x,\lambda,u,v) := \begin{cases} f(x,u) + (\lambda g)(x,v), & \text{if } x \in C, \lambda \in K^*, u \in \mathscr{U}, v \in \mathscr{V}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Definition 2.3. [37] We say that $(\lambda,u,v) \in K^* \times \mathscr{U} \times \mathscr{V}$ is an ε-Kuhn-Tucker vector of (UP), iff $\inf_{x\in C} \mathscr{L}(x,\lambda,u,v) \geq \inf(\text{UP}) - \varepsilon$. The set of all ε-Kuhn-Tucker vectors for (UP) is denoted by KT_ε.

3. Lagrangian-Type Function Properties

In this section, we present some new properties of the Lagrangian-type function associated with (UP). It is observed that the Lagrangian-type function associated with a Lagrange multiplier corresponding to a robust ε-optimal solution may not be constant on the ε-robust optimal solution set.

Proposition 3.1. *For Problem* (UP), *let* $\varepsilon \geq 0$. *Assume that* $\bar{x} \in \mathbb{S}_\varepsilon(\mathrm{UP})$ *and there exist* $\bar{u} \in \mathscr{U}$, $\bar{v} \in \mathscr{V}$, $\bar{\lambda} \in K^*$, *and* $\varepsilon_i \geq 0$, $i = 1,2,3$, *such that conditions* (2.1) *and* (2.2) *hold. Moreover, assume that* $f(\bar{x},\bar{u}) = \max_{u\in\mathscr{U}} f(\bar{x},u)$. *Then,*

(i) $|\mathscr{L}\left(x,\bar{\lambda},\bar{u},\bar{v}\right) - f(\bar{x},\bar{u})| \leq \varepsilon, \forall x \in \mathbb{S}_\varepsilon(\mathrm{UP})$.

(ii) $-\varepsilon \leq \bar{\lambda} g(\bar{x},\bar{v}) \leq 0$.

(iii) *If* $x \in \mathbb{S}_\varepsilon(\mathrm{UP})$, *then* $-2\varepsilon \leq f(x,\bar{u}) - \max_{u\in\mathscr{U}} f(x,u) \leq 0$ *and* $-2\varepsilon \leq \left(\bar{\lambda} g\right)(x,\bar{v}) \leq 0$.

Proof. (i). By (2.1), there exist $u^* \in \partial_{\varepsilon_1} f(\cdot,\bar{u})(\bar{x})$, $v^* \in \partial_{\varepsilon_2}\left(\left(\bar{\lambda} g\right)(\cdot,\bar{v})\right)(\bar{x})$, and $w^* \in \partial_{\varepsilon_3}\delta_C(\bar{x})$ such that

$$u^* + v^* + w^* = 0. \tag{3.1}$$

In view of $u^* \in \partial_{\varepsilon_1} f(\cdot,\bar{u})(\bar{x})$, $v^* \in \partial_{\varepsilon_2}\left(\left(\bar{\lambda} g\right)(\cdot,\bar{v})\right)(\bar{x})$, and $w^* \in \partial_{\varepsilon_3}\delta_C(\bar{x})$, we obtain that, for any $x \in \mathscr{F}$, $f(x,\bar{u}) - f(\bar{x},\bar{u}) \geq \langle u^*, x-\bar{x}\rangle - \varepsilon_1$, $\left(\bar{\lambda} g\right)(x,\bar{v}) - \left(\bar{\lambda} g\right)(\bar{x},\bar{v}) \geq \langle v^*, x-\bar{x}\rangle - \varepsilon_2$, and $\delta_C(x) - \delta_C(\bar{x}) \geq \langle w^*, x-\bar{x}\rangle - \varepsilon_3$. Then, for any $x \in \mathscr{F}$,

$$\begin{aligned} f(x,\bar{u}) + \left(\bar{\lambda} g\right)(x,\bar{v}) - \left(f(\bar{x},\bar{u}) + \left(\bar{\lambda} g\right)(\bar{x},\bar{v})\right) &\geq \langle u^* + v^* + w^*, x-\bar{x}\rangle - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 \\ &= -\varepsilon_1 - \varepsilon_2 - \varepsilon_3, \end{aligned}$$

where the equality is derived from (3.1). Combining this with (2.2), we have

$$\mathscr{L}\left(x,\bar{\lambda},\bar{u},\bar{v}\right) - f(\bar{x},\bar{u}) \geq \left(\bar{\lambda} g\right)(\bar{x},\bar{v}) - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 \geq -\varepsilon, \ \forall x \in \mathscr{F}. \tag{3.2}$$

On the other hand, if $x \in \mathbb{S}_\varepsilon(\mathrm{UP})$, then $\max_{u\in\mathscr{U}} f(x,u) \leq \max_{u\in\mathscr{U}} f(z,u) + \varepsilon$ for all $z \in \mathscr{F}$. Hence,

$$\mathscr{L}\left(x,\bar{\lambda},\bar{u},\bar{v}\right) \leq f(x,\bar{u}) \leq \max_{u\in\mathscr{U}} f(x,u) \leq \max_{u\in\mathscr{U}} f(\bar{x},u) + \varepsilon = f(\bar{x},\bar{u}) + \varepsilon, \ \forall x \in \mathbb{S}_\varepsilon(\mathrm{UP}),$$

which together with (3.2) gives $|\mathscr{L}\left(x,\bar{\lambda},\bar{u},\bar{v}\right) - f(\bar{x},\bar{u})| \leq \varepsilon, \forall x \in \mathbb{S}_\varepsilon(\mathrm{UP})$. Thus (i) holds.

(ii): Obviously, by $\bar{x} \in \mathscr{F}$ and (3.2), we can easily have $-\varepsilon \leq \left(\bar{\lambda} g\right)(\bar{x},\bar{v}) \leq 0$. Thus (ii) holds.

(iii). Clearly, $\max_{u\in\mathscr{U}} f(x,u) \leq \max_{u\in\mathscr{U}} f(\bar{x},u) + \varepsilon$ for all $x \in \mathbb{S}_\varepsilon(\mathrm{UP})$, which together with $f(x,\bar{u}) \leq \max_{u\in\mathscr{U}} f(x,u)$ and $f(\bar{x},\bar{u}) = \max_{u\in\mathscr{U}} f(\bar{x},u)$ yields that

$$f(x,\bar{u}) \leq f(\bar{x},\bar{u}) + \varepsilon, \ \forall x \in \mathbb{S}_\varepsilon(\mathrm{UP}). \tag{3.3}$$

Moreover, from (3.2), we have

$$\left(\bar{\lambda} g\right)(x,\bar{v}) \geq f(\bar{x},\bar{u}) - f(x,\bar{u}) - \varepsilon, \ \forall x \in \mathbb{S}_\varepsilon(\mathrm{UP}). \tag{3.4}$$

Then, it follows from (3.3) and (3.4) that $-2\varepsilon \leq \left(\bar{\lambda} g\right)(x,\bar{v}) \leq 0$ for all $x \in \mathbb{S}_\varepsilon(\mathrm{UP})$. In view of $\left(\bar{\lambda} g\right)(x,\bar{v}) \leq 0$ and (3.2), we have

$$f(x,\bar{u}) \geq f(\bar{x},\bar{u}) - \varepsilon = \max_{u\in\mathscr{U}} f(\bar{x},u) - \varepsilon \geq \max_{u\in\mathscr{U}} f(x,u) - 2\varepsilon, \ \forall x \in \mathbb{S}_\varepsilon(\mathrm{UP}).$$

where the last inequality holds due to $x \in \mathbb{S}_\varepsilon(\mathrm{UP})$. This, together with $\max_{u\in\mathscr{U}} f(x,u) \geq f(x,\bar{u})$, gives $-2\varepsilon \leq f(x,\bar{u}) - \max_{u\in\mathscr{U}} f(x,u) \leq 0$ for all $x \in \mathbb{S}_\varepsilon(\mathrm{UP})$. Thus (iii) holds and the proof is complete. □

Inspired by [25, Example 3.1] and [31, Example 5.1], we give an example to explain Proposition 3.1.

Example 3.1. Let $X = \mathbb{R}^2$, $Y = Z = \overline{Z} = \mathbb{R}$, $C = \mathbb{R}^2_+$,and $K = \mathbb{R}_+$. Consider the following uncertain optimization problem

$$\begin{cases} \inf\ x_1 + x_2^2 u \\ \text{ s.t. } x_1^2 - 2vx_1 \leq 0, \\ \qquad (x_1, x_2) \in \mathbb{R}^2_+, \end{cases}$$

where u and v are uncertain parameters belonging to their respective uncertainty sets $\mathscr{U} = [0,1]$ and $\mathscr{V} = [-1,1]$. Its robust counterpart is given by

$$\begin{cases} \inf\ \max\limits_{u \in \mathscr{U}} x_1 + x_2^2 u \\ \text{ s.t. } x_1^2 - 2vx_1 \leq 0, \forall v \in \mathscr{V}, \\ \qquad (x_1, x_2) \in \mathbb{R}^2_+. \end{cases}$$

Obviously, $f(x,u) = x_1 + x_2^2 u$ and $g(x,v) = x_1^2 - 2vx_1$. It is easy to see that

$$\mathscr{F} = \{0\} \times [0, +\infty) \quad \text{and} \quad \mathbb{S}_\varepsilon(\text{UP}) = \{0\} \times [0, \sqrt{\varepsilon}].$$

Set $\varepsilon = \frac{1}{4}$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{64}$, $\bar{u} = 1$, $\bar{v} = \bar{\lambda} = \frac{1}{4}$, and $\bar{x} = \left(0, \frac{1}{8}\right)$. Obviously, $\bar{x} \in \mathbb{S}_\varepsilon(\text{UP})$ and

$$f(\bar{x}, \bar{u}) = \max_{u \in \mathscr{U}} f(\bar{x}, u) = \frac{1}{64}.$$

Moreover, we can show that

$$\partial_{\varepsilon_1} f(\cdot, \bar{u})(\bar{x}) = \{1\} \times \left[0, \frac{1}{2}\right], \quad \partial_{\varepsilon_2} \left(\left(\bar{\lambda} g \right) (\cdot, \bar{v}) \right) (\bar{x}) = \left[-\frac{1}{4}, 0 \right] \times \{0\},$$

and

$$\partial_{\varepsilon_3} \delta_C(\bar{x}) = [-\infty, 0] \times \left[-\frac{1}{8}, 0 \right].$$

Clearly,

$$\partial_{\varepsilon_1} f(\cdot, \bar{u})(\bar{x}) + \partial_{\varepsilon_2} \left(\left(\bar{\lambda} g \right) (\cdot, \bar{v}) \right) (\bar{x}) + \partial_{\varepsilon_3} \delta_C(\bar{x}) = [-\infty, 1] \times \left[-\frac{1}{8}, \frac{1}{2} \right]$$

and $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon \leq \left(\bar{\lambda} g \right) (\bar{x}, \bar{v})$. Thus (2.1) and (2.2) hold.

We now verify the Lagrangian-type function properties obtained in Proposition 3.1. Indeed, for any $(x, \lambda, u, v) \in \mathbb{R}^2_+ \times \mathbb{R}_+ \times \mathscr{U} \times \mathscr{V}$, $\mathscr{L}(x, \lambda, u, v) = x_1 + x_2^2 u + \lambda (x_1^2 - 2vx_1)$. Then,

$$\mathscr{L}\left(x, \bar{\lambda}, \bar{u}, \bar{v}\right) = \frac{1}{4} x_1^2 + \frac{7}{8} x_1 + x_2^2.$$

Obviously, $\left|\mathscr{L}\left(x, \bar{\lambda}, \bar{u}, \bar{v}\right) - f(\bar{x}, \bar{u})\right| \leq \varepsilon$ for all $x \in \mathbb{S}_\varepsilon(\text{UP})$. Moreover, $-\varepsilon \leq \left(\bar{\lambda} g \right) (\bar{x}, \bar{v}) \leq 0$.

On the other hand, for any $x \in \mathbb{S}_\varepsilon(\text{UP}) = \{0\} \times [0, \frac{1}{2}]$, we have

$$\left(\bar{\lambda} g \right) (x, \bar{v}) = \frac{1}{4} x_1^2 - \frac{1}{8} x_1 = 0$$

and

$$f(x, \bar{u}) - \max_{u \in \mathscr{U}} f(x, u) = x_1 + x_2^2 - x_2^2 = x_1 = 0.$$

Clearly,

$$-2\varepsilon \le f(x,\bar{u}) - \max_{u\in\mathscr{U}} f(x,u) \le 0 \text{ and } -2\varepsilon \le \left(\bar{\lambda} g\right)(x,\bar{v}) \le 0,\ \ \forall x \in \{0\}\times\left[0,\frac{1}{2}\right],$$

Thus Proposition 3.1 is applicable.

Set $\varepsilon = 0$ in Proposition 3.1, the following constant Lagrangian property associate with (UP) can be easily established.

Corollary 3.1. *For Problem* (UP)*, assume that* $\bar{x} \in \mathbb{S}(\text{UP})$ *and there exist* $\bar{u} \in \mathscr{U}$*,* $\bar{v} \in \mathscr{V}$*, and* $\bar{\lambda} \in K^*$ *such that conditions* (2.3) *and* (2.4) *hold. Then, for any* $x \in \mathbb{S}(\text{UP})$*,* $\left(\bar{\lambda} g\right)(x,\bar{v}) = 0$*,* $f(x,\bar{u}) = \max_{u\in\mathscr{U}} f(x,u)$*, and* $\mathscr{L}\left(x,\bar{\lambda},\bar{u},\bar{v}\right)$ *is a constant on* $\mathbb{S}(\text{UP})$*.*

Remark 3.1. It is worth mentioning that Corollary 3.1 was established in [15, Theorem 3.1], [16, Theorem 3.1], and [18, Theorem 3.3].

4. Characterizations of Robust ε-Optimal Solution Sets

In this section, by using the Lagrangian-type function properties, we present some new characterizations of the robust ε-optimal solution set for (UP) in terms of subdifferentials and Lagrange multipliers.

Lemma 4.1. *For Problem* (UP)*, let* $\varepsilon \ge 0$*. Assume that* $\bar{x} \in \mathbb{S}_\varepsilon(\text{UP})$ *and there exist* $\bar{u} \in \mathscr{U}$*,* $\bar{v} \in \mathscr{V}$*,* $\bar{\lambda} \in K^*$*, and* $\varepsilon_i \ge 0$*,* $i = 1,2,3$*, such that conditions* (2.1) *and* (2.2) *hold. Then there exists* $u^* \in \partial_{\varepsilon_1} f(\cdot,\bar{u})(\bar{x})$ *such that* $\langle u^*, x-\bar{x}\rangle \ge \varepsilon_1 - \varepsilon - \left(\bar{\lambda} g\right)(\bar{x},\bar{v})$ *for all* $x \in \mathscr{F}$*.*

Proof. By (2.1), one sees that there exist $u^* \in \partial_{\varepsilon_1} f(\cdot,\bar{u})(\bar{x})$, $v^* \in \partial_{\varepsilon_2}\left(\left(\bar{\lambda} g\right)(\cdot,\bar{v})\right)(\bar{x})$, and $w^* \in \partial_{\varepsilon_3}\delta_C(\bar{x})$ such that

$$u^* + v^* + w^* = 0. \tag{4.1}$$

From $v^* \in \partial_{\varepsilon_2}\left(\left(\bar{\lambda} g\right)(\cdot,\bar{v})\right)(\bar{x})$ and $w^* \in \partial_{\varepsilon_3}\delta_C(\bar{x})$, we obtain that, for any $x \in \mathscr{F}$,

$$\left(\bar{\lambda} g\right)(x,\bar{v}) - \left(\bar{\lambda} g\right)(\bar{x},\bar{v}) \ge \langle v^*, x-\bar{x}\rangle - \varepsilon_2,$$

and $\delta_C(x) - \delta_C(\bar{x}) \ge \langle w^*, x-\bar{x}\rangle - \varepsilon_3$. Thus $\left(\bar{\lambda} g\right)(x,\bar{v}) - \left(\bar{\lambda} g\right)(\bar{x},\bar{v}) \ge \langle v^*+w^*, x-\bar{x}\rangle - \varepsilon_2 - \varepsilon_3$ for all $x \in \mathscr{F}$, which indicates that

$$-\langle v^*+w^*, x-\bar{x}\rangle \ge -\left(\bar{\lambda} g\right)(x,\bar{v}) + \left(\bar{\lambda} g\right)(\bar{x},\bar{v}) - \varepsilon_2 - \varepsilon_3,\ \ \forall x \in \mathscr{F}.$$

Then, it follows from (2.2) and (4.1) that $\langle u^*, x-\bar{x}\rangle \ge -\left(\bar{\lambda} g\right)(x,\bar{v}) + \varepsilon_1 - \varepsilon$ for all $x \in \mathscr{F}$. This completes the proof. □

Lemma 4.2. *For Problem* (UP)*, let* $\varepsilon \ge 0$ *and* $\bar{x} \in \mathscr{F}$*. For* $0 \le \varepsilon_1 \le \varepsilon$*, assume that there exist* $\bar{u} \in \mathscr{U}$*,* $\bar{v} \in \mathscr{V}$*,* $\bar{\lambda} \in K^*$*, and* $\xi \in \partial_{\varepsilon_1} f(\cdot,\bar{u})(\bar{x})$ *such that* $f(\bar{x},\bar{u}) = \max_{u\in\mathscr{U}} f(\bar{x},u)$ *and* $\langle \xi, x-\bar{x}\rangle \ge \varepsilon_1 - \varepsilon - \left(\bar{\lambda} g\right)(\bar{x},\bar{v})$ *for all* $x \in \mathscr{F}$*. Then* $\bar{x} \in \mathbb{S}_\varepsilon(\text{UP})$*.*

Proof. By $\xi \in \partial_{\varepsilon_1} f(\cdot,\bar{u})(\bar{x})$, we have $f(x,\bar{u}) - f(\bar{x},\bar{u}) \ge \langle \xi, x-\bar{x}\rangle - \varepsilon_1$ for all $x \in \mathscr{F}$. It follows that $f(x,\bar{u}) - f(\bar{x},\bar{u}) \ge -\varepsilon - \left(\bar{\lambda} g\right)(\bar{x},\bar{v}) \ge -\varepsilon,\quad \forall x \in \mathscr{F}$. This implies that

$$\max_{u\in\mathscr{U}} f(x,u) \le \max_{u\in\mathscr{U}} f(\bar{x},u) + \varepsilon,\quad \forall x \in \mathscr{F}.$$

Thus $x \in \mathbb{S}_\varepsilon(\text{UP})$. This completes the proof. □

Give $0 \leq \varepsilon_1 \leq \varepsilon$. The robust ε-optimal solution set for (UP) corresponding to $(\bar{\lambda}, \bar{u}, \bar{v}, \varepsilon_1)$ mentioned above is denoted by $\mathbb{S}(\bar{\lambda}, \bar{u}, \bar{v}, \varepsilon_1)$. Obviously, $\mathbb{S}(\bar{\lambda}, \bar{u}, \bar{v}, \varepsilon_1) \subseteq \mathbb{S}_\varepsilon(\text{UP})$, where $0 \leq \varepsilon_1 \leq \varepsilon$. By Proposition 3.1(iii), Lemmas 4.1, and 4.2, we have

$$\mathbb{S}(\bar{\lambda}, \bar{u}, \bar{v}, \varepsilon_1) := \Big\{x \in C \mid \xi \in \partial_{\varepsilon_1} f(\cdot, \bar{u})(x), \langle \xi, y - x\rangle \geq \varepsilon_1 - \varepsilon - \left(\bar{\lambda} g\right)(y, \bar{v}), \forall y \in \mathscr{F},$$

$$-2\varepsilon \leq f(x, \bar{u}) - \max_{u \in \mathscr{U}} f(x, u) \leq 0, -2\varepsilon \leq \left(\bar{\lambda} g\right)(x, \bar{v}) \leq 0\Big\}.$$

Let

$$\mathbb{S}_\varepsilon(\bar{\lambda}, \bar{u}, \bar{v}) := \bigcup_{0 \leq \varepsilon_1 \leq \varepsilon} \mathbb{S}(\bar{\lambda}, \bar{u}, \bar{v}, \varepsilon_1).$$

Clearly, $\mathbb{S}_\varepsilon(\bar{\lambda}, \bar{u}, \bar{v}) \subseteq \mathbb{S}_\varepsilon(\text{UP})$. In what follows, we introduce a new approach to characterize $\mathbb{S}_\varepsilon(\text{UP})$. To do this, let $y \in C$ and $\lambda \in K^*$. We consider the following function $\varphi(\lambda, u, v) := \inf_{x \in C} L(x, \lambda, u, v)$. For each fixed $u \in \mathscr{U}$ and $v \in \mathscr{V}$, the conventional Lagrange type dual problem of (UP) is given by

$$\text{(UD)} \qquad\qquad \max_{\lambda \in K^*} \varphi(\lambda, u, v).$$

The optimistic counterpart of (UD), called optimistic dual optimization problem, is given by

$$\text{(OUD)} \qquad \max_{\lambda \in K^*} \max_{u \in \mathscr{U}, v \in \mathscr{V}} \varphi(\lambda, u, v).$$

where the maximization is also over all the parameters $u \in \mathscr{U}$ and $v \in \mathscr{V}$.

In what follows, the optimal value and the feasible set of (OUD) are denoted by $\sup(\text{OUD})$ and $\mathscr{F}^{(\text{OUD})}$, respectively. Note that the strong duality between (RUP) and (OUD) was given in [13, Theorem 3] in terms of a robust basic subdifferential constraint qualification condition.

Definition 4.1. Let $\varepsilon \geq 0$. $(\bar{\lambda}, \bar{u}, \bar{v}) \in \mathscr{F}^{(\text{OUD})}$ is said to be a robust ε-optimal solution to (UD) iff it is an ε-optimal solution to (OUD), i.e., $\varphi(\bar{\lambda}, \bar{u}, \bar{v}) + \varepsilon \geq \varphi(\lambda, u, v)$ for all $(\lambda, u, v) \in K^* \times \mathscr{U} \times \mathscr{V}$. The set of all robust ε-optimal solutions for (UD) is denoted by

$$\mathbb{S}_\varepsilon(\text{UD}) := \left\{(\bar{\lambda}, \bar{u}, \bar{v}) \in \mathscr{F}^{(\text{OUD})} \mid \varphi(\bar{\lambda}, \bar{u}, \bar{v}) + \varepsilon \geq \varphi(\lambda, u, v), \quad \forall (\lambda, u, v) \in K^* \times \mathscr{U} \times \mathscr{V}\right\}.$$

The following proposition gives the relationships between (UD) and KT_ε.

Proposition 4.1. *Suppose that the strong duality between* (RUP) *and* (OUD) *holds. Then,* $\mathbb{S}_\varepsilon(\text{UD}) = \text{KT}_\varepsilon$.

Proof. Letting $(\bar{\lambda}, \bar{u}, \bar{v}) \in \text{KT}_\varepsilon$, we have $\varphi(\bar{\lambda}, \bar{u}, \bar{v}) = \inf_{x \in C} L(x, \bar{\lambda}, \bar{u}, \bar{v}) \geq \inf(\text{UP}) - \varepsilon$. In view of the weak duality, we deduce that $\varphi(\lambda, u, v) \leq \inf(\text{UP})$ for all $(\lambda, u, v) \in K^* \times \mathscr{U} \times \mathscr{V}$. Hence,

$$\varphi(\bar{\lambda}, \bar{u}, \bar{v}) \geq \varphi(\lambda, u, v) - \varepsilon, \quad \forall (\lambda, u, v) \in K^* \times \mathscr{U} \times \mathscr{V}.$$

Thus $(\bar{\lambda}, \bar{u}, \bar{v}) \in \mathbb{S}_\varepsilon(\text{UD})$.

Conversely, let $(\bar{\lambda}, \bar{u}, \bar{v}) \in \mathbb{S}_\varepsilon(\text{UD})$. Then, $\varphi(\bar{\lambda}, \bar{u}, \bar{v}) + \varepsilon \geq \varphi(\lambda, u, v)$ for all $(\lambda, u, v) \in K^* \times \mathscr{U} \times \mathscr{V}$. Since the robust strong duality between (RUP) and (OUD) holds, we have $\varphi(\bar{\lambda}, \bar{u}, \bar{v}) + \varepsilon \geq \inf(\text{UP})$, which means that $\inf_{x \in C} L(x, \bar{\lambda}, \bar{u}, \bar{v}) + \varepsilon \geq \inf(\text{UP})$. Thus $(\bar{\lambda}, \bar{u}, \bar{v}) \in \text{KT}_\varepsilon$. This completes the proof. □

Proposition 4.2. *Let* $\bar{\lambda} \in K^*$*,* $\bar{u} \in \mathscr{U}$ *and* $\bar{v} \in \mathscr{V}$ *satisfy the conditions* (2.1) *and* (2.2) *corresponding to* $\bar{x} \in \mathbb{S}_\varepsilon(\text{UP})$*. Then,* $(\bar{\lambda}, \bar{u}, \bar{v}) \in KT_\varepsilon$.

Proof. Following similar arguments to those used in Proposition 3.1, we see that

$$\mathscr{L}\left(x,\bar{\lambda},\bar{u},\bar{v}\right)-f(\bar{x},\bar{u})\geq-\varepsilon. \tag{4.2}$$

Moreover, from the weak duality between (RUP) and (OUD), we have

$$\max_{u\in\mathscr{U}}f(\bar{x},u)\geq\varphi(\lambda,u,v),\quad\forall(\lambda,u,v)\in K^*\times\mathscr{U}\times\mathscr{V}. \tag{4.3}$$

Note that $f(\bar{x},\bar{u})=\max_{u\in\mathscr{U}}f(\bar{x},u)$. Then, it follows from (4.2) and (4.3) that $\mathscr{L}\left(x,\bar{\lambda},\bar{u},\bar{v}\right)\geq\varphi(\lambda,u,v)-\varepsilon$ for all $(\lambda,u,v)\in K^*\times\mathscr{U}\times\mathscr{V}$ and $x\in C$. Thus,

$$\varphi(\bar{\lambda},\bar{u},\bar{v})=\inf_{x\in C}\mathscr{L}\left(x,\bar{\lambda},\bar{u},\bar{v}\right)\geq\varphi(\lambda,u,v)-\varepsilon,\quad\forall(\lambda,u,v)\in K^*\times\mathscr{U}\times\mathscr{V}.$$

Therefore, $(\bar{\lambda},\bar{u},\bar{v})\in\mathbb{S}_{\varepsilon}(\mathrm{UD})$. It follows from Proposition 4.1 that $(\bar{\lambda},\bar{u},\bar{v})\in KT_{\varepsilon}$. This completes the proof. □

Now, by using the ε-subdifferentials of convex functions, we present a formulation to describe the constructions of robust ε-optimal solution sets of (UP). In what follows, for an ε-optimal solution of (UP), we assume that $\bar{\lambda}\in K^*$, $\bar{u}\in\mathscr{U}$, and $\bar{v}\in\mathscr{V}$ satisfy conditions (2.1) and (2.2).

Theorem 4.1. *Let the strong duality between* (RUP) *and* (OUD) *hold. Then,*

$$\mathbb{S}_{\varepsilon}(\mathrm{UP})=\bigcup_{(\bar{\lambda},\bar{u},\bar{v})\in KT_{\varepsilon}}\mathbb{S}_{\varepsilon}(\bar{\lambda},\bar{u},\bar{v}),$$

where

$$\mathbb{S}_{\varepsilon}(\bar{\lambda},\bar{u},\bar{v})=\bigcup_{0\leq\varepsilon_1\leq\varepsilon}\mathbb{S}(\bar{\lambda},\bar{u},\bar{v},\varepsilon_1).$$

That is

$$\mathbb{S}_{\varepsilon}(\mathrm{UP})=\bigcup_{(\bar{\lambda},\bar{u},\bar{v})\in KT_{\varepsilon}}\bigcup_{0\leq\varepsilon_1\leq\varepsilon}\Big\{x\in C\mid\xi\in\partial_{\varepsilon_1}f(\cdot,\bar{u})(x),\langle\xi,y-x\rangle\geq\varepsilon_1-\varepsilon-\left(\bar{\lambda}g\right)(x,\bar{v}),\forall y\in\mathscr{F},$$
$$-2\varepsilon\leq f(x,\bar{u})-\max_{u\in\mathscr{U}}f(x,u)\leq0,-2\varepsilon\leq\left(\bar{\lambda}g\right)(x,\bar{v})\leq0\Big\}.$$

Proof. ($\subseteq$): Let $z\in\mathbb{S}_{\varepsilon}(\mathrm{UP})$. Then, from Lemma 4.1, there exists $\xi\in\partial_{\varepsilon_1}f(\cdot,\bar{u})(z)$ such that

$$\langle\xi,y-z\rangle\geq\varepsilon_1-\varepsilon-\left(\bar{\lambda}g\right)(z,\bar{v}),\quad\forall y\in\mathscr{F}.$$

Moreover, we find from Proposition 3.1 (iii) that $-2\varepsilon\leq f(z,\bar{u})-\max_{u\in\mathscr{U}}f(z,u)\leq0$ and $-2\varepsilon\leq\bar{\lambda}g(z,\bar{v})\leq0$. This means that $z\in\mathbb{S}(\bar{\lambda},\bar{u},\bar{v},\varepsilon_1)$. Therefore, $z\in\mathbb{S}_{\varepsilon}(\bar{\lambda},\bar{u},\bar{v})$. By Proposition 4.2, we have $(\bar{\lambda},\bar{u},\bar{v})\in KT_{\varepsilon}$. Thus

$$\mathbb{S}_{\varepsilon}(\mathrm{UP})\subseteq\bigcup_{(\bar{\lambda},\bar{u},\bar{v})\in KT_{\varepsilon}}\mathbb{S}_{\varepsilon}(\bar{\lambda},\bar{u},\bar{v}).$$

($\supseteq$): Let $z\in\bigcup_{(\bar{\lambda},\bar{u},\bar{v})\in KT_{\varepsilon}}\mathbb{S}_{\varepsilon}(\bar{\lambda},\bar{u},\bar{v})$. Then, there exists $(\bar{\lambda},\bar{u},\bar{v})\in KT_{\varepsilon}$ such that $z\in\mathbb{S}_{\varepsilon}(\bar{\lambda},\bar{u},\bar{v})$, which means that there exists $0\leq\varepsilon_1\leq\varepsilon$ such that $z\in\mathbb{S}(\bar{\lambda},\bar{u},\bar{v},\varepsilon_1)$.

On the other hand, by Proposition 4.1, we have $\mathbb{S}_{\varepsilon}(\mathrm{UD})=\mathrm{KT}_{\varepsilon}$. In view of $(\bar{\lambda},\bar{u},\bar{v})\in KT_{\varepsilon}$, one has $(\bar{\lambda},\bar{u},\bar{v})\in\mathbb{S}_{\varepsilon}(\mathrm{UD})$. Thus $(\bar{\lambda},\bar{u},\bar{v})\in K^*\times\mathscr{U}\times\mathscr{V}$. By Lemma 4.2, we deduce $z\in\mathbb{S}_{\varepsilon}(\mathrm{UP})$. The proof is complete. □

Remark 4.1. Note that (UP) is finite. Then, for any minimizing sequence $\{y_k\} \in \mathscr{F}$ of (UP), $f(y_k) \to \inf(\mathrm{UP})$. Thus, the formula of $\mathbb{S}_\varepsilon(\mathrm{UP})$ established in Theorem 4.1 can be rewritten as

$$\mathbb{S}_\varepsilon(\mathrm{UP}) = \bigcup_{(\bar\lambda,\bar u,\bar v)\in KT_\varepsilon} \bigcup_{0\le\varepsilon_1\le\varepsilon} \Big\{x \in C \mid \xi \in \partial_{\varepsilon_1} f(\cdot,\bar u)(x), \langle \xi, y_k - x\rangle \ge \varepsilon_1 - \varepsilon - \bar\lambda g(x,\bar v), \forall k \in \mathbb{N},$$
$$-2\varepsilon \le f(x,\bar u) - \max_{u\in\mathscr{U}} f(x,u) \le 0, -2\varepsilon \le \bar\lambda g(x,\bar v) \le 0\Big\}.$$

Remark 4.2. Clearly, by virtue of Example 3.1, we can also illustrate the conclusion of Theorem 4.1.

5. Conclusions

In this paper, following the framework of robust optimization, we considered the characterization of robust ε-optimal solution sets of (UP) in locally convex vector spaces. We first demonstrated that the Lagrangian-type function associated with a Lagrange multiplier corresponding to a robust ε-optimal solution may not be constant on $\mathbb{S}_\varepsilon(\mathrm{UP})$. Then, based on the Lagrange-type function properties, we obtained a representation of the robust ε-optimal solution set $\mathbb{S}_\varepsilon(\mathrm{UP})$ in terms of ε-subdifferential and Lagrange multipliers. It is worth noting that some existing results in [15, 16, 18, 22] can be regarded as special cases of our results.

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