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IMAGE REGULARITY CONDITIONS BASED ON NONCONVEX SEPARATION WITH APPLICATIONS

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Abstract. This paper focuses on the investigation of regularity conditions by virtue of the image space analysis, based on nonconvex separation. We consider a scalar constrained optimization problem and employ the Gerstewitz separation function, which is well known from scalarization of vector optimization problems to present and investigate a collection of nonlinear weak separation functions in connection with methods of image space analysis. With the separation theorems associated with the Gerstewitz function, some image regularity conditions which guarantee the existence of a weak separation function, conducting a regular separation, are studied. In addition, a Lagrange type function and a penalty function are constructed, while the existence of saddle points and exact penalty functions are established by means of the image regularity conditions.

Keywords. Exact penalty function; Gerstewitz function; Image space analysis; Nonlinear separation function; Regularity condition; Saddle point.

1. Introduction

Optimality conditions constitute a fundamental topic in both the theoretical and algorithmic aspects of optimization. Good optimality conditions, which contain adequate problem information and are computationally tractable, are typically required to fulfill certain types of structures. In general sense, the assumptions under which optimality conditions are guaranteed to satisfy such requirements are called regularity conditions or constraint qualifications, the difference in the terminology depending on whether the assumption involves the objective function or not. The best known example, originated in [1–3], are the assumptions which ensure a nonzero multiplier for the objective function under Fritz-John optimality conditions so that one can achieve the Karush-Kuhn-Tucker ones. Numerous regularity conditions and constraint qualifications were proposed and studied in the literature. For different circumstances, a suitable choice of these conditions enables a more straightforward proof and establishment of the corresponding optimality. That is why the various kinds of regularity conditions and constraint qualifications are of great significance and practical utility.

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The unifying scheme of image space analysis, initiated by Castellani and Giannessi in [4] and Giannessi in [5, 6], has been proven to be a very powerful method to handle constrained optimization problems; see, e.g., [7–12]. The emergence and development of this theory provide an approach to unify various regularity conditions and constraint qualifications, studying them and building new ones in a more systemic way. In the image space analysis, the optimality is transformed to the impossibility of a parametric system, and subsequently expressed as the disjointness between two suitable sets $\mathscr{K}$ and $\mathscr{H}$ in the image space associated with the constrained optimization, where $\mathscr{K}$ is determined by the images of the functions relevant to the problem, while $\mathscr{H}$ depends only on the basic structure of the optimization model. With a similar idea, the regularity condition can also be presented as a regular disjointness of $\mathscr{K}$ and $\mathscr{H}$. Since the set $\mathscr{K}$ contains the information of the objective function, the conditions obtained through image space analysis are usually called image regularity conditions.

The techniques of investigating regularity conditions through image space analysis go back to some early work, such as [13–15]. These ideas were further employed in [16–19], where the image regularity condition is characterized as

$$\mathrm{cl\ cone\ conv}\,(\mathscr{K} - \mathrm{cl}\,\mathscr{H}) \cap \mathscr{H}_u = \emptyset, \tag{1.1}$$

where $\mathscr{H}_u$ is the projection of $\mathscr{H}$ on the image space of the objective function. [20, 21] investigated another image regularity condition formulated by the notion of quasi-interiors, which were proved to be equivalent to (1.1) in [22] under some appropriate assumptions. In general, verifying a geometric disjointness directly is impracticable. Therefore, separation functions are needed to conduct this geometric feature in a more analytical way. Although the optimality of the constrained problem can be characterized by the empty intersection between $\mathscr{K}$ and $\mathscr{H}$, a stronger disjointness, which can be understood as regular disjointness, is required to guarantee the existence of a separation function. The reason for considering the closure of a convex conical hull in (1.1) is that linear separation is used.

The existence of a separation function not only provides a handy form for the image regularity, but also is an indispensable connection between regularity and many other related topics, like saddle point, duality theory, and penalization; see [16–19, 23]. However, condition (1.1) may not be fulfilled in the general case of constrained nonconvex optimization, for which the idea of nonlinear separation can be an effective way to overcome this problem. Various nonlinear separation functions in the sense of image space analysis are proposed by several authors, such as [24–26], where the construction is based on the Hiriart-Urruty oriented distance function ([27]), while [28–30] are based on the Gerstewitz function. The investigation of image regularity condition via a nonlinear separation function was also been considered in some literature; see, e.g., [18, 31, 32]. However, some special structures need to be imposed to the constrained problem to ensure the existence of a nonlinear separation function (like [31]), or the basic tool employed is still classical convex separation theorem, and therefore some properties of the separation function $\underline{w} : \mathbb{R}^m \times (\mathbb{R}^p_+ \times \mathbb{R}^{m-p}) \to \mathbb{R}$, like

$$\exists\, \alpha > 0 \text{ s.t. } \underline{w}(v,\lambda) \leq \alpha \langle \lambda, v \rangle \text{ for all } v \in \mathbb{R}^m \text{ and } \lambda \in \mathbb{R}^p_+ \times \mathbb{R}^{m-p}$$

in [18,32], are required to transform the existence of a linear separation function to the existence of a nonlinear one.

The motivation of this paper is to eliminate these special requirements and establish some image regularity conditions for general scalar constrained optimization problems, based directly on results from nonlinear separation. The essential tools applied here are the Gerstewitz function and a separation theorem associated with it, which date back to the work in [33, 34]. Gerstewitz function was introduced as a nonlinear separation tool for not necessarily convex sets, providing the vital foundation for the achievement of numerous theories for nonconvex problems. This function has abundant properties and can be formulated from different perspectives, which were studied in detail in [34–38], and then, together with the corresponding separation theorem, extended further to more general situations; see [39–41]. In addition to its relevance to many other important topics, such as operator theory, mathematical finance and abstract convex analysis ([42–45]), Gerstewitz function is also well known for its tremendous performance in scalarization techniques, developing risk measures, and unifying approaches in optimization under uncertainty. It provides a uniform idea for various kinds of scalarization techniques that are frequently used in vector optimization and relevant problems. The relationships among this function and its dual minimax reformulation, as well as the oriented distance function, which are all widely used types of scalarization, are investigated in [46–48].

The paper is organized as follows. In Section 2, some preliminaries are recalled, particularly the definition and properties of the Gerstewitz function, as well as the concept of separation functions in image space analysis. In Section 3, we propose a collection of nonlinear weak separation functions via the Gerstewitz function. In Section 4, based on the nonconvex separation theorems given by the Gerstewitz function, some image regularity conditions are considered to ensure the existence of a regular separation. Moreover, saddle points and exact penalty functions for a constrained optimization problem formulated using a Lagrange type function involving nonlinear separation functions are investigated. In Section 5, the last section, some conclusions are given.

2. Preliminaries

Let X and Z be real Banach spaces equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, while K_X, and K_Z are closed, convex, and pointed cones with nonempty interior, which induce the partial order in X and Z, respectively, in the following way:

$$x_1 \succeq_{K_X} x_2 :\Longleftrightarrow x_1 \in x_2 + K_X, \qquad z_1 \succeq_{K_Z} z_2 :\Longleftrightarrow z_1 \in z_2 + K_Z.$$

For a subset $A \subset X$, the topological interior, boundary, and closure of A are denoted by $\operatorname{int} A$, $\operatorname{bd} A$ and $\operatorname{cl} A$ respectively, while the convex hull and the cone hull of A are denoted by $\operatorname{conv} A$ and $\operatorname{cone} A$, respectively. Denote the closed ball with the center x and the radius ε by $B(x,\varepsilon)$. We denote the set of all nonnegative real numbers by $\mathbb{R}_+$ and the set of all positive real numbers by $\mathbb{R}_{++}$. For a function $f : X \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, the level set of f w.r.t. the level $t \in \mathbb{R}$ is defined by

$$\operatorname{lev}_{\Re t} f := \{x \in X :\ f(x) \,\Re\, t\},$$

where $\Re \in \{\leq, <, =, \geq, >\}$.

The essential tool of this paper is the Gerstewitz function and the separation theorem associated with it; see, e.g., [33, 35]). There are various types of generalization of such theories. The one employed here is the following version, in which the topological structure is involved.

Definition 2.1. ([40]) Let A be a proper subset of X and $e \in \operatorname{int} K_X$ such that, for all $\alpha > 0$, $\operatorname{cl} A - \alpha e \subseteq \operatorname{int} A$, the Gerstewitz function $\xi_{A,e} : X \to \bar{\mathbb{R}}$ be defined as,

$$\forall x \in X : \quad \xi_{A,e}(x) := \inf\{\lambda \in \mathbb{R} : x \in \lambda e + A\}. \tag{2.1}$$

In this case, the following properties are fulfilled by the Gerstewitz function (see [40, Theorem 4.6.6, Proposition 4.6.13]).

Proposition 2.1. *Let A be a proper subset of X and $e \in \operatorname{int} K_X$ with*

$$\forall \alpha > 0 : \ \operatorname{cl} A - \alpha e \subseteq \operatorname{int} A, \tag{2.2}$$

and $\xi_{A,e} : X \to \bar{\mathbb{R}}$ be defined as above. Then,

(i) $\operatorname{cl} A = \operatorname{cl} \operatorname{int} A$.
(ii) $\operatorname{int} A = \bigcup\limits_{\alpha > 0} (A - \alpha e)$.
(iii) $\xi_{A,e}$ *is continuous.*
(iv) $\forall t \in \mathbb{R} : \operatorname{lev}_{\leq t} \xi_{A,e} = \operatorname{cl} A + te$, $\operatorname{lev}_{< t} \xi_{A,e} = \operatorname{int} A + te$, $\operatorname{lev}_{= t} \xi_{A,e} = \operatorname{bd} A + te$.
(v) $\xi_{A,e} = \xi_{\operatorname{cl} A, e}$.
(vi) *If, in addition,* $e \in -\operatorname{int} 0^+ A$, *then* $\xi_{A,e}$ *is finite-valued, where* $0^+ A$ *is the recession cone of the set* A, *defined by* $0^+ A := \{k \in X : A + tk \subseteq A \ \text{ for all } t \geq 0\}$.
(vii) *For any set* $B \subset X$ *with* $\operatorname{cl} A - B \subseteq \operatorname{cl} A$, $\xi_{A,e}$ *is B-monotone, i.e., for any* $x_1, x_2 \in X$, $x_2 - x_1 \in B \Longrightarrow \xi_{A,e}(x_1) \leq \xi_{A,e}(x_2)$.
(viii) *For any* $x \in X$ *and* $\lambda \in \mathbb{R}$, $\xi_{A,e}(x + \lambda e) = \xi_{A,e}(x) + \lambda$.
(ix) $\xi_{A,e}$ *is positively homogeneous if and only if* $\operatorname{cl} A$ *is a cone.*

The corresponding separation theorem is given below (see [40, Theorem 4.11.4]).

Theorem 2.1. *Letting A be a proper subset of X and $e \in \operatorname{int} K_X$ with $\forall \lambda > 0 : \ \operatorname{cl} A - \lambda e \subseteq \operatorname{int} A$, and $\xi_{A,e} : X \to \bar{\mathbb{R}}$ be defined as above. Then, for each set $D \subseteq X$,*

(i) $\operatorname{int} A \cap D = \emptyset \Longleftrightarrow \forall d \in D : \xi_{A,e}(d) \geq 0$.
(ii) $\operatorname{cl} A \cap \operatorname{int} D = \emptyset \Longleftrightarrow \forall d \in \operatorname{int} D : \xi_{A,e}(d) > 0$.
(iii) $\operatorname{int} A \cap D = \emptyset \Longrightarrow \ \operatorname{cl} A \cap \operatorname{int} D = \emptyset$.
(vi) *If* $D \subseteq \operatorname{cl} \operatorname{int} D$, *then:* $\operatorname{int} A \cap D = \emptyset \Longleftrightarrow \operatorname{cl} A \cap \operatorname{int} D = \emptyset$.

We consider a general constrained optimization problem

$$\min\{f(x) : \ x \in \mathscr{R}\}, \tag{P}$$

where $f : X \to \mathbb{R}$, $g : X \to Z$, and $\mathscr{R} := \{x \in X : \ g(x) \succeq_{K_Z} 0_Z\}$.

In the following, we introduce the main notions of image space analysis; see [5, 6, 49] and the references therein. For an arbitrary $\bar{x} \in X$, define the sets

$$\begin{aligned} \mathscr{K} &:= \{(u,v) \in \mathbb{R} \times Z : u = f(\bar{x}) - f(x), \ v = g(x), \ x \in X\}, \\ \mathscr{H} &:= \{(u,v) \in \mathbb{R} \times Z : u > 0, \ v \in K_Z\}. \end{aligned} \tag{2.3}$$

$\mathbb{R} \times Z$ is the image space associated with (P) and $\mathscr{K}$ is called the image of (P), while $\mathscr{H} \cup \{0_{\mathbb{R} \times Z}\}$ is a convex cone that depends only on the ordering structures involved in (P). For any $\bar{x} \in \mathscr{R}$, the optimality of $\bar{x}$ concerning (P) can be characterized by the disjointness between $\mathscr{K}$ and $\mathscr{H}$. We also recall the conical extension of the image of $\mathscr{K}$, defined by

$$\mathscr{E} := \mathscr{K} - \operatorname{cl} \mathscr{H}. \tag{2.4}$$

Since $\mathrm{cl}\,\mathscr{H}+\mathscr{H}\subseteq\mathscr{H}$, $\mathscr{K}\cap\mathscr{H}=\emptyset$ is equivalent to $\mathscr{E}\cap\mathscr{H}=\emptyset$, or to $\mathscr{E}\cap\mathscr{H}_u=\emptyset$, where

$$\mathscr{H}_u := \{(u,v)\in\mathbb{R}\times Z : u>0,\ v=0_{K_Z}\}.$$

The advantage of such an extension is that $\mathscr{E}$ not only preserves the optimality structure, but also enjoys better properties than the set $\mathscr{K}$. For instance, $\mathscr{E}$ fulfills a crucial property, for constructing the Gerstewitz type function, that, for any $e\in\mathrm{int}\,\mathscr{H}$ (see Section 4 for the proof) $\mathrm{cl}\,\mathscr{E}-e\subset\mathrm{int\,cl}\ \mathscr{E}$, while it does not need to be fulfilled for $\mathscr{K}$ in place of $\mathscr{E}$.

To conduct the disjointness between two sets in the image space, separation functions are needed as a crucial tool.

Definition 2.2. The class of functions $w_\pi:\mathbb{R}\times Z\to\mathbb{R}$, denoted by W_Π, is called a class of weak separation functions if, for all $w_\pi\in W_\Pi$, $\mathscr{H}\subseteq\mathrm{lev}_{\geq 0}\,w_\pi$ and $\bigcap\limits_{w_\pi\in W_\Pi}\mathrm{lev}_{>0}\,w_\pi\subseteq\mathscr{H}$, while, it is called a class of regular weak separation functions if

$$\bigcap_{w_\pi\in W_\Pi}\mathrm{lev}_{>0}\,w_\pi=\mathscr{H}, \tag{2.5}$$

where $\pi\in\Pi$ is the parameter.

Definition 2.3. The sets $\mathscr{K}$ and $\mathscr{H}$ are said to admit a separation by some $w_\pi\in W_\Pi$ if

$$\forall(u,v)\in\mathscr{K}:\ w_\pi(u,v)\leq 0 \quad\text{and}\quad \forall(u,v)\in\mathscr{H}:\ w_\pi(u,v)\geq 0. \tag{2.6}$$

The separation is said to be regular if

$$\forall(u,v)\in\mathscr{K}:\ w_\pi(u,v)\leq 0 \quad\text{and}\quad \forall(u,v)\in\mathscr{H}:\ w_\pi(u,v)> 0. \tag{2.7}$$

It can be deduced from (2.5) that if W_Π is a collection of regular weak separation functions, then the separation conducted by any $w_\pi\in W_\Pi$ is a regular separation.

3. Nonlinear Weak Separation Function

Gerstewitz function offers a powerful tool to construct nonlinear separation functions, see for instance [28–30], for which the element $e\in\mathrm{int}\,K_X$ is often regarded as a key parameter. The abundant features of these nonlinear separation functions come not only from the useful properties enjoyed by the Gerstewitz function, but also from its good behavior with respect to the variation of the parameter e (like stability, see [30, 50]).

In this section, we propose another way to construct nonlinear separation functions based on the Gerstewitz function, in which the shifted set A in (2.1) is also considered as parameter in the formulation of the functional $\xi_{A,e}$. Define the set

$$\begin{aligned}\mathscr{A}:=\{A\subset\mathbb{R}\times Z:\quad &(i)\quad A \text{ is proper and closed},\\ &(ii)\quad A-\mathrm{int}\,\mathscr{H}\subseteq\mathrm{int}\,A,\\ &(iii)\quad \mathrm{int}\,A\cap\mathscr{H}=\emptyset\}.\end{aligned}$$

Remark 3.1. (i) Since $\mathscr{H}=\mathbb{R}_{++}\times K_Z$, $\mathrm{int}\,K_Z\neq\emptyset$ implies $\mathrm{int}\,\mathscr{H}=\mathbb{R}_{++}\times\mathrm{int}\,K_Z\neq\emptyset$.

(ii) Since $\mathscr{H}$ is convex with $\mathrm{int}\,\mathscr{H}\neq\emptyset$, we have $\mathrm{cl\ int}\,\mathscr{H}=\mathrm{cl}\,\mathscr{H}$. Together with the fact that A is closed, $A-\mathrm{cl}\,\mathscr{H}\subseteq A$ follows from $A-\mathrm{int}\,\mathscr{H}\subseteq\mathrm{int}\,A$.

(iii) $\mathrm{int}\,A\cap\mathscr{H}=\emptyset$ indicates that $(\mathbb{R}\times Z)\setminus\mathrm{int}\,A$ is a closed set containing $\mathscr{H}$. Therefore, $\mathrm{int}\,A\cap\mathrm{cl}\,\mathscr{H}=\emptyset$.

With the help of $\mathscr{A}$, we are able to give the following collection of nonlinear functionals:

$$\mathscr{W}_\Pi = \{\xi_\pi : \mathbb{R}\times Z \to \mathbb{R} : \pi \in \Pi\}, \tag{3.1}$$

where

$$\Pi = \{\pi = (A, \alpha, e_Z) : A \in \mathscr{A}, \alpha \in \mathbb{R}_{++}, e_Z \in \operatorname{int} K_Z\}$$

and

$$\forall (u,v) \in \mathbb{R}\times Z : \ \xi_\pi(u,v) := \xi_{A,\alpha,e_Z}(u,v) = \inf\{\lambda \in \mathbb{R} : (u,v) \in \lambda(\alpha, e_Z) + A\}. \tag{3.2}$$

To prove that $\mathscr{W}_\Pi$ is a class of weak separation functions, we give the following lemma.

Lemma 3.1. *Consider $A_0 := (\mathbb{R}\times Z)\setminus \operatorname{int}\mathscr{H}$, then $A_0 \in \mathscr{A}$ and A_0 is the upper bound of $\mathscr{A}$, that is $A \subseteq A_0$ for all $A \in \mathscr{A}$.*

Proof. First, $\operatorname{int}\mathscr{H} \neq \emptyset$ provides that A_0 is proper and closed. We claim that $A_0 - \operatorname{int}\mathscr{H} \subset A_0$, i.e.,

$$(\mathbb{R}\times Z)\setminus \operatorname{int}\ \mathscr{H} - \operatorname{int}\ \mathscr{H} \subset \ (\mathbb{R}\times Z)\setminus \operatorname{int}\ \mathscr{H}.$$

Otherwise, there exist $(u,v) \in (\mathbb{R}\times Z)\setminus \operatorname{int}\mathscr{H}$ and $(q_1,q_2) \in \operatorname{int}\mathscr{H}$ such that $(u,v)-(q_1,q_2) \in \operatorname{int}\mathscr{H}$. Then, $(u,v) \in \operatorname{int}\mathscr{H} + \operatorname{int}\mathscr{H} = \operatorname{int}\mathscr{H}$, contradicting $(u,v) \in (\mathbb{R}\times Z)\setminus \operatorname{int}\mathscr{H}$. Thus, $A_0 - \operatorname{int}\mathscr{H}$ is an open set contained in A_0, which shows $A_0 - \operatorname{int}\mathscr{H} \subset \operatorname{int} A_0$.

On the other hand, since A_0 is closed and $\mathscr{H}$ is convex with nonempty interior, we obtain

$$A_0 = \operatorname{cl} A_0 = \operatorname{int} A_0 \sqcup \operatorname{bd}((\mathbb{R}\times Z)\setminus \operatorname{int}\mathscr{H}) = \operatorname{int} A_0 \sqcup \operatorname{bd}(\operatorname{int}\mathscr{H}),$$

where $\sqcup$ denotes the disjoint union. Therefore,

$$\operatorname{int} A_0 \cap \mathscr{H} \subset \operatorname{int} A_0 \cap \operatorname{cl}\operatorname{int}\ \mathscr{H} = \operatorname{int} A_0 \cap (\operatorname{int}\mathscr{H} \sqcup \operatorname{bd}\operatorname{int}\ \mathscr{H}) = \emptyset.$$

Thus $A_0 \in \mathscr{A}$. For any $A \in \mathscr{A}$, it holds that $A - \operatorname{int}\mathscr{H} \subset \operatorname{int} A$ and $\operatorname{int} A \cap \mathscr{H} = \emptyset$ taking into account the definition of $\mathscr{A}$. This ensures

$$\operatorname{int} A \subset (\mathbb{R}\times Z)\setminus \mathscr{H} \subset (\mathbb{R}\times Z)\setminus \operatorname{int}\mathscr{H} = A_0.$$

Applying Proposition 2.1 (i) for set A and taking into account the fact that A_0 is closed, we obtain $A = \operatorname{cl}\operatorname{int}\ A \subseteq A_0$. $\square$

Using Lemma 3.1, we are able to present the weak separation characteristic of $\mathscr{W}_\Pi$.

Proposition 3.1. *The collection $\mathscr{W}_\Pi$ defined in (3.1) is a class of weak separation functions, i.e.,*

$$\forall \pi \in \Pi : \ \mathscr{H} \subseteq \operatorname{lev}_{\geq 0} \xi_\pi, \tag{3.3}$$

and

$$\bigcap_{\pi\in\Pi} \operatorname{lev}_{>0} \xi_\pi \subseteq \mathscr{H}. \tag{3.4}$$

Proof. For every $\pi = (A, \alpha, e_Z) \in \Pi$, we have, for all $t > 0$, $A - t(\alpha, e_Z) \subset A - \operatorname{int}\mathscr{H} \subseteq \operatorname{int} A$ such that the assumption (2.2) in Proposition 2.1 is fulfilled. Then, taking into account the definition of ξ_{A,α,e_Z} in (3.2) and Proposition 2.1 (iv), we see that $\xi_{A,\alpha,e_Z}(u,v) < 0$ if and only if $(u,v) \in \operatorname{int} A$. Therefore, it follows from $\operatorname{int} A \cap \mathscr{H} = \emptyset$ that, for all $(u,v) \in \mathscr{H}$, $\xi_{A,\alpha,e_Z}(u,v) \geqslant 0$. This means that (3.3) holds. Picking arbitrary $\alpha > 0$ and $e_Z \in \operatorname{int} K_Z$, together with $A_0 = (\mathbb{R}\times Z)\setminus \operatorname{int}\mathscr{H}$, we have $\pi_0 = (A_0, \alpha, e_Z) \in \Pi$ due to Lemma 3.1. Hence, $(u,v) \in \bigcap_{\pi\in\Pi} \operatorname{lev}_{>0} \xi_\pi$

implies $\xi_{\pi_0}(u,v) = \xi_{A_0,\alpha,e_Z}(u,v) > 0$. Applying Proposition 2.1 (iv), we conclude $(u,v) \notin A_0$, meaning that $(u,v) \in \operatorname{int}\mathscr{H} \subseteq \mathscr{H}$ such that we get (3.4). □

The infimum of the collection of functions $\mathscr{W}_\Pi$, with respect to the parameter $\pi \in \Pi$, is calculated below.

Proposition 3.2. *For the collection $\mathscr{W}_\Pi$, one has*

$$\inf_{\pi\in\Pi} \xi_\pi(u,v) = \begin{cases} 0 & \text{if } (u,v) \in \operatorname{cl}\mathscr{H}, \\ -\infty & \text{otherwise.} \end{cases}$$

Proof. Suppose $(\bar{u},\bar{v}) \in \operatorname{cl}\mathscr{H}$. As a result of Proposition 3.1, we obtain, for all $\pi \in \Pi$ and $(u,v) \in \mathscr{H}$, $\xi_\pi(u,v) \geq 0$. Then, taking into account Proposition 2.1 (iii) and $(\bar{u},\bar{v}) \in \operatorname{cl}\mathscr{H}$, see that, for all $\pi \in \Pi$, $\xi_\pi(\bar{u},\bar{v}) \geq 0$. For any $\pi = (A,\alpha,e_Z) \in \Pi$, setting $\pi_k = (A,k\alpha,ke_Z)$, then $\pi_k \in \Pi$ for every $k \in \mathbb{N}$ and

$$\xi_{\pi_k}(\bar{u},\bar{v}) = \xi_{A,k\alpha,ke_Z}(\bar{u},\bar{v}) = \frac{1}{k}\xi_{A,\alpha,e_Z}(\bar{u},\bar{v}) \longrightarrow 0$$

as $k \to +\infty$. Thus $\inf_{\pi\in\Pi} \xi_\pi(\bar{u},\bar{v}) = 0$ in this case.

For the case that $(\bar{u},\bar{v}) \notin \operatorname{cl}\mathscr{H}$, considering $A_0 = (\mathbb{R}\times Z) \setminus \operatorname{int}\mathscr{H} \in \mathscr{A}$, we have $(\bar{u},\bar{v}) \in \operatorname{int} A_0$. Fix some $\alpha > 0$ and $e_Z \in \operatorname{int} K_Z$. According to Proposition 2.1 (iv), one has $\xi_{A_0,\alpha,e_Z}(\bar{u},\bar{v}) < 0$, which leads to

$$\xi_{A_0,\frac{1}{k}\alpha,\frac{1}{k}e_Z}(\bar{u},\bar{v}) = k\xi_{A_0,\alpha,e_Z}(\bar{u},\bar{v}) \longrightarrow -\infty,$$

as $k \to +\infty$. Thus $\inf_{\pi\in\Pi} \xi_\pi(\bar{u},\bar{v}) = -\infty$ when $(\bar{u},\bar{v}) \notin \operatorname{cl}\mathscr{H}$. □

The above proposition gives a way to detect elements of $\operatorname{cl}\mathscr{H}$ using the functional (3.2). This kind of property, which is critical in the investigation of other problems related to constrained optimization, has also been studied in works like [51, 52] and [30, 53]. In our paper, it will be used to establish the existence of saddle points in Section 4.

4. Image Regularity Conditions

In the proof of Proposition 3.1, we have demonstrated that

$$\bigcap_{\pi\in\Pi} \operatorname{lev}_{>0} \xi_\pi \subseteq \operatorname{int}\mathscr{H} \subseteq \mathscr{H}$$

for functionals ξ_π (see (3.2)) belonging to the collection of nonlinear functions $\mathscr{W}_\Pi$. This shows that $\mathscr{W}_\Pi$ is only a class of weak separation functions, not a regular one. However, it is still possible that there exist an element in $\mathscr{W}_\Pi$ that performs a regular separation between $\mathscr{K}$ and $\mathscr{H}$. In [54], Giannessi proposed a characterization of a regular separation based on the zero level set. (In (4.1), we give a formulation of regular separation w.r.t. the general weak separation functions belonging to a class W_Π. $\mathscr{W}_\Pi$ is the notation of the specific class of separation function when using ξ_π; see (3.1)). For any class of weak separation functions W_Π such that

$$\forall (u,v) \in \mathscr{H},\ \forall \pi \in \Pi : \qquad w_\pi(u,v) \geq w_\pi(u,0_Z), \tag{4.1}$$

the separation conducted by some $w_{\bar{\pi}} \in W_\Pi$, i.e., (see (2.6))

$$\forall (u,v) \in \mathscr{K} : \quad w_{\bar{\pi}}(u,v) \leq 0 \quad \text{and} \quad \forall (u,v) \in \mathscr{H} : \quad w_{\bar{\pi}}(u,v) \geq 0 \tag{4.2}$$

is regular if and only if

$$\operatorname{lev}_{=0} w_{\bar{\pi}} \cap \mathscr{H}_u = \emptyset. \tag{4.3}$$

Of interest is the question under which assumptions on the sets $\mathscr{K}$ and $\mathscr{H}$, which can be regarded as image regularity, the existence of parameters $\pi \in \Pi$ satisfying conditions (4.2) and (4.3) can be guaranteed.

This problem has been studied in the literature like in [16, 18], where linear functionals are used as the main separation tool. Even though some special nonlinear functions are considered, the basic tools applied in these references are classical convex separation results, and therefore the image regularity condition proposed there is $\operatorname{cl}\operatorname{cone}\operatorname{conv}\mathscr{E}\cap\mathscr{H}_u=\emptyset$, in which the convex structure is indispensable.

In this section, we mainly consider two types of image regularity conditions that do not require a convex structure, namely

$$\operatorname{cl}\mathscr{E}\cap\mathscr{H}_u=\emptyset \tag{IRC1}$$

and

$$\operatorname{cl}\operatorname{cone}\ \mathscr{E}\cap\mathscr{H}_u=\emptyset. \tag{IRC2}$$

(IRC2) is studied in [31] for a scalar constrained optimization problem in the finite dimensional case, while some local versions of (IRC2) with respect to multiobjective optimization problems are considered in [19, 22].

Note that although $\operatorname{cone}\mathscr{E}\cap H_u=\emptyset$ is not actually different from $\mathscr{E}\cap H_u=\emptyset$, since $\mathscr{H}$ is a cone, $\operatorname{cl}\mathscr{E}\cap H_u=\emptyset$ is strictly weaker than $\operatorname{cl}\operatorname{cone}\mathscr{E}\cap H_u=\emptyset$. We give the following example to illustrate this fact.

Example 4.1. In (P), let $X=Z=\mathbb{R}$, $K_X=K_Z=\mathbb{R}_+$, and let $f(x)=x$,

$$g(x)=\begin{cases}-1 & \text{if } x\le 0,\\ x-1 & \text{if } x>0\end{cases}$$

for all $x\in\mathbb{R}$. Then, $\mathscr{H}_u=\{(u,v)\in\mathbb{R}^2: u>0, v=0\}$. It is not hard to observe that $\bar{x}=1\in\mathscr{R}$ is a minimum point of (P), while the corresponding sets associated with $\bar{x}$ are

$$\mathscr{K}=\{(u,v)\in\mathbb{R}^2: u\ge 1, v=-1\}\cup\{(u,v)\in\mathbb{R}^2: u<1, v=-u\},$$

and

$$\mathscr{E}=\{(u,v)\in\mathbb{R}^2: u\ge 1, v\le -1\}\cup\{(u,v)\in\mathbb{R}^2: u<1, v\le -u\}.$$

Then,

$$\operatorname{cone}\ \mathscr{E}=\{(u,v)\in\mathbb{R}^2: u>0, v<0\}\cup\{(u,v)\in\mathbb{R}^2: u\le 0, v\le -u\},$$

and

$$\operatorname{cl}\operatorname{cone}\ \mathscr{E}=\{(u,v)\in\mathbb{R}^2: u\ge 0, v\le 0\}\cup\{(u,v)\in\mathbb{R}^2: u\le 0, v\le -u\}.$$

Therefore, $\mathscr{H}_u=\{(u,v)\in\mathbb{R}^2: u>0, v=0\}\subset\operatorname{cl}\operatorname{cone}\ \mathscr{E}$, but $\operatorname{cl}\mathscr{E}\cap\mathscr{H}_u=\emptyset$.

When the class of weak separation functions is specified as the one given in (3.1), due to Proposition 2.1 (vii) and Remark 3.1, (4.1) is satisfied. The characterization of regular separation given by (4.2) and (4.3) via the nonlinear functional (3.2) with $\pi=(A,\alpha,e_Z)$ becomes

$$\forall (u,v)\in\mathscr{K}:\qquad \xi_\pi(u,v)=\xi_{A,\alpha,e_Z}(u,v)\le 0 \tag{4.4}$$

and

$$\operatorname{lev}_{=0}\xi_\pi\cap\mathscr{H}_u=\operatorname{lev}_{=0}\xi_{A,\alpha,e_Z}\cap\mathscr{H}_u=\emptyset, \tag{4.5}$$

which, by Proposition 2.1 (iv), is equivalent to $\mathscr{K} \subseteq A$ and bd $A \cap \mathscr{H}_u = \emptyset$. Therefore, we can obtain the following result immediately.

Proposition 4.1. *If* (IRC1) *holds, then there exists some* $\bar{\pi} = (\bar{A}, \bar{\alpha}, \bar{e}_Z) \in \Pi$ *such that* (4.4) *and* (4.5) *are fulfilled by* $\xi_{\bar{A},\bar{\alpha},\bar{e}_Z}$.

Proof. If $\mathrm{cl}\,\mathscr{E} \cap \mathscr{H}_u = \emptyset$, then $\mathrm{cl}\,\mathscr{E}$ is a proper closed subset of $\mathbb{R} \times Z$ with $\mathrm{int\,cl}\,\mathscr{E} \cap \mathscr{H} = \emptyset$. Moreover,

$$\mathrm{cl}\,\mathscr{E} - \mathrm{int}\,\mathscr{H} \subseteq \mathrm{cl}(\mathscr{E} - \mathrm{int}\,\mathscr{H}) \subseteq \mathrm{cl}\,\mathscr{E},$$

which implies $\mathrm{cl}\,\mathscr{E} - \mathrm{int}\,\mathscr{H} \subseteq \mathrm{int\,cl}\,\mathscr{E}$ since $\mathrm{cl}\,\mathscr{E} - \mathrm{int}\,\mathscr{H}$ is an open set. Therefore, $\mathrm{cl}\,\mathscr{E} \in \mathscr{A}$. Picking some $\bar{\alpha} > 0$, $\bar{e}_Z \in \mathrm{int}\,K_Z$, and letting $\bar{A} = \mathrm{cl}\,\mathscr{E}$, one has $\xi_{\bar{A},\bar{\alpha},\bar{e}_Z} \in \mathscr{W}_\Pi$, while (4.4) and (4.5) associated with $\xi_{\bar{A},\bar{\alpha},\bar{e}_Z}$ follow from Proposition 2.1. □

Since (IRC2) implies (IRC1), $\xi_{\bar{A},\bar{\alpha},\bar{e}_Z}$ that satisfies (4.4) and (4.5) can also be guaranteed by (IRC2).

For the case where only $\mathscr{E} \cap \mathscr{H}_u = \emptyset$ holds, we first give the following lemma.

Lemma 4.1. *If* $\mathscr{E} \cap \mathscr{H}_u = \emptyset$, *then* $\mathrm{int\,cl}\,\mathscr{E} \cap \mathscr{H} = \emptyset$.

Proof. If there exists some $(u,v) \in \mathrm{int\,cl}\,\mathscr{E} \cap \mathscr{H}$, then $(u,v) \in \mathscr{H}$ and there exists some $\delta > 0$ such that $(u,v) + B(0_{\mathbb{R}\times Z}, \delta) \subseteq \mathrm{cl}\,\mathscr{E}$. Since $\mathscr{E} - \mathrm{cl}\,\mathscr{H} \subseteq \mathscr{E}$, then $((u,v) + \mathrm{cl}\,\mathscr{H}) \cap \mathscr{E} = \emptyset$. Consider the set $((u,v) + \mathrm{cl}\,\mathscr{H}) \cap ((u,v) + B(0_{\mathbb{R}\times Z}, \delta))$, and pick some $(q_1,q_2) \in \mathrm{int}\,\mathscr{H}$ such that $\|(q_1,q_2)\|_{\mathbb{R}\times Z} < \delta$. Then there exists $(u,v)+(q_1,q_2) \in ((u,v)+\mathrm{cl}\,\mathscr{H}) \cap ((u,v)+B(0_{\mathbb{R}\times Z},\delta))$. It follows from $(u,v)+B(0_{\mathbb{R}\times Z},\delta) \subseteq \mathrm{cl}\,\mathscr{E}$ and $((u,v)+\mathrm{cl}\,\mathscr{H}) \cap \mathscr{E} = \emptyset$ that

$$((u,v)+\mathrm{cl}\,\mathscr{H}) \cap ((u,v)+B(0_{\mathbb{R}\times Z},\delta)) \subseteq \mathrm{cl}\,\mathscr{E} \setminus \mathscr{E}.$$

Choosing some $\eta > 0$ such that $(q_1,q_2)+B(0_{\mathbb{R}\times Z},\eta) \subseteq \mathrm{int}\,\mathscr{H}$ and $\|(q_1,q_2)\|_{\mathbb{R}\times Z} + \eta < \delta$, one sees that

$$(u,v)+(q_1,q_2)+B(0_{\mathbb{R}\times Z},\eta) \subseteq ((u,v)+\mathrm{cl}\,\mathscr{H}) \cap ((u,v)+B(0_{\mathbb{R}\times Z},\delta)) \subseteq \mathrm{cl}\,\mathscr{E} \setminus \mathscr{E}.$$

However, $(u,v)+(q_1,q_2)+B(0_{\mathbb{R}\times Z},\eta) \subseteq \mathrm{cl}\,\mathscr{E} \setminus \mathscr{E}$ is not possible. □

With Lemma 4.1 and Theorem 2.1, when $\mathscr{E} \cap \mathscr{H}_u = \emptyset$, we can still use $\mathrm{cl}\,\mathscr{E}$ as an element in $\mathscr{A}$ to define a $\xi_{\bar{A},\bar{\alpha},\bar{e}_Z} \in \mathscr{W}_\Pi$ separating $\mathscr{K}$ and $\mathscr{H}$. However, the regularity of the separation cannot be guaranteed. This is presented in the following proposition.

Proposition 4.2. *If* $\mathscr{E} \cap \mathscr{H}_u = \emptyset$ *holds, then there exists some* $\bar{\pi} = (\bar{A}, \bar{\alpha}, \bar{e}_Z) \in \Pi$ *such that, for all* $(u,v) \in \mathscr{K}$, $\xi_{\bar{A},\bar{\alpha},\bar{e}_Z}(u,v) \leq 0$.

Given a class of weak separation functions W_Π and some $(\hat{x},\hat{\pi}) \in X \times \Pi$, the Lagrange-type function $L_{W_\Pi} : X \times \Pi \to \mathbb{R}$ for (P) with respect to W_Π and $(\hat{x},\hat{\pi}) \in X \times \Pi$ is defined by

$$\forall (x,\pi) \in X \times \Pi: \qquad L_{W_\Pi}(x,\pi;\hat{x},\hat{\pi}) := w_{\hat{\pi}}(f(\hat{x}),0_Z) - w_\pi(f(\hat{x})-f(x),g(x)). \tag{4.6}$$

Definition 4.1. For the Lagrange-type function given in (4.6), a pair $(\bar{x},\bar{\pi}) \in X \times \Pi$ is said to be a Fritz John (for short, FJ) saddle point for (P) if there exists some $(\hat{x},\hat{\pi}) \in X \times \Pi$ such that

$$\forall x \in X,\ \forall \pi \in \Pi: \qquad L_{W_\Pi}(\bar{x},\pi;\hat{x},\hat{\pi}) \leq L_{W_\Pi}(\bar{x},\bar{\pi};\hat{x},\hat{\pi}) \leq L_{W_\Pi}(x,\bar{\pi};\hat{x},\hat{\pi}), \tag{4.7}$$

while it is said to be a Karush-Kuhn-Tucker (for short, KKT) saddle point for (P) if, in addition,

$$\forall u > 0: \qquad w_{\bar{\pi}}(u,0_Z) > 0.$$

Some approaches for constructing the Lagrange-type function and defining the corresponding saddle points are also studied in [19, 22]. In [19], the first term of the Lagrange-type function $w_{\hat{\pi}}(f(\hat{x}), 0_Z)$ is actually equal to $f(\hat{x})$ under the assumption therein that $w_\pi(u, 0_Z) = u$ for all $\pi \in \Pi$, while in [22], the Lagrange-type function is based on linear separation functions. Here, we try to remove such an assumption on w_π as in [19] and consider the nonlinear separation functions in a more general situation.

Next, we pay attention to the relationship between the image regularity conditions (IRC1), (IRC2) and the existence of JF and KKT saddle points for (P).

Theorem 4.1. *Let the Lagrange-type function be defined as in (4.6), and $W_\Pi = \{w_\pi : \mathbb{R} \times Z \to \mathbb{R} : \pi \in \Pi\}$ be a collection of weak separation functions such that*

(a) *For every $\pi \in \Pi$, $w_\pi(u, v)$ is continuous w.r.t. v, or $\operatorname{cl} \mathscr{H} \subseteq \operatorname{lev}_{\geq 0} w_\pi$.*

(b)
$$\inf_{\pi \in \Pi} w_\pi(u, v) = \begin{cases} 0 & \text{if } (u, v) \in \operatorname{cl} \mathscr{H}, \\ -\infty & \text{otherwise.} \end{cases}$$

Then, the following assertions hold.

(i) *Given $\bar{x} \in \mathscr{R}$ and the corresponding set $\mathscr{K}$ defined by (2.3), if there exists some $\bar{\pi} \in \Pi$ such that, for all $(u, v) \in \mathscr{K}$, $w_{\bar{\pi}}(u, v) \leq 0$, then $(\bar{x}, \bar{\pi})$ is a FJ saddle point for (P).*

(ii) *If $(\bar{x}, \bar{\pi})$ is a FJ saddle point for (P) for which the element $\hat{x}$ in (4.10) is specified as $\bar{x}$, then $\bar{x} \in \mathscr{R}$ and, for all $(u, v) \in \mathscr{K}$, $w_{\bar{\pi}}(u, v) \leq 0$.*

Proof. (i) Suppose that $\bar{x} \in \mathscr{R}$ and, for all $(u, v) \in \mathscr{K}$, $w_{\bar{\pi}}(u, v) \leq 0$, i.e.,

$$\forall x \in X : \qquad -w_{\bar{\pi}}(f(\bar{x}) - f(x), g(x)) \geq 0 \tag{4.8}$$

for some $\bar{\pi} \in \Pi$. Then

$$\forall x \in X : \qquad w_{\bar{\pi}}(f(\bar{x}), 0_Z) - w_{\bar{\pi}}(f(\bar{x}) - f(x), g(x)) \geq w_{\bar{\pi}}(f(\bar{x}), 0_Z). \tag{4.9}$$

Letting $x = \bar{x}$ in (4.8) gives $w_{\bar{\pi}}(0, g(\bar{x})) \leq 0$. Since W_Π is a collection of weak separation functions, $w_{\bar{\pi}}(u, v) \geq 0$ for all $(u, v) \in \mathscr{H}$. Therefore, it can be deduced from $\bar{x} \in \mathscr{R}$ and condition (a) that $w_{\bar{\pi}}(0, g(\bar{x})) = 0$. Together with (4.9), we obtain

$$\forall x \in X : \quad w_{\bar{\pi}}(f(\bar{x}), 0_Z) - w_{\bar{\pi}}(f(\bar{x}) - f(x), g(x)) \geq w_{\bar{\pi}}(f(\bar{x}), 0_Z) - w_{\bar{\pi}}(f(\bar{x}) - f(\bar{x}), g(\bar{x})),$$

which means, for all $x \in X$, $L_{W_\Pi}(\bar{x}, \bar{\pi}; \bar{x}, \bar{\pi}) \leq L_{W_\Pi}(x, \bar{\pi}; \bar{x}, \bar{\pi})$.

On the other hand, for any $\pi \in \Pi$, with a similar argument as for $\bar{\pi}$, we have $w_\pi(0, g(\bar{x})) \geq 0 = w_{\bar{\pi}}(0, g(\bar{x}))$. It follows that

$$\forall \pi \in \Pi : \quad w_{\bar{\pi}}(f(\bar{x}), 0_Z) - w_\pi(f(\bar{x}) - f(\bar{x}), g(\bar{x})) \leq w_{\bar{\pi}}(f(\bar{x}), 0_Z) - w_{\bar{\pi}}(f(\bar{x}) - f(\bar{x}), g(\bar{x})),$$

namely, for all $\pi \in \Pi$, $L_{W_\Pi}(\bar{x}, \pi; \bar{x}, \bar{\pi}) \leq L_{W_\Pi}(\bar{x}, \bar{\pi}; \bar{x}, \bar{\pi})$. Thus $(\bar{x}, \bar{\pi})$ is a FJ saddle point for (P) and the proof of (i) is completed.

(ii) Assume that $(\bar{x}, \bar{\pi})$ is a FJ saddle point for (P), for which $\hat{x}$ is specified as $\bar{x}$. This means that there exists some $\hat{\pi} \in \Pi$ such that

$$\forall x \in X, \ \forall \pi \in \Pi : \quad L_{W_\Pi}(\bar{x}, \pi; \bar{x}, \hat{\pi}) \leq L_{W_\Pi}(\bar{x}, \bar{\pi}; \bar{x}, \hat{\pi}) \leq L_{W_\Pi}(x, \bar{\pi}; \bar{x}, \hat{\pi}). \tag{4.10}$$

From the first inequality $L_{W_\Pi}(\bar{x}, \pi; \bar{x}, \hat{\pi}) \leq L_{W_\Pi}(\bar{x}, \bar{\pi}; \bar{x}, \hat{\pi})$ for all $\pi \in \Pi$ in (4.10), i.e.,

$$\forall \pi \in \Pi : \quad w_{\hat{\pi}}(f(\bar{x}), 0_Z) - w_\pi(f(\bar{x}) - f(\bar{x}), g(\bar{x})) \leq w_{\hat{\pi}}(f(\bar{x}), 0_Z) - w_{\bar{\pi}}(f(\bar{x}) - f(\bar{x}), g(\bar{x})),$$

we have, for all $\pi \in \Pi$, $w_{\pi}(0, g(\bar{x})) \geq w_{\bar{\pi}}(0, g(\bar{x}))$. Then, it follows from condition (b) that

$$w_{\bar{\pi}}(0, g(\bar{x})) = 0 \tag{4.11}$$

and $\bar{x} \in \mathscr{R}$. The second inequality in (4.10)

$$\forall x \in X: \quad L_{W_\Pi}(\bar{x}, \bar{\pi}; \bar{x}, \hat{\pi}) \leq L_{W_\Pi}(x, \bar{\pi}; \bar{x}, \hat{\pi}),$$

means

$$\forall x \in X: \quad w_{\hat{\pi}}(f(\bar{x}), 0_Z) - w_{\bar{\pi}}(f(\bar{x}) - f(\bar{x}), g(\bar{x})) \leq w_{\hat{\pi}}(f(\bar{x}), 0_Z) - w_{\bar{\pi}}(f(\bar{x}) - f(x), g(x)).$$

This together with (4.11) yields, for all $x \in X$, $w_{\bar{\pi}}(f(\bar{x}) - f(x), g(x)) \leq 0$ and (ii) is proved. □

Theorem 4.1 presents a general formulation of the relationship between separation and FJ saddle points. We can also consider the specific collection of weak separation functions $\mathscr{W}_\Pi$ constructed in (3.1). According to Propositions 2.1 and 3.2, conditions (a) and (b) in Theorem 4.1 are fulfilled by $\mathscr{W}_\Pi$, and therefore the formulation by $\mathscr{W}_\Pi$ is given below.

Theorem 4.2. *Let the Lagrange-type function be defined as in (4.6), and put W_Π as the collection of weak separation functions $\mathscr{W}_\Pi$ defined in (3.1). Then, the following assertions hold.*

(i) *Given $\bar{x} \in \mathscr{R}$ and the corresponding set $\mathscr{E}$ defined by (2.4). If $\mathscr{E} \cap \mathscr{H}_u = \emptyset$ holds, then there exists some $\bar{\pi} = (\bar{A}, \bar{\alpha}, \bar{e}_Z) \in \Pi$ such that $(\bar{x}, \bar{\pi})$ is a FJ saddle point for (P).*

(ii) *If $(\bar{x}, \bar{\pi}) = (\bar{x}, (\bar{A}, \bar{\alpha}, \bar{e}_Z))$ is a FJ saddle point for (P), for which the element $\hat{x}$ in (4.10) is specified as $\bar{x}$, then $\bar{x} \in \mathscr{R}$ and, for all $(u, v) \in \mathscr{K}$, $\xi_{\bar{A}, \bar{\alpha}, \bar{e}_Z}(u, v) \leq 0$.*

Proof. (i) If $\bar{x} \in \mathscr{R}$ and $\mathscr{E} \cap \mathscr{H}_u = \emptyset$, we obtain from Proposition 4.2 the existence of some $\bar{\pi} = (\bar{A}, \bar{\alpha}, \bar{e}_Z) \in \Pi$ such that, for all $(u, v) \in \mathscr{K}$, $\xi_{\bar{A}, \bar{\alpha}, \bar{e}_Z}(u, v) \leq 0$. Then, the assertion follows immediately from Theorem 4.1 (i).

(ii) can be obtained directly from Theorem 4.1 (ii). □

Next, we consider the existence of a KKT saddle point for (P), for which the image regularity conditions are required.

Theorem 4.3. *Let the Lagrange-type function be defined as in (4.6), and $W_\Pi = \{w_\pi : \mathbb{R} \times Z \to \mathbb{R} : \ \pi \in \Pi\}$ be a collection of weak separation functions satisfying conditions (a) and (b) of Theorem 4.1. Then, one has the following assertions.*

(i) *Given $\bar{x} \in \mathscr{R}$ and the corresponding set $\mathscr{K}$ defined by (2.3). If there exists some $\bar{\pi} \in \Pi$ such that $\mathscr{K}$ and $\mathscr{H}$ admit a regular separation by $w_{\bar{\pi}}$, then $(\bar{x}, \bar{\pi})$ is a KKT saddle point for (P).*

(ii) *If $(\bar{x}, \bar{\pi})$ is a KKT saddle point for (P), for which the element $\hat{x}$ in (4.10) is specified as $\bar{x}$, then $\bar{x} \in \mathscr{R}$, while $\mathscr{K}$ and $\mathscr{H}$ admit regular separation by $w_{\bar{\pi}}$.*

Proof. Note that if $\mathscr{K}$ and $\mathscr{H}$ admit a regular separation by $w_{\bar{\pi}}$, that is, for all $(u, v) \in \mathscr{K}$, $w_{\bar{\pi}}(u, v) \leq 0$ and $w_{\bar{\pi}}(u, v) > 0$, then, for all $u > 0$, $w_{\bar{\pi}}(u, 0_Z) > 0$, provided $\mathscr{H}_u \subseteq \mathscr{H}$. Therefore, the assertions can be verified by similar arguments as in the proof of Theorem 4.1. □

If the collection $\mathscr{W}_\Pi$ is employed, applying Proposition 4.1, we can demonstrate that the existence of a KKT saddle point can be guaranteed by the image regularity conditions, which is presented in the following result.

Theorem 4.4. *Let the Lagrange-type function be defined as in (4.6), and put W_Π as the collection of weak separation functions $\mathscr{W}_\Pi$ defined in (3.1). Then, the following assertions hold.*

(i) *Given* $\bar{x} \in \mathscr{R}$ *and the corresponding set* $\mathscr{E}$ *defined by* (2.4) *. If* (IRC1) *or* (IRC2) *hold, then there exists some* $\bar{\pi} = (\bar{A}, \bar{\alpha}, \bar{e}_Z) \in \Pi$ *such that* $(\bar{x}, \bar{\pi})$ *is a KKT saddle point for* (P).

(ii) *If* $(\bar{x}, \bar{\pi}) = (\bar{x}, (\bar{A}, \bar{\alpha}, \bar{e}_Z))$ *is a KKT saddle point for* (P)*, for which the element* $\hat{x}$ *in (4.10) is specified as* $\bar{x}$*, then* $\bar{x} \in \mathscr{R}$ *and* (IRC1) *hold.*

Proof. (i) Since (IRC2) implies (IRC1), we only need to prove the scenario that $\bar{x} \in \mathscr{R}$ and (IRC1) are satisfied. Note that condition (4.5) is equivalent to, $\xi_{A,\alpha,e_Z}(u, 0_Z) > 0$ for all $u > 0$ by Proposition 2.1 (vii). Then, Remark 3.1 (ii) leads to, for all $(u,v) \in \mathscr{H}$, $\xi_{A,\alpha,e_Z}(u,v) > 0$. Thus the existence of a KKT saddle point follows from Proposition 4.1 and Theorem 4.3 (i).

(ii) Suppose that $(\bar{x}, \bar{\pi}) = (\bar{x}, (\bar{A}, \bar{\alpha}, \bar{e}_Z))$ is a KKT saddle point for (P), for which the element $\hat{x}$ is specified as $\bar{x}$. In view of Theorem 4.3 (ii), we deduce that $\bar{x} \in \mathscr{R}$ and $\xi_{\bar{A},\bar{\alpha},\bar{e}_Z}(u,v) \le 0$ and $\xi_{\bar{A},\bar{\alpha},\bar{e}_Z}(u,v) > 0$. In the light of Proposition 2.1 (iii), (vii), and Remark 3.1 (ii), this indicates

$$\forall (u,v) \in \operatorname{cl} \mathscr{E} : \ \xi_{\bar{A},\bar{\alpha},\bar{e}_Z}(u,v) \le 0, \ \text{ and } \ \forall (u,v) \in \mathscr{H} : \ \xi_{\bar{A},\bar{\alpha},\bar{e}_Z}(u,v) > 0,$$

which shows $\operatorname{cl} \mathscr{E} \cap \mathscr{H} = \emptyset$. □

Subsequently, by means of the collection of weak separation functions $\mathscr{W}_\Pi$, we try to construct a penalty function for problem (P) and discuss the existence of an exact penalty function for (P) based on the image regularity condition (IRC2).

The relationship between (IRC2) and the exact penalty function for (P) were considered in [31] and [19], under some special assumptions on the structure of constrained optimization problem (P). For example, the model studied in [31] is

$$\min\{f(x) : \ x \in S, \ g(x) \succeq_{K_Z} 0_Z\},$$

where $S \subseteq \mathbb{R}^m$, $Z = \mathbb{R}^n$, $f : \mathbb{R}^m \to \mathbb{R}$, $g : \mathbb{R}^m \to \mathbb{R}^n$ and K_Z is a closed convex cone in $\mathbb{R}^n$, while in [19], the formulation of the penalty function as well as the assumption imposed on the separation functions is highly dependent on the coordinate representation of the constraint function g, and therefore, the image space of g has to be finite dimensional.

In this work, we consider the general constrained scalar optimization problem (P). For this problem, the penalty function is introduced in the next definition.

Definition 4.2. The penalty function $L^P_{\mathscr{W}_\Pi} : X \times \Pi \to \mathbb{R}$ with respect to $\mathscr{W}_\Pi$ is given by

$$\forall (x,\pi) = (x,(A,\alpha,e_Z)) \in \Pi : \ L^P_{\mathscr{W}_\Pi}(x,(A,\alpha,e_Z)) := f(x) - \min\{\xi_{A,\alpha,e_Z}(0,g(x)),0\}. \quad (4.12)$$

For the parameter set $\Pi = \{\pi = (A,\alpha,e_Z) : A \in \mathscr{A}, \alpha \in \mathbb{R}_{++}, e_Z \in \operatorname{int} K_Z\}$, we define a binary relation, called the penalty order and denoted by $\preceq_P$, in the sense that for arbitrary $\pi_1 = (A_1,\alpha_1,e_Z^1)$, $\pi_2 = (A_2,\alpha_2,e_Z^2) \in \Pi$,

$$\pi_1 \preceq_P \pi_2 \ \text{ if and only if } \ A_1 \subseteq A_2, \alpha_1 \ge \alpha_2, e_Z^1 \succeq_{K_Z} e_Z^2.$$

Lemma 4.2. *For arbitrary* $\pi_1 = (A_1,\alpha_1,e_Z^1)$ *and* $\pi_2 = (A_2,\alpha_2,e_Z^2) \in \Pi$*, if* $\pi_1 \preceq_P \pi_2$*, then*

$$\forall (u,v) \in \mathbb{R} \times Z : \qquad \min\{\xi_{\pi_1}(u,v),0\} \ge \min\{\xi_{\pi_2}(u,v),0\}.$$

Proof. Suppose that $\pi_1 \preceq_P \pi_2$ and (u,v) is arbitrary in $\mathbb{R} \times Z$, then $A_1 \subseteq A_2, \alpha_1 \ge \alpha_2, e_Z^1 \succeq_{K_Z} e_Z^2$. We divide it into three cases.

Case 1: If $(u,v) \in A_1$, then $(u,v) \in A_2$ as well. By Proposition 2.1 (iv), we obtain $\xi_{\pi_1}(u,v) \leq 0$ and $\xi_{\pi_2}(u,v) \leq 0$. Then,

$$\begin{aligned}(u,v) &\in \xi_{\pi_1}(u,v)(\alpha_1, e_Z^1) + A_1 \\ &\subseteq \xi_{\pi_1}(u,v)(\alpha_2, e_Z^2) + \xi_{\pi_1}(u,v)((\alpha_1, e_Z^1) - (\alpha_2, e_Z^2)) + A_2 \\ &\subseteq \xi_{\pi_1}(u,v)(\alpha_2, e_Z^2) + A_2 - \operatorname{cl}\mathscr{H} \\ &\subseteq \xi_{\pi_1}(u,v)(\alpha_2, e_Z^2) + A_2.\end{aligned}$$

Hence,

$$\min\{\xi_{\pi_1}(u,v), 0\} = \xi_{\pi_1}(u,v) \geq \xi_{\pi_2}(u,v) = \min\{\xi_{\pi_2}(u,v), 0\}.$$

Case 2: If $(u,v) \in A_2$, but $(u,v) \notin A_1$, which implies $\xi_{\pi_1}(u,v) > 0$ and $\xi_{\pi_2}(u,v) \leq 0$, then

$$\min\{\xi_{\pi_2}(u,v), 0\} = \xi_{\pi_2}(u,v) \leq 0 = \min\{\xi_{\pi_1}(u,v), 0\}.$$

Case 3: If $(u,v) \notin A_2$, then $(u,v) \notin A_1$ and therefore $\xi_{\pi_i}(u,v) > 0$, for $i = 1,2$. Then,

$$\min\{\xi_{\pi_1}(u,v), 0\} = 0 = \min\{\xi_{\pi_2}((u,v), 0\}.$$

□

It can be observed from Lemma 4.2 that, intuitively speaking, $\pi_1 \preceq_P \pi_2$ indicates that ξ_{π_2} imposes a heavier penalty than ξ_{π_1}.

Definition 4.3. For $\bar{x} \in \mathscr{R}$, the penalty function given in (4.12) is said to be an exact penalty function for (P) at $\bar{x}$ if there exists $\bar{\pi} = (\bar{A}, \bar{\alpha}, \bar{e}_Z) \in \Pi$ such that, for any $\pi \in \Pi$ with $\pi \succeq_P \bar{\pi}$, $\bar{x}$ is an optimal solution to the penalty problem given by

$$\min\{L^P_{\mathscr{W}_\Pi}(x,\pi) : \ x \in X\}, \tag{P_π}$$

when $\bar{x}$ is an optimal solution to (P).

Theorem 4.5. *Let $\bar{x} \in \mathscr{R}$ and the corresponding set $\mathscr{E}$ be defined by (2.4). Then condition (IRC2) holds for (P) if and only if the penalty function given by (4.12) is an exact penalty function for (P) at $\bar{x}$.*

Proof. Suppose that condition (IRC2) holds for (P), but $L^P_{\mathscr{W}_\Pi}$ is not an exact penalty function at $\bar{x}$. Picking some $(\alpha_0, e_Z^0) \in \mathbb{R}_{++} \times \operatorname{int} K_Z$ and letting $A_0 = (\mathbb{R} \times Z) \setminus \operatorname{int}\mathscr{H}$, then, for every $(A_0, \frac{1}{k}\alpha_0, \frac{1}{k}e_Z^0) \in \Pi$ $(k \in \mathbb{N})$, there exists $\pi_k = (A_k, \alpha_k, e_Z^k) \in \Pi$ with $(A_k, \alpha_k, e_Z^k) \succeq_P (A_0, \frac{1}{k}\alpha_0, \frac{1}{k}e_Z^0)$ and $x_k \in X$ such that $L^P_{\mathscr{W}_\Pi}(x_k, \pi_k) < L^P_{\mathscr{W}_\Pi}(\bar{x}, \pi_k)$, i.e.,

$$f(x_k) - \min\{\xi_{A_k,\alpha_k,e_Z^k}(0, g(x_k)), 0\} < f(\bar{x}) - \min\{\xi_{A_k,\alpha_k,e_Z^k}(0, g(\bar{x})), 0\}. \tag{4.13}$$

Note that $(A_k, \alpha_k, e_Z^k) \succeq_P (A_0, \frac{1}{k}\alpha_0, \frac{1}{k}e_Z^0)$ means $A_0 \subseteq A_k$, $\frac{1}{k}\alpha_0 \geq \alpha_k$ and $\frac{1}{k}e_Z^0 \succeq_{K_Z} e_Z^k$. It can be deduced from Lemma 3.1 that $A_k = A_0$ for all $k \in \mathbb{N}$. We claim that $x_k \notin \mathscr{R}$ for every $k \in \mathbb{N}$. Otherwise, there exists some $x_k \in \mathscr{R}$, which means $g(x_k) \in K_Z$. Then,

$$\xi_{A_k,\alpha_k,e_Z^k}(0, g(\bar{x})) = \xi_{A_k,\alpha_k,e_Z^k}(0, g(x_k)) = 0$$

follows from Proposition 2.1 (iv), and (4.13) becomes $f(x_k) < f(\bar{x})$, contradicting that $\bar{x}$ is an optimal solution to (P). Hence, $(0, g(x_k)) \in \operatorname{int} A_k$ and $\xi_{A_k,\alpha_k,e_Z^k}(0, g(x_k)) < 0$, while (4.13) gives

$$f(x_k) - \xi_{A_k,\alpha_k,e_Z^k}(0, g(x_k)) < f(\bar{x}).$$

Since $(A_k,\alpha_k,e_Z^k)\succeq_P(A_0,\frac{1}{k}\alpha_0,\frac{1}{k}e_Z^0)$, due to Lemma 4.2, we can further obtain

$$f(\bar{x})-f(x_k)>-\xi_{A_0,\frac{1}{k}\alpha_0,\frac{1}{k}e_Z^0}(0,g(x_k))=-k\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k))>0.$$

Setting $\lambda_k=\frac{1}{f(\bar{x})-f(x_k)}$, we have

$$\frac{1}{k}>-\lambda_k\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k))>0,$$

which yields that $\lambda_k\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k))\to 0$ as $k\to+\infty$. Set

$$(u_k,v_k):=(0,g(x_k))-((0,g(x_k))-\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k))(\alpha_0,e_Z^0)).$$

Since

$$\xi_{A_0,\alpha_0,e_Z^0}((0,g(x_k))-\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k))(\alpha_0,e_Z^0))=\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k))-\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k))=0,$$

we see that there exists

$$(0,g(x_k))-\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k))(\alpha_0,e_Z^0)\in\operatorname{cl}\mathscr{H},$$

indicating $(u_k,v_k)\in(0,g(x_k))-\operatorname{cl}\mathscr{H}$. Then, $\lambda_k(f(\bar{x})-f(x_k)+u_k,v_k)\in\operatorname{cone}\mathscr{E}$, while

$$\begin{aligned}&\lambda_k(f(\bar{x})-f(x_k)+u_k,v_k)\\=\ &(1+\lambda_k\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k))\alpha_0,\lambda_k\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k))e_Z^0)\\\to\ &(1,0_Z)\in\mathscr{H}_u\end{aligned}$$

as $k\to+\infty$, which is a contradiction to (IRC2).

Conversely, assume that $L^P_{\mathscr{W}_\Pi}$ is an exact penalty function at $\bar{x}$, but condition (IRC2) is violated. Then, there exist sequences $\{\lambda_k\}_{k=1}^{+\infty}\subseteq\mathbb{R}_{++}$, $\{u_k\}_{k=1}^{+\infty}\subseteq\mathbb{R}_+$, $\{x_k\}_{k=1}^{+\infty}\subseteq X$ and $\{v_k\}_{k=1}^{+\infty}\subseteq K_Z$ such that

$$\lambda_k(f(\bar{x})-f(x_k)-u_k,v_k)\to(1,0_Z)\tag{4.14}$$

as $k\to+\infty$. Since $\lambda_k(f(\bar{x})-f(x_k)-u_k)\to 1$ and $\lambda_k>0$, we might assume that $f(\bar{x})-f(x_k)-u_k>0$ for every k. Consider the parameter $\pi_0=(A_0,\alpha_0,e_Z^0)$. Then, A_0 is a cone, which ensures that ξ_{A_0,α_0,e_Z^0} is positively homogeneous and continuous, by Proposition 2.1 (iii) and (ix). It follows from (4.14) that

$$\frac{\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k)-v_k)}{f(\bar{x})-f(x_k)-u_k}=\frac{\xi_{A_0,\alpha_0,e_Z^0}(0,\lambda_k(g(x_k)-v_k))}{\lambda_k(f(\bar{x})-f(x_k)-u_k)}\to 0$$

as $k\to+\infty$. Hence, for arbitrary $n\in\mathbb{N}$, there exists some k with

$$\left|\frac{\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k)-v_k)}{f(\bar{x})-f(x_k)-u_k}\right|<\frac{1}{n},$$

i.e.,

$$|\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k)-v_k)|<\frac{1}{n}(f(\bar{x})-f(x_k)-u_k)\le\frac{1}{n}(f(\bar{x})-f(x_k)).\tag{4.15}$$

Next, we divide it into two cases.

Case 1: If $x_k\in\mathscr{R}$, then $g(x_k)\in K_Z$ and $\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k))=0$. Therefore,

$$f(\bar{x})-f(x_k)>0=-\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k)).$$

Case 2: If $x_k \notin \mathscr{R}$, then $g(x_k) \notin K_Z$ and $\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k)) < 0$. It can be deduced from (4.15) that

$$-\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k)) \leq |\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k)-v_k)| < \frac{1}{n}(f(\bar{x})-f(x_k)).$$

Both cases provide that

$$f(x_k) - n\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k)) < f(\bar{x}),$$

and $\xi_{A_0,\alpha_0,e_Z^0}(0,g(x_k)) \leq 0$. Then, it follows that

$$f(x_k) - \min\{\xi_{A_0,\frac{1}{n}\alpha_0,\frac{1}{n}e_Z^0}(0,g(x_k)),0\} < f(\bar{x}) - \min\{\xi_{A_0,\frac{1}{n}\alpha_0,\frac{1}{n}e_Z^0}(0,g(\bar{x})),0\},$$

i.e.,

$$L^P_{\mathscr{W}_\Pi}(x_k,(A_0,\frac{1}{n}\alpha_0,\frac{1}{n}e_Z^0)) < L^P_{\mathscr{W}_\Pi}(\bar{x},(A_0,\frac{1}{n}\alpha_0,\frac{1}{n}e_Z^0)). \tag{4.16}$$

This shows that, for any $\pi = (A,\alpha,e_Z) \in \Pi$, there exist some $x_k \in X$ and $n \in \mathbb{N}$ with $(A_0,\frac{1}{n}\alpha_0,\frac{1}{n}e_Z^0) \succeq_P (A,\alpha,e_Z)$, such that (4.16) holds, which contradicts that $L^P_{\mathscr{W}_\Pi}$ is an exact penalty function for (P) at $\bar{x}$. □

Remark 4.1. If we consider a local optimal solution of (P), then only a local version of the image regularity condition is needed. The local image regularity condition holds at some $\bar{x} \in \mathscr{R}$ for (P) if there exists some $\delta > 0$ such that

$$\text{cl cone } \mathscr{E}^\delta \cap \mathscr{H}_u = \emptyset, \tag{LIRC2}$$

where

$$\mathscr{K}^\delta := \{(u,v) \in \mathbb{R} \times Z: \ u = f(\bar{x}) - f(x), v = g(x), x \in X \cap B(\bar{x},\delta)\}$$

and $\mathscr{E}^\delta := \mathscr{K}^\delta - \text{cl}\,\mathscr{H}$. With an argument similar to the one used in the proof of Theorem 4.5, it can be proved that, for some $\bar{x} \in \mathscr{R}$, condition (LIRC2) holds if and only if $L^P_{\mathscr{W}_\Pi}$ is an exact penalty function for (P) at $\bar{x}$, that is, if $\bar{x}$ is a local optimal solution for (P), then there exists $\bar{\pi} \in \Pi$ such that $\bar{x}$ is a local optimal solution of (P_π) for all $\pi \in \Pi$ with $\pi \succeq_P \bar{\pi}$.

5. Conclusions

Image space analysis initiated by Castellani and Giannessi in [4] and Giannessi in [5, 6] has been proven to be instrumental in studying different aspects of constrained optimizations. This paper aims at exploiting this approach to investigate the regularity conditions. The method of image space analysis is based on the idea of separation, which is why the concept of separation functions and various separation theorems are crucial for it. In our paper, the essential tools applied to a scalar constrained problem (P) are the Gerstewitz function and the corresponding separation results. With the help of that, a collection of nonlinear weak separation functions, along with a Lagrange type function and a penalty function with respect to it, was proposed. Then, we introduced some image regularity conditions. By virtue of the conditions, the existence of regular separation functions, the existence of saddle points, and an exact penalty function for (P) were established. For further research, it is of interest to use the existence results for saddle points and the exact penalty function in order to derive strong duality statements for the primal problem (P) and a suitable dual problem under the image regularity conditions.

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