J. Nonlinear Var. Anal. 9 (2025), No. 4, pp. 561-568
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.9.2025.4.06

SOME REMARKS ON THE ENVELOPE OF A FAMILY OF CURVES

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Abstract. By means of Clarke's differentiability theory, a generalization of the notion of envelope for a nondifferentiable family of curves is considered. In particular, we propose an extension of the celebrated conditions fulfilled by an envelope curve in the continuously differentiable case. An application in the theory of long-run cost curves in Economics, is also provided.

Keywords. Clarke directional derivative; Envelope; Subdifferential.

1. Introduction

Consider the family $\mathscr{F}$ of curves

$$f(x,y,t) = 0, \qquad t \in T, \tag{1.1}$$

where $T \subseteq \mathbf{R}$ and $f : \mathbf{R} \times \mathbf{R} \times T \longrightarrow \mathbf{R}$.

The envelope of $\mathscr{F}$ is a curve that is tangent, at each of its non singular points, to a curve of the family $\mathscr{F}$. The notion of envelope finds applications in many fields, such as Physics, the Huygens principle in geometric optics [1], and Economics, the theory of long-run cost curves [2–4]. Recently, the applications to problems arising in matrix theory or hyperbolic geometry were proposed in [5]. Moreover, there are practical applications in robotics and gear constructions; see [6].

In the literature, it is in general assumed that each curve of $\mathscr{F}$ is smooth and that f is continuously differentiable at each (x,y,t) such that $f(x,y,t) = 0$. In this paper, we consider the problem of extending the definition of the envelope to a nondifferentiable family of curves. To this aim, we make use of the generalized differentiability theory of Clarke [7]. In particular, in the third section of this paper, we state a generalization of the classic conditions fulfilled by the envelope in the differentiable case:

$$f(x(t),y(t),t) = 0, \quad f'_t(x(t),y(t),t) = 0, \qquad \forall t \in T, \tag{1.2}$$

where f'_t denotes the standard partial derivative of f with respect to t.

Section 4 provides an application to the theory of long-run cost curves in economics. In fact, it is known that a long-run cost curves can be obtained as the envelope of the family of short-run cost curves [2,3]. In this context, we present an example where the function f which defines the family (1.1) of short-run cost curves is nondifferentiable. Section 5, the last section, is devoted to further developments of the analysis and concluding remarks.

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Received 5 January 2025; Accepted 17 February 2025; Published online 15 May 2025.

2. Preliminary Results

Let us introduce the main notations and definitions that are used in the sequel.

Let $X \subseteq \mathbf{R}^n$. Recall that X is said to be a cone iff $\lambda X \subseteq X$, with $\lambda > 0$. The closure and the convex hull of X are denoted, by $cl\,X$ and $conv\,X$, respectively.

The normal cone to X at x, denoted by $N(X;x)$, is defined by

$$N(X;x) := \{y \in \mathbf{R}^n : \ \langle y, z-x\rangle \le 0, \quad \forall z \in X\},$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbf{R}^n$.

Let $\phi : X \longrightarrow \mathbf{R}$. We recall that ϕ is locally Lipschitz at $x \in X$ iff there exist a constant $K > 0$ and a neighbourhood V of x, such that

$$|\phi(y) - \phi(y')| \le K\|y - y'\|, \qquad \forall (y,y') \in V \cap X.$$

Let ϕ be locally Lipschitz at $x \in X$ and let $v \in \mathbf{R}^n$. The generalized directional derivative of ϕ at x in the direction v is defined by

$$\phi^0(x;v) := \lim_{y\to x,\ t\downarrow 0} \sup \frac{\phi(y+tv) - \phi(y)}{t}.$$

The Clarke generalized subdifferential of ϕ at x [7], denoted by $\partial\phi(x)$, is the set

$$\partial\phi(x) := \{\xi \in \mathbf{R}^n : \phi^0(x;v) \le \langle\xi, v\rangle, \forall v \in \mathbf{R}^n\}.$$

The previous definitions were given in [7] in the more general setting where X is a subset of a Banach space. In the finite-dimensional case, $\partial\phi(x)$ is a convex compact set for every $x \in X$. If ϕ is continuously differentiable, then $\phi^0(x;v) = \langle\nabla\phi(x), v\rangle$ and $\partial\phi(x) = \nabla\phi(x)$. We recall that, for a differentiable function which is not C^1, then the Clarke generalized gradient does not necessarily coincide with the classic gradient.

When ϕ is a convex locally Lipschitz function on the convex set X, $\phi^0(x;v)$ coincides with the one sided directional derivative, in the direction v

$$\phi'(x;v) := \lim_{t\downarrow 0} \frac{\phi(x+tv) - \phi(x)}{t}$$

and $\partial\phi(x)$ collapses to the classic subdifferential of convex analysis [8], that is the set of $\xi \in \mathbf{R}^n$ such that

$$\phi(y) - \phi(x) \ge \langle\xi, y-x\rangle, \qquad \forall y \in X. \tag{2.1}$$

When f is differentiable at (x,y,t), we denote by $(f'_x(x,y,t), f'_y(x,y,t), f'_t(x,y,t))$, the gradient $\nabla f(x,y,t)$ and by $\partial_{(x,y)} f(x,y,t)$ and $\partial_t f(x,y,t)$ the partial subdifferentials, at (x,y,t), of $f(\cdot,\cdot,t)$ and $f(x,y,\cdot)$, respectively.

Throughout the paper, we assume that the following assumptions are fulfilled:

(A1) $f(x,y,t)$ is a Lipschitz function $\forall (x,y,t) \in \mathbf{R}\times\mathbf{R}\times T$.

(A2) $f(x,y,t)$ is regular in the sense of Clarke [7] i.e., for all $(x,y,t) \in \mathbf{R}\times\mathbf{R}\times T$, the usual one sided directional derivative $f'(x,y,t;v)$ exists, and $f'(x,y,t;v) = f^0(x,y,t;v)$, the Clarke directional derivative, for all $v \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}$.

Remark 2.1. If f is a C^1 function or if it is convex on $\mathbf{R}\times\mathbf{R}\times T$, provided that T is convex, then f is regular [7].

For completeness, we report the following results concerning the main properties of Clarke subdifferential that are used in what follows.

Let $\phi = g \circ h$, where $h : X \longrightarrow \mathbf{R}^n$ and $g : \mathbf{R}^n \longrightarrow \mathbf{R}$ are given function spaces. The components of h are denoted by $h_i, i = 1, \ldots, n$. We assume that each h_i is locally Lipschitz at x and that g is locally Lipschitz at $h(x)$.

Theorem 2.1. *[7, Theorem 2.3.9] $\partial\phi(x) \subset$ cl conv$\{\sum_{i=1}^n \alpha_i \xi_i : \xi_i \in \partial h_i(x),\ \alpha \in \partial g(h(x))\}$, where cl conv denotes the closed convex hull and $\alpha := (\alpha_1, \ldots, \alpha_n)$. Equality holds under the following additional hypotheses: g is regular at $h(x)$ and h is strictly differentiable at x. (In this case, it follows that f is regular at x and the cl conv is superfluous.*

Remark 2.2. We recall that h is strictly differentiable at x if there exists an element of the topological dual of X, denoted by $D_s h(x)$, such that, for each $v \in X$,

$$\lim_{y \to x,\ t \downarrow 0} \frac{h(y + tv) - h(y)}{t} = \langle D_s h(x), v \rangle.$$

If h is continuously differentiable at x, then it is strictly differentiable at x [7].

Let $X = X_1 \times X_2$, where X_1 and X_2 are Banach spaces, and let $\phi(x_1, x_2) : X \to \mathbf{R}$ be locally Lipschitz at (x_1, x_2).

Proposition 2.1. *[7, Proposition 2.3.15] If ϕ is regular at (x_1, x_2), then*

$$\partial\phi(x_1, x_2) \subset \partial_{x_1}\phi(x_1, x_2) \times \partial_{x_2}\phi(x_1, x_2).$$

3. A Generalized Definition of Envelope

We introduce the following generalization of the definition of the envelope of a family of curves.

Definition 3.1. The envelope of the family $\mathscr{F}$ of curves, defined by (1.1), is a differentiable curve γ of parametric equations $x = x(t), y = y(t),\ t \in T$, such that the following conditions are fulfilled:

(a) $f(x(t), y(t), t) = 0$, for every $t \in T$.

(b) There exists $\xi(t) \in \partial_{(x,y)} f(x(t), y(t), t)$ such that $\langle (x'(t), y'(t)), \xi(t) \rangle = 0$, for every $t \in T$ with $\xi(t) \neq 0$ a.e. on T.

Remark 3.1. The set $\partial_{(x,y)} f(x(t), y(t), t)$ can be considered as a set of generalized normals to the curve $\delta_t := \{(x, y) \in \mathbf{R}^2 : f(x, y, t) = 0\}$ at the point $(x(t), y(t))$, for each $t \in T$.

Condition (b) of Definition 3.1 can be interpreted as a generalized tangency condition of the envelope γ to a generic curve of the family $\mathscr{F}$.

When f is a continuously differentiable function, condition (b) becomes

$$\langle (x'(t), y'(t)), (f'_x(x(t), y(t), t), f'_y(x(t), y(t), t)) \rangle = 0,$$

which is equivalent to say that the envelope γ is tangent, in the classic sense, to the curve δ_t, at the point $(x(t), y(t))$, for every $t \in T$. Therefore, in such a case, Definition 3.1 collapses to the classic notion of envelope.

By means of the next proposition, we provide a characterization of condition (b) of Definition 3.1 in the convex case.

Proposition 3.1. *If $f(x,y,t)$ is a convex function, with respect to (x,y) jointly, for each $t \in T$, then*

$$\xi \in \partial_{(x,y)} f(x(t),y(t),t) \quad \Leftrightarrow \quad \xi \in N(\delta_t,(x(t),y(t)).$$

Proof. By (2.1), we have that $\xi \in \partial_{(x,y)} f(x(t),y(t),t)$ if and only if the following inequality holds:

$$f(x,y,t) - f(x(t),y(t),t) \geq \langle \xi,(x,y)-(x(t),y(t))\rangle, \qquad \forall (x,y) \in \delta_t. \tag{3.1}$$

Since the left-hand side of (3.1) is 0, for each $(x,y) \in \delta_t$, then (3.1) holds if and only if $\xi \in N(\delta_t,(x(t),y(t))$. This completes the proof. ☐

Therefore, in the convex case, condition (b) is equivalent to the fact that the tangent to the envelope at $(x(t),y(t))$ is a supporting line for the curve δ_t, for almost every $t \in T$.

We aim to state a generalization of the classic conditions (1.2) to the case of a nonsmooth family of curves. We need two preliminary lemmas.

Lemma 3.1. *Let $F(t) := f(x(t),y(t),t)$, where $x(t)$ and $y(t)$ are continuously differentiable functions for all $t \in T$. Then $\partial F(t) = \{\langle \alpha,\beta\rangle : \alpha \in \partial f(x(t),y(t),t), \beta = (x'(t),y'(t),1)\}$.*

Proof. Taking into account that continuous differentiability implies strict differentiability, we conclude from Theorem 2.1 the desired conclusion immediately. ☐

Lemma 3.2. *If f is regular at (x,y,t), then $\partial f(x,y,t) \subset \partial_{(x,y)} f(x,y,t) \times \partial_t f(x,y,t)$.*

Proof. The proof follows from Proposition 2.1. ☐

We are now in a position to state a generalization of conditions (1.2) to a nondifferentiable family of curves.

Theorem 3.1. *Let $T \subseteq \mathbf{R}$ be an open set. If a continuously differentiable curve $\gamma(t) := (x(t),y(t))$, $t \in T$, is an envelope of the family of curves $f(x,y,t) = 0$, $t \in T$, then it is a solution to the following system*

$$\begin{cases} f(x(t),y(t),t) = 0, & \forall t \in T, \\ 0 \in \partial_t f(x(t),y(t),t), & \forall t \in T. \end{cases} \tag{3.2}$$

Proof. By condition (a) of Definition 3.1, for every $t \in T$ and Δt sufficiently small, it results that $f(x(t+\Delta t),y(t+\Delta t),t+\Delta t) = 0$. Therefore, for all $t \in T$,

$$\frac{dF}{dt}(t) := \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t),y(t+\Delta t),t+\Delta t) - f(x(t),y(t),t)}{\Delta t} = 0.$$

By Lemma 3.1, it follows that $0 = \{\langle \alpha,\beta\rangle : \alpha \in \partial f(x(t),y(t),t), \beta = (x'(t),y'(t),1)\}$. By Lemma 3.2, we see that $\partial f(x(t),y(t),t) \subset \partial_{(x,y)} f(x(t),y(t),t) \times \partial_t f(x(t),y(t),t)$, so that each element of $\partial f(x(t),y(t),t)$ can be expressed in the form (ξ,γ) with $\xi \in \partial_{(x,y)} f(x(t),y(t),t)$ and $\gamma \in \partial_t f(x(t),y(t),t)$ and the envelope $(x(t),y(t))$ fulfils the relation

$$\langle \xi,(x'(t),y'(t))\rangle \; + \; \gamma = 0.$$

By the condition (b) of Definition 3.1, it must be $\langle \xi,(x'(t),y'(t))\rangle = 0$ for a suitable $\xi \in \partial_{(x,y)} f(x(t),y(t),t)$. Thus $0 \in \partial_t f(x(t),y(t),t)$ is fulfilled. ☐

If $f(x,y,\cdot)$ is a C^1 function, then Clarke subdifferential collapses to classic gradient [7], so that $\partial_t f(x(t),y(t),t) = f_t'(x(t),y(t),t)$ and (3.2) collapses to (1.2).

Example 3.1. Consider the family of curves

$$f(x,y,t) = t^2 + (x-t)^2 + |x-t| - y = 0, \quad t \in T := \mathbf{R}. \tag{3.3}$$

Observe that f is a convex Lipschitz function, so that the Clarke subdifferential collapses to the classic subdifferential of convex analysis.

$$\partial_t f(x,y,t) = 2t - 2(x-t) + \begin{cases} [-1,1], & if \quad t = x, \\ 1, & if \quad t > x, \\ -1, & if \quad t < x. \end{cases}$$

Consider the condition $0 \in \partial_t f(x(t), y(t), t), \ t \in T$. It follows that

$$\begin{cases} 0 \in 2t - 2(x-t) + [-1,1], & t = x, \\ 0 = 4t - 2x + 1, & t > x, \\ 0 = 4t - 2x - 1, & t < x. \end{cases}$$

Therefore

$$\begin{cases} t = x, & x \in \left[-\frac{1}{2}, \frac{1}{2}\right], \\ t = \frac{2x-1}{4}, & x < -\frac{1}{2}, \\ t = \frac{2x+1}{4}, & x > \frac{1}{2}. \end{cases}$$

Substituting t in the equation $f(x,y,t) = 0$, we obtain

$$y = \begin{cases} x^2, & x \in \left[-\frac{1}{2}, \frac{1}{2}\right], \\ \frac{x^2}{2} - \frac{x}{2} - \frac{1}{8}, & x < -\frac{1}{2}, \\ \frac{x^2}{2} + \frac{x}{2} - \frac{1}{8}, & x > \frac{1}{2}. \end{cases} \tag{3.4}$$

Observe that

$$y'(x) = \begin{cases} 2x, & x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ x - \frac{1}{2}, & x < -\frac{1}{2} \\ x + \frac{1}{2}, & x > \frac{1}{2} \end{cases}$$

so that the envelope of the family of curves (3.3) is a C^1 function.

The following example shows a case where Theorem 3.1 cannot be applied although an envelope of the considered family of curves exists.

Example 3.2. Let $1 < \alpha < 2$ and $T := \mathbf{R}$. Consider the family of curves

$$0 = f(x,y,t) = \begin{cases} y - |t|^\alpha \sin(1/t) - (x-t)[\alpha |t|^{\alpha-1} \sin(1/t) - |t|^{\alpha-2} \cos(1/t)], & x \geq t, \ t \neq 0, \\ y - |t|^\alpha \sin(1/t) + (x-t)[\alpha |t|^{\alpha-1} \sin(1/t) - |t|^{\alpha-2} \cos(1/t)], & x \leq t, \ t \neq 0, \\ y, & for \quad t = 0. \end{cases}$$

An envelope of such a family of curves is given by the function

$$\phi(x) = \begin{cases} |x|^\alpha \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

for $1 < \alpha < 2$.

Actually, for fixed $t \neq 0$, the curve $f(x,y,t) = 0$ consists of two halflines passing through the point $(\bar{x},\bar{y}) := (t,\phi(t))$, one of which is the tangent to the graph of the function ϕ at $(\bar{x},\bar{y})$. Observe that ϕ is a differentiable function for every $x \in \mathbf{R}$, but not locally Lipschitz at $x = 0$ since

$$\phi'(x) = \begin{cases} \alpha|x|^{\alpha-1}\sin(1/x) - |x|^{\alpha-2}\cos(1/x), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

and $\limsup_{x\to 0} \phi'(x) = +\infty$, (consider the sequence $x_k = 1/(2k+1)\pi, k \in \mathbb{N}$). Thus we also see that $f(x,y,t)$ is not Lipschitz at $(x,y,0)$. Therefore the basic assumption (A1) does not hold and Theorem 3.1 cannot be applied.

4. An Application in Economics

The notion of envelope finds an important application in economics, in the theory of long-run cost curves. Actually, a long-run cost curve turns out to be the envelope of the family of short-run cost curves [2, 3]. The family of short-run cost curves provides the cost of production as a function of output with the level of a given input as the parameter of the family. The long-run cost curve is defined by the function that selects the level of input that minimizes the cost.

Let $C : X \times T \longrightarrow \mathbf{R}$. If $y = C(x,t)$, $t \in T$, denotes the family of short-run curves, with T being the set of input parameters, then the long-run function is given by $\psi(x) := \min_{t\in T} C(x,t)$. If we assume that $C(x,\cdot)$ is convex on the open convex set $T \subseteq \mathbf{R}$, then, for all $x \in X$,

$$\psi(x) = C(x,t), \quad \text{where} \quad 0 \in \partial_t C(x,t) \tag{4.1}$$

The previous condition is fulfilled by the envelope of the family of curves

$$f(x,y,t) := C(x,t) - y = 0, \quad t \in T. \tag{4.2}$$

Applying Theorem 3.1, we obtain that the envelope of (4.2) fulfils the system

$$\begin{cases} 0 \in \partial_t C(x,t) \\ y = C(x,t),\ t \in T \end{cases} \tag{4.3}$$

Observe that (4.3) is equivalent to (4.1).

Example 4.1. Let L be the amount of labour required for production of a certain good, $K > 0$ the level of capital input, and $Q = Q(L,K)$ the quantity of product obtained. Moreover, let $r > 0$ be the unit cost of capital, and let $w > 0$ be the wage rate. Let $X := \mathbf{R}_+$ and $T := \{t \in \mathbf{R} : t > 0\}$. Assume that the total cost of production is defined by the function $C : X \times T \longrightarrow \mathbf{R}$, where

$$C(L,K) := \begin{cases} rK + wL, & for \quad 0 < K < K^0, \\ 2r(K - K^0/2) + wL, & for \quad K \geq K^0, \end{cases} \tag{4.4}$$

and that $Q : X \times T \longrightarrow \mathbf{R}$, where

$$Q(L,K) := a\sqrt{LK}, \tag{4.5}$$

where K^0 and a are suitable positive constants. By (4.5), we obtain $L = Q^2/(Ka^2)$, and substitute in (4.4) that

$$C(Q,K) = \begin{cases} rK + \dfrac{wQ^2}{Ka^2}, & for\ 0 < K < K^0, \\ 2r(K - K^0/2) + \dfrac{wQ^2}{Ka^2}, & for\ K \geq K^0. \end{cases} \tag{4.6}$$

Let us rewrite (4.6) in (1.1)

$$0 = f(Q,C,K) := \begin{cases} rK + \dfrac{wQ^2}{Ka^2} - C, & for\ 0 < K < K^0, \\ 2r(K - K^0/2) + \dfrac{wQ^2}{Ka^2} - C, & for\ K \geq K^0. \end{cases} \tag{4.7}$$

Note that f is not differentiable w.r.t. the parameter K and that the envelope of the family of the short-run cost curves (4.7) fulfils the system

$$\begin{cases} 0 \in \partial_K f(Q,C,K), \\ f(Q,C,K) = 0,\ K > 0. \end{cases}$$

The subdifferential of f w.r.t. K is given by

$$\partial_K f(Q,C,K) = \begin{cases} r - \dfrac{wQ^2}{K^2a^2}, & for\ 0 < K < K^0, \\ [r - \dfrac{wQ^2}{K^2a^2}, 2r - \dfrac{wQ^2}{K^2a^2}], & for\ K = K^0, \\ 2r - \dfrac{wQ^2}{K^2a^2}, & for\ K > K^0. \end{cases}$$

Thus $0 \in \partial_K f(Q,C,K)$ iff

$$\begin{cases} K = \dfrac{Q}{a}\sqrt{\dfrac{w}{r}}, & for \quad 0 < K < K^0, \\ K \in \Big[\dfrac{Q}{a}\sqrt{\dfrac{w}{2r}}, \dfrac{Q}{a}\sqrt{\dfrac{w}{r}}\Big], & for \quad K = K^0, \\ K = \dfrac{Q}{a}\sqrt{\dfrac{w}{2r}}, & for \quad K > K^0. \end{cases}$$

Substituting the previous relations in (4.7), we obtain

$$C(Q) = \begin{cases} \dfrac{2Q}{a}\sqrt{wr}, & 0 < Q < K^0 a\sqrt{\dfrac{r}{w}}, \\ rK^0 + \dfrac{wQ^2}{K^0a^2}, & K^0 a\sqrt{\dfrac{r}{w}} \leq Q \leq K^0 a\sqrt{\dfrac{2r}{w}}, \\ \dfrac{2Q}{a}\sqrt{2wr} - rK^0, & Q > K^0 a\sqrt{\dfrac{2r}{w}}. \end{cases}$$

We observe that $C(Q)$ is a C^1 function and coincides with the envelope of the family of the short-run cost curves defined by (4.7).

5. Further Developments and Concluding Remarks

We considered a generalization of the notion of envelope to a nondifferentiable family of curves. The analysis was carried out by exploiting Clarke's generalized differentiability theory and the calculus rules of the Clarke subdifferential. Further extensions can be obtained by using suitable subdifferentials that enjoy analogous calculus rules. The presented results were stated for a family of curves in the plane but it can be shown that they can be generalized to the case of a family of surfaces. In Section, 4 we considered an application in economics which shows that we can associate with the solution of a parametric optimization problem the envelope of a suitable family of curves. Conversely, in [9], it was pointed out that, in the differentiable case, the problem of finding the envelope of the family of curves (1.1) can be associated with a parametric optimization problem $P(x)$ defined by

$$\begin{cases} \min\limits_{y,t} \ (\max\limits_{y,t}) \ y \\ f(x,y,t) = 0, \ t \in T \end{cases} \tag{5.1}$$

The first order optimality conditions for (5.1) are actually equivalent to (1.2) assuming that $f_y \neq 0$ and that T is an open set. It would be of interest to generalize such an equivalence to the nonsmooth case. To this aim, the Image Space Analysis [10] could be a powerful tool, in order to derive optimality conditions for (5.1).

Acknowledgments

The author wishes to express his gratitude to Prof. Valentino Magnani for useful discussions.

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