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A UNIFIED VARIATIONAL FRAMEWORK FOR GLOBAL AGRICULTURAL SUPPLY CHAINS UNDER UNCERTAIN EXCHANGE RATES: NEW CLASSES OF COST AND PRICE FUNCTIONS

MAURO PASSACANTANDO[1], FABIO RACITI[2,*]

[1]*Department of Business and Law, University of Milano-Bicocca, Via Bicocca degli Arcimboldi 8, 20126 Milan, Italy*
[2]*Department of Mathematics and Computer Science, V.le A. Doria 6, 95125-Catania, Italy*

Abstract. We investigate a random variational inequality which was put forward recently to describe a network equilibrium model of international agricultural trade, where the exchange rates were modeled as random variables. We prove a convergence theorem for the approximated mean values and variances of the solution flows. Moreover, we introduce a new class of non-separable price and cost functions, and prove related monotonicity properties. Finally, we compare two different measures which describe the performance of the road network and use them to assess the vulnerability of the different routes of the supply chain.

Keywords. Network equilibrium; Network vulnerability; Random variational inequality; Supply chain.

1. Introduction

The international food supply chain was severely disrupted in recent years for a variety of reasons, from the COVID-19 pandemic to natural disaster such as earthquakes and floods.the Russia-Ukraine war in February 2022 had a huge impact on the cereal supply. In this paper, we enlarge the variational inequality model that was proposed recently in [16] and [19] to describe an international food supply chain, subject to possible disruptions, in a scenario of uncertain exchange rates. Specifically, the mathematical framework is that of random (or stochastic) variational inequalities, developed in the last 15 years (see, e.g., [7]). With respect to these papers, we provide the following improvements. In the first place, we provide a rigorous convergence analysis of the approximated mean values and variances of the solutions of the random variational inequality which governs our model, that represent the equilibrium flows of commodities. Secondly, we put forward a class of *non-separable* price and cost functions, and prove related monotonicity properties, thus enlarging the modeling power of our previous approach. Finally, we compare two different measures of performance of the road network, which are used to identify the group of routes whose disruption has the greatest impact on the supply chain.

*Corresponding author.
E-mail address: mauro.passacantando@unimib.it (M. Passacantando), fabio.raciti@unict.it (F. Raciti).
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We now highlight several key contributions that form the foundation of the mathematical models employed in this paper. The optimization-based framework for modeling spatially distributed markets was first introduced by Samuelson in 1952 [21], and subsequently extended by Takayama and Judge in a series of works [22, 23]. A broader formulation using variational inequalities was later developed through multiple contributions; see, e.g., [5, 6, 11, 18]. Specific applications to agricultural spatial price equilibrium problems can be found in [3] and [24]. For a detailed treatment of network vulnerability analysis, comparable to that in Section 5, the foundational works [13, 14] as well as the comprehensive book [15] are recommended. The incorporation of random exchange rates into the model is achieved through the framework of random (or stochastic) variational inequalities. A thorough presentation of various methodologies in this area can also be found in the [7]. In this work, we adopt the rigorous L^p-based approach, following the developments in, for example, [4, 8, 9]. An alternative stochastic spatial equilibrium model, which does not rely on variational inequalities, is presented in [20].

The paper is organized as follows. In Section 2, we describe the basic model which is essentially the one investigated in [19]. In Section 3, we prove the convergence theorem for the approximated mean values and variances of the solution. In Section 4, we put forward the new class of non-separable price and cost functions and investigate their monotonicity properties. Section 5 describes the two network performance measures which are used to assess the importance of the different routes in the supply chain, and in Section 6 we provide some illustrative numerical examples. We summarize our results and outline some possible future research perspectives in the small concluding section, Section 7.

2. Model

We now recall the model put forward in [19]. In what follows, vectors in $\mathbb{R}^s$ are thought of as columns, and when involved in matrix operations, $a^\top$ denotes the transpose of vector a and $a^\top b$ the canonical scalar product in $\mathbb{R}^s$. There are m countries which are supply markets, which produce K commodities, and n countries which purchase the commodities. We assume that, for each pair (i, j) of supply-consumer countries, trade takes place through L routes. We assume that all commercial transactions are carried out in an international currency, i.e., in US dollars. Thus, for each supply market i, we introduce the exchange rate e_i to US dollars, which we model as a random variable $e_i : \Omega \to \mathbb{R}$. Analogously, $h_j : \Omega \to \mathbb{R}$ represents the exchange rate to US dollars of the demand market j. The flow of commodity k produced at supply market i and shipped through route l to demand market j is denoted with Q^{kl}_{ij}, where we understand that the superscript l denotes a route which depends on the pair of countries (i, j). All Q^{kl}_{ij} are grouped into a vector $Q \in \mathbb{R}^{KLmn}$ according to a prescribed order. The supply of commodity k produced at supply market i is denoted with s^k_i, while the demand for commodity k at demand market j is denoted with d^k_j. Flows are nonnegative and must satisfy the following flow conservation equations:

$$s^k_i = \sum_{j=1}^{n} \sum_{l=1}^{L} Q^{kl}_{ij}, \qquad \forall\, k = 1, \dots, K,\ i = 1, \dots, m, \tag{2.1}$$

$$d^k_j = \sum_{i=1}^{m} \sum_{l=1}^{L} Q^{kl}_{ij}, \qquad \forall\, k = 1, \dots, K,\ j = 1, \dots, n. \tag{2.2}$$

Following [16], we assume that the commodities are homogeneous and denote with $\bar{Q}^l_{ij}$ the transportation capacity of route l connecting the pair (i,j):

$$\sum_{k=1}^{K} Q^{kl}_{ij} \leq \bar{Q}^l_{ij}, \qquad \forall\, l=1,\dots,L,\ i=1,\dots,m,\ j=1,\dots,n.$$

Moreover, we denote by $\bar{S}_i$ the overall production capacity of supply market i such that $\sum_{k=1}^{K} s^k_i \leq \bar{S}_i$ for all $i=1,\dots,m$, which, exploiting (2.1), can be transformed as $\sum_{k=1}^{K}\sum_{j=1}^{n}\sum_{l=1}^{L} Q^{kl}_{ij} \leq \bar{S}_i$ for all $i=1,\dots,m$. We now introduce the country supply price functions: $P^k_i = P^k_i(s)$ for all $i=1,\dots,m$, $k=1,\dots,K$, which can be expressed as function of the flows by using (2.1): $\pi^k_i(Q) := P^k_i(s)$ for all $i=1,\dots,m$, $k=1,\dots,K$. The unit transportation cost of commodity k from country i to country j using route l is denoted with $c^{kl}_{ij} = c^{kl}_{ij}(Q)$ for all $k=1,\dots,K$, $l=1,\dots,L$, $i=1,\dots,m$, $j=1,\dots,n$. The demand price of commodity k in country j is $R^k_j = R^k_j(d)$ for all $k=1,\dots,K$, $j=1,\dots,n$, which can be expressed as function of the flows by using (2.2): $\rho^k_j(Q) := R^k_j(d)$ for all $k=1,\dots,K$, $j=1,\dots,n$. The set of feasible flows is denoted by $\mathscr{K}$, such that

$$\mathscr{K} = \Bigg\{ Q \in \mathbb{R}^{KLmn}_+ : \sum_{k=1}^{K} Q^{kl}_{ij} \leq \bar{Q}^l_{ij}, \quad \forall\, l=1,\dots,L,\ i=1,\dots,m,\ j=1,\dots,n,$$
$$\sum_{k=1}^{K}\sum_{j=1}^{n}\sum_{l=1}^{L} Q^{kl}_{ij} \leq \bar{S}_i, \quad \forall\, i=1,\dots,m. \Bigg\}.$$

In our model the exchange rates $e_i, h_j : \Omega \to \mathbb{R}$, for any $i=1,\dots,m$ and $j=1,\dots,n$, are random variables. We thus introduce a probability measure P on the sample space Ω and consider the following variational inequality problem:

Find a random vector $Q^* : \Omega \to \mathbb{R}^{KLmn}$ such that $Q^*(\omega) \in \mathscr{K}$, P-a.s., and for P-almost every elementary event $\omega \in \Omega$:

$$\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{i=1}^{m}\sum_{j=1}^{n} \left\{ \left[\pi^k_i(Q^*(\omega)) + c^{kl}_{ij}(Q^*(\omega)) \right] e_i(\omega) - \rho^k_j(Q^*(\omega)) h_j(\omega) \right\} \cdot \left[Q^{kl}_{ij} - Q^{kl*}_{ij}(\omega) \right] \geq 0 \tag{2.3}$$

holds for any $Q \in \mathscr{K}$.

Theorem 2.1. *Let $\lambda : \Omega \to \mathbb{R}^{Lmn}$ and $\mu : \Omega \to \mathbb{R}^m$ two random vectors. The variational inequality (2.3) is equivalent to the following equilibrium conditions, satisfied P-a.s.:*

$$\left[\pi^k_i(Q^*(\omega)) + c^{kl}_{ij}(Q^*(\omega)) \right] e_i(\omega) + \lambda^{l*}_{ij}(\omega) + \mu^*_i(\omega) \begin{cases} = \rho^k_j(Q^*(\omega)) h_j(\omega) & \text{if } Q^{kl*}_{ij}(\omega) > 0, \\ \geq \rho^k_j(Q^*(\omega)) h_j(\omega) & \text{if } Q^{kl*}_{ij}(\omega) = 0, \end{cases}$$

for any $k=1,\dots,K$, $l=1,\dots,L$, $i=1,\dots,m$, $j=1,\dots,n$;

$$\lambda^{l*}_{ij}(\omega) \begin{cases} \geq 0 & \text{if } \sum_{k=1}^{K} Q^{kl*}_{ij}(\omega) = \bar{Q}^l_{ij}, \\ = 0 & \text{if } \sum_{k=1}^{K} Q^{kl*}_{ij}(\omega) < \bar{Q}^l_{ij}, \end{cases}$$

for any $l = 1,\dots,L,\ i = 1,\dots,m,\ j = 1,\dots,n$;

$$\mu_i^*(\omega)\begin{cases} \geq 0 & \text{if } \sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{j=1}^{n} Q_{ij}^{kl*}(\omega) = \bar{S}_i, \\ = 0 & \text{if } \sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{j=1}^{n} Q_{ij}^{kl*}(\omega) < \bar{S}_i, \end{cases}$$

for any $i = 1,\dots,m$.

For the proof, we refer to [19].

In order to write variational inequality (2.3) in compact form, we introduce the following maps:

- $F_1 : \Omega \times \mathbb{R}^{KLmn} \to \mathbb{R}^{KLmn}$ is the map such that $F_1(\omega, Q)$ is the vector with components $\left(\pi_i^k(Q)e_i(\omega)\right)_{ijkl}$ (ordered in the same manner as Q_{ij}^{kl});
- $F_2 : \Omega \times \mathbb{R}^{KLmn} \to \mathbb{R}^{KLmn}$ is the map such that $F_2(\omega, Q)$ is the vector with components $\left(c_{ij}^{kl}(Q)e_i(\omega)\right)_{ijkl}$;
- $F_3 : \Omega \times \mathbb{R}^{KLmn} \to \mathbb{R}^{KLmn}$ is the map such that $F_3(\omega, Q)$ is the vector with components $-\left(\rho_j^k(Q)h_j(\omega)\right)_{ijkl}$.

Denoting with F the sum $F_1 + F_2 + F_3$, variational inequality (2.3) reads as:

Find $Q^* : \Omega \to \mathbb{R}^{KLmn}$ such that $Q^*(\omega) \in \mathscr{K}$, P-a.s., and for P-almost every elementary event $\omega \in \Omega$:

$$F(\omega, Q^*(\omega))^\top (Q - Q^*(\omega)) \geq 0 \tag{2.4}$$

holds for any $Q \in \mathscr{K}$.

Remark 2.1. In the maps F_1, F_2, and F_3 the deterministic and the random variables are separated. Moreover, if we assume that π_i^k, c_{ij}^{kl} and ρ_j^k are continuous, it follows that F_1, F_2 and F_3 are Carathéodory functions, i.e. $F_1(\omega,\cdot)$, $F_2(\omega,\cdot)$ and $F_3(\omega,\cdot)$ are continuous for P-almost every $\omega \in \Omega$, and $F_1(\cdot, Q)$, $F_2(\cdot, Q)$ and $F_3(\cdot, Q)$ are P-measurable, for each Q.

Theorem 2.2. *Let $F(\omega,\cdot)$ be continuous, for any $\omega \in \Omega$. Since the set $\mathscr{K}$ is compact and convex, the existence of a solution of* (2.4) *follows from the Hartman-Stampacchia theorem. Moreover, if $F(\omega,\cdot)$ is strictly monotone, the solution is unique (see, e.g., [10, 12]).*

We are interested in computing the mean values and the variances of solutions of (2.3). In order to select only solutions with finite first and second moments, we consider an integral version of (2.3), under additional assumptions.

Monotonicity Assumption. We assume that F is strongly monotone on $\mathbb{R}^{KLmn}$, uniformly with respect to $\omega \in \Omega$, i.e.,

$$\exists\, \sigma > 0 :\ [F(\omega, Q) - F(\omega, \tilde{Q})]^\top (Q - \tilde{Q}) \geq \sigma \|Q - \tilde{Q}\|^2, \qquad \forall\, Q, \tilde{Q} \in \mathbb{R}^{KLmn},\ \forall\, \omega \in \Omega.$$

Linear Assumption. We assume that $F_1(\omega,\cdot)$, $F_2(\omega,\cdot)$, and $F_3(\omega,\cdot)$ are linear.

Define now the set:

$$\mathscr{K}^P = \left\{U \in L^2(\Omega, P, \mathbb{R}^{KLmn}) \text{ such that } U(\omega) \in \mathscr{K}, P\text{-a.s.}\right\}.$$

Under the above assumptions, the following integral variational inequality is well-posed:

Find $U^* \in \mathscr{K}^P$ such that for all $U \in \mathscr{K}^P$ we have:

$$\int_\Omega F(\omega, U^*(\omega))^\top (U - U^*(\omega))\, dP(\omega) \geq 0. \tag{2.5}$$

Indeed, under the linearity and Carathéodory assumptions on F, the Nemitsky operator (see, e.g., [1]):

$$N_F(Q)(\omega) := F(\omega, Q(\omega))$$

is such that $N_F : L^2(\Omega, P, \mathbb{R}^{KLmn}) \to L^2(\Omega, P, \mathbb{R}^{KLmn})$. In extended form, (2.5) reads as:

Find $U^* \in \mathscr{K}^P$ such that for all $U \in \mathscr{K}^P$:

$$\int_\Omega \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} \sum_{l=1}^{L} \left\{ \left[\pi_i^k(U^*(\omega)) + c_{ij}^{kl}(U^*(\omega)) \right] e_i(\omega) - \rho_j^k(U^*(\omega)) h_j(\omega) \right\} \cdot [U_{ij}^{kl} - U_{ij}^{kl*}(\omega)]\, dP(\omega) \geq 0. \tag{2.6}$$

The following theorem ensures existence and uniqueness of the solution of the integral variational inequalities above.

Theorem 2.3. *Let F be Carathéodory and $F(\omega, \cdot)$ linear. Moreover, let F be strongly monotone, uniformly with respect to $\omega \in \Omega$. Then, (2.5) has a unique solution.*

Proof. The set $\mathscr{K}^P$ is a closed and convex subset of $L^2(\Omega, P, \mathbb{R}^{KLmn})$. Moreover, it is norm-bounded, hence weakly compact. Since F is Carathéodeory, the Nemitsky operator N_F is continuous from $L^2(\Omega, P, \mathbb{R}^{KLmn})$ to $L^2(\Omega, P, \mathbb{R}^{KLmn})$. At last, the uniform strong monotonicity of F implies the strong monotonicity of N_F. The Lions-Stampacchia theorem (see, e.g., [10]) then applies and the existence and uniqueness of the solution follows. $\square$

The random variables $E_i = e_i(\omega)$ and $H_j = h_j(\omega)$ have values which are distributed according to the induced probabilities:

$$d\mathbb{P}(E_i) = \delta_i(E_i) dE_i, \quad i = 1, \dots, m, \qquad d\mathbb{P}(H_j) = \theta_j(H_j) dH_j, \quad j = 1, \dots, n,$$

where δ_i and θ_j are the probability densities of $\mathbb{P}(E_i)$ and $\mathbb{P}(H_j)$, respectively. The transformed feasible set is thus:

$$\mathscr{K}^{\mathbb{P}} := \left\{ U \in L^2(\mathbb{R}^{m+n}, \mathbb{P}, \mathbb{R}^{KLmn}) : \, U(E,H) \in \mathscr{K}, \ \mathbb{P} - a.s. \right\}.$$

We can then transform the integral variational inequality (2.6) to the following:

Find $U^* \in \mathscr{K}^{\mathbb{P}}$ such that, for all $U \in \mathscr{K}^{\mathbb{P}}$, we have:

$$\begin{aligned} \int_{\mathbb{R}^{m+n}} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} \sum_{l=1}^{L} & \left\{ \left[\pi_i^k(U^*(E,H)) + c_{ij}^{kl}(U^*(E,H)) \right] E_i - \rho_j^k(U^*(E,H)) H_j \right\} \\ & \cdot \left[U_{ij}^{kl} - U_{ij}^{kl*}(E,H) \right] d\mathbb{P}(E)\, d\mathbb{P}(H) \geq 0. \end{aligned} \tag{2.7}$$

3. Approximation Procedure

Variational inequality (2.7) can be approximated with a procedure which was used, for instance, in [4, 8, 9]. In the following, we outline the approximation scheme and also provide a convergence theorem for the mean values and variances of the solution of the specific variational inequality (2.7). For further details on the general functional analytic tools, we refer to [7, Chapter 6].

First of all, the functions E_i can be approximated, for each i, by sequences $\{\varphi_i^\nu\}_\nu$ of step functions, with $\varphi_i^\nu \to \varphi_i$, in L^∞ subordinated to partitions of their support $[\underline{e}_i, \bar{e}_i]$, where $\varphi_i(E,H) =$

E_i. Moreover, the functions H_j can be approximated, for each j, by sequences $\{\psi_j^\nu\}_\nu$ of step functions, with $\psi_j^\nu \to \psi_j$, in L^∞ subordinated to partitions of their support $[\underline{h}_j, \bar{h}_j]$, where $\psi_j(E,H) = h_j$. Let us now denote by $L^{2,\nu}(\mathbb{R}^{m+n}, \mathbb{P}, \mathbb{R}^{KLmn})$ the finite-dimensional subspaces of $L^2(\mathbb{R}^{m+n}, \mathbb{P}, \mathbb{R}^{KLmn})$ made up with the step functions corresponding to the above mentioned partitions, and consider the following closed convex sets, for any $\nu \in \mathbb{N}$:

$$\mathscr{K}_\nu^{\mathbb{P}} := \left\{ U^\nu \in L^{2,\nu}(\mathbb{R}^{m+n}, \mathbb{P}, \mathbb{R}^{KLmn}) : \; U^\nu(E,H) \in \mathscr{K}, \; \mathbb{P} - \text{a.s.} \right\}.$$

We can then consider the following finite-dimensional variational inequalities, for each $\nu \in \mathbb{N}$:

Find $U^{*\nu} \in K_\nu^{\mathbb{P}}$ such that, for all $U^\nu \in K_\nu^{\mathbb{P}}$,

$$\int_{\mathbb{R}^{m+n}} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} \sum_{l=1}^{L} \left\{ \left[\pi_i^k(U^{*,\nu}(E,H)) + c_{ij}^{kl}(U^{*,\nu}(E,H)) \right] \varphi_i^\nu - \rho_j^k(U^{*,\nu}(E,H)) \psi_j^\nu \right\} \cdot \left[U_{ij}^{kl,\nu} - U_{ij}^{kl*,\nu}(E,H) \right] d\mathbb{P}(E)\, d\mathbb{P}(H) \geq 0. \tag{3.1}$$

Solutions of (3.1) can be used to approximate the mean values and the variances of the random equilibrium flows. We first need to recall a lemma which describes in which sense the sets $\mathscr{K}_\nu^{\mathbb{P}}$ approximate the set $\mathscr{K}^{\mathbb{P}}$.

Lemma 3.1. *For $\nu \to \infty$, the sequence of closed convex sets $\{\mathscr{K}_\nu^{\mathbb{P}}\}_\nu$ converges to the set $\mathscr{K}^{\mathbb{P}}$ in the sense of Mosco, that is,*

(1) For each point $U \in \mathscr{K}^{\mathbb{P}}$, there exists a sequence $\{U^\nu\}$, with $U^\nu \in \mathscr{K}_\nu^{\mathbb{P}}$, such that, for $\nu \to \infty$, $U^\nu \to U$, strongly in L^2.

(2) If U is a weak accumulation point of $\{U^\nu\}_\nu$, with $U^\nu \in \mathscr{K}_\nu^{\mathbb{P}}$ for each ν, then $U \in \mathscr{K}^{\mathbb{P}}$.

Proof. This lemma is a special case of [7, Theorem 6.1.5]. □

Theorem 3.1. *For each $\nu \in \mathbb{N}$, let $U^{*,\nu}$ be the unique solution to (3.1). Then,*

$$\lim_{\nu \to \infty} E_{\mathbb{P}} \left[U_{ij}^{kl*,\nu}(E,H) \right] = E_{\mathbb{P}} \left[U_{ij}^{kl*}(E,H) \right]$$

and

$$\lim_{\nu \to \infty} Var_{\mathbb{P}} \left[U_{ij}^{kl*,\nu}(E,H) \right] = Var_{\mathbb{P}} \left[U_{ij}^{kl*}(E,H) \right],$$

where U^* is the solution of (2.7), and $E_{\mathbb{P}}$ and $Var_{\mathbb{P}}$ denote the expectation and the variance.

Proof. For each $\nu \in \mathbb{N}$, $U^{*,\nu} \in \mathscr{K}^{\mathbb{P}}$, which is closed, convex and bounded, hence weakly compact. Therefore, it exists a weak accumulation point U^*, such that $U^{*,\nu} \to U^*$, weakly in L^2, as $\nu \to \infty$. Under the monotonicity assumption, Minty's lemma holds and it is thus enough to prove that U^* satisfies

$$\int_{\mathbb{R}^{m+n}} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} \sum_{l=1}^{L} \left\{ \left[\pi_i^k(U(E,H)) + c_{ij}kl(U(E,H)) \right] E_i - \rho_j^k(U^(E,H)) H_j \right\} \cdot \left[U_{ij}^{kl} - U_{ij}^{kl*}(E,H) \right] d\mathbb{P}(E)\, d\mathbb{P}(H) \geq 0. \tag{3.2}$$

Let us choose $U \in \mathscr{K}^{\mathbb{P}}$ arbitrarily. Because of Mosco-convercence, there exists $V^\nu \in \mathscr{K}_\nu^{\mathbb{P}}$ such that $\lim_{\nu \to \infty} V^\nu = U$, strongly in L^2. We can apply Minty's lemma to (3.1), choose $U^\nu = V^\nu$, and obtain

$$\begin{aligned}\int_{\mathbb{R}^{m+n}} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{l=1}^{L} \Big\{ \Big[\pi_i^k(V^\nu(E,H)) + c_{ij}^{kl}(V^\nu(E,H))\Big]\varphi_i^\nu - \rho_j^k(V^\nu(E,H))\psi_j^\nu \Big\} \\ \cdot \Big[V_{ij}^{kl,\nu} - U_{ij}^{kl*,\nu}(E,H)\Big]\, d\mathbb{P}(E)\, d\mathbb{P}(H) \geq 0.\end{aligned} \tag{3.3}$$

Since, for any i, j, $\varphi_i^\nu \to \varphi_i$ in L^∞ and $\psi_i^\nu \to \psi_i$ in L^∞, then $V^\nu - U^{\nu,*}$ converges to $U - U^*$ weakly in L^2, and V^ν converges to U strongly in L^2. Hence, passing to the limit for $\nu \to \infty$ (3.3) yields (3.2). We thus prove the weak convergence of $U^{*,\nu}$ to U^*, which immediately implies the convergence of the sequence of the approximated mean values to the exact mean value.

Under the strong monotonicity assumption, we now show that $U^{*,\nu}$ converges to U^* strongly in L^2. Let us consider a sequence $\{U^\nu\}$ such that $U^\nu \in \mathscr{K}_\nu^{\mathbb{P}}$ and $U^\nu \to U^*$ strongly in L^2 as $\nu \to \infty$. Such a sequence exists because of Mosco convergence. We prove that $\|U^\nu - U^{*,\nu}\|_2 \to 0$, because $\|U^{*,\nu} - U^*\|_2 \leq \|U^{*,\nu} - U^\nu\|_2 + \|U^\nu - U^*\|_2$. We have

$$\begin{aligned} 0 &\leq \alpha \|U^{*,\nu} - U^*\|_2 \\ &\leq \int_{\mathbb{R}^{m+n}} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{l=1}^{L} \Big\{ \Big[\Big(\pi_i^k(U^\nu(E,H)) + c_{ij}^{kl}(U^\nu(E,H))\Big)\varphi_i^\nu - \rho_j^k(U^\nu(E,H))\psi_j^\nu\Big] \\ &\quad - \Big[\Big(\pi_i^k(U^{*,\nu}(E,H)) + c_{ij}^{kl}(U^{*,\nu}(E,H))\varphi_i^\nu\Big) - \rho_j^k(U^{*,\nu}(E,H))\psi_j^\nu\Big]\Big\} \\ &\qquad \cdot \Big[U_{ij}^{kl,\nu} - U_{ij}^{kl*,\nu}(E,H)\Big]\, d\mathbb{P}(E)\, d\mathbb{P}(H) \\ &\leq \int_{\mathbb{R}^{m+n}} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{l=1}^{L} \Big[\Big(\pi_i^k(U^\nu(E,H)) + c_{ij}^{kl}(U^\nu(E,H))\Big)\varphi_i^\nu - \rho_j^k(U^\nu(E,H))\psi_j^\nu\Big] \\ &\qquad \cdot \Big[U_{ij}^{kl,\nu} - U_{ij}^{kl*,\nu}(E,H)\Big]\, d\mathbb{P}(E)\, d\mathbb{P}(H). \end{aligned}$$

Now, we have for each component $U_{ij}^{kl,\nu} \rightharpoonup U_{ij}^{kl,*}$, (strongly, hence weakly) in L^2, and also $U_{ij}^{kl*,\nu} \rightharpoonup U_{ij}^{kl*}$ in L^2, whence $U_{ij}^{kl,\nu} - U_{ij}^{kl*,\nu} \rightharpoonup 0$. Moreover, the Nemitsky operators induced by π, c, ρ are strong-strong continuous. Hence,

$$\pi_i(U^\nu) \to \pi_i(U^*), \qquad c_{ij}^{kl}(U^\nu) \to c_{ij}^{kl}(U^*), \quad \rho_j^k(U^\nu) \to \rho_j^k(U^*), \text{ in } L^2, \text{ when } \nu \to \infty.$$

Moreover, for $\nu \to \infty$, $\varphi_i^\nu \to \varphi_i$ and $\psi_j^\nu \to \psi_j$ in L^∞. It follows then that the sequence defined by the last integral converges to 0 for $\nu \to \infty$. The convergence of variances then follows immediately because, for $\nu \to \infty$, $(E_{\mathbb{P}}[U^\nu])^2 \to (E_{\mathbb{P}}[U])^2$ for the convergence of the mean values, while $E_{\mathbb{P}}[(U^\nu)^2] \to E_{\mathbb{P}}[U^2]$ for the continuity of the L^2-norm. □

4. MONOTONICITY

In the following result, we prove that a wide class of non-separable linear price and cost functions guarantee the strong monotonicity of F.

Theorem 4.1. *Assume, for any $i = 1, \ldots, m$, that there exist $\underline{e}_i, \bar{e}_i > 0$ such that $\underline{e}_i \leq e_i(\omega) \leq \bar{e}_i$ for any $\omega \in \Omega$, and, for any $j = 1, \ldots, n$, that there exist $\underline{h}_j, \bar{h}_j > 0$ such that $\underline{h}_j \leq h_j(\omega) \leq \bar{h}_j$ for any $\omega \in \Omega$. Let $\underline{e} = \min_i \underline{e}_i$, $\bar{e} = \max_i \bar{e}_i$, $\underline{h} = \min_j \underline{h}_j$, $\bar{h} = \max_j \bar{h}_j$. Moreover, assume that* *any supply price function is affine, i.e.,*

$$P_i^k(s) = \sum_{i'=1}^{m} \sum_{k'=1}^{K} \alpha_{ii'}^{kk'} s_{i'}^{k'} + a_i^k, \tag{4.1}$$

with $\alpha_{ii'}^{kk'} > 0$*, for any* $i, i' = 1, \dots, m$*,* $k, k' = 1, \dots, K$*, and the condition*

$$2\underline{e}\, \alpha_{ii}^{kk} \geq \bar{e} \sum_{(i',k') \neq (i,k)} \left(\alpha_{ii'}^{kk'} + \alpha_{i'i}^{k'k} \right) \tag{4.2}$$

holds for any $i = 1, \dots, m$*,* $k = 1, \dots, K$*; any transportation cost function is affine, i.e.,*

$$c_{ij}^{kl}(Q) = \sum_{i'=1}^{m} \sum_{j'=1}^{n} \sum_{k'=1}^{K} \sum_{l'=1}^{L} \beta_{iji'j'}^{klk'l'} Q_{i'j'}^{k'l'} + b_{ij}^{kl}, \tag{4.3}$$

with $\beta_{iji'j'}^{klk'l'} > 0$*, for any* $i, i' = 1, \dots, m$*,* $j, j' = 1, \dots, n$*,* $k, k' = 1, \dots, K$*,* $l, l' = 1, \dots, L$*, and the condition*

$$2\underline{e}\, \beta_{ijij}^{klkl} > \bar{e} \sum_{(i',j',k',l') \neq (i,j,k,l)} \left(\beta_{iji'j'}^{klk'l'} + \beta_{i'j'ij}^{k'l'kl} \right) \tag{4.4}$$

holds for any $i = 1, \dots, m$*,* $j = 1, \dots, n$*,* $k = 1, \dots, K$*,* $l = 1, \dots, L$*; any demand price function is affine, i.e.,*

$$R_j^k(d) = -\sum_{j'=1}^{n} \sum_{k'=1}^{K} \gamma_{jj'}^{kk'} d_{j'}^{k'} + g_i^k, \tag{4.5}$$

with $\gamma_{jj'}^{kk'} > 0$*, for any* $j, j' = 1, \dots, n$*,* $k, k' = 1, \dots, K$*, and the condition*

$$2\underline{h}\, \gamma_{jj}^{kk} \geq \bar{h} \sum_{(j',k') \neq (j,k)} \left(\gamma_{jj'}^{kk'} + \gamma_{j'j}^{k'k} \right) \tag{4.6}$$

holds for any $j = 1, \dots, n$ *and* $k = 1, \dots, K$*. Then the map* F *is strongly monotone, uniformly with respect to* $\omega \in \Omega$*.*

Proof. We first prove that F_1 is monotone on $\mathbb{R}^{mnKL}$ for any $\omega \in \Omega$. We order the components Q_{ij}^{kl} of the vector Q according to the lexicographical order on the indices (j, l, i, k) and the components s_i^k of s according to the lexicographical order on the indices (i, k). Then, the linear relation (2.1) between s and Q reads $s = SQ$, where S is the $mK \times mKnL$ block-matrix defined as

$$S = \left(\; I_{mK} \; \middle| \; \cdots \; \middle| \; I_{mK} \; \right),$$

and I_{mK} is the identity matrix of order mK. We denote by A the $mK \times mK$ matrix whose element at row (i, k) and column (i', k') is equal to $\alpha_{ii'}^{kk'}$. Then, equation (4.1) reads $P(s) = As + a$, where $a = (a_i^k)$ and its components are ordered according to the lexicographical order on the indices (i, k). Hence, the affine map F_1 can be written as $F_1(\omega, Q) = E(\omega)(ASQ + a)$, where $E(\omega)$ is the $mKnL \times mK$ block-matrix defined as follows:

$$E(\omega) = \begin{pmatrix} D(\omega) \\ \hline \vdots \\ \hline D(\omega) \end{pmatrix},$$

and

$$D(\omega)=\left(\begin{array}{c|c|c|c} e_1(\omega)I_K & 0 & \cdots & 0 \\ \hline 0 & e_2(\omega)I_K & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & e_m(\omega)I_K \end{array}\right).$$

Therefore, the Jacobian matrix of F_1 is the following block-matrix

$$E(\omega)AS=\left(\begin{array}{c|c|c} D(\omega)A & \cdots & D(\omega)A \\ \hline \vdots & \ddots & \vdots \\ \hline D(\omega)A & \cdots & D(\omega)A \end{array}\right).$$

The generic element of the matrix $D(\omega)A$ at row (i,k) and column (i',k') is equal to $e_i(\omega)\alpha_{ii'}^{kk'}$. Thus, the generic element of the symmetric part of $D(\omega)A$, that is $(D(\omega)A+(D(\omega)A)^\top)/2$, at row (i,k) and column (i',k') is equal to

$$\frac{e_i(\omega)\alpha_{ii'}^{kk'}+e_{i'}(\omega)\alpha_{i'i}^{k'k}}{2}.$$

The assumptions on $e_i(\omega)$ and (4.2) guarantee that, for any $i=1,\dots,m$ and $k=1,\dots,K$, the following chain of inequalities holds for any $\omega\in\Omega$:

$$e_i(\omega)\alpha_{ii}^{kk}\geq \underline{e}\,\alpha_{ii}^{kk}\geq \sum_{(i',k')\neq(i,k)}\frac{\bar{e}}{2}\left(\alpha_{ii'}^{kk'}+\alpha_{i'i}^{k'k}\right)\geq \sum_{(i',k')\neq(i,k)}\frac{e_i(\omega)\alpha_{ii'}^{kk'}+e_{i'}(\omega)\alpha_{i'i}^{k'k}}{2}.$$

Hence, the symmetric part of $D(\omega)A$ is a diagonally dominant matrix for any $\omega\in\Omega$. Therefore, the symmetric part of $D(\omega)A$ and $D(\omega)A$ itself are positive semidefinite for any $\omega\in\Omega$. Finally, given any $Q,\tilde{Q}\in\mathbb{R}^{mnKL}$, we have

$$\begin{aligned}(Q-\tilde{Q})^\top[F_1(\omega,Q)-F_1(\omega,\tilde{Q})]&=(Q-\tilde{Q})^\top E(\omega)AS(Q-\tilde{Q})\\ &=\left[\sum_{j=1}^{n}\sum_{l=1}^{L}(Q_j^l-\tilde{Q}_j^l)\right]^\top D(\omega)A\left[\sum_{j=1}^{n}\sum_{l=1}^{L}(Q_j^l-\tilde{Q}_j^l)\right]\geq 0,\end{aligned}$$

where Q_j^l denotes the subvector of Q with components Q_{ij}^{kl} for $i=1,\dots,m$ and $k=1,\dots,K$. Thus, F_1 is monotone on $\mathbb{R}^{mnKL}$ for any $\omega\in\Omega$.

We now prove that F_2 is strongly monotone on $\mathbb{R}^{mnKL}$, uniformly with respect to $\omega\in\Omega$. We order the components Q_{ij}^{kl} of Q according to the lexicographical order on the indices (i,j,k,l). We denote by B the $mnKL\times mnKL$ matrix whose element at row (i,j,k,l) and column (i',j',k',l') is equal to $\beta_{iji'j'}^{klk'l'}$. Then, equation (4.3) reads $c(Q)=BQ+b$, where $b=(b_{ij}^{kl})$ and its components are ordered according to the lexicographical order on the indices (i,j,k,l). Hence, the affine map F_2 can be written as $F_2(\omega,Q)=\Delta(\omega)(BQ+b)$, where $\Delta(\omega)$ is the $mnKL\times mnKL$ block-matrix defined as follows:

$$\Delta(\omega)=\left(\begin{array}{c|c|c|c} e_1(\omega)I_{nKL} & 0 & \cdots & 0 \\ \hline 0 & e_2(\omega)I_{nKL} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & e_m(\omega)I_{nKL} \end{array}\right).$$

Therefore, the Jacobian matrix of F_2 is equal to $\Delta(\omega)B$. The generic element of the matrix $\Delta(\omega)B$ at row (i,j,k,l) and column (i',j',k',l') is equal to $e_i(\omega)\beta_{iji'j'}^{klk'l'}$. Thus, the generic element of the symmetric part of $\Delta(\omega)B$, that is $(\Delta(\omega)B+(\Delta(\omega)B)^\top)/2$, at row (i,j,k,l) and column (i',j',k',l') is equal to

$$\frac{e_i(\omega)\beta_{iji'j'}^{klk'l'}+e_{i'}(\omega)\beta_{i'j'ij}^{k'l'kl}}{2}.$$

The assumptions on $e_i(\omega)$ and (4.4) guarantee that, for any $i=1,\dots,m$, $j=1,\dots,n$, $k=1,\dots,K$ and $l=1,\dots,L$, the following chain of inequalities holds for any $\omega\in\Omega$:

$$\begin{aligned} e_i(\omega)\beta_{ijij}^{klkl} \ge \underline{e}\,\beta_{ijij}^{klkl} &> \sum_{(i',j',k',l')\neq(i,j,k,l)} \left(\frac{\bar{e}\beta_{iji'j'}^{klk'l'}+\bar{e}\beta_{i'j'ij}^{k'l'kl}}{2}\right) \\ &\ge \sum_{(i',j',k',l')\neq(i,j,k,l)} \left(\frac{e_i(\omega)\beta_{iji'j'}^{klk'l'}+e_{i'}(\omega)\beta_{i'j'ij}^{k'l'kl}}{2}\right). \end{aligned}$$

Hence, the Gershgorin circle theorem guarantees that there exists $\tau>0$ such that

$$\lambda_{min}\left(\frac{\Delta(\omega)B+(\Delta(\omega)B)^\top}{2}\right)\ge\tau$$

holds for any $\omega\in\Omega$, where $\lambda_{min}((\Delta(\omega)B+(\Delta(\omega)B)^\top)/2)$ denotes the minimum eigenvalue of the symmetric part of $\Delta(\omega)B$. Finally, given any $Q,\tilde{Q}\in\mathbb{R}^{mnKL}$ and $\omega\in\Omega$, we have

$$\begin{aligned} (Q-\tilde{Q})^\top[F_2(\omega,Q)-F_2(\omega,\tilde{Q})] &= (Q-\tilde{Q})^\top\Delta(\omega)B(Q-\tilde{Q}) \\ &\quad (Q-\tilde{Q})^\top\left(\frac{\Delta(\omega)B+(\Delta(\omega)B)^\top}{2}\right)(Q-\tilde{Q}) \\ &\ge \lambda_{min}\left(\frac{\Delta(\omega)B+(\Delta(\omega)B)^\top}{2}\right)\|Q-\tilde{Q}\|^2 \\ &\ge \tau\|Q-\tilde{Q}\|^2, \end{aligned}$$

thus F_2 is strongly monotone on $\mathbb{R}^{mnKL}$, uniformly with respect to $\omega\in\Omega$.

Finally, we prove that F_3 is monotone on $\mathbb{R}^{mnKL}$ for any $\omega\in\Omega$. We order the components Q_{ij}^{kl} of Q according to the lexicographical order on the indices (i,l,j,k) and the components d_j^k of d according to the lexicographical order on the indices (j,k). Then, the linear relation (2.2) between d and Q reads $d=TQ$, where T is the $nK\times nKmL$ block-matrix defined as

$$T=\left(\begin{array}{c|c|c} I_{nK} & \cdots & I_{nK}\end{array}\right),$$

and I_{nK} is the identity matrix of order nK. We denote by Γ the $nK\times nK$ matrix whose element at row (j,k) and column (j',k') is equal to $\gamma_{jj'}^{kk'}$. Then, equation (4.5) reads $R(d)=-\Gamma d+g$, where $g=(g_j^k)$ and its components are ordered according to the lexicographical order on the indices (j,k). Hence, the affine map F_3 can be written as $F_3(\omega,Q)=-H(\omega)(-\Gamma TQ+g)$, where $H(\omega)$ is the $nKmL\times nK$ block-matrix defined as follows:

$$H(\omega)=\left(\begin{array}{c} R(\omega) \\ \hline \vdots \\ \hline R(\omega)\end{array}\right),$$

and

$$R(\omega) = \left(\begin{array}{c|c|c|c} h_1(\omega)I_K & 0 & \cdots & 0 \\ \hline 0 & h_2(\omega)I_K & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & h_n(\omega)I_K \end{array}\right).$$

Therefore, the Jacobian matrix of F_3 is the following block-matrix

$$H(\omega)\Gamma T = \left(\begin{array}{c|c|c} R(\omega)\Gamma & \cdots & R(\omega)\Gamma \\ \hline \vdots & \ddots & \vdots \\ \hline R(\omega)\Gamma & \cdots & R(\omega)\Gamma \end{array}\right).$$

The generic element of the matrix $R(\omega)\Gamma$ at row (j,k) and column (j',k') is equal to $h_j(\omega)\gamma_{jj'}^{kk'}$. Thus, the generic element of the symmetric part of $R(\omega)\Gamma$, that is $(R(\omega)\Gamma + (R(\omega)\Gamma)^\top)/2$, at row (j,k) and column (j',k') is equal to

$$\frac{h_j(\omega)\gamma_{jj'}^{kk'} + h_{j'}(\omega)\gamma_{j'j}^{k'k}}{2}.$$

The assumptions on $h_j(\omega)$ and (4.6) guarantee that, for any $j = 1,\ldots,n$ and $k = 1,\ldots,K$, the following chain of inequalities holds for any $\omega \in \Omega$:

$$h_j(\omega)\gamma_{jj}^{kk} \geq \underline{h}\,\gamma_{jj}^{kk} \geq \sum_{(j',k')\neq(j,k)} \frac{\bar{h}}{2}\left(\gamma_{jj'}^{kk'} + \gamma_{j'j}^{k'k}\right) \geq \sum_{(j',k')\neq(j,k)} \frac{h_j(\omega)\gamma_{jj'}^{kk'} + h_{j'}(\omega)\gamma_{j'j}^{k'k}}{2}.$$

Hence, the symmetric part of $R(\omega)\Gamma$ is a diagonally dominant matrix for any $\omega \in \Omega$. Therefore, the symmetric part of $R(\omega)\Gamma$ and $R(\omega)\Gamma$ itself are positive semidefinite for any $\omega \in \Omega$. Finally, given any $Q, \tilde{Q} \in \mathbb{R}^{mnKL}$, we have

$$\begin{aligned}(Q-\tilde{Q})^\top[F_3(\omega,Q) - F_3(\omega,\tilde{Q})] &= (Q-\tilde{Q})^\top H(\omega)\Gamma T(Q-\tilde{Q}) \\ &= \left[\sum_{j=1}^{n}\sum_{l=1}^{L}(Q_j^l - \tilde{Q}_j^l)\right]^\top R(\omega)\Gamma \left[\sum_{j=1}^{n}\sum_{l=1}^{L}(Q_j^l - \tilde{Q}_j^l)\right] \geq 0,\end{aligned}$$

where Q_i^l denotes the subvector of Q with components Q_{ij}^{kl} for $j = 1,\ldots,n$ and $k = 1,\ldots,K$. Thus, F_3 is monotone on $\mathbb{R}^{mnKL}$ for any $\omega \in \Omega$. It then follows that $F = F_1 + F_2 + F_3$ is strongly monotone on $\mathbb{R}^{mnKL}$, uniformly with respect to $\omega \in \Omega$. □

Theorem 4.1 deals with the most general case in which any price or cost function depends on all the possible variables. However, in applications it is often interesting to consider the special case in which the supply price functions P_i^k (resp. demand price functions R_j^k) depend only on the supply of commodities produced at market i (resp. the demand for commodities in market j). The following result provides, in this particular case, weaker conditions than (4.2), (4.4) and (4.6) for F to be strongly monotone. The proof is omitted because it is very similar to that of Theorem 4.1.

Corollary 4.1. *Assume, for any $i = 1,\ldots,m$, that there exist $\underline{e}_i, \bar{e}_i > 0$ such that $\underline{e}_i \leq e_i(\omega) \leq \bar{e}_i$ for any $\omega \in \Omega$, and, for any $j = 1,\ldots,n$, that there exist $\underline{h}_j, \bar{h}_j > 0$ such that $\underline{h}_j \leq h_j(\omega) \leq \bar{h}_j$ for any $\omega \in \Omega$. Moreover, assume that any supply price function is $P_i^k(s) = \sum_{k'=1}^{K} \alpha_i^{kk'} s_i^{k'} + a_i^k$, with*

$\alpha_i^{kk'} > 0$, for any $i, = 1,\dots,m$, $k,k' = 1,\dots,K$, and $2\underline{e}_i \alpha_i^{kk} \ge \bar{e}_i \sum_{k' \neq k} \left(\alpha_i^{kk'} + \alpha_i^{k'k}\right)$ holds for any $i = 1,\dots,m$, $k = 1,\dots,K$; any transportation cost function is $c_{ij}^{kl}(Q) = \sum_{k'=1}^{K} \beta_{ij}^{kk'l} Q_{ij}^{k'l} + b_{ij}^{kl}$, with $\beta_{ij}^{kk'l} > 0$, for any $i = 1,\dots,m$, $j = 1,\dots,n$, $k,k' = 1,\dots,K$, $l = 1,\dots,L$, and $2\underline{e}_i \beta_{ij}^{kkl} > \bar{e}_i \sum_{k' \neq k} \left(\beta_{ij}^{kk'l} + \beta_{ij}^{k'kl}\right)$ holds for any $i = 1,\dots,m$, $j = 1,\dots,n$, $k = 1,\dots,K$, $l = 1,\dots,L$; any demand price function is $R_j^k(d) = -\sum_{k'=1}^{K} \gamma_j^{kk'} d_j^{k'} + g_i^k$, with $\gamma_j^{kk'} > 0$, for any $j = 1,\dots,n$, $k,k' = 1,\dots,K$, and $2\underline{h}_j \gamma_j^{kk} \ge \bar{h}_j \sum_{k' \neq k} \left(\gamma_j^{kk'} + \gamma_j^{k'k}\right)$ holds for any $j = 1,\dots,n$ and $k = 1,\dots,K$. Then the map F is strongly monotone, uniformly with respect to $\omega \in \Omega$.

5. Vulnerability Analysis

For each pair (i,j) of producer and consumer countries, we denote with

$$\mathscr{R}^{(i,j)} := \left\{ r_1^{(i,j)}, r_2^{(i,j)}, \dots, r_L^{(i,j)} \right\}, \qquad i = 1,\dots,m,\ j = 1,\dots,n,$$

the set of all the routes which connect i and j, while the set of all routes in the network is denoted by $\mathscr{R} := \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \mathscr{R}^{(i,j)}$. Let $A \subset \mathscr{R}$ be a given subset of routes which are no longer available in the network. The disruptions may have been caused by natural disasters or, for instance, by military attacks. We denote $\mathscr{R}_{-A} := \mathscr{R} \setminus A$ and $\mathscr{A}$ the family of all the subsets A which will be included in our analysis.

In order to address the network vulnerability, we consider two measures: the global expected trade volume, at equilibrium, corresponding to the baseline route configuration $\mathscr{R}$, defined as

$$W(\mathscr{R}) := \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} \sum_{l=1}^{L} E_{\mathbb{P}} \left[U_{ij}^{kl*}(\mathscr{R}) \right],$$

where $U^*(\mathscr{R})$ is the equilibrium flow corresponding to the configuration $\mathscr{R}$, and the expected network performance (see [17]) corresponding to the baseline route configuration $\mathscr{R}$, defined as

$$P(\mathscr{R}) := \frac{1}{Kn} \sum_{k=1}^{K} \sum_{j=1}^{n} \frac{E_{\mathbb{P}} \left[\sum\limits_{i=1}^{m} \sum\limits_{l=1}^{L} U_{ij}^{kl*}(\mathscr{R}) \right]}{E_{\mathbb{P}} \left[\rho_j^k(U^*(\mathscr{R})) H_j \right]}.$$

The latter measure implies that the international trade network performs better if, on the average, it can handle higher commodity demands at lower prices. Now, for each $A \in \mathscr{A}$, we compute

$$W(\mathscr{R}_{-A}) := \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} \sum_{l=1}^{L} E_{\mathbb{P}} \left[U_{ij}^{kl*}(\mathscr{R}_{-A}) \right]$$

and

$$P(\mathscr{R}_{-A}) := \frac{1}{Kn} \sum_{k=1}^{K} \sum_{j=1}^{n} \frac{E_{\mathbb{P}} \left[\sum\limits_{i=1}^{m} \sum\limits_{l=1}^{L} U_{ij}^{kl*}(\mathscr{R}_{-A}) \right]}{E_{\mathbb{P}} \left[\rho_j^k(U^*(\mathscr{R}_{-A})) H_j \right]},$$

where $U^*(\mathscr{R}_{-A})$ is the equilibrium flow corresponding to the new network configuration $\mathscr{R}_{-A}$. Then, the quantities

$$\frac{W(\mathscr{R}) - W(\mathscr{R}_{-A})}{W(\mathscr{R})} \qquad \text{and} \qquad \frac{P(\mathscr{R}) - P(\mathscr{R}_{-A})}{P(\mathscr{R})}$$

express the relative variation of the global expected trade volumes and the expected network performance with respect to the baseline scenario, respectively. We are thus naturally led to search the set of routes A whose disruption will affect the most the global trade volume or the network performance. These routes are then the ones which need most protection and for which recovery plans are a priority.

6. Numerical Experiments

This section presents numerical results obtained using the approximation method detailed in Section 3. The procedure was implemented in MATLAB R2024b. At each iteration, a finite-dimensional variational inequality, as defined in equation (3.1), was solved. Leveraging the affine and strongly monotone nature of these inequalities, they were reformulated as equivalent convex quadratic optimization problems (following [2]). These problems were then solved using MATLAB's `quadprog` function from the optimization toolbox.

We divide this section into two parts. The first part analyzes a simplified trade network with one supply country, two commodities, two demand countries, and two routes per supply-demand pair, showing the convergence of the mean values and standard deviations of the approximated solutions in two scenarios. The second part evaluates network vulnerability by quantifying the impact of route disruptions on global trade volumes. We illustrate this with a numerical example based on [17], modeling Ukraine's wheat and corn exports to Lebanon and Egypt via two routes: Black Sea ports and overland via Romania. The model accounts for Ukrainian hryvnia (UAH), Lebanese pound (LBP), Egyptian pound (EGP) and United States dollar (USD) currencies.

Adopting a pre-war scenario, we used supply, transportation, and demand price functions based on [17], structured as in Corollary 4.1. This ensures F is strongly monotone, uniformly with respect to $\omega \in \Omega$. The supply price functions for wheat and corn (UAH per ton) are then defined as

$$P_1^1(s) = 0.000136 s_1^1 + 0.000058 s_1^2 + 7{,}001.6, \qquad P_1^2(s) = 0.000063 s_1^1 + 0.000142 s_1^2 + 6{,}728.2.$$

The unit transportation cost functions for wheat and corn (UAH per ton) are:

$$\begin{aligned}
c_{11}^{11}(Q) &= 0.000556 Q_{11}^{11} + 2{,}046.8, & c_{11}^{12}(Q) &= 0.007512 Q_{11}^{12} + 1{,}984.6,\\
c_{11}^{21}(Q) &= 0.005566 Q_{11}^{21} + 2{,}046.8, & c_{11}^{22}(Q) &= 0.006812 Q_{11}^{22} + 1{,}984.6,\\
c_{12}^{11}(Q) &= 0.000185 Q_{12}^{11} + 2{,}046.8, & c_{12}^{12}(Q) &= 0.007312 Q_{12}^{12} + 1{,}984.6,\\
c_{12}^{21}(Q) &= 0.001259 Q_{12}^{21} + 2{,}046.8, & c_{12}^{22}(Q) &= 0.007012 Q_{12}^{22} + 1{,}984.6.
\end{aligned}$$

The demand price functions for wheat and corn, in LBP per ton, are given by:

$$R_1^1(d) = -0.15 d_1^1 - 0.02 d_1^2 + 602{,}344, \qquad R_1^2(d) = -0.1 d_1^1 - 0.68 d_1^2 + 574{,}560,$$

while the demand price functions for wheat and corn, in EGP per ton, are given by:

$$R_2^1(d) = -0.000475 d_2^1 + 0.0001 d_2^2 + 6{,}290, \qquad R_2^2(d) = -0.0002 d_2^1 - 0.000758 d_2^2 + 5{,}980.$$

The supply capacity in Ukraine is $\bar{S}_1 = 5{,}000{,}000$ tons, while the transportation capacities (in tons) of routes are:

$$\bar{Q}_{11}^1 = 5{,}000{,}000, \qquad \bar{Q}_{11}^2 = 500{,}000, \qquad \bar{Q}_{12}^1 = 5{,}000{,}000, \qquad \bar{Q}_{12}^2 = 500{,}000.$$

TABLE 1. Scenario 1 with uniform distribution: mean values and standard deviations of the approximated solution.

	Mean values			
Variables	$N=10$	$N=20$	$N=50$	$N=100$
U_{11}^{11}	448,705.018	444,695.582	442,326.406	441,541.360
U_{11}^{12}	41,490.946	41,194.188	41,021.091	40,963.044
U_{11}^{21}	17,451.176	17,276.475	17,177.668	17,143.886
U_{11}^{22}	23,356.983	23,206.316	23,118.530	23,089.822
U_{12}^{11}	1,885,748.546	1,902,453.951	1,912,485.901	1,915,835.340
U_{12}^{12}	56,217.653	56,640.315	56,900.589	56,985.586
U_{12}^{21}	432,447.007	436,855.146	439,489.377	440,367.188
U_{12}^{22}	86,516.084	87,300.212	87,774.047	87,931.398
	Standard deviations			
Variables	$N=10$	$N=20$	$N=50$	$N=100$
U_{11}^{11}	160,051.188	160,092.823	159,952.099	159,870.569
U_{11}^{12}	11,846.174	11,849.256	11,856.958	11,848.457
U_{11}^{21}	8,381.774	8,374.217	8,460.901	8,457.802
U_{11}^{22}	6,924.019	6,936.044	6,934.704	6,932.887
U_{12}^{11}	701,843.939	699,111.747	696,718.796	695,796.645
U_{12}^{12}	17,757.266	17,688.139	17,685.011	17,657.217
U_{12}^{21}	189,745.997	189,074.401	188,519.796	188,299.874
U_{12}^{22}	34,068.769	33,965.485	33,870.862	33,824.800

We examined two scenarios with varying exchange rate volatility. In the first scenario, the UAH/USD exchange rate (E_1) ranged from 1/30 to 1/25, the LBP/USD rate (H_1) from 1/1600 to 1/1400, and the EGP/USD rate (H_2) from 1/17 to 1/13. The second scenario featured higher volatility, with E_1 ranging from 1/50 to 1/25, H_1 from 1/1800 to 1/1400, and H_2 from 1/30 to 1/13. Each interval was divided into N subintervals ($N = 10, 20, 50, 100$) for the approximation procedure. We considered both uniform and truncated normal distributions for the random variables in each scenario.

For Scenario 1, Table 1 presents the convergence of mean values and standard deviations for a uniform distribution of exchange rates. The results indicate that Black Sea port transport is generally preferred over the Romanian route for both wheat and corn, though this preference is less pronounced for corn exports to Lebanon due to the smaller volumes involved.

For Scenario 1 with truncated normal exchange rate distributions, Table 2 presents the convergence of mean values and standard deviations. The truncated normal distributions were defined with means at the interval midpoints and standard deviations equal to one-tenth of the interval widths. Notably, the mean values of the approximate solutions are similar to those obtained with uniform distributions, while the standard deviations are approximately one-third of the uniform distribution results.

TABLE 2. Scenario 1 with truncated normal distribution: mean values and standard deviations of the approximated solution.

Mean values				
Variables	$N = 10$	$N = 20$	$N = 50$	$N = 100$
U_{11}^{11}	448,920.665	444,986.377	442,640.715	441,861.373
U_{11}^{12}	41,506.908	41,215.712	41,042.099	40,984.458
U_{11}^{21}	17,602.284	17,427.988	17,323.544	17,288.844
U_{11}^{22}	23,513.551	23,371.136	23,285.795	23,257.377
U_{12}^{11}	1,911,198.172	1,928,154.188	1,938,159.654	1,941,467.082
U_{12}^{12}	56,861.551	57,290.553	57,543.704	57,627.504
U_{12}^{21}	437,028.587	441,480.241	444,114.331	444,986.446
U_{12}^{22}	87,338.704	88,137.995	88,610.943	88,767.493
Standard deviations				
Variables	$N = 10$	$N = 20$	$N = 50$	$N = 100$
U_{11}^{11}	58,114.406	56,236.838	55,648.627	55,545.032
U_{11}^{12}	4,301.333	4,162.364	4,118.845	4,111.917
U_{11}^{21}	3,061.078	2,963.319	2,933.083	2,931.754
U_{11}^{22}	2,501.169	2,421.291	2,396.523	2,392.302
U_{12}^{11}	253,997.583	244,770.952	241,615.947	240,969.447
U_{12}^{12}	6,426.361	6,192.919	6,113.183	6,099.338
U_{12}^{21}	68,813.132	66,363.486	65,536.916	65,370.906
U_{12}^{22}	12,355.353	11,915.520	11,767.114	11,737.471

Analogously, for Scenario 2, Table 3 displays the convergence of mean values and standard deviations for uniform exchange rate distributions, while Table 4 presents the results for truncated normal distributions. Consistent with Scenario 1, direct Black Sea port transport of wheat remains preferable to the Romanian route for both Lebanon and Egypt. Notably, Scenario 2 exhibits higher wheat and corn exports to Lebanon but lower exports to Egypt compared to Scenario 1. This difference likely stems from the greater volatility increase of the Egyptian pound relative to the Lebanese pound in Scenario 2.

Table 5 presents the percentage ratios of standard deviation to mean for the approximated solutions in both scenarios, using uniform and truncated normal distributions. The results demonstrate that, as expected, the higher exchange rate volatility in Scenario 2 leads to significantly larger standard deviation-to-mean ratios compared to Scenario 1.

To assess network vulnerability, we examined six possible route disruption scenarios, represented by sets A: individual route 1 or route 2 closures for each supply-demand pair, and simultaneous closures of route 1 or route 2 for both pairs. Table 6 displays the global expected trade volumes and network performance for each disruption, along with their relative variations compared to the baseline scenario. The results highlight the critical importance of route 1 (Black Sea port exports) over route 2 (Romanian route). For example, closing route 1 between Ukraine and Lebanon reduces trade volumes by approximately 8%, while closing route 2 reduces them by less than 1%. This effect is more pronounced for Ukraine-Egypt trade, where

TABLE 3. Scenario 2 with uniform distribution: mean values and standard deviations of the approximated solution.

	Mean values			
Variables	$N = 10$	$N = 20$	$N = 50$	$N = 100$
U_{11}^{11}	755,534.618	731,890.717	718,049.545	713,506.014
U_{11}^{12}	64,045.943	62,283.775	61,213.537	60,870.305
U_{11}^{21}	30,703.377	29,684.094	29,073.545	28,875.734
U_{11}^{22}	33,815.828	32,918.227	32,397.017	32,224.722
U_{12}^{11}	1,627,552.463	1,648,651.755	1,660,890.145	1,664,955.739
U_{12}^{12}	47,509.988	48,096.812	48,454.817	48,566.882
U_{12}^{21}	390,026.419	395,008.674	397,866.394	398,816.117
U_{12}^{22}	76,504.458	77,429.896	77,975.717	78,153.980
	Standard deviations			
Variables	$N = 10$	$N = 20$	$N = 50$	$N = 100$
U_{11}^{11}	403,101.002	399,989.149	397,639.391	396,801.095
U_{11}^{12}	30,144.036	30,082.703	29,879.191	29,827.597
U_{11}^{21}	18,978.454	18,871.979	18,746.165	18,707.090
U_{11}^{22}	16,239.734	16,203.877	16,170.956	16,152.557
U_{12}^{11}	1,277,806.681	1,279,060.889	1,279,473.924	1,279,488.937
U_{12}^{12}	35,185.323	35,178.344	35,182.807	35,159.424
U_{12}^{21}	320,699.671	321,469.704	321,803.992	321,891.312
U_{12}^{22}	60,552.229	60,781.721	60,748.452	60,746.459

route 1 closure leads to a 65% decrease, and route 2 closure only a 1% decrease. Similarly, simultaneous closure of route 1 for both destinations results in an 77% trade volume reduction, compared to a 2% reduction for simultaneous route 2 closures. The findings regarding network performance corroborate the trends seen in expected trade volumes.

7. CONCLUSIONS

In this paper, we have improved a previous variational inequality model of the international food supply chain. In particular, the investigation of a large class of non-separable price and cost functions greatly enhances the flexibility of the model, allowing the description of competitive behavior at all levels of the supply chain. A vulnerability analysis of the supply chain is carried out using two different measures of the road network and illustrated by some simple examples. This research can be developed in several directions: investigation of nonlinear price and cost functions; study of optimal investments in infrastructure or policy mechanisms to increase resilience or reduce vulnerability; calibration of models using real trade and exchange rate data from international agencies (e.g. FAO, WTO, IMF); development of scalable solvers for large networks.

Acknowledgments

TABLE 4. Scenario 2 with truncated normal distribution: mean values and standard deviations of the approximated solution.

	Mean values			
Variables	$N=10$	$N=20$	$N=50$	$N=100$
U_{11}^{11}	862,073.733	836,289.106	820,967.292	815,862.020
U_{11}^{12}	72,086.394	70,177.950	69,047.372	68,667.539
U_{11}^{21}	37,415.204	36,251.620	35,553.041	35,318.878
U_{11}^{22}	39,702.423	38,751.675	38,182.602	37,990.005
U_{12}^{11}	1,745,932.550	1,777,210.003	1,795,167.070	1,800,952.975
U_{12}^{12}	52,458.114	53,323.255	53,803.255	53,954.243
U_{12}^{21}	407,159.425	414,149.550	418,364.802	419,730.344
U_{12}^{22}	81,673.647	82,990.111	83,782.736	84,033.547
	Standard deviations			
Variables	$N=10$	$N=20$	$N=50$	$N=100$
U_{11}^{11}	156,395.473	150,919.617	149,114.039	148,745.786
U_{11}^{12}	11,575.597	11,170.327	11,055.335	11,016.216
U_{11}^{21}	7,779.144	7,538.522	7,470.433	7,458.453
U_{11}^{22}	6,356.335	6,159.704	6,118.870	6,098.026
U_{12}^{11}	779,751.038	748,878.273	737,063.424	734,094.429
U_{12}^{12}	20,257.529	19,326.235	18,978.043	18,888.938
U_{12}^{21}	205,174.032	198,174.452	195,442.720	194,778.593
U_{12}^{22}	37,468.062	36,110.252	35,564.740	35,424.863

TABLE 5. Percentage ratio between the standard deviation and the mean of the approximated solution in the two scenarios with uniform and truncated normal distributions.

	Scenario 1		Scenario 2	
Variables	uniform	truncated normal	uniform	truncated normal
U_{11}^{11}	36.21%	12.57%	55.61%	18.23%
U_{11}^{12}	28.92%	10.03%	49.00%	16.04%
U_{11}^{21}	49.33%	16.96%	64.78%	21.12%
U_{11}^{22}	30.03%	10.29%	50.12%	16.05%
U_{12}^{11}	36.32%	12.41%	76.85%	40.76%
U_{12}^{12}	30.99%	10.58%	72.39%	35.01%
U_{12}^{21}	42.76%	14.69%	80.71%	46.41%
U_{12}^{22}	38.47%	13.22%	77.73%	42.16%

TABLE 6. Vulnerability analysis: global expected trade volumes and network performance corresponding to six possible route removals and their relative variations with respect to the baseline scenario.

Route removals		Trade	Rel. variation of trade	Network	Rel. variation of network
Ukraine-Lebanon	Ukraine-Egypt	volumes	volumes	performance	performance
route 1	–	2,763,646	8.50%	1,940	8.79%
route 2	–	3,009,545	0.36%	2,118	0.42%
–	route 1	1,030,717	65.87%	701	67.04%
–	route 2	2,982,505	1.25%	2,097	1.41%
route 1	route 1	694,442	77.01%	444	79.13%
route 2	route 2	2,971,573	1.61%	2,088	1.83%
baseline scenario		3,020,293		2,127	

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