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SEQUENTIAL SPLITTING ALGORITHMS WITH BREGMAN DISTANCE FOR SOLVING EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we propose a splitting algorithm involving the Bregman distance for solving pseudomonotone equilibrium problems where the equilibrium bifunction is the sum of two bifunctions. The convergence of the sequence generated by the proposed algorithm is established under some suitable assumptions on the Bregman function and the equilibrium function. We also give some numerical examples to support our algorithm and convergence results.

Keywords. Bregman distance; Hölder continuity; Pseudomonotone equilibrium problems; Splitting algorithm; Weak convergence.

1. Introduction

Let $\mathscr{H}$ be a real Hilbert space, C be a nonempty, closed, and convex subset of $\mathscr{H}$, and $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem (in short, EP) is to find $\bar{x} \in C$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

We denote the solution set of the EP (1.1) by $S(f,C)$. Some important problems such as variational inequalities, saddle point problems, fixed point problems, and Nash equilibrium are also particular cases of the EP (1.1); see, for example, [1–3] and the references therein. For historical details and applications, we refer to [1, Chapter 1]. Various solution methods were developed to solve EPs; see, for example, [4–13] and the references therein.

For a function $g : C \to \mathbb{R}$ and $\lambda > 0$, the proximal operator of g is the mapping $\mathrm{Prox}_g : C \to C$ defined by

$$\mathrm{Prox}_g(x) = \operatorname*{argmin}_{y \in C} \left\{ \lambda g(y) + \frac{1}{2}\|x - y\|^2 \right\}, \quad \forall x \in C.$$

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By using the proximal operator, the following extragradient algorithm for solving EP (1.1) was proposed in [8, 13]:

$$\begin{cases} x^0 \in C, \\ y^k = \operatorname*{argmin}_{y\in C} \left\{ \lambda f(x^k,y) + \frac{1}{2}\|y-x^k\|^2 \right\}, \\ x^{k+1} = \operatorname*{argmin}_{y\in C} \left\{ \lambda f(y^k,y) + \frac{1}{2}\|y-x^k\|^2 \right\}. \end{cases} \tag{1.2}$$

Go back to the paper of Blum and Oettli [2], where they considered equilibrium bifunction f as a sum of two bifunctions, say, f_1 and f_2, that is,

$$f(x,y) := f_1(x,y) + f_2(x,y), \quad \forall x,y \in C.$$

In particular, they considered the following form of the equilibrium problem, also known as mixed equilibrium problem:

$$\text{Find } \bar{x} \in C \text{ such that } f(\bar{x},y) := f_1(\bar{x},y) + f_2(\bar{x},y) \geq 0, \quad \forall y \in C, \tag{1.3}$$

where $f_1, f_2 : C \times C \to \mathbb{R}$ are bifunctions.

It is worth to mention that the eignvalue problems, hemivariational inequalities, the problem of finding anti-period solutions for nonlinear evolution equations, boundary value problems related to the stationary Navier-Stokes equations for heat-conducting fluids, nonlinear implicit differential equations associated to a time-dependent operator, and so on can be formulated in the form of an equilibrium problem (1.3); see, for example, [14–16] and the references therein. Several solution methods for solving equilibrium problem (1.3) were proposed and analyzed in the literature. In [17], Zeng et al. proposed hybrid proximal-type algorithms for finding a common solution of the EP (1.1) and mixed equilibrium problem (1.3) in the setting of Banach spaces.

Some splitting algorithms, similar to the extragradient algorithm (1.2), were proposed in [4, 7, 18, 19] to investigate equilibrium problems whose objective bifunction f can be decomposed into the sum of two bifunctions, say f_1 and f_2, that is, to deal with the mixed equilibrium problem (1.3). Pham and Trinh [4] proposed splitting algorithms where one of the component bifunctions, say, f_1, needs to be Hölder continuous and the other, say, f_2, needs to be Lipschitz continuous. With the Euclidean distance as the regularized term in the algorithms, the convergence of the sequence generated by their proposed algorithm to a solution of the EP (1.3) was obtained in [4] under the assumptions that bifunction f is strongly pseudomonotone and with respect to the second argument, the component bifunctions f_1 and f_2 are convex and lower semicontinuous. Recently, Hai and Vinh [19] proposed the following splitting algorithm to solve EP (1.3), which is different from the one studied in [4]

$$\begin{cases} \text{Take } x^0 \in \mathscr{H} \text{ and set } z^0 = x^0. \text{ Choose} \{\lambda_k\} \subset (0,1). \\ y^k = \operatorname*{argmin}_{t\in C} \left\{ \lambda_k f_1(x^k,t) + \frac{1}{2}\|t-x^k\|^2 \right\}, \\ x^{k+1} = \operatorname*{argmin}_{t\in C} \left\{ \lambda_k f_2(y^k,t) + \frac{1}{2}\|t-y^k\|^2 \right\}, \\ z^{k+1} = \frac{\sum_{n=1}^{k+1} \lambda_n x^n}{\sum_{n=1}^{k+1} \lambda_n}. \end{cases} \tag{1.4}$$

They also used the Euclidean distance as the regularized term in algorithm (1.4). They studied the convergence of the sequence generated by algorithm (1.4) under the assumptions that f is pseudomonotone and with respect to its first argument, concave and weakly upper semicontinuous, while the component bifunctions f_1 and f_2 are convex and lower semicontinuous with respect to the second variable. For the convergence of this algorithm, both f_1 and f_2 need to be Hölder continuous.

Recently, the quadratic regularization term $\|\cdot\|^2$ has successfully been replaced by a distance-like function $d(.,.)$ and applied to solve many problems. A proper choice of $d(.,.)$ helps the iterative steps to be in the interior of the constraint set. Bregman distance is a well-known example of such a regularization. The concept of the Bregman function was first used by Bregman [20] to find common point of convex sets. But the name "Bregman function" was first used by Censor and Lent in [21]. Bregman function is now used in various ways in order to design and analyze iterative algorithms for solving feasibility problems, optimization problems, variational inequalities and also for computing fixed points of nonlinear mappings; see, for example, [5,22–26] and the references therein.

In this paper, We use the Bregman distance in place of quadratic regularized term $\|\cdot\|^2$ in algorithm (1.4) and study the convergence of the generated sequence under some suitable assumptions. At each iteration, our algorithm deals with component bifunctions separately instead of solving subproblems involving the objective bifunction. We show that, for a particular choice of the Bregman function, our method reduces to the one proposed in [4]. So, our method is more general and helpful to solve a larger class of equilibrium problems.

The remaining part of this paper is arranged as follows. In Section 2, we mention some notations, basic definitions, and results, which are required in the sequel. The concepts of the Bregman function and its properties are also discussed in this section. In Section 3, we propose our algorithm and prove the convergence for equilibrium problem (1.3). In Section 4, the last section, some numerical examples are given to show the applicability of our proposed method.

2. Preliminaries

2.1. **Basic definitions and results.** Let $\mathscr{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|\cdot\|$, respectively. The strong convergence and the weak convergence of a sequence $\{x^k\}_{k\in\mathbb{N}}$ to $x \in \mathscr{H}$ are denoted by $x^k \to x$ and $x^k \rightharpoonup x$, respectively.

For a nonempty subset Z in $\mathscr{H}$, the normal cone at $x \in Z$, denoted by $N_Z(x)$, is defined by

$$N_Z(x) = \{\mu \in \mathscr{H} : \langle \mu, y-x \rangle \leq 0, \forall y \in Z\}.$$

If Z is a nonempty, closed, and convex subset in $\mathscr{H}$, then $N_Z(x)$ is nonempty, closed, and convex. Let $g : \mathscr{H} \to \mathbb{R} \cup \{\pm\infty\}$ be an extended real-valued function. The effective domain of g is defined by $\mathrm{Dom}(g) = \{x \in \mathscr{H} \ : \ g(x) < +\infty\}$. Moreover, g is said to be proper if $g(x) > -\infty$ for all $x \in \mathscr{H}$, and there exists at least one $x \in \mathscr{H}$ such that $g(x) < +\infty$.

Let C be a nonempty and convex subset of $\mathscr{H}$. Recall that a function $g : \mathscr{H} \to \mathbb{R}$ is said to be

(a) convex on C if $g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda) g(y)$ for all $x, y \in C$ and for all $\lambda \in [0,1]$;

(b) strongly convex on C with modulus α if there exists $\alpha > 0$ such that

$$g(\lambda x+(1-\lambda)y) \leq \lambda g(x)+(1-\lambda)g(y)-\lambda(1-\lambda)\frac{\alpha}{2}\|x-y\|^2, \ \forall x,y \in C, \forall \lambda \in [0,1].$$

If the inequality in (a) is strict whenever $x \neq y$, then g is said to be strictly convex on C.

For a function $g : C \to \mathbb{R}$, the usual subdifferential, also known as Fenchel subdifferential, at a point $x \in C$ is defined by

$$\partial g(x) := \{u \in \mathscr{H} : \langle u, y-x\rangle \leq g(y)-g(x), \forall y \in C\}.$$

The elements in $\partial g(x)$ are called the subgradients of g at x. Function g is called subdifferentiable at x if $\partial g(x) \neq \emptyset$; and subdifferentiable on C if $\partial g(x) \neq \emptyset$ for all $x \in C$. It is known that if g is convex, then $\partial g(x)$ is nonempty and bounded.

Lemma 2.1. [27] *Let $f : C \to \mathbb{R}$ be convex and subdifferentiable on C. Then, x^* is a solution to the minimization problem:* $\min\{f(x) : x \in C\}$ *if and only if* $\mathbf{0} \in \partial f(x^*) + N_C(x^*)$*, where* $\mathbf{0}$ *denotes the zero vector of* $\mathscr{H}$.

Let us now recall the following definitions of lower semicontinuity and weakly lower semicontinuity. A function $\psi : \mathscr{H} \to \mathbb{R}\cup\{+\infty\}$ is said to be:

(a) lower (upper) semicontinuous at $x_0 \in \mathscr{H}$ if

$$\liminf_{k\to\infty} \psi(x^k) \geq \psi(x_0) \quad \left(\limsup_{k\to\infty} \psi(x^k) \leq \psi(x_0)\right),$$

for any sequence $\{x^k\}_{k\in\mathbb{N}}$ converging to x_0 in $\mathscr{H}$.

(b) weakly lower (upper) semicontinuous at $x_0 \in \mathscr{H}$ if

$$\liminf_{k\to\infty} \psi(x^k) \geq \psi(x_0) \quad \left(\limsup_{k\to\infty} \psi(x^k) \leq \psi(x_0)\right),$$

for any sequence $\{x^k\}_{k\in\mathbb{N}}$ that weakly converges to x_0.

The following proposition gives the relation between lower semicontinuity and weakly lower semicontinuity of a function $\psi : \mathscr{H} \to \mathbb{R}\cup\{+\infty\}$.

Proposition 2.1. [28, Proposition 5.4.4] *A convex and lower semicontinuous function* $\psi : \mathscr{H} \to \mathbb{R}\cup\{+\infty\}$ *is weakly lower semicontinuous.*

Recall that an operator $B : \mathscr{H} \to \mathscr{H}$ is said to be weakly-weakly continuous on a nonempty, closed, and convex set $C \subseteq \mathscr{H}$ if

$$\forall \{x^k\}_{k\in\mathbb{N}} \subset C, \quad x^k \rightharpoonup x \implies B(x^k) \rightharpoonup B(x).$$

Let C be a nonempty set in $\mathscr{H}$. A bifunction $f : C \times C \to \mathbb{R}$ is said to be:

(a) monotone if, for all $x,y \in C$, $f(x,y)+f(y,x) \leq 0$;

(b) pseudomonotone if, for all $x,y \in C$, $f(x,y) \geq 0 \quad \Rightarrow \quad f(y,x) \leq 0$;

(c) β-strongly pseudomonotone if there exists $\beta > 0$ such that, for all $x,y \in C$,

$$f(x,y) \geq 0 \quad \Rightarrow \quad f(y,x) \leq -\beta\|x-y\|^2.$$

It is clear that (a) $\implies$ (b) and (c) $\implies$ (b). There are strongly pseudomonotone bifunctions which are not necessarily monotone; see, for example, [29]. It is important to note that when the EP (1.1) corresponding to a strongly pseudomonotone bifunction has a solution, then the solution is unique.

Definition 2.1. A bifunction $f : C \times C \to \mathbb{R}$ is said to be τ-Hölder continuous in the first argument (respectively, second argument) if there exist constants $L > 0$ and $\tau \in (0,1]$ such that, for all $x,y,z \in C$,

$$|f(x,y) - f(z,y)| \leq L\|x-z\|^{\tau} \quad (\text{respectively, } |f(x,y) - f(x,z)| \leq L\|y-z\|^{\tau}).$$

We need the following lemmas to prove the main results of this paper.

Lemma 2.2. [30] *If $\{a_k\}$ and $\{b_k\}$ are two sequences of nonnegative real numbers such that $a_{k+1} \leq a_k + b_k$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k < +\infty$, then $\lim_{k\to\infty} a_k$ exists.*

Lemma 2.3. [31] *Let $\{a_k\}$ and $\{\lambda_k\}$ be two sequences of nonnegative real numbers such that*

$$\lim_{k\to\infty} a_k = a, \quad \lim_{k\to\infty} \lambda_k = 0, \quad \text{and} \quad \sum_{k=1}^{\infty} \lambda_k = \infty.$$

Then,

$$\lim_{k\to\infty} \frac{\sum_{n=1}^{k} \lambda_n a_n}{\sum_{n=1}^{k} \lambda_n} = a.$$

2.2. **Basics of Bregman distance.** Recall that a function $\psi : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$ is said to be Gâteaux differentiable at a point $x \in \operatorname{int}(\operatorname{Dom}(\psi))$ if the limit

$$\psi^{G}(x,d) = \lim_{t\to 0} \frac{\psi(x+td) - \psi(x)}{t} \tag{2.1}$$

exists for any $d \in \mathscr{H}$, where $\operatorname{int}(\operatorname{Dom}(\psi))$ denotes the interior of the domain of ψ, denoted by $\operatorname{Dom}(\psi)$. Recall that ψ is said to be Gâteaux differentiable if it is Gâteaux differentiable at every $x \in \operatorname{int}(\operatorname{Dom}(\psi))$. In addition, if the limit in (2.1) is uniformly attained for all d with $\|d\| = 1$, then ψ is called Fréchet differentiable at x. ψ is said to be uniformly Fréchet differentiable on a subset C of $\mathscr{H}$ if the limit is attained uniformly for all $x \in C$ and all $d \in \mathscr{H}$ with $\|d\| = 1$.

Remark 2.1. If the function $\psi : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$ is Gâteaux differentiable at x, then $\psi^{G}(x,d)$ coincides with $\langle \nabla\psi(x), d\rangle$, where $\nabla\psi(x)$ is the gradient of ψ at x.

Let $\psi : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$ be strictly convex, proper, and lower semicontinuous with closed domain $\mathscr{D} := \operatorname{Dom}(\psi) \subset \mathscr{H}$, and Gâteaux differentiable on $\operatorname{Dom}(\nabla\psi) = \operatorname{int}(\operatorname{Dom}(\psi))$, where $\nabla\psi$ denotes the gradient of ψ. The Bregman distance $d_\psi : \operatorname{Dom}(\psi) \times \operatorname{int}\big(\operatorname{Dom}(\psi)\big) \to [0,+\infty)$ is defined by

$$d_\psi(x,y) = \psi(x) - \psi(y) - \langle \nabla\psi(y), x-y\rangle, \quad \forall x,y \in \mathscr{H}. \tag{2.2}$$

The function $\psi : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$ is known as the Bregman function. The set of all Bregman functions is denoted by $\mathscr{B}$. We mention that the bifunction d_ψ does not satisfy all the properties of a metric. For example, it is not symmetric and does not satisfy the triangle inequality, and hence, it is not a metric. However, the Bregman distance satisfies the following identities; see (2.2): For all $x \in \operatorname{Dom}(\psi)$ and all $y,z \in \operatorname{int}(\operatorname{Dom}(\psi))$,

$$d_\psi(u,v) + d_\psi(v,u) = \langle \nabla\psi(u) - \nabla\psi(v), u-v\rangle, \quad \forall u,v \in \operatorname{int}(\operatorname{Dom}(\psi)), \tag{2.3}$$

and

$$d_\psi(x,y) + d_\psi(y,z) - d_\psi(x,z) = \langle \nabla\psi(z) - \nabla\psi(y), x-y\rangle. \tag{2.4}$$

Identity (2.4) is known as the three point identity. The examples of Bregman functions can be found in [26, 32, 33].

Lemma 2.4. [21, Lemma 2.1] *For every $\psi \in \mathscr{B}$, $d_\psi(x,y) \geq 0$; and $d_\psi(x,y) = 0$ if and only if $x = y$.*

For the partial level sets of $d_\psi(x,y)$, we adopt the following notations for $\alpha \in (0,+\infty)$:

$$L_1(\alpha,y) = \left\{x \in \mathscr{D} : d_\psi(x,y) \leq \alpha\right\},\ L_2(x,\alpha) = \left\{y \in \operatorname{int}\mathscr{D} : d_\psi(x,y) \leq \alpha\right\}.$$

The sets $L_1(\alpha,y)$ and $L_2(x,\alpha)$ are called left level set of $d_\psi(\cdot,y)$ and right level set of $d_\psi(x,\cdot)$, respectively.

The concept of quasi-Fejér convergence is known in the scalar case with the Euclidean norm. Let us define the concept of quasi-Fejér convergence to a set using Bregman distance. A sequence $\{x^k\} \subset \operatorname{int}\mathscr{D}$ is said to be d_ψ-quasi-Fejér convergent to a nonempty set $U \subset \operatorname{int}\mathscr{D}$ if, for all $k \in \mathbb{N}$, $d_\psi(x,x^{k+1}) \leq d_\psi(x,x^k) + \gamma_k$, where γ_k is a sequence of nonnegative real numbers such that $\sum_{k=1}^\infty \gamma_k < \infty$.

We now provide the conditions under which a d_ψ-quasi-Fejér convergent sequence is bounded.

Lemma 2.5. *Let $\psi : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper, strictly convex, lower semicontinuous, and Gâteaux differentiable function on $\operatorname{int}(\operatorname{Dom}(\psi))$. Suppose that $U \subset \operatorname{int}(\operatorname{Dom}(\psi))$ is a nonempty set and $\{x^k\} \subset \operatorname{int}(\operatorname{Dom}(\psi))$ is d_ψ-quasi-Fejér convergent to U. If the right level set $L_2(x,\alpha)$ is bounded for all $x \in \operatorname{Dom}(\psi)$ and for all $\alpha \geq 0$, then $\{x^k\}$ is bounded.*

Proof. Let $z \in U$ be arbitrary fixed. Since $\{x^k\} \subset \operatorname{int}(\operatorname{Dom}(\psi))$ is d_ψ-quasi-Fejér convergent to U, then there exists a sequence $\{\gamma_k\}$ of nonnegative real numbers with $\sum_{k=1}^\infty \gamma_k < \infty$ such that

$$\begin{aligned} d_\psi(z,x^{k+1}) &\leq d_\psi(z,x^k) + \gamma_k \\ &\leq d_\psi(z,x^{k-1}) + \sum_{n=k-1}^{k} \gamma_n \\ &\cdots \\ &\leq d_\psi(z,x^1) + \sum_{n=1}^{k} \gamma_n \\ &\leq d_\psi(z,x^1) + s, \quad \forall k \in \mathbb{N}, \end{aligned}$$

where $s = \sum_{n=1}^\infty \gamma_n$. Therefore, $\{x^k\}$ is bounded as $x^k \in L_2(z,\alpha)$ for all $k \in \mathbb{N}$ with $\alpha = d_\psi(z,x^1) + s$. $\square$

As mentioned by Butnariu and Resmerita [34] that an intrinsic feature of the Bregman distance in designing iterative algorithms consists of ensuring that the sequences generated by these algorithms lie in the domain of a convex function ψ. The convergence can be guaranteed if we have

$$\lim_{k\to\infty} d_\psi(x^k,x) = 0 \implies \lim_{k\to\infty} \|x^k - x\| = 0. \tag{2.5}$$

Butnariu et al. [35] pointed out that a special type of functions which satisfy (2.5) are known as totally convex functions.

Let $\psi : \mathscr{H} \to (-\infty,+\infty]$ be a proper convex function. The modulus of total convexity of ψ at $x \in \operatorname{int}(\operatorname{Dom}(\psi))$ is the function $\nu_\psi(x,\cdot) : [0,+\infty) \to [0,+\infty)$ defined by

$$\nu_\psi(x,t) = \inf\left\{d_\psi(y,x) : y \in \operatorname{Dom}(\psi),\ \|y - x\| = t\right\}.$$

ψ is called totally convex at $x \in \mathrm{int}(\mathrm{Dom}(\psi))$ if $\nu_\psi(x,t) > 0$ for all $t \in (0,\infty)$. It is called totally convex when it is totally convex at every point $x \in \mathrm{int}(\mathrm{Dom}(\psi))$. The following result shows that the total convexity is a strong form of strict convexity.

Proposition 2.2. [24]

(a) *If ψ is totally convex at any $x \in \mathrm{int}(\mathrm{dom}(\psi))$, then it is strictly convex on the set* $\mathrm{int}(\mathrm{dom}(\psi))$.
(b) *If $\dim \mathscr{H} < +\infty$, $\mathrm{Dom}(\psi)$ is closed and the function ψ restricted to $\mathrm{Dom}(\psi)$ is continuous and strictly convex, then ψ is totally convex at any $x \in \mathrm{int}(\mathrm{Dom}(\psi))$.*

Examples of totally convex functions can be found, for instance, in [24, 34]. Resmerita [36] demonstrated that the totally convex functions are the only convex functions which satisfy (2.5).

The following proposition regarding the characterization of total convexity is crucial to the proof of our main results.

Proposition 2.3. [34] *Let $\mathscr{H}$ be a real Hilbert space and $\psi : \mathscr{H} \to (-\infty,+\infty]$ be a convex function. If $x \in \mathrm{int}(\mathrm{Dom}(\psi))$, then the following statements are equivalent:*

(a) *ψ is totally convex at x.*
(b) $\lim_{k\to\infty} d_\psi(y^k,x) = 0 \implies \lim_{k\to\infty} \|y^k - x\| = 0$ *for any sequence* $\{y^k\}_{k\in\mathbb{N}} \subseteq \mathrm{Dom}(\psi)$.
(c) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $y \in \mathrm{Dom}(\psi)$ and $d_\psi(y,x) \le \delta$, then* $\|x-y\| \le \varepsilon$.

Proposition 2.4. [23, Proposition 3.2] *Let $\psi : \mathscr{H} \to (-\infty,+\infty]$ be proper, closed, and convex with $\mathrm{int}\big(\mathrm{Dom}(\psi)\big) \neq \emptyset$. If ψ is Gâteaux differentiable on $\mathrm{int}\,(\mathrm{Dom}(\psi))$, then*

$$\{y^k\} \subseteq \mathrm{int}\big(\mathrm{Dom}(\psi)\big),\ y^k \to y \in \mathrm{int}\big(\mathrm{Dom}(\psi)\big) \implies d_\psi(y,y^k) \to 0.$$

Proposition 2.5. [9] *Let $\psi : \mathscr{H} \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. If ψ is strongly convex with modulus β on a convex set $C \subset \mathrm{int}\big(\mathrm{Dom}(\psi)\big)$, then $d_\psi(x,y) \ge \frac{\beta}{2}\|x-y\|^2$ for all $x,y \in C$.*

Definition 2.2. [27] A proper convex function $\psi : \mathscr{H} \to (-\infty,+\infty]$ is called essentially strictly convex if it is strictly convex on every convex subset of $\mathrm{Dom}(\nabla\psi)$.

Theorem 2.1. [23, Theorem 3.7] *Let $\psi : \mathscr{H} \to (-\infty,+\infty]$ be proper, convex, and lower semicontinuous, Gâteaux differentiable on $\mathrm{int}(\mathrm{Dom}(\psi)) \neq \emptyset$ and essentially strictly convex. Then, for any $y \in \mathrm{int}(\mathrm{Dom}(\psi))$, $d_\psi(\cdot,y)$ is proper, convex, and lower semicontinuous on $\mathscr{H}$, differentiable on $\mathrm{int}(\mathrm{Dom}(\psi))$, and essentially strictly convex.*

Now we state the following lemma regarding the characterization of strong convexity of function $d_\psi(\cdot,y)$ for every $y \in \mathrm{int}\mathscr{D}$, where $\mathscr{D} := \mathrm{Dom}(\psi)$.

Lemma 2.6. [7, Lemma 3.8] *Let $\psi : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$ be proper, α-strongly convex, and Gâteaux differentiable on $\mathrm{int}\mathscr{D}$. Then, the function $d_\psi(\cdot,y)$ is also α-strongly convex for every $y \in \mathrm{int}\mathscr{D}$.*

Throughout the paper, unless otherwise specified, we consider the following assumptions on the Bregman function ψ.

Assumption 2.1. Let $\psi : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semicontinuous function with closed domain $\mathscr{D} = \mathrm{Dom}(\psi)$ and uniformly Fréchet differentiable with its weakly-weakly continuous gradient $\nabla\psi$ on $\mathrm{Dom}(\nabla\psi) = \mathrm{int}\mathscr{D}$. Moreover, let C be a nonempty, closed, and convex set in $\mathscr{H}$ such that $C \subset \mathrm{int}\mathscr{D}$.

(A1) ψ is β-strongly convex on C with modulus $\beta > 0$. Then, $d_\psi(x,y) \geq \frac{\beta}{2}\|x-y\|^2$ for all $x, y \in C$.
(A2) The right level set $L_2(x,\alpha)$ is bounded for all $x \in \mathrm{Dom}(\psi)$ and for all $\alpha \geq 0$.
(A3) The function ψ is totally convex.
(A4) The gradient of the function ψ is Lipschitz continuous with constant $L > 0$, that is,

$$\|\nabla\psi(x) - \nabla\psi(y)\| \leq L\|x-y\|, \quad \forall x, y \in \mathrm{int}\mathscr{D}.$$

If ψ satisfies assumption (A1), then it is strictly convex on every convex subset of $\mathrm{int}\mathscr{D}$. From Theorem 2.1 and Lemma 2.6, it follows that $d_\psi(\cdot, y)$ is proper, β-strongly convex, and lower semicontinuous for every $y \in \mathrm{int}\mathscr{D}$. Moreover, ψ is lower semicontinuous and by (A1), it is convex. Hence, by Proposition 2.1, ψ is weakly lower semicontinuous.

The Bregman projection of $x \in \mathrm{int}\mathscr{D}$ onto a nonempty, closed, and convex set $C \subset \mathrm{Dom}(\psi)$ with respect to function ψ is defined by $P_C^\psi(x) = \mathrm{argmin}\left\{d_\psi(y,x) : y \in C\right\}$. Since $\mathscr{H}$ is reflexive, ψ is totally convex, and Gâteaux differentiable on $\mathrm{int}\mathscr{D}$ and lower semicontinuous, then there exists a unique minimizer of the function $d_\psi(\cdot, x)$ in C for every $x \in \mathrm{int}\mathscr{D}$ (see [24, Proposition 2.1.5]).

The Bregman projection mapping $P_C^\psi(\cdot)$ is characterized by the following proposition.

Proposition 2.6. [37] *Assume that the function $\psi : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$ satisfies Assumption 2.1 and $C \subset \mathscr{H}$ is a nonempty, closed, and convex set such that $C \cap \overline{\mathrm{int}\mathscr{D}} \neq \emptyset$, where $\overline{\mathrm{int}\mathscr{D}}$ denotes the closure of $\mathrm{int}\mathscr{D}$. Let $y \in \mathrm{int}\mathscr{D}$ and $P_C^\psi(y) \in \mathrm{int}\mathscr{D}$, Then, for any $z \in C \cap \overline{\mathrm{int}\mathscr{D}}$,*

$$d_\psi\big(P_C^\psi(y), y\big) \leq d_\psi(z,y) - d_\psi\big(z, P_C^\psi(y)\big).$$

3. Iterative Algorithms and their Convergence Analysis

For a bifunction $f : C \times C \to \mathbb{R}$, we assume that $f(x,y) = f_1(x,y) + f_2(x,y)$ for every $x, y \in C$. We adopt the following assumptions on f, f_1, and f_2 to study equilibrium problem (1.1):

Assumption 3.1. (B1) f is pseudomonotone, and $f(\cdot, y)$ is concave, and weakly upper semicontinuous for every $y \in C$.
(B2) For each $x \in C$, $f_i(x,\cdot)$ are convex, lower semicontinuous, and $f_i(x,x) = 0$, $i = 1,2$.
(B3) $S(f,C)$, the solution set, is nonempty.

We state following proposition which is used in the proof of our main results.

Proposition 3.1. [19, Proposition 2] *Under the Assumption 3.1,*

(a) $S(f,C) = \{y \in C : f(x,y) \leq 0, \ \forall x \in C\}$,
(b) *$S(f,C)$ is closed and convex.*

If $\psi(x) = \|x\|^2$, then $d_\psi(y, x^k) = \|y - x^k\|^2$ and $d_\psi(y, y^k) = \|y - y^k\|^2$. Therefore, for this particular choice of Bregman function, Algorithm 1 reduces to the one proposed in [19, Algorithm 1]. Moreover, since, for every $x \in C$, $f_i(x,\cdot)$, $i = 1,2$, are proper, convex, and lower semicontinuous, and $d_\psi(\cdot, x)$ is proper, lower semicontinuous, and β-strongly convex, for every $\lambda > 0$,

Algorithm 1 Sequential Splitting Algorithm with Bregman Distance

Step 0: Choose the sequence $\{\lambda_k\} \subset (0,1)$. Take $x^0 \in \mathscr{H}$ and set $z^0 = x^0$, $k = 0$.
Step 1: Given x^k, compute y^k, x^{k+1} and z^{k+1} as

$$y^k := \underset{y\in C}{\operatorname{argmin}} \left\{ \lambda_k f_1\left(x^k, y\right) + \frac{1}{2} d_\psi(y, x^k) \right\}, \tag{3.1}$$

$$x^{k+1} := \underset{y\in C}{\operatorname{argmin}} \left\{ \lambda_k f_2\left(y^k, y\right) + \frac{1}{2} d_\psi(y, y^k) \right\}. \tag{3.2}$$

and

$$z^{k+1} = \frac{\sum_{n=1}^{k+1} \lambda_n x^n}{\sum_{n=1}^{k+1} \lambda_n}.$$

Step 2: Update $k := k+1$ and go to **Step 1**.

we have that $\lambda f_i(x,\cdot) + \frac{1}{2} d_\psi(\cdot, x)$, $i = 1,2$, are proper, lower semicontinuous, and β-strongly convex. Therefore, for every $k \in \mathbb{N}$, y^k in (3.1) and x^{k+1} in (3.2) are uniquely determined (see, e.g., [38, Corollary 11.17]). Thus Algorithm 1 is well-defined.

To prove the convergence of the sequence generated by Algorithm 1, we need the following lemma.

Lemma 3.1. *Let C be a nonempty, convex, and closed subset of $\mathscr{H}$ and $f : C \times C \to \mathbb{R}$ be a pseudomonotone bifunction such that $f := f_1 + f_2$, where f, f_1, and f_2 satisfy Assumption 3.1. Further, assume that f_1 is τ_1-Hölder continuous in the first or second argument while f_2 is τ_2-Hölder continuous in the first one. Then, the sequences $\{x^k\}$ and $\{y^k\}$ generated by Algorithm 1 have the following properties:*

(S1) *There exists $M > 0$ such that*

$$d_\psi(x^k, y^k) \le M \lambda_k^{\frac{2}{2-\tau_1}} \quad \textit{and} \quad d_\psi(x^{k+1}, y^k) \le M \lambda_k^{\frac{2}{2-\tau_2}}, \quad \forall n \in \mathbb{N};$$

(S2) *If $\tau := \min\{\tau_1, \tau_2\}$, then there exists $L > 0$ such that*

$$d_\psi(x, x^{k+1}) \le d_\psi(x, x^k) + 2\lambda_k f(x^k, x) + L \lambda_k^{\frac{2}{2-\tau}}, \quad \forall x \in C.$$

Proof. (S1) Since y^k is the unique solution to (3.1), we see by Lemma 2.1 that there exist $u_k \in \partial f_1(x^k,\cdot)(y^k)$ and $q^k \in N_C(y^k)$ such that $0 = \lambda_k u_k + \frac{1}{2}\left(\nabla\psi(y^k) - \nabla\psi(x^k)\right) + q^k$, that is,

$$q^k = \frac{1}{2}\left(\nabla\psi(x^k) - \nabla\psi(y^k)\right) - \lambda_k u_k.$$

From the definition of $N_C(y^k)$, we have

$$\frac{1}{2}\left\langle \left(\nabla\psi(x^k) - \nabla\psi(y^k)\right), y - y^k \right\rangle \le \lambda_k \left\langle u_k, y - y^k \right\rangle, \quad \forall y \in C. \tag{3.3}$$

Since $u_k \in \partial f_1(x^k,\cdot)(y^k)$, we have

$$\left\langle u_k, y - y^k \right\rangle \le f_1(x^k, y) - f_1(x^k, y^k), \quad \forall y \in C. \tag{3.4}$$

Combining (3.3) and (3.4) yields

$$\left\langle \nabla\psi(x^k)-\nabla\psi(y^k), y-y^k\right\rangle \le 2\lambda_k\left(f_1(x^k,y)-f_1(x^k,y^k)\right), \quad \forall y\in C,$$

equivalently, it follows from three point identity (2.4) that

$$d_\psi(y,y^k)+d_\psi(y^k,x^k)-d_\psi(y,x^k) \le 2\lambda_k\left(f_1(x^k,y)-f_1(x^k,y^k)\right), \quad \forall y\in C. \tag{3.5}$$

Taking $y=x^k\in C$ in (3.5), we obtain

$$d_\psi(x^k,y^k)+d_\psi(y^k,x^k) \le -2\lambda_k f_1(x^k,y^k) \le 2\lambda_k\left|f_1(x^k,y^k)\right|. \tag{3.6}$$

Since f_1 is τ_1-Hölder continuous in the first or second argument and $f_1(x,x)=0$ for all $x\in C$, then there exists a constant $Q_1>0$ such that

$$\left|f_1(x^k,y^k)\right| \le Q_1\|x^k-y^k\|^{\tau_1} \le Q_1\left(\frac{2}{\beta}\right)^{\frac{\tau_1}{2}}\left[d_\psi(x^k,y^k)\right]^{\frac{\tau_1}{2}}, \tag{3.7}$$

where the last inequality follows from Assumption 2.1 (A1). Combining (3.6) and (3.7), we obtain

$$d_\psi(x^k,y^k) \le 2\lambda_k Q_1\left(\frac{2}{\beta}\right)^{\frac{\tau_1}{2}}\left[d_\psi(x^k,y^k)\right]^{\frac{\tau_1}{2}},$$

that is,

$$d_\psi(x^k,y^k) \le (2Q_1\lambda_k)^{\frac{2}{2-\tau_1}}\left(\frac{2}{\beta}\right)^{\frac{\tau_1}{2-\tau_1}}. \tag{3.8}$$

Since x^{k+1} is the unique solution to (3.2), we also have

$$d_\psi(y,x^{k+1})+d_\psi(x^{k+1},y^k)-d_\psi(y,y^k) \le 2\lambda_k\left(f_2(y^k,y)-f_2(y^k,x^{k+1})\right), \quad \forall y\in C. \tag{3.9}$$

Letting $y=y^k\in C$ in (3.9) and using the assumption that $f_2(x,x)=0$ for all $x\in C$, we arrive at

$$d_\psi(y^k,x^{k+1})+d_\psi(x^{k+1},y^k) \le -2\lambda_k f_2(y^k,x^{k+1}) \le 2\lambda_k|f_2(y^k,x^{k+1})|. \tag{3.10}$$

Using τ_2-Hölder continuity in the first argument of f_2 and proceeding as in the case of f_1, we obtain

$$d_\psi(x^{k+1},y^k) \le (2Q_2\lambda_k)^{\frac{2}{2-\tau_2}}\left(\frac{2}{\beta}\right)^{\frac{\tau_2}{2-\tau_2}}. \tag{3.11}$$

Taking

$$M := \max\left\{(2Q_1)^{\frac{2}{2-\tau_1}}\left(\frac{2}{\beta}\right)^{\frac{\tau_1}{2-\tau_1}}, (2Q_2)^{\frac{2}{2-\tau_2}}\left(\frac{2}{\beta}\right)^{\frac{\tau_2}{2-\tau_2}}\right\},$$

we find the desired results from (3.8) and (3.11), respectively.

(S2) By adding (3.5) and (3.9), and using τ_i-Hölder continuity of f_i, $i=1,2$, we conclude

$$
\begin{aligned}
&d_\psi(y,x^{k+1})\\
&\le d_\psi(y,x^k)-d_\psi(x^{k+1},y^k)-d_\psi(y^k,x^k)\\
&\qquad+2\lambda_k\left(f_1(x^k,y)+f_2(y^k,y)-f_1(x^k,y^k)-f_2(y^k,x^{k+1})\right)\\
&\le d_\psi(y,x^k)+2\lambda_k\left(f(x^k,y)+\left|f_2(y^k,y)-f_2(x^k,y)\right|+\left|f_1(x^k,y^k)\right|+\left|f_2(y^k,x^{k+1})\right|\right)\\
&\le d_\psi(y,x^k)+2\lambda_k\left(f(x^k,y)+Q_2\|y^k-x^k\|^{\tau_2}+Q_1\|x^k-y^k\|^{\tau_1}+Q_2\|y^k-x^{k+1}\|^{\tau_2}\right).
\end{aligned}
\tag{3.12}
$$

From Assumption 2.1 (A1), (3.8) and (3.11), for all $y\in C$, we have

$$
\begin{aligned}
&2\lambda_k\left(f(x^k,y)+Q_2\|y^k-x^k\|^{\tau_2}+Q_1\|x^k-y^k\|^{\tau_1}+Q_2\|y^k-x^{k+1}\|^{\tau_2}\right)\\
&\le 2\lambda_k f(x^k,y)+2\lambda_k\left(Q_2\left(\frac{2}{\beta}\right)^{\frac{\tau_2}{2}}\left[d_\psi(x^k,y^k)\right]^{\frac{\tau_2}{2}}+Q_1\left(\frac{2}{\beta}\right)^{\frac{\tau_1}{2}}\left[d_\psi(x^k,y^k)\right]^{\frac{\tau_1}{2}}\right.\\
&\qquad\left.+Q_2\left(\frac{2}{\beta}\right)^{\frac{\tau_2}{2}}\left[d_\psi(x^{k+1},y^k)\right]^{\frac{\tau_2}{2}}\right)\\
&\le 2\lambda_k f(x^k,y)+2\lambda_k\left(Q_2\left(\frac{2}{\beta}\right)^{\frac{\tau_2}{2}}(2Q_1\lambda_k)^{\frac{\tau_2}{2-\tau_1}}\left(\frac{2}{\beta}\right)^{\frac{\tau_1\tau_2}{2(2-\tau_1)}}\right.\\
&\qquad\left.+Q_1\left(\frac{2}{\beta}\right)^{\frac{\tau_1}{2}}(2Q_1\lambda_k)^{\frac{\tau_1}{2-\tau_1}}\left(\frac{2}{\beta}\right)^{\frac{\tau_1^2}{2(2-\tau_1)}}\right)+2\lambda_kQ_2\left(\frac{2}{\beta}\right)^{\frac{\tau_2}{2}}(2Q_2\lambda_k)^{\frac{\tau_2}{2-\tau_2}}\left(\frac{2}{\beta}\right)^{\frac{\tau_2^2}{2(2-\tau_2)}}\\
&\le 2\lambda_k f(x^k,y)+2\lambda_k\left(Q_2Q_1^{\frac{\tau_2}{2-\tau_1}}\left(\frac{4}{\beta}\right)^{\frac{\tau_2}{2-\tau_1}}\lambda_k^{\frac{\tau}{2-\tau}}+Q_1^{\frac{2}{2-\tau_1}}\left(\frac{4}{\beta}\right)^{\frac{\tau_1}{2-\tau_1}}\lambda_k^{\frac{\tau}{2-\tau}}\right)\\
&\qquad+2\lambda_kQ_2^{\frac{2}{2-\tau_2}}\left(\frac{4}{\beta}\right)^{\frac{\tau_2}{2-\tau_2}}\lambda_k^{\frac{\tau}{2-\tau}},\quad\text{where}\quad\tau=\min\{\tau_1,\tau_2\}.
\end{aligned}
$$

From (3.12), we obtain

$$
d_\psi(y,x^{k+1})\le d_\psi(y,x^k)+2\lambda_k f(x^k,y)+L\lambda_k^{\frac{2}{2-\tau}},\quad\forall y\in C,
\tag{3.13}
$$

where $L=2\left(Q_2Q_1^{\frac{\tau_2}{2-\tau_1}}\left(\frac{4}{\beta}\right)^{\frac{\tau_2}{2-\tau_1}}+Q_1^{\frac{2}{2-\tau_1}}\left(\frac{4}{\beta}\right)^{\frac{\tau_1}{2-\tau_1}}+Q_2^{\frac{2}{2-\tau_2}}\left(\frac{4}{\beta}\right)^{\frac{\tau_2}{2-\tau_2}}\right)$. This completes the proof. □

We are now ready to prove the weak convergence of the sequence generated by Algorithm 1.

Theorem 3.1. *Let C be a nonempty, convex, and closed subset of $\mathscr{H}$, and let $f:C\times C\to\mathbb{R}$ satisfy all the conditions of Lemma 3.1. Suppose that $\{\lambda_k\}$ satisfies $\sum_{k=1}^\infty\lambda_k=\infty$ and $\sum_{k=1}^\infty\lambda_k^{\frac{2}{2-\tau}}<\infty$, where $\tau=\min\{\tau_1,\tau_2\}$. Then, the sequence $\{z^k\}$ generated by Algorithm 1 converges weakly to a solution of EP (1.1).*

Proof. We divide the proof into four steps.

STEP 1. We prove the boundedness of $\{x^k\}$ and $\{z^k\}$.

Since f is pseudomonotone, for any $\bar{x} \in S(f,C)$, we have $f(x^k,\bar{x}) \leq 0$ for all $k \geq 1$. Taking $y = \bar{x} \in S(f,C) \subset C$ in (3.13), we see that

$$d_\psi(\bar{x},x^{k+1}) \leq d_\psi(\bar{x},x^k) + L\lambda_k^{\frac{2}{2-\tau}}, \quad \forall k \geq 1.$$

Thus $\{x^k\}$ is d_ψ-quasi-Fejér convergent to $S(f,C)$. Hence, by Assumption 2.1 (A2) and Lemma 2.5, we conclude that $\{x^k\}$ is bounded. Then, there exists $N > 0$ such that $\|x^k\| \leq N$ for all $k \geq 1$. Observe that the definition of $\{z^k\}$ yields $\|z^k\| \leq \frac{\sum_{n=1}^k \lambda_n \|x^n\|}{\sum_{n=1}^k \lambda_n} \leq N$, which implies that $\{z^k\}$ is bounded. Hence, there exists a subsequence $\{z^{k_i}\} \subset \{z^k\}$ such that $z^{k_i} \rightharpoonup \bar{z} \in C$.

STEP 2. We show that $\bar{z} \in S(f,C)$.

We re-write (3.13) as $d_\psi(y,x^{k+1}) - d_\psi(y,x^k) \leq 2\lambda_k f(x^k,y) + L\lambda_k^{\frac{2}{2-\tau}}$ for all $y \in C$ and $k \in \mathbb{N}$. Taking $k = 1,\ldots,k_i$ and summing all the inequalities, we obtain

$$d_\psi(y,x^{k_i+1}) - d_\psi(y,x^1) \leq 2\sum_{n=1}^{k_i} \lambda_n f(x^n,y) + L\sum_{n=1}^{k_i} \lambda_n^{\frac{2}{2-\tau}}, \quad \forall y \in C.$$

By using the concavity of $f(\cdot,y)$, we have

$$\begin{aligned}
\frac{d_\psi(y,x^{k_i+1}) - d_\psi(y,x^1)}{\sum_{n=1}^{k_i} \lambda_n} &\leq 2\frac{\sum_{n=1}^{k_i} \lambda_n f(x^n,y)}{\sum_{n=1}^{k_i} \lambda_n} + L\frac{\sum_{n=1}^{k_i} \lambda_n^{\frac{2}{2-\tau}}}{\sum_{n=1}^{k_i} \lambda_n} \\
&\leq 2f\left(\frac{\sum_{n=1}^{k_i} \lambda_n x^n}{\sum_{n=1}^{k_i} \lambda_n}, y\right) + L\frac{\sum_{n=1}^{k_i} \lambda_n^{\frac{2}{2-\tau}}}{\sum_{n=1}^{k_i} \lambda_n} = 2f\left(z^{k_i},y\right) + L\frac{\sum_{n=1}^{k_i} \lambda_n^{\frac{2}{2-\tau}}}{\sum_{n=1}^{k_i} \lambda_n}.
\end{aligned}$$

Since $\sum_{k=1}^\infty \lambda_k = \infty$, $\sum_{k=1}^\infty \lambda_k^{\frac{2}{2-\tau}} < \infty$, $z^{k_i} \rightharpoonup \bar{z}$, and $f(\cdot,y)$ is weakly upper semicontinuous, we have $f(\bar{z},y) \geq \limsup_{i\to\infty} f\left(z^{k_i},y\right) \geq 0$ for all $y \in C$. Thus $\bar{z} \in S(f,C)$. We take Bregman projection of x^k on the nonempty, closed, and convex set $S(f,C)$ for each $k \in \mathbb{N}$ and define $u^k = P^\psi_{S(f,C)}(x^k)$. Then, u^k is uniquely determined for each x^k. If we show that $u^k \to \bar{z}$, then all the weak limit points of the sequence $\{z^k\}$ are equal to $\bar{z}$ and the whole sequence $\{z^k\}$ weakly converges to $\bar{z}$.

STEP 3. We prove the convergence of the sequence $\{u^k\}$.

Since $u^k \in S(f,C)$ and f is pseudomonotone, we find from (3.13) that

$$d_\psi(u^k,x^{k+p}) \leq d_\psi(u^k,x^k) + L\sum_{n=k}^{p-1} \lambda_n^{\frac{2}{2-\tau}}, \quad \forall k \in \mathbb{N},\ p \geq 1. \tag{3.14}$$

Since $u^{k+p} = P^\psi_{S(f,C)}(x^{k+p}) = \operatorname{argmin}\left\{d_\psi(y,x^{k+p}) : y \in S(f,C)\right\}$, we have

$$d_\psi(u^{k+p},x^{k+p}) \leq d_\psi\left(\frac{1}{2}(u^k+u^{k+p}),x^{k+p}\right) \leq \frac{1}{2}d_\psi(u^k,x^{k+p}) + \frac{1}{2}d_\psi(u^{k+p},x^{k+p}),$$

that is,

$$d_\psi(u^{k+p},x^{k+p}) \leq d_\psi(u^k,x^{k+p}), \quad \forall k \in \mathbb{N},\ p \geq 1. \tag{3.15}$$

By combining (3.14) and (3.15), we obtain

$$d_\psi(u^{k+p}, x^{k+p}) \le d_\psi(u^k, x^k) + L\sum_{n=k}^{p-1} \lambda_n^{\frac{2}{2-\tau}}, \quad \forall k \in \mathbb{N},\ p \ge 1. \tag{3.16}$$

Taking $p = 1$ in (3.16), we obtain $d_\psi(u^{k+1}, x^{k+1}) \le d_\psi(u^k, x^k) + L\lambda_k^{\frac{2}{2-\tau}}$, and hence, by Lemma 2.2, $\lim_{k\to\infty} d_\psi(u^k, x^k)$ exists. Since u^{k+p} is the Bregman projection of x^{k+p} onto $S(f,C)$, Proposition 2.6 implies

$$d_\psi(u^{k+p}, x^{k+p}) \le d_\psi(z, x^{k+p}) - d_\psi(z, u^{k+p}), \quad \forall z \in S(f,C).$$

Taking $z = u^k \in S(f,C)$ in the above inequality and using (3.14), we obtain

$$\begin{aligned}
0 \le d_\psi(u^k, u^{k+p}) &\le d_\psi(u^k, x^{k+p}) - d_\psi(u^{k+p}, x^{k+p}) \\
&\le d_\psi(u^k, x^k) + L\sum_{n=k}^{p-1} \lambda_n^{\frac{2}{2-\tau}} - d_\psi(u^{k+p}, x^{k+p}) \\
&\le d_\psi(u^k, x^k) - d_\psi(u^{k+p}, x^{k+p}) + L\sum_{n=k}^{\infty} \lambda_n^{\frac{2}{2-\tau}}, \quad \forall k \in \mathbb{N},\ p \ge 1.
\end{aligned}$$

Since $\lim_{k\to\infty} d_\psi(u^k, x^k)$ exists and $\lim_{k\to\infty} \sum_{n=k}^{\infty} \lambda_n^{\frac{2}{2-\tau}} = 0$, we conclude that $\lim_{k\to\infty} d_\psi(u^k, u^{k+p}) = 0$ for all $p \ge 1$. Since the Bregman function ψ is totally convex, by using Proposition 2.3 (c), we obtain $\lim_{k\to\infty} \|u^{k+p} - u^k\| = 0$ for all $p \ge 1$, which shows that $\{u^k\}$ is a Cauchy sequence in a complete space, and hence converges to $\hat{z}$ for some $\hat{z} \in S(f,C)$.

STEP 4: We prove $\hat{z} = \bar{z}$.

Since $u^k = P^{\psi}_{S(f,C)}(x^k)$, by Proposition 2.6, we have $d_\psi(y, u^k) + d_\psi(u^k, x^k) \le d_\psi(y, x^k)$ for all $y \in S(f,C)$. Since $\bar{x}, \hat{z} \in S(f,C)$, we have

$$d_\psi(\bar{x}, u^k) + d_\psi(u^k, x^k) \le d_\psi(\bar{x}, x^k),\ d_\psi(\hat{z}, u^k) + d_\psi(u^k, x^k) \le d_\psi(\hat{z}, x^k).$$

In view of (A1) and (A4) of Assumption 2.1, we obtain by choosing $\beta \ge 2L$ that

$$\begin{aligned}
d_\psi(x^k, \hat{z}) &\le \left\langle \nabla\psi(x^k) - \nabla\psi(\hat{z}), x^k - \hat{z} \right\rangle - d_\psi(\hat{z}, u^k) - d_\psi(u^k, x^k) \\
&\le \|\nabla\psi(x^k) - \nabla\psi(\hat{z})\|\|x^k - \hat{z}\| - d_\psi(\hat{z}, u^k) - d_\psi(u^k, x^k) \\
&\le L\|x^k - \hat{z}\|^2 - d_\psi(\hat{z}, u^k) - d_\psi(u^k, x^k) \\
&\le L\|x^k - u^k + u^k - \hat{z}\|^2 - d_\psi(\hat{z}, u^k) - d_\psi(u^k, x^k) \\
&\le L\|x^k - u^k\|^2 + L\|u^k - \hat{z}\|^2 + 2L\|x^k - u^k\|\|u^k - \hat{z}\| - d_\psi(\hat{z}, u^k) - d_\psi(u^k, x^k) \\
&\le \left(\frac{2L}{\beta} - 1\right) d_\psi(u^k, x^k) + \left(\frac{2L}{\beta} - 1\right) d_\psi(\hat{z}, u^k) + 2L\|x^k - u^k\|\|u^k - \hat{z}\| \\
&\le 2L\|x^k - u^k\|\|u^k - \hat{z}\| \\
&\le 2L\sqrt{\frac{2}{\beta} d_\psi(u^k, x^k)}\sqrt{\frac{2}{\beta} d_\psi(\hat{z}, u^k)} \\
&\le \frac{4LM}{\beta} d_\psi(\hat{z}, u^k),
\end{aligned}$$

where $M^2 = \sup\left\{d_\psi(u^k, x^k) : k \geq 1\right\} < +\infty$. Writing the above inequality with an index n and summing from $n = 1$ to k_i, we obtain

$$\sum_{n=1}^{k_i} \lambda_n d_\psi(x^n, \hat{z}) \leq \frac{4LM}{\beta} \sum_{n=1}^{k_i} \lambda_n d_\psi(\hat{z}, u^n).$$

Therefore,

$$d_\psi(z^{k_i}, \hat{z}) = d_\psi\left(\frac{\sum_{n=1}^{k_i} \lambda_n x^n}{\sum_{n=1}^{k_i} \lambda_n}, \hat{z}\right) \leq \frac{\sum_{n=1}^{k_i} \lambda_n d_\psi(x^n, \hat{z})}{\sum_{n=1}^{k_i} \lambda_n} \leq \frac{4LM}{\beta} \frac{\sum_{n=1}^{k_i} \lambda_n d_\psi(\hat{z}, u^n)}{\sum_{n=1}^{k_i} \lambda_n}. \tag{3.17}$$

Since $u^k \to \hat{z}$, we have $d_\psi(\hat{z}, u^n) \to d_\psi(\hat{z}, \hat{z})$, that is, $d_\psi(\hat{z}, u^n) \to 0$. Since $\sum_{n=1}^{\infty} \lambda_n = \infty$, by applying Lemma 2.3 with $a_k = d_\psi(\hat{z}, u^k)$, we obtain from inequality (3.17) that

$$\lim_{i\to\infty} d_\psi(z^{k_i}, \hat{z}) \leq 0, \quad \text{that is,} \quad d_\psi(\bar{z}, \hat{z}) \leq 0,$$

which shows $\bar{z} = \hat{z}$, and this completes the proof. □

When $f_2 \equiv 0$ in Algorithm 1, we obtain the following algorithm.

Algorithm 2 One-projection Sequential Algorithm with Bregman Distance

Step 0: Choose $x^0 \in \mathscr{H}$.
Step 1: Given x^k, compute x^{k+1}, z^{k+1} as

$$x^{k+1} := \underset{y\in C}{\operatorname{argmin}}\left\{\lambda_k f(x^k, y) + \frac{1}{2} d_\psi(y, x^k)\right\}, \; z^{k+1} := \frac{\sum_{n=1}^{k+1} \lambda_k x^k}{\sum_{n=1}^{k+1} \lambda_k}.$$

We derive the following corollary from Theorem 3.1 which provides the weak convergence of the sequence generated by the Algorithm 2 to a solution of the EP (1.1).

Corollary 3.1. *For a nonempty, convex, and closed subset C of $\mathscr{H}$, suppose that $f : C \times C \to \mathbb{R}$ satisfies the following conditions:*

(i) *f is pseudomonotone and τ-Hölder continuous in the first or second argument;*
(ii) *For each $x \in C$, $f(x, \cdot)$ is convex, lower semicontinuous, $f(\cdot, x)$ is concave and weakly upper semicontinuous, and $f(x, x) = 0$;*
(iii) *$S(f, C) \neq \emptyset$.*

Assume that the sequence $\{\lambda_k\}$ of positive numbers satisfies $\sum_{k=1}^{\infty} \lambda_k = \infty$ and $\sum_{k=1}^{\infty} \lambda_k^{\frac{2}{2-\tau}} < \infty$. Then, the sequence $\{z^k\}$ generated by Algorithm 2 converges weakly to a point in $S(f, C)$.

If we choose $\psi(x) = \|x\|^2$, then Algorithm 2 and Corollary 3.1 reduces to the corollary given in [19, Corollary 1].

4. Numerical Examples

In this section, we give an example to show the efficiency of our method. We also compare our algorithm for different choices of Bregman functions. The computational programming were performed in MATLAB R2024a running on Acer Swift 3 PC with 11th Gen Intel(R) Core(TM) i5 @ 2.42 GHz processor and RAM 16.0 GB.

We consider the following Bregman functions ψ_1, ψ_2, ψ_3, and ψ_4 to show the convergence of the sequence generated by Algorithm 1.

(i) $\psi_1 : \mathbb{R}^n \to \mathbb{R}$ defined by $\psi_1(x) = \|x\|^2$.
(ii) $\psi_2 : \mathbb{R}^n \to \mathbb{R}$ defined by $\psi_2(x) = \frac{1}{2}x^\top Ax$, where $A \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix.
(iii) $\psi_3 : \mathbb{R}^n_+ \to \mathbb{R}$ defined by $\psi_3(x) = \sum_{i=1}^n x_i \log(x_i)$, where $x = (x_1,\ldots,x_n) \in \mathrm{Dom}(\psi_3) = \mathbb{R}^n_+$.
(iv) $\psi_4 : \mathbb{R}^n_+ \to \mathbb{R}$ defined by $\psi_4(x) = -\sum_{i=1}^n \log(x_i)$, where $x = (x_1,\ldots,x_n) \in \mathrm{Dom}(\psi_4) = \mathbb{R}^n_+$.

The corresponding Bregman distances are given by

(i) $d_{\psi_1}(x,y) = \|x-y\|^2$, known as the Euclidean distance (in short, ED),
(ii) $d_{\psi_2}(x,y) = \frac{1}{2}(x-y)^\top A(x-y)$, known as Mahalanobis distance (in short, MD),
(iii) $d_{\psi_3}(x,y) = \sum_{i=1}^n \left(x_i \log\left(\frac{x_i}{y_i}\right) + y_i - x_i\right)$, known as Kullback-Leibler distance (in short, KLD),
(iv) $d_{\psi_4}(x,y) = \sum_{i=1}^n \left(\frac{x_i}{y_i} - \log\left(\frac{x_i}{y_i}\right) - 1\right)$, known as Itakura-Saito distance (in short, ISD),

respectively.

Example 4.1. Let f_1, $f_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be bifunctions defined by

$$f_1(x,y) = \langle Px + Qy + q, y - x\rangle, \quad \forall x,y \in \mathbb{R}^n,$$

and

$$f_2(x,y) = \|y\|^4 - \|x\|^4, \quad \forall x,y \in \mathbb{R}^n,$$

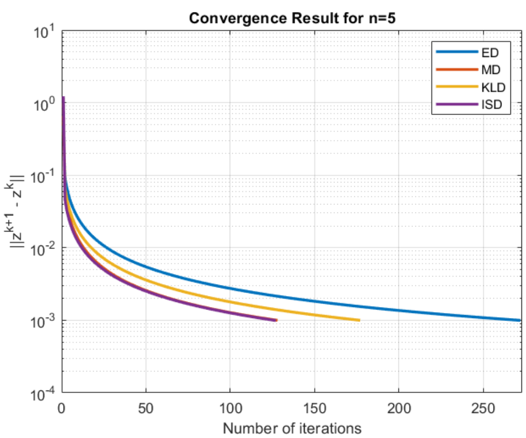
where $P = M^\top M$, $Q = N^\top N$, M, N are $n\times n$ matrices, $M(i,j)$, $N(i,j)$ and $q(i) \in [0,1]$ are randomly generated. Consider the equilibrium problem (1.1) in which the bifunction f is of the form $f = f_1 + f_2$ whose feasible set is

$$C = \left\{ x \in \mathbb{R}^n_+ : \sum_{k=1}^n x_k \le n \right\}.$$

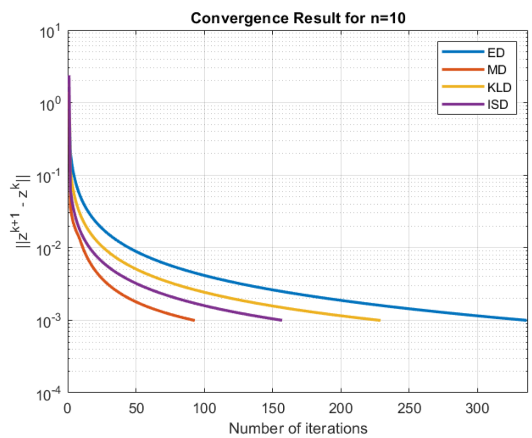
It can be easily seen that P and Q are positive definite and all the conditions of Assumption 3.1 are satisfied. We use randomly generated point x^0 as our starting point. The $n\times n$ positive definite matrix A for Mahalanobis distance (MD) is randomly generated. We set $\|z^{k+1} - z^k\| < 10^{-3}$ as stopping criterion to compare the convergence of the sequence $\{z^k\}$ generated by Algorithm 1 corresponding to ED, MD, KLD, and ISD. For different choices of Bregman function, we choose $\lambda_k = \frac{1}{10^{2.5}k^{0.7}}$ and the convergence results are shown in Figure 1 (A), (B), (C), and (D) for $n = 5, 10, 50$ and 100, respectively. Number of iterations and CPU times (secs) for different choices of Bregman functions are displayed in Table 1.

n	Observations	ED	MD	KLD	ISD
5	No. of Iterations	272	128	177	127
	CPU Times (Secs)	4.1364	3.0883	4.7968	2.6620
10	No. of Iterations	336	93	229	157
	CPU Times (Secs)	5.5512	3.3780	8.3058	5.6937
50	No. of Iterations	159	97	124	297
	CPU Times (Secs)	6.1853	15.0329	20.5170	47.9316
100	No. of Iterations	175	296	96	290
	CPU Times (Secs)	12.0500	66.4745	14.9310	46.0127

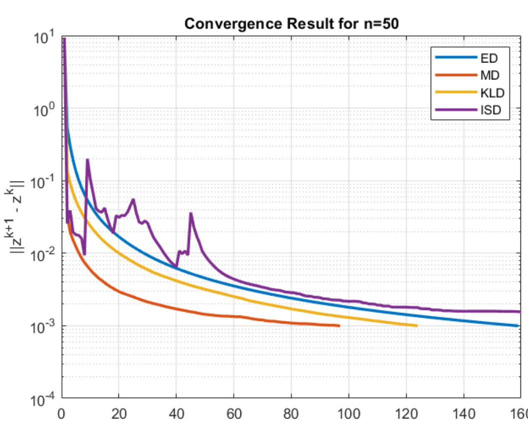
TABLE 1. Performance of Algorithm 1 for different Bregman distances



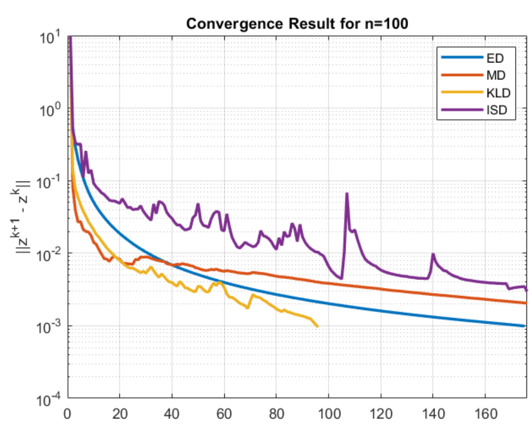
(A) $n = 5$ and $x_0 = rand(1,5)$



(B) $n = 10$ and $x_0 = rand(1,10)$



(C) $n = 50$ and $x_0 = rand(1,50)$



(D) $n = 100$ and $x_0 = rand(1,100)$

FIGURE 1. Convergence of Example 4.1

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REFERENCES

[1] Q. H. Ansari, E. Köbis, J.-C. Yao, Vector Variational Inequalities and Vector Optimization: Theory and Applications, Springer, Cham, 2018.

[2] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994), 123-145.

[3] F. Flores-Bazán, Existence theorems for generalized noncoercive equilibrium problems: the quasi-convex case, SIAM J. Optim. 11 (2000), 675-690.

[4] K. A. Pham, N. H. Trinh, Splitting extragradient-like algorithms for strongly pseudomonotone equilibrium problems, Numer. Algo. 76 (1997), 67-91.

[5] Q. H. Ansari, F. Babu, M. S. Raju, Proximal point method with Bregman distance for quasiconvex pseudomonotone equilibrium problems, Optimization, (2023) 10.1080/02331934.2023.2252430

[6] Q. H. Ansari, J. Balooee, Regularized general nonconvex mixed equilibrium problems: Predictor–corrector methods and algorithms, J. Nonlinear Convex Anal. 17 (2016), 599-619.

[7] Q. H. Ansari, M. S. Raju, Splitting extragradient-like algorithms with Bregman distance for solving equilibrium problems, J. Nonlinear Convex Anal. 26 (2025), 325-347.

[8] S. D. Flåm, A. S. Antipin, Equilibrium programming using proximal-like algorithms, Math. Program. 78 (1997), 29-41.

[9] R. Maluleka, G. C. Ugwunnadi, C. C. Okeke, M. Aphane, On pseudomonotone equilibrium and fixed point problems in Hadamard spaces with applications, J. Nonlinear Funct. Anal. 2024 (2024) 7.

[10] X. Qin, Y. J. Cho, S. M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, J. Comput. Appl. Math. 225 (2009), 20-30.

[11] D. V. Hieu, N. H. Ha, The relaxed regularized method of extragradient type for equilibrium problems, Appl. Set-Valued Anal. Optim. 6 (2024), 385-412.

[12] N. H. Ha, D. V. Hieu, L. D. Muu, An accelerated regularization method for variational inclusions, J. Appl. Numer. Optim. 6 (2024), 97-113.

[13] X. Qin, A weakly convergent method for splitting problems with nonexnpansive mappings, J. Nonlinear Convex Anal. 24 (2023), 1033-1043.

[14] S. Al-Homidan, Q. H. Ansari, O. Chadli, Noncoercive mixed equilibrium problems under pseudomonotone perturbations and applications to nonlinear evolution equations with lack of coercivity, Appl. Anal. 98 (2019), 1199-1223.

[15] S. Al-Homidan, Q. H. Ansari, O. Chadli, Noncoercive stationary Navier-Stokes equations of heat-conducting fluids modeled by hemivariational inequalities: An equilibrium problem approach, Results Math. 74 (2019), Article ID 132.

[16] O. Chadli, Q. H. Ansari, S. Al-Homidan, Existence of solutions for nonlinear implicit differential equations: An equilibrium problem approach, Numer. Funct. Anal. Optim. 37 (2016), 1385-1419.

[17] L. C. Zeng, S. Al-Homidan, Q. H. Ansari, Hybrid proximal-type algorithms for generalized equilibrium problems, maximal monotone operators and relatively nonexpansive mappings, Fixed Point Theory Appl. 2011 (2011), Article ID 973028.

[18] A. Moudafi, On the convergence of splitting proximal methods for equilibrium problems in Hilbert spaces, J. Math. Anal. Appl. 359 (2009), 508-513.

[19] T. N. Hai, N. T. Vinh, Two new splitting algorithms for equilibrium problems, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 111 (2017), 1051-1069.

[20] L. M. Brègman, A relaxation method of finding a common point of convex sets and its application to the solution of problems in convex programming, Ž. Vyčisl. Mat i Mat. Fiz. 7 (1967), 620-631.

[21] Y. Censor, A. Lent, An iterative row-action method for interval convex programming, J. Optim. Theory Appl. 34 (1981), 321-353.

[22] Y. Alber, D. Butnariu, Convergence of Bregman projection methods for solving consistent convex feasibility problems in reflexive Banach spaces, J. Optim. Theory Appl. 92 (1997), 33-61.
[23] H. H. Bauschke, J. M. Borwein, Legendre functions and the method of random Bregman projections, J. Convex Anal. 4 (1997), 27-67.
[24] D. Butnariu, A. N. Iusem, Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization, Kluwer Academic Publishers, Dordrecht, 2000.
[25] K. C. Kiwiel, Proximal minimization methods with generalized Bregman functions, SIAM J. Control Optim. 35 (1997), 1142-1168.
[26] F. Lara, R. T. Marcavillaca, Bregman proximal point type algorithms for quasiconvex minimization, Optimization, 73 (2024), 497-515.
[27] R. T. Rockafellar, Convex Analysis, volume No. 28 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1970.
[28] P. Neittaanmäki, S. Repin, Reliable Methods for Computer Simulation, vol. 33 of Studies in Mathematics and its Applications, Elsevier Science B.V., Amsterdam, 2004.
[29] P. T. Vuong, J. J. Strodiot, A dynamical system for strongly pseudo-monotone equilibrium problems, J. Optim. Theory Appl. 185 (2020), 767-784.
[30] K.-K. Tan, H. K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993), 301-308.
[31] R. E. Bruck, A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces, Israel J. Math. 32 (1979), 107-116.
[32] D. Van Hieu, P. Cholamjiak, Modified extragradient method with Bregman distance for variational inequalities, Appl. Anal. 101 (2022), 655-670.
[33] D. Reem, S. Reich, A. De Pierro, Re-examination of Bregman functions and new properties of their divergences, Optimization, 68 (2019), 279-348.
[34] D. Butnariu, E. Resmerita, Bregman distances, totally convex functions, and a method for solving operator equations in Banach spaces, Abstr. Appl. Anal. 2006 (2006) Art. ID 84919.
[35] D. Butnariu, Y. Censor, S. Reich, Iterative averaging of entropic projections for solving stochastic convex feasibility problems, Comput. Optim. Appl. 8 (1997), 21-39.
[36] E. Resmerita, On total convexity, Bregman projections and stability in Banach spaces, J. Convex Anal. 11 (2004), 1-16.
[37] Y. Censor, S. A. Zenios, Proximal minimization algorithm with D-functions, J. Optim. Theory Appl.73 (1992), 451-464.
[38] H. H. Bauschke, P. L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer, New York, 2011.