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ON THE EXISTENCE OF BILATERAL SOLUTIONS TO AN ANISOTROPIC NONLINEAR COUPLED ELLIPTIC SYSTEM

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Abstract. The existence of a bilateral solution at a given height to the strongly nonlinear and degenerate problem $A(u) = \rho(u)|\nabla\varphi|^2$, $\operatorname{div}(\rho(u)\nabla\varphi) = 0$ in Ω, $u = 0$ and $\varphi = \varphi_0$ on $\partial\Omega$, where A is a Leray-Lions operator, is proved in the framework of anisotropic Sobolev space. The bilateral solution is obtained through a double approximation process, with the first one being a penalization technique.

Keywords. Anisotropic Sobolev spaces; Bilateral solution; Coupled system; Strongly nonlinear elliptic equation; Thermistor problem.

1. INTRODUCTION

We consider a nonlinear coupled system of elliptic type in the framework of anisotropic Sobolev spaces. This system is given as follows.

$$\begin{cases} A(u) = \rho(u)|\nabla\varphi|^2 & \text{in } \Omega, \\ \operatorname{div}(\rho(u)\nabla\varphi) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \varphi = \varphi_0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where A is a strongly nonlinear operator in divergence form, namely,

$$A(u) = -\operatorname{div}(a(x,u,\nabla u)). \tag{1.2}$$

As a model example, operator (1.2) includes the particular case of the $\vec{p}$-Laplacian,

$$-\Delta_{\vec{p}}u = -\sum_{\nu=1}^{N} \partial_\nu(|\partial_\nu u|^{p_\nu-2}\partial_\nu u), \quad (\partial_\nu = \partial/\partial x_\nu).$$

Notice that system (1.1) is a generalized model of the well-known thermistor problem. In that setting, u stands for the temperature, φ is the electric potential, $\rho(u)$ is the temperature dependent electric conductivity, $\Omega \subset \mathbb{R}^N$ is the region in the space occupied by the semiconductor device (thermistor), and $N \geq 1$ is the space dimension. The first equation of (1.1) expresses the

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diffusion of heat inside the semiconductor which is generated from the source term $\rho(u)|\nabla\varphi|^2$, which is Joule's effect. The second equation describes the conservation of electric charges.

The difficulty in the mathematical analysis of a problem like (1.1) does not only come from the nonlinearities appearing in both partial differential equations but from the nonuniformly elliptic character of the second equation as well. Indeed, in most practical situations, one has $\rho \in C(\mathbb{R})$ is such that $\rho(s) > 0$ with $\rho(s) \to 0$ as $s \to +\infty$. In order to deal with this difficulty, Xu ([1]) introduced the notion of capacity solutions to the thermistor problem (evolution case) in the framework of Sobolev spaces with $p = 2$. He proved the existence of a capacity solution. Afterwards, other existence results were obtained by many authors, In particular, the authors in [2, 3] proved an existence result of a capacity solution to a nonstationary thermistor problem in the classical Sobolev spaces $W^{1,p}(\Omega)$ for any $p \geq 2$. Also, the authors in [4, 5] studied the existence of a solution for both the steady-state and the evolution thermistor problem in the context of Orlicz-Sobolev spaces.

In [6, 7], Ortegón Gallego, Rhoudaf and Talbi analyzed the existence of a capacity solution to a coupled nonlinear elliptic or parabolic-elliptic systems in the context of anisotropic Sobolev spaces. This result was generalized to the anisotropic Orlicz-Sobolev space by Ortegón Gallego, Ouyahya and Rhoudaf in [8]. In all these results, the authors have just considered the case of a Sobolev space with $p \geq 2$ or the Orlicz-Sobolev space with an equivalent condition, that is, the N-function M admits the representation $M(s) = \int_0^{|s|} m(t)\,\mathrm{d}t$ with $m(t) \geq t$ for all $t \geq 0$. Due to this assumption, we may deduce that the right-hand side of our first equation lies in a 'good' dual space. Moreover, by considering $\rho(u)\nabla\varphi$ as a single function, we may derive a new variational formulation and the solution to this new formulation is called a capacity solution. For the other cases, namely $1 < p < 2$, or $m(t) < t$ in the context of the Orlicz spaces, the right hand side of the first equation does not belong to the adequate dual space, even for capacity solutions. The introduction of the notion of bilateral solutions may deal with these situations so that this analysis fills the gap $1 < p < 2$ or $m(t) < t$.

The notion of bilateral solutions to problem (1.1) was recently introduced by the authors in [9] in the context of isotropic Sobolev spaces, $W^{1,p}(\Omega)$, $1 < p < \infty$. In this paper, under more restrictive assumptions on the data, we demonstrate the existence of a bilateral solution to problem (1.1) in the framework of anisotropic Sobolev spaces. In order to prove an existence result of a solution for a certain approximate problem, we make use of the so-called penalization technique, which was firstly introduced by Boccardo and Murat in [10]. These authors approximated the following variational inequality

$$\begin{cases} \text{To find } u \in \mathscr{K} \text{ such that,} \\ \langle A(u), v-u\rangle \geq \langle f, v-u\rangle, \text{ for all } v \in \mathscr{K}, \end{cases}$$

where the convex set $\mathscr{K}$ is defined by $\mathscr{K} = \{v \in W_0^{1,p}(\Omega) \colon |v(x)| \leq 1 \text{ a.e. in } \Omega\}$, the mapping A is given by $A(u) = -\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $f \in W^{-1,p'}(\Omega)$, and by the sequence of problems

$$\begin{cases} A(u_n) + |u_n|^{n-2}u_n = f & \text{in } \mathscr{D}'(\Omega), \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Inspired by this approach, we investigate certain bilateral solutions to system (1.1) at a given height $M > 0$ in the sense of the Definition 3.1 given below. Notice that the existence, in the framework of anisotropic Sobolev spaces of order r and exponent $\vec{p} := (p_0, \ldots, p_N)$, $W_0^{r,\vec{p}}(\Omega)$, with $rp_0 > N$, $p_0 = \min_{1\le\nu\le N}\{p_\nu\}$, of weak solutions to the following problem

$$\begin{cases} A(u)+g(x,u)=f & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

where A is a strongly nonlinear operator satisfying the monotonicity, coerciveness, and growth conditions, and where the nonlinear perturbation term g is a Carathéodory function and satisfies the sign condition $g(x,t)t \ge 0$, for all $t \in \mathbb{R}$, but without any restriction on its growth, was proved in [11] in the following sense

$$\begin{cases} u \in W_0^{1,\vec{p}}(\Omega), g(x,u) \in L^1(\Omega) \text{ and } g(x,u)u \in L^1(\Omega) \\ \langle A(u),v\rangle + \int_\Omega g(x,u)v\,\mathrm{d}x = \langle f,v\rangle, \text{ for all } v \in W_0^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega). \end{cases} \tag{1.4}$$

If $rp_0 > N$, we have $W_0^{r,\vec{p}}(\Omega) \subset L^\infty(\Omega)$ and the test functions in (1.4) run the whole space $W_0^{r,\vec{p}}(\Omega)$. Problem (1.3) was also considered in [12, 13] for $r=1$, $\vec{p}$ being an admissible vector and with a less restrictive assumption on the domain Ω. In these works, the authors proved, by using the Hedberg-type's approximation, that the solution, u, of (1.3) could be taken as a test function in (1.4), and then problem (1.3) has a solution in the following sense

$$\begin{cases} u \in W_0^{1,\vec{p}}(\Omega), g(x,u) \in L^1(\Omega) \text{ and } g(x,u)u \in L^1(\Omega) \\ \langle A(u),u-v\rangle + \int_\Omega g(x,u)(u-v)\,\mathrm{d}x = \langle f,u-v\rangle, \text{for all } v \in W_0^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega). \end{cases} \tag{1.5}$$

In this paper, we focus on $r=1$ and any exponents $\vec{p}$ such that $1 < p_\nu < \infty$, for all $\nu = 0,\dots,N$. In particular, this covers the case $1 < p_\nu < 2$ for at least one $\nu \in \{0,\dots,N\}$, so that the required condition above $p_0 > N$, which may not be satisfied, is included as well. In fact, without assuming that $p_0 > N$, we will show that problem (1.3) still has a solution in the sense (1.5), for any $f \in W^{-1,\vec{p}'}(\Omega)$. For the particular case when $p_\nu = p$ for all $\nu = 1,\dots,N$, we refer to [14]. The case $0 < r < 1$, where fractional derivatives spaces are involved ([15]), may be the subject of future works.

In [16], where the dependence of the solution with respect to different choices of the three exponents p_1, p_2, and p_3 was remarked, we can find some 3D numerical simulations of problem (1.1). Moreover, these numerical simulations seem to infer that the problem may have multiple solutions when one the exponents is close enough to 1 while the others two are kept constant. Thus the question of the uniqueness of the solution is still open.

In order to study the problem (1.1) under the assumptions given below, we adopt the following organization. In Section 2, we introduce the definition of the anisotropic Sobolev spaces, recall some of their properties, and give some technical lemmas. In Section 3, we enumerate the assumptions on the data and introduce the notion of bilateral solutions adapted to our context. In Section 4, we present our main results. In the last section, Section 5, we present the proof of the main results.

2. Preliminaries

In this section, we begin by recalling the definition of the anisotropic Sobolev spaces, and giving some of their properties.

Let Ω be an open bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with boundary $\partial\Omega$. We denote by $\vec{p} := (p_1,\dots,p_N)$ and by $p_0 = \min_{1\leq\nu\leq N}\{p_\nu\}$, and we introduce the anisotropic Sobolev space of exponent $\vec{p}$, denoted by $W^{1,\vec{p}}(\Omega)$, as

$$W^{1,\vec{p}}(\Omega) = \left\{u \in L^{p_0}(\Omega)/\ \partial_\nu u \in L^{p_\nu}(\Omega),\ \text{for all } \nu = 1,\dots,N\right\}.$$

The space $W^{1,\vec{p}}(\Omega)$ is a Banach space equipped with the norm $\|u\|_{\vec{p}} = \sum_{\nu=0}^{N}\|\partial_\nu u\|_{p_\nu}$, where $\partial_0 u = u, \partial_\nu u = \frac{\partial u}{\partial x_\nu}$, and $\|\cdot\|_{p_\nu}$ is the standard norm in $L^{p_\nu}(\Omega)$. Since we will consider homogenous Dirichlet boundary conditions, we use the functional space $W_0^{1,\vec{p}}(\Omega)$ defined as the closure of $\mathscr{D}(\Omega)$ in $W^{1,\vec{p}}(\Omega)$.

Both spaces $W^{1,\vec{p}}(\Omega)$ and $W_0^{1,\vec{p}}(\Omega)$ are separable when $1 \leq p_\nu < \infty$ for all $\nu = 1,\dots,N$, and reflexive when $1 < p_\nu < \infty$ for all $\nu = 1,\dots,N$ (the proof is an adaptation from Adams [17]). When $\vec{p} \in [1,+\infty)^N$, the dual of $W_0^{1,\vec{p}}(\Omega)$ is denoted by $W^{-1,\vec{p}'}(\Omega)$, $\vec{p}' := (p'_1,\dots,p'_N)$, $p'_\nu \in (1,+\infty)\cup\{+\infty\}$ being the conjugate of p_ν, i.e., $1/p'_\nu + 1/p_\nu = 1$ for all $\nu = 1,\dots,N$. In $W_0^{1,\vec{p}}(\Omega)$, the seminorm $|\cdot|_{\vec{p}}$ defined as $|u|_{\vec{p}} = \sum_{\nu=1}^{N}\|\partial_\nu u\|_{p_\nu}$, $u \in W_0^{1,\vec{p}}(\Omega)$, is a norm in $W_0^{1,\vec{p}}(\Omega)$, which is equivalent to the norm $\|\cdot\|_{\vec{p}}$, that is, there exists a constant $C_0 = C_0(\Omega,\vec{p})$ such that

$$|u|_{\vec{p}} \leq \|u\|_{\vec{p}} \leq C_0|u|_{\vec{p}},\ \text{for all } u \in W_0^{1,\vec{p}}(\Omega). \tag{2.1}$$

Lemma 2.1. *Let Ω be a bounded open set of $\mathbb{R}^N$. Then, the natural injection $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^{p_0}(\Omega)$ is compact.*

The proof of this lemma follows immediately from the classical embedding theorems of Sobolev spaces and the fact that $W_0^{1,\vec{p}}(\Omega) \hookrightarrow W_0^{1,p_0}(\Omega)$ with continuous injection. In particular, since $W_0^{1,p_0}(\Omega) \subset L^q(\Omega)$ with $q = p_0^* = Np_0/(N-p_0)$ if $p_0 < N$, any $q \in [1,\infty)$ if $p_0 = N$, or $q = \infty$ if $p_0 > N$, the same is true for $W_0^{1,\vec{p}}(\Omega)$. The next result tells us that we may improve this L^q-regularity for the functions in $W_0^{1,\vec{p}}(\Omega)$.

Lemma 2.2 ([18, 19, 20])**.** *Let $p_1,\dots,p_N \in \mathbb{R}$ such that $p_0 = \min_{1\leq\nu\leq N}\{p_\nu\} > 1$. Let $\bar{p}$ be the harmonic mean of these numbers, that is $1/\bar{p} = 1/N\sum_{\nu=1}^{N} 1/p_\nu$. Then, there exists a constant $C > 0$ such that*

$$\|u\|_q \leq C\prod_{\nu=1}^{N}\|\partial_\nu u\|_{p_\nu}^{1/N},\ \textit{for all } u \in W_0^{1,\vec{p}}(\Omega), \tag{2.2}$$

where $q = \bar{p}^* = N\bar{p}/(N-\bar{p})$ if $\bar{p} < N$, any $q \in [1,\infty)$ if $\bar{p} \geq N$.

A straightforward consequence of Lemma 2.2 is the continuous injection $W_0^{1,\vec{p}}(\Omega) \subset L^q(\Omega)$ where q means the same as in this last result. Indeed, this is due to (2.2) together with the celebrated inequality relating the geometric mean and the arithmetic mean, namely,

$$\prod_{\nu=1}^{N} a_\nu^{1/N} \leq \frac{1}{N}\sum_{\nu=1}^{N} a_\nu,\ \text{for all } a_1,\dots,a_N \in [0,+\infty).$$

The next lemma is useful in combination with the assumption (3.4) below.

Lemma 2.3. *Let $p_1,\dots,p_N,a_1,\dots,a_N$ be $2N$ real numbers such that $p_0 = \min_{1\leq\nu\leq N}\{p_\nu\} \geq 1$ and $\min_{1\leq\nu\leq N}\{a_\nu\} \geq 0$. Then,*

$$a_1^{p_1} + \cdots + a_N^{p_N} \geq \frac{1}{N^{p_0-1}}(a_1 + \cdots + a_N)^{p_0} - (N-1).$$

Proof. Since $p_0 \geq 1$, we see that $s \in [0,+\infty) \mapsto s^{p_0}$ is convex. Thus

$$\left(\frac{a_1+\cdots+a_N}{N}\right)^{p_0} \leq \frac{1}{N}(a_1^{p_0}+\cdots+a_N^{p_0}),$$

and

$$a_1^{p_0}+\cdots+a_N^{p_0} \geq \frac{1}{N^{p_0-1}}(a_1+\cdots+a_N)^{p_0}.$$

On the other hand, the following inequalities hold

$$a_\nu^{p_\nu}+1 \geq a_\nu^{p_0}, \text{ for all } \nu = 1,\ldots,N,$$

with $a_\nu^{p_\nu} = a_\nu^{p_0}$ for at least one $\nu \in \{1,\ldots,N\}$. Summing up these inequalities, we obtain

$$a_1^{p_1}+\cdots+a_N^{p_N} \geq a_1^{p_0}+\cdots+a_N^{p_0}-(N-1) \geq \frac{1}{N^{p_0-1}}(a_1+\cdots+a_N)^{p_0}-(N-1).$$

□

Corollary 2.1. *Let $\vec{p} = (p_1,\ldots,p_N) \in \mathbb{R}^N$ such that $p_0 = \min_{1\leq\nu\leq N}\{p_\nu\} \geq 1$. Then,*

$$\sum_{\nu=1}^{N}\int_\Omega |\partial_\nu u|^{p_\nu} \geq \frac{1}{C_0^{p_0}N^{p_0-1}}\|u\|_{\vec{p}}^{p_0}-(N-1), \text{ for all } u \in W_0^{1,\vec{p}}(\Omega),$$

where $C_0 > 0$ is the constant appearing in (2.1).

Proof. Let $u \in W_0^{1,\vec{p}}(\Omega)$ and put $a_\nu = \|\partial_\nu u\|_{p_\nu} = (\int_\Omega |\partial_\nu u|^{p_\nu})^{1/p_\nu}$ for $\nu = 1,\ldots,N$. According to Lemma 2.3, we have

$$\sum_{\nu=1}^{N}\int_\Omega |\partial_\nu u|^{p_\nu} = a_1^{p_1}+\cdots+a_N^{p_N} \geq \frac{1}{N^{p_0-1}}(a_1+\cdots+a_N)^{p_0}-(N-1) = \frac{1}{N^{p_0-1}}|u|_{\vec{p}}^{p_0}-(N-1).$$

Using (2.1) we obtain the desired result immediately. □

For the sake of completeness, we recall the definition of some useful concepts in abstract Banach spaces. In these definitions, X stands for a reflexive Banach space with norm $\|\cdot\|_X$ and X^* its dual space ([21]).

Definition 2.1. Let $G\colon X \mapsto X^*$ be a mapping.

(1) G is bounded if it transforms any bounded set of X in a bounded set of X^*.

(2) G is said to be coercive if

$$\lim_{\|u\|_X\to\infty}\frac{\langle G(u),u\rangle_{X^*,X}}{\|u\|_X} = +\infty,$$

where $\langle\cdot,\cdot\rangle_{X^*,X}$ is the duality product between X^* and X.

(3) G is monotone if

$$\langle G(u)-G(v),u-v\rangle_{X^*,X} \geq 0 \text{ for any } u,v \in X. \tag{2.3}$$

and strictly monotone if inequality (2.3) is strict whenever $u \neq v$.

(4) G is said to be hemicontinuous if, for all $u,v,w \in X$, $\lim_{t\to 0^+}\langle G(u+tv),w\rangle_{X^*,X} = \langle G(u),w\rangle_{X^*,X}$.

Another important concept in this context is the notion of a pseudo-monotone mapping. This can be done through two equivalent definitions.

Theorem/Definition 2.1. Let $G : X \mapsto X^*$ be a bounded mapping. We say that G is pseudo-monotone if it satisfies one of the two following equivalent conditions:

(PM1) For any sequence $(u_n) \subset X$ such that $u_n \to u$ weakly in X and $\limsup_{n\to\infty}\langle G(u_n), u_n - u\rangle_{X^*,X} \le 0$, it follows that $\liminf_{n\to\infty}\langle G(u_n), u_n - v\rangle_{X^*,X} \ge \langle G(u), u - v\rangle_{X^*,X}$ for all $v \in X$.

(PM2) For any sequence $(u_n) \subset X$ such that $u_n \to u$ weakly in X, $G(u_n) \to \chi$ weakly in X^* and $\limsup_{n\to\infty}\langle G(u_n), u_n - u\rangle_{X^*,X} \le 0$, it follows that $G(u) = \chi$ and $\lim_{n\to\infty}\langle G(u_n), u_n - u\rangle_{X^*,X} = 0$.

Proof. (PM1) ⇒ (PM2) Let $(u_n) \subset X$, $u \in X$, and $\chi \in X^*$ such that $u_n \to u$ weakly in X, $G(u_n) \to \chi$ weakly in X^*, and $\limsup_{n\to\infty}\langle G(u_n), u_n - u\rangle_{X^*,X} \le 0$. Letting $v \in X$, we have

$$\langle G(u_n), u_n - v\rangle_{X^*,X} = \langle G(u_n), u_n - u\rangle_{X^*,X} + \langle G(u_n), u - v\rangle_{X^*,X}.$$

Using this identity and (PM1) yields

$$\begin{cases} \langle G(u), u - v\rangle_{X^*,X} \le \liminf\limits_{n\to\infty}\langle G(u_n), u_n - v\rangle_{X^*,X} \\ \qquad = \liminf\limits_{n\to\infty}\langle G(u_n), u_n - u\rangle_{X^*,X} + \langle \chi, u - v\rangle_{X^*,X} \\ \qquad \le \limsup\limits_{n\to\infty}\langle G(u_n), u_n - u\rangle_{X^*,X} + \langle \chi, u - v\rangle_{X^*,X} \le \langle \chi, u - v\rangle_{X^*,X}. \end{cases} \tag{2.4}$$

Consequently, $\langle G(u) - \chi, u - v\rangle_{X^*,X} \le 0$ for any $v \in X$. Hence $G(u) = \chi$ and all the inequalities in (2.4) are in fact equalities. In particular, $\langle G(u), u - v\rangle_{X^*,X} = \liminf_{n\to\infty}\langle G(u_n), u_n - v\rangle_{X^*,X}$ for all $v \in X$. Taking $v = u$, we deduce

$$\liminf_{n\to\infty}\langle G(u_n), u_n - u\rangle_{X^*,X} = 0 \ge \limsup_{n\to\infty}\langle G(u_n), u_n - u\rangle_{X^*,X},$$

which implies that $\lim_{n\to\infty}\langle G(u_n), u_n - u\rangle_{X^*,X} = 0$.

(PM2) ⇒ (PM1) Let $(u_n) \subset X$, $u \in X$ such that $u_n \to u$ weakly in X and $\limsup_{n\to\infty}\langle G(u_n), u_n - u\rangle_{X^*,X} \le 0$. Since $(u_n) \subset X$ is bounded and G is a bounded mapping, one has that $(G(u_n)) \subset X^*$ is bounded. Due to the reflexivity of X, one deduce thats, for some subsequence $(G(u_k)) \subset (G(u_n))$, there exists $\chi \in X^*$ such that $G(u_k) \to \chi$ weakly in X^*. From (PM2), we obtain that $\chi = G(u)$ and that it is the whole sequence $(G(u_n))$, which is weakly convergent to $G(u)$. Moreover, $\lim_{n\to\infty}\langle G(u_n), u_n - u\rangle_{X^*,X} = 0$. Now, letting $v \in X$, we have $\langle G(u_n), u_n - v\rangle_{X^*,X} = \langle G(u_n), u_n - u\rangle_{X^*,X} + \langle G(u_n), u - v\rangle_{X^*,X}$. Thus $\lim_{n\to\infty}\langle G(u_n), u_n - v\rangle_{X^*,X} = \langle G(u), u - v\rangle_{X^*,X}$. □

Remark 2.1. The definition of a pseudo-monotone mapping given in (PM1) is used in [11, 21] and the equivalent condition (PM2) appears in [14]. The advantage of (PM2) is that, under these assumptions, one can readily identify the weak limit of the sequence $(G(u_n))$ as $G(u)$.

A well-known result relates the concepts of monotonicity and pseudo-monotonicity under certain conditions.

Lemma 2.4. *Let $G\colon X \mapsto X^*$ be a bounded, hemicontinuous, and monotone mapping. Then, G is pseudo-monotone.*

Proof. Let $(u_n) \subset X$, $u \in X$, and $\chi \in X^*$ such that $u_n \to u$ weakly in X, $G(u_n) \to \chi$ weakly in X^* and $\limsup_{n\to\infty}\langle G(u_n), u_n - u\rangle_{X^*,X} \le 0$. Then, for any $v \in X$,

$$0 \le \langle G(u_n) - G(v), u_n - v\rangle_{X^*,X} = \langle G(u_n), u_n - u\rangle_{X^*,X} + \langle G(u_n), u - v\rangle_{X^*,X} - \langle G(v), u_n - v\rangle_{X^*,X}.$$

Thus

$$0 \leq \limsup_{n\to\infty} \langle G(u_n), u_n - u\rangle_{X^*,X} + \langle \chi, u - v\rangle_{X^*,X} - \langle G(v), u - v\rangle_{X^*,X}, \tag{2.5}$$

which yields $0 \leq \langle \chi - G(v), u - v\rangle_{X^*,X}$, for all $v \in X$. Taking $v = u + tw$ for $t \in (0,1)$ and $w \in X$ in this last inequality, we obtain $0 \leq \langle \chi - G(u+tw), w\rangle_{X^*,X}$ for any $t \in (0,1)$ and $w \in X$. Making $t \to 0^+$, and using the hemicontinuity of G, we have $0 \leq \langle \chi - G(u), w\rangle_{X^*,X}$ for all $w \in X$. Consequently, $\chi = G(u)$. Owing to (2.5), we see that

$$\limsup_{n\to\infty} \langle G(u_n), u_n - u\rangle_{X^*,X} = 0. \tag{2.6}$$

On the other hand, we have

$$0 \leq \langle G(u_n) - G(u), u_n - u\rangle_{X^*,X} = \langle G(u_n), u_n - u\rangle_{X^*,X} - \langle G(u), u_n - u\rangle_{X^*,X}.$$

Taking the $\liminf$ in this inequality, we readily obtain $\liminf_{n\to\infty} \langle G(u_n), u_n - u\rangle_{X^*,X} \geq 0$, which, together with (2.6), leads to $\lim_{n\to\infty} \langle G(u_n), u_n - u\rangle_{X^*,X} = 0$. □

The most interesting result on pseudo-monotone mappings is that, under certain assumptions, they are surjective.

Theorem 2.1 ([21])**.** *Let X be a real reflexive Banach space and suppose $G\colon X \mapsto X^*$ is continuous, coercive and pseudo-monotone. Then, for every $f \in X^*$ there exists a solution $u \in X$ of the equation $G(u) = f$. Moreover, if G is strictly monotone, then this solution is unique.*

The following lemma involves a sequence of monotone operators. In fact, it is a generalization of the so-called monotonicity trick.

Lemma 2.5 ([22])**.** *Let X be a Banach space, X^* its dual and $\mathscr{A}_j\colon X \mapsto X^*$, $j \geq 1$, a sequence of mappings. Assume that the sequences $(\mathscr{A}_j)$ and $(u_j) \subset X$ fulfill the following conditions:*

(a) *$\mathscr{A}_j$ is monotone for each $j \geq 1$;*
(b) *$u_j \to u$ weakly in X, for some $u \in X$;*
(c) *$\mathscr{A}_j(u_j) \to \chi$ weakly in X^*, for some $\chi \in X^*$;*
(d) *$\langle \mathscr{A}_j(u_j), u_j\rangle \to \langle \chi, u\rangle$;*
(e) *there exists a mapping $\mathscr{A} : X \mapsto X^*$ such that $\langle \mathscr{A}_j(v), u_j\rangle \to \langle \mathscr{A}(v), u\rangle$ for all $v \in X$;*
(f) *$\mathscr{A}$ is hemicontinuous.*

Then, $\mathscr{A}(u) = \chi$.

We go back to the framework of anisotropic Sobolev spaces. Let $\vec{p} = (p_1, \dots, p_N) \in \mathbb{R}^N$ and assume $p_0 = \min_{1\leq \nu\leq N}\{p_\nu\} > 1$. We then consider the reflexive Banach space $X = W_0^{1,\vec{p}}(\Omega)$ and denote by $\langle\cdot,\cdot\rangle$ the duality product between $W^{-1,\vec{p}'}(\Omega)$ and $W_0^{1,\vec{p}}(\Omega)$. Let $A\colon W_0^{1,\vec{p}}(\Omega) \mapsto W^{-1,\vec{p}'}(\Omega)$ be a continuous, coercive and pseudo-monotone operator, and let $g\colon \Omega\times\mathbb{R} \mapsto \mathbb{R}$ be a Carathédorory function such that, for any $s > 0$, there exists $h_s \in L^1(\Omega)$ such that $\sup_{|t|\leq s} |g(x,t)| \leq h_s(x)$ a.e. in Ω. We assume also the sign condition on g, namely,

$$g(x,t)t \geq 0, \text{for a.e. } x \in \Omega \text{ and for all } t \in \mathbb{R}.$$

Lemma 2.6. *For every $f \in W^{-1,\vec{p}'}(\Omega)$, there exists $u \in W_0^{1,\vec{p}}(\Omega)$ solution to*

$$\begin{cases} A(u) + g(x,u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense of (1.5).

Proof. We follow the same arguments as in [13, 14]. Let $g_k(x,u) = T_k(g(x,u))$, where $T_k(s)$ is the truncation function at height k, $T_k(s) = \min(k, \max(s,-k))$, for all $s \in \mathbb{R}$. For a given $k > 0$ and a function $u \in W_0^{1,\vec{p}}(\Omega)$, we associate the element $G_k u \in W^{-1,\vec{p}'}(\Omega)$ defined as

$$G_k u\colon W_0^{1,\vec{p}}(\Omega) \mapsto \mathbb{R}$$
$$v \mapsto \int_\Omega g_k(x,u) v \,\mathrm{d}x.$$

It is easy to check that $G_k\colon W_0^{1,\vec{p}}(\Omega) \mapsto W^{-1,\vec{p}'}(\Omega)$ is continuous. Due to the coerciveness of A and the sign condition on g, it is clear that $A + G_k$ is also coercive. Moreover, since, for any sequence $(u_n) \subset W_0^{1,\vec{p}}(\Omega)$, $u_n \to u$ weakly in $W_0^{1,\vec{p}}(\Omega)$, we have $\lim_{n\to\infty}\langle G_k u_n, u_n - u\rangle = 0$, which is straightforward to show that $A + G_k$ is pseudo-monotone. Thus, we may apply Theorem 2.1 to deduce the existence of a function $u_k \in W_0^{1,\vec{p}}(\Omega)$ such that $A(u_k) + g_k(x,u_k) = f$ or in its variational formulation,

$$\langle A(u_k), v\rangle + \int_\Omega g_k(x,u_k) v \,\mathrm{d}x = \langle f, v\rangle \text{ for all } v \in W_0^{1,\vec{p}}(\Omega). \tag{2.7}$$

Due to the assumptions on A and g, by taking $v = u_k$, we may deduce that (u_k) is bounded in $W_0^{1,\vec{p}}(\Omega)$ and $(A(u_k))$ is bounded in $W^{-1,\vec{p}'}(\Omega)$. Thus, there exist some $u \in W_0^{1,\vec{p}}(\Omega)$ and $\chi \in W^{-1,\vec{p}'}(\Omega)$, and a subsequence still denoted in the same way such that

$$u_k \to u \text{ weakly in } W_0^{1,\vec{p}}(\Omega), \quad A(u_k) \to \chi \text{ weakly in } W^{-1,\vec{p}'}(\Omega).$$

On the other hand, we have that $\int_\Omega g_k(x,u_k) u_k \,\mathrm{d}x \le C$ for all $k \ge 1$, where C is a constant not depending on k. Then, we can also deduce that ([14])

$$g(x,u) \in L^1(\Omega),\ g(x,u)u \in L^1(\Omega) \text{ and } g_k(x,u_k) \to g(x,u) \text{ in } L^1(\Omega).$$

Consequently, by passing to the limit in (2.7), we obtain

$$\langle \chi, v\rangle + \int_\Omega g(x,u) v \,\mathrm{d}x = \langle f, v\rangle \text{ for all } v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega). \tag{2.8}$$

Taking $v = T_k(u)$ as a test function in the last equation, we have

$$\langle \chi, T_k(u)\rangle + \int_\Omega g(x,u) T_k(u) \,\mathrm{d}x = \langle f, T_k(u)\rangle.$$

It is clear that

$$\langle \chi, T_k(u)\rangle \to \langle \chi, u\rangle \quad \text{ and } \quad \langle f, T_k(u)\rangle \to \langle f, u\rangle \tag{2.9}$$

On the other hand, we have $g(x,u)T_k(u) \to g(x,u)u$ a.e. in Ω and $0 \le g(x,u)T_k(u) \le g(x,u)u \in L^1(\Omega)$. Thus, by the Lebesgue dominated convergence theorem, we obtain

$$g(x,u)T_k(u) \to g(x,u)u \text{ in } L^1(\Omega). \tag{2.10}$$

By (2.9) and (2.10), we conclude that equation (2.8) is still valid for $v = u$. It remains to prove that $\chi = A(u)$. Indeed, by using Fatou's lemma, we have

$$\limsup_{k\to\infty}\langle A(u_k), u_k - u\rangle = \limsup_{k\to\infty}\langle A(u_k), u_k\rangle - \langle \chi, u\rangle \le \langle f, u\rangle - \int_\Omega g(x,u) u \,\mathrm{d}x - \langle \chi, u\rangle = 0.$$

Since A is pseudo-monotone, we obtain $\chi = A(u)$. Thus,

$$\langle A(u), u-v\rangle + \int_\Omega g(x,u)(u-v)\,\mathrm{d}x = \langle f, u-v\rangle \text{ for all } v \in W_0^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega).$$

□

3. Assumptions and Definitions

We consider the following nonlinear elliptic coupled system

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = \rho(u)|\nabla\varphi|^2 & \text{in } \Omega, \\ \operatorname{div}(\rho(u)\nabla\varphi) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \varphi = \varphi_0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is an open, bounded, connected and smooth enough set of $\mathbb{R}^N$ with $N \geq 2$ being an integer. Let $\vec{p} = (p_1, \dots, p_N) \in (1, +\infty)^N$, $p'_\nu = \frac{p_\nu}{p_\nu - 1}$, and $\nu = 1, \dots, N$. The assumptions on the data are the following.

(A1) The vector field $a\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \mapsto \mathbb{R}^N$, $a(x,u,\nabla u) = (a_1(x,u,\nabla u), \dots, a_N(x,u,\nabla u))$, is a Carathéodory vector function and such that, for all $\xi, \eta \in \mathbb{R}^N$, $\xi \neq \eta$, $s \in \mathbb{R}$, and for a.e. $x \in \Omega$,

$$\sum_{\nu=1}^N \big(a_\nu(x,s,\xi) - a_\nu(x,s,\eta)\big)(\xi_\nu - \eta_\nu) > 0, \tag{3.2}$$

$$|a_\nu(x,s,\xi)| \leq \beta\left[c_\nu(x) + |s|^{p_0/p'_\nu} + |\xi_\nu|^{p_\nu - 1}\right], \text{ for all } \nu = 1, \dots, N, \tag{3.3}$$

$$\sum_{\nu=1}^N a_\nu(x,s,\xi)\xi_\nu \geq \alpha \sum_{\nu=1}^N |\xi_\nu|^{p_\nu}, \tag{3.4}$$

where $\xi = (\xi_1, \dots, \xi_N)$, $\eta = (\eta_1, \dots, \eta_N)$, α and β are positive constants, and $c_\nu \in L^{p'_\nu}(\Omega)$, $\nu = 1, \dots, N$ are nonnegative functions.

(A2) $\rho \in C(\mathbb{R})$ and

$$0 < \rho(s) \text{ for all } s \in \mathbb{R}. \tag{3.5}$$

(A3) $\varphi_0 \in H^1(\Omega)$ and it is not a constant function on $\partial\Omega$.

For any $M > 0$, we denote by $\mathscr{K}_M$ the closed and convex set in $W_0^{1,\vec{p}}(\Omega)$ is given as

$$\mathscr{K}_M := \{v \in W_0^{1,\vec{p}}(\Omega) \,/\, |v(x)| \leq M \text{ a.e. in } \Omega\}.$$

Now we introduce the definition of a bilateral solution to problem (3.1).

Definition 3.1. Let M be a positive real number. A pair (u, φ) is called a bilateral solution to problem (3.1) at height M if the following conditions are fulfilled

(C_1) $u \in \mathscr{K}_M$ and $\varphi - \varphi_0 \in H_0^1(\Omega)$.

(C_2) For all $v \in \mathscr{K}_M$,

$$\sum_{\nu=1}^N \int_\Omega a_\nu(x,u,\nabla u)\partial_\nu(u-v) \leq \int_\Omega \rho(u)|\nabla\varphi|^2(u-v). \tag{3.6}$$

(C_3)

$$\int_\Omega \rho(u)\nabla\varphi\nabla\psi = 0,\ \forall \psi \in H_0^1(\Omega).$$

Remark 3.1. Assume that (u,φ) is a bilateral solution at a certain height $M>0$ such that, for some constant $\gamma>0$, $|u(x)|\le\gamma<M$ for almost everywhere $x\in\Omega$. Then, (u,φ) is a weak solution to problem (3.1), that is,

$$\begin{cases} u\in W_0^{1,\vec{p}}(\Omega), \quad \varphi-\varphi_0\in H_0^1(\Omega), \\ \displaystyle\sum_{\nu=1}^N\int_\Omega a_\nu(x,u,\nabla u)\partial_\nu v=\int_\Omega \rho(u)|\nabla\varphi|^2 v, \text{ for all } v\in W_0^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega), \\ \displaystyle\int_\Omega \rho(u)\nabla\varphi\nabla\psi=0, \text{ for all } \psi\in H_0^1(\Omega). \end{cases} \tag{3.7}$$

Indeed, if $w\in W_0^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega)$, then, for $\delta=(M-\gamma)/(1+\|w\|_\infty)$, $v_\pm=u\pm\delta w$ belong to $\mathscr{K}_M$. Plugging $v_\pm$ in (3.6), we readily obtain (3.7).

Conversely, any weak solution (u,φ) with $u\in L^\infty(\Omega)$ is readily a bilateral solution at any height $M>\|u\|_\infty$. In the case $p_0=\min_{1\le\nu\le N}p_\nu\ge 2$, where the existence of a capacity solution was shown ([6]), any $u\in L^\infty(\Omega)$ leads to a weak solution as well. Thus, in this case, the notions of weak, capacity, and bilateral solution coincide.

Remark 3.2. In general, we cannot assure that if (u,φ) is a weak solution, then u is bounded. Thus, we may interpret a bilateral solution at a given height M as the solution of the projection problem on the convex set $\mathscr{K}_M$ given by conditions (C_1)-(C_3).

4. Main Results

The nature of a bilateral solution (u,φ) is by approximation. This means that (u,φ) is obtained as the limit of the solutions of certain approximate problems. To do so, for any two integers $n\ge1$ and $m\ge1$, we first consider the following problem

$$\begin{cases} -\operatorname{div} a(x,u_m^n,\nabla u_m^n)+\left|\dfrac{u_m^n}{M}\right|^{n-2}\dfrac{u_m^n}{M}=T_m(\rho_m(u_m^n)|\nabla\varphi_m^n|^2) & \text{in } \Omega, \\ \operatorname{div}(\rho_m(u_m^n)\nabla\varphi_m^n)=0 & \text{in } \Omega, \\ u_m^n=0, & \text{on } \partial\Omega, \\ \varphi_m^n=\varphi_0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where the regularized function, $\rho_m(s)$, is given by

$$\rho_m(s)=\rho(T_m(s)) \text{ for all } s\in\mathbb{R}. \tag{4.2}$$

The existence of a solution (u_m^n,φ_m^n) to this approximate problem is guaranteed by the following result, which is proved in the next section.

Lemma 4.1. *There exists (u_m^n,φ_m^n) solution to problem (4.1) in the following sense*

$$\begin{cases} u_m^n\in W_0^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega), \quad \varphi_m^n-\varphi_0\in H_0^1(\Omega) \textit{ and} \\ \displaystyle\sum_{\nu=1}^N\int_\Omega a_\nu(x,u_m^n,\nabla u_m^n)\partial_\nu(u_m^n-v)+\int_\Omega\left|\frac{u_m^n}{M}\right|^{n-2}\frac{u_m^n}{M}(u_m^n-v) \\ \qquad =\displaystyle\int_\Omega T_m(\rho_m(u_m^n)|\nabla\varphi_m^n|^2)(u_m^n-v), \textit{ for all } v\in W_0^{1,\vec{p}}(\Omega), \end{cases} \tag{4.3}$$

and

$$\int_\Omega \rho_m(u_m^n)\nabla\varphi_m^n\nabla\psi=0 \textit{ for all } \psi\in H_0^1(\Omega). \tag{4.4}$$

Now, we fix $m \geq 1$ and consider the sequences $(u_m^n)_{n\geq 1} \subset W_0^{1,\vec{p}}(\Omega)$ and $(\varphi_m^n)_{n\geq 1} \subset H_0^1(\Omega)$. We also show in the next section the following result.

Lemma 4.2. *Let $(u_m^n, \varphi_m^n) \in W_0^{1,\vec{p}}(\Omega) \times H^1(\Omega)$ be a solution to (4.3)-(4.4). Then, there exist $u_m \in \mathscr{K}_M$ and $\varphi_m \in H^1(\Omega)$ with $\varphi_{m|\partial\Omega} = \varphi_0$, and subsequences, still denoted in the same way, such that $u_m^n \to u_m$ weakly in $W_0^{1,\vec{p}}(\Omega)$ and $\varphi_m^n \to \varphi_m$ in $H^1(\Omega)$ as $n \to \infty$, and (u_m, φ_m) satisfies the approximate bilateral problem*

$$\sum_{\nu=1}^{N} \int_\Omega a_\nu(x, u_m, \nabla u_m)\partial_\nu(u_m - v) \leq \int_\Omega T_m(\rho_m(u_m)|\nabla\varphi_m|^2)(u_m - v), \text{ for all } v \in \mathscr{K}_M, \tag{4.5}$$

$$\int_\Omega \rho_m(u_m)\nabla\varphi_m\nabla\psi = 0, \text{ for all } \psi \in H_0^1(\Omega). \tag{4.6}$$

The main result of this work now follows.

Theorem 4.1. *Assume (A1)-(A3) and let (u_m, φ_m) be a solution to (4.5)-(4.6). Then, there exists a subsequence of (u_m, φ_m) that converges to a bilateral solution (u, φ) of the problem (3.1).*

5. Proof of the Main Results

This section is devoted to the proof of lemmas 4.1 and 4.2 and Theorem 4.1.

Remark 5.1. From now on, we denote by C (respectively, C_m) any positive constant, which may depend on the data of our problem but not on n or m (respectively, on n), and whose value may differ from one occurrence to another.

Proof of Lemma 4.1. The proof is based on Schauder's fixed point theorem. Let $\omega_m^n \in L^{p_0}(\Omega)$, and consider the following elliptic problem

$$\begin{cases} \operatorname{div}(\rho_m(\omega_m^n)\nabla\varphi_m^n) = 0 & \text{in } \Omega, \\ \varphi_m^n = \varphi_0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Since $0 < \min_{|s|\leq m} \rho(s) \leq \rho_m(\omega_m^n) \leq \max_{|s|\leq m} \rho(s)$, we find by Lax-Milgram's theorem that (5.1) has a unique solution $\varphi_m^n \in H^1(\Omega)$.

Now we consider the following monotone elliptic problem

$$\begin{cases} -\operatorname{div} a(x, \omega_m^n, \nabla u_m^n) + \left|\dfrac{u_m^n}{M}\right|^{n-2} \dfrac{u_m^n}{M} = T_m(\rho_m(\omega_m^n)|\nabla\varphi_m^n|^2) & \text{in } \Omega, \\ u_m^n = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

By the definition of the truncation function, T_m, it is clear that the right hand side of (5.2) belongs to $L^\infty(\Omega) \subset W^{-1,\vec{p}'}(\Omega)$. Also, the mapping $A\colon W_0^{1,\vec{p}}(\Omega) \mapsto W^{-1,\vec{p}'}(\Omega)$ defined by $A(u) = -\operatorname{div} a(x, u, \nabla u)$ is (i) bounded and continuous, thanks to (3.3); (ii) strictly monotone, thanks to (3.2); and (iii) coercive, by a direct application of (3.4) and Corollary 2.1 since $p_0 > 1$. In view of Lemma 2.4, A is a pseudo-monotone mapping. Thus Lemma 2.6 implies that there exists at least a solution $u_m^n \in W_0^{1,\vec{p}}(\Omega) \cap L^n(\Omega)$ to (5.2) in the following sense

$$\begin{cases} \displaystyle\sum_{\nu=1}^{N} \int_\Omega a_\nu(x, \omega_m^n, \nabla u_m^n)\partial_\nu(u_m^n - v) + \int_\Omega \left|\frac{u_m^n}{M}\right|^{n-2} \frac{u_m^n}{M}(u_m^n - v) \\ \displaystyle\quad = \int_\Omega T_m(\rho_m(\omega_m^n)|\nabla\varphi_m^n|^2)(u_m^n - v), \text{ for all } v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{5.3}$$

Since $t \mapsto |t|^{s-2}t$ is non-decreasing, it follows from condition (3.2) that this solution to (5.2) is unique. It remains to show the regularity $u_m^n \in L^\infty(\Omega)$ and then the test functions v in (5.3) may be taken in the whole space $W_0^{1,\vec{p}}(\Omega)$. In order to show that u is bounded in Ω ,we use the Stampacchia method of truncations. This is based in the following result due to Stampacchia ([23, Lemma 4.1]).

Lemma 5.1. *Let $k_0 \in \mathbb{R}$ and $\Phi\colon [k_0,+\infty) \mapsto [0,+\infty)$ be a non-increasing function such that*

$$\Phi(h) \le \frac{\boldsymbol{c}}{(h-k)^{\boldsymbol{\alpha}}}\big(\Phi(k)\big)^{\boldsymbol{\beta}} \text{ for all } h > k \ge k_0,$$

where $\boldsymbol{c}$ and $\boldsymbol{\alpha}$ are positive constants and $\boldsymbol{\beta} > 1$. Then,

$$\Phi(k_0+d) = 0, \tag{5.4}$$

where $d^{\boldsymbol{\alpha}} = \boldsymbol{c}\big(\Phi(k_0)\big)^{\boldsymbol{\beta}-1} 2^{\boldsymbol{\alpha}\boldsymbol{\beta}/(\boldsymbol{\beta}-1)}$.

The original result of Stampacchia in [23] also gives estimates for Φ when $\boldsymbol{\beta} = 1$ and $\boldsymbol{\beta} < 1$ (in this case, with $k_0 > 0$), but we do not use them in this presentation. Obviously, condition (5.4) implies that $\Phi(k) = 0$ for all $k \ge k_0 + d$.

Now, let $h > k > 0$ and take $v = u_m^n - T_{h-k}(R_k(u_m^n))$ in (5.3), where $R_k(s) = s - T_k(s)$ ([18, 23]). This yields the estimate

$$\sum_{\nu=1}^N \int_\Omega a_\nu(x,\omega_m^n,\nabla u_m^n)\partial_\nu T_{h-k}(R_k(u_m^n)) \le \int_\Omega T_m(\rho_m(\omega_m^n)|\nabla\varphi_m^n|^2)T_{h-k}(R_k(u_m^n)).$$

Hence,

$$\sum_{\nu=1}^N \int_\Omega a_\nu(x,\omega_m^n,\nabla T_{h-k}(R_k(u_m^n)))\partial_\nu T_{h-k}(R_k(u_m^n)) \le m(h-k)|A_k|,$$

where $A_k = \{|u_m^n| > k\}$ and $|A_k| = \int_{A_k} \mathrm{d}x$ is the Lebesgue measure of the set A_k. From (3.4), we obtain

$$\alpha \int_\Omega |\partial_j T_{h-k}(R_k(u_m^n))|^{p_j} \le \alpha \sum_{\nu=1}^N \int_\Omega |\partial_\nu T_{h-k}(R_k(u_m^n))|^{p_\nu} \le m(h-k)|A_k|, \text{ for each } j \in \{1,\dots,N\}.$$

Therefore,

$$\alpha^{1/\bar{p}} \prod_{j=1}^N \|\partial_j T_{h-k}(R_k(u_m^n))\|_{p_j}^{1/N} \le m^{1/\bar{p}}(h-k)^{1/\bar{p}}|A_k|^{1/\bar{p}}.$$

In view (2.2), we deduce the existence of $q > \bar{p}$ and a constant $C_m > 0$ such that

$$\|T_{h-k}(R_k(u_m^n))\|_q \le C_m(h-k)^{1/\bar{p}}|A_k|^{1/\bar{p}}.$$

Since

$$\|T_{h-k}(R_k(u_m^n))\|_q^q = \int_\Omega |T_{h-k}(R_k(u_m^n))|^q \ge \int_{\{|u_m^n|>h\}} |T_{h-k}(R_k(u_m^n))|^q = (h-k)^q|A_h|,$$

we finally deduce

$$|A_h| \le \frac{C_m^q}{(h-k)^{q(1-1/\bar{p})}}|A_k|^{q/\bar{p}} \text{ for all } h > k > 0.$$

Putting $\Phi(k)=|A_k|$, $k_0=0$, $\boldsymbol{c}=C_m^q$, $\boldsymbol{\alpha}=q(1-1/\bar{p})$, and $\boldsymbol{\beta}=q/\bar{p}$, we see that this function Φ satisfies the assumptions of Lemma 5.1. Consequently, there exists $K_m>0$ such that $\Phi(s)=0$ for all $s\geq K_m$ and this means that $|u_m^n|\leq K_m$ a.e. in Ω. Thus $u_m^n\in L^\infty(\Omega)$ and

$$\|u_m^n\|_\infty\leq K_m \text{ for all } m,n\geq 1. \tag{5.5}$$

Now we introduce a mapping G from $L^{p_0}(\Omega)$ into itself, namely,

$$G\colon\ \omega_m^n\in L^{p_0}(\Omega)\longmapsto G(\omega_m^n)=u_m^n\in W_0^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega)\subset L^{p_0}(\Omega),$$

with u_m^n being the unique solution to (5.3). We have the following lemma which is proved in the Appendix below.

Lemma 5.2. *The mapping G satisfies the hypotheses of Schauder's fixed point theorem.*

By Lemma 5.2 and the Schauder's fixed point theorem, we conclude that G has at least one fixed point $u_m^n=G(u_m^n)$, which means that

$$-\operatorname{div}a(x,u_m^n,\nabla u_m^n)+\left|\frac{u_m^n}{M}\right|^{n-2}\frac{u_m^n}{M}=T_m(\rho_m(u_m^n)|\nabla\varphi_m^n|^2) \text{ in } \Omega. \tag{5.6}$$

Thus we have the existence of a solution (u_m^n,φ_m^n) to the approximate problem (4.1), where u_m^n belongs to the Sobolev space $W_0^{1,\vec{p}}(\Omega)$ with the extra regularity $u_m^n\in L^\infty(\Omega)$, $\varphi_m^n-\varphi_0$ belongs to $H_0^1(\Omega)$ and $\operatorname{div}(\rho_m(u_m^n)\nabla\varphi_m^n)=0$ in Ω. $\square$

Proof of Lemma 4.2. We first deduce some estimates on the sequences $(u_m^n)_{n\geq 1}$ and $(\varphi_m^n)_{n\geq 1}$. By taking $\varphi_m^n-\varphi_0\in H_0^1(\Omega)$ as a test function in (4.4) and using (A2), we have

$$\int_\Omega\rho_m(u_m^n)|\nabla\varphi_m^n|^2=\int_\Omega\rho_m(u_m^n)\nabla\varphi_m^n\nabla\varphi_0\leq\frac{1}{2}\int_\Omega\rho_m(u_m^n)|\nabla\varphi_m^n|^2+\frac{1}{2}\int_\Omega\rho_m(u_m^n)|\nabla\varphi_0|^2.$$

Since ρ is continuous, we see from the definition of ρ_m given in (4.2) that

$$\min_{|s|\leq m}\rho(s)\int_\Omega|\nabla\varphi_m^n|^2\leq\int_\Omega\rho_m(u_m^n)\,|\nabla\varphi_m^n|^2\leq\max_{|s|\leq m}\rho(s)\int_\Omega|\nabla\varphi_0|^2.$$

Thus, we deduce the estimate

$$\int_\Omega|\nabla\varphi_m^n|^2\leq C(m,\varphi_0)=C_m, \tag{5.7}$$

where C_m does not depends on n.

On the other hand, it is known that if $\Gamma\subset\bar{\Omega}$ is a smooth enough hypersurface, then the norm

$$\|\varphi\|_\Gamma=\left(\int_\Omega|\nabla\varphi|^2+\int_\Gamma\varphi^2\right)^{1/2},\ \varphi\in H^1(\Omega), \tag{5.8}$$

is equivalent to the usual norm of $H^1(\Omega)$. In particular, taking $\Gamma=\partial\Omega$, we deduce from (5.7) that $\|\varphi_m^n\|_{\partial\Omega}^2\leq C_m+\int_{\partial\Omega}\varphi_0^2$ for all $m,n\geq 1$. Consequently, for every $m\geq 1$, $(\varphi_m^n)_{n\geq 1}$ is bounded in $H^1(\Omega)$. Hence, there exists a function $\varphi_m\in H^1(\Omega)$ and a subsequence, still denoted in the same way, such that

$$\varphi_m^n\to\varphi_m \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^2(\Omega) \text{ as } n\to\infty. \tag{5.9}$$

As for $(u_m^n)_{n\geq 1}$, taking $v=0$ as a test function in (4.3), we obtain

$$\sum_{\nu=1}^{N}\int_\Omega a_\nu(x,u_m^n,\nabla u_m^n)\partial_\nu u_m^n+\int_\Omega M\left|\frac{u_m^n}{M}\right|^n=\int_\Omega T_m(\rho_m(u_m^n)|\nabla\varphi_m^n|^2)u_m^n.$$

Using (3.4) yields

$$\alpha\sum_{\nu=1}^{N}\int_\Omega|\partial_\nu u_m^n|^{p_\nu}+\int_\Omega M\left|\frac{u_m^n}{M}\right|^n\leq m\int_\Omega|u_m^n|.$$

In view of Corollary 2.1, we deduce $\alpha_0\|u_m^n\|_{\vec{p}}^{p_0}+\int_\Omega M\left|\frac{u_m^n}{M}\right|^n\leq m\int_\Omega|u_m^n|+\alpha_1$, for some positive constants α_0 and α_1. From Young's inequality, we obtain

$$\alpha_0\|u_m^n\|_{\vec{p}}^{p_0}+\int_\Omega M\left|\frac{u_m^n}{M}\right|^n\leq\frac{\alpha_0}{2}\int_\Omega|u_m^n|^{p_0}+C_m.$$

Consequently, we have the following estimates, for all $m,n\geq 1$,

$$\|u_m^n\|_{\vec{p}}\leq C_m, \tag{5.10}$$

and $0\leq\int_\Omega\left|\frac{u_m^n}{M}\right|^n\leq C_m$. By (5.10), we may extract a subsequence, still denoted in the same way, such that

$$u_m^n\to u_m \text{ weakly in } W_0^{1,\vec{p}}(\Omega),\ \text{strongly in } L^{p_0}(\Omega) \text{ and a.e. in } \Omega,\ \text{as } n\to\infty. \tag{5.11}$$

Moreover, we have the following lemma, whose proof can be found in [9].

Lemma 5.3. *The weak limit u_m appearing in (5.11) verifies $|u_m|\leq M$ almost everywhere in Ω.*

From (4.4), (5.9) and the almost everywhere convergence of $(u_m^n)_{n\geq 1}$ to u_m, we readily obtain the equation for φ_m, namely,

$$\int_\Omega\rho_m(u_m)\nabla\varphi_m\nabla\psi=0,\ \text{for all } \psi\in H_0^1(\Omega). \tag{5.12}$$

Now, it is easy to deduce that the convergence of $(\varphi_m^n)_{n\geq 1}$ to φ_m in $H^1(\Omega)$ is, in fact, strongly. Indeed, from (4.4) and (4.6), we have, for all $m,n\geq 1$,

$$\int_\Omega\rho_m(u_m^n)\nabla\varphi_m^n\nabla\psi=\int_\Omega\rho_m(u_m)\nabla\varphi_m\nabla\psi,\ \text{for all } \psi\in H_0^1(\Omega).$$

Taking $\psi=\varphi_m^n-\varphi_m$ in the last equality, we obtain

$$\int_\Omega\rho_m(u_m^n)\nabla\varphi_m^n\nabla(\varphi_m^n-\varphi_m)=\int_\Omega\rho_m(u_m)\nabla\varphi_m\nabla(\varphi_m^n-\varphi_m).$$

Inserting $-\rho_m(u_m^n)\nabla\varphi_m\nabla(\varphi_m^n-\varphi_m)$ in the integral above, we have

$$\int_\Omega\rho_m(u_m^n)|\nabla(\varphi_m^n-\varphi_m)|^2=\int_\Omega(\rho_m(u_m)-\rho_m(u_m^n))\nabla\varphi_m\nabla(\varphi_m^n-\varphi_m).$$

Using Hölder's inequality yields $\int_\Omega|\nabla(\varphi_m^n-\varphi_m)|^2\leq C_m\int_\Omega|\rho_m(u_m)-\rho_m(u_m^n)|^2|\nabla\varphi_m|^2$. By the continuity of ρ, we have $\rho_m(u_m^n)-\rho_m(u_m)\to 0$ a.e. in Ω, and

$$|\rho_m(u_m)-\rho_m(u_m^n)|^2|\nabla\varphi_m|^2\leq C_m|\nabla\varphi_m|^2.$$

Since $\nabla\varphi_m\in L^2(\Omega)$, then we conclude by the Lebesgue convergence theorem that

$$\int_\Omega|\rho_m(u_m)-\rho_m(u_m^n)|^2|\nabla\varphi_m|^2\to 0 \text{ as } n\to\infty.$$

Thus $\int_\Omega |\nabla(\varphi_m^n - \varphi_m)|^2 \to 0$ as $n \to \infty$, that is, $\nabla\varphi_m^n \to \nabla\varphi_m$ strongly in $L^2(\Omega)$.

Now, it remains to show that u_m satisfies variational inequatity (4.5). To do so, we take θv as a test function in (5.6), with $v \in \mathscr{K}_M$ and θ is a real number such that $0 < \theta < 1$. This yields

$$\left\{ \begin{aligned} &\sum_{\nu=1}^{N} \int_\Omega a_\nu(x, u_m^n, \nabla u_m^n)\partial_\nu(u_m^n - \theta v) + \int_\Omega \left|\frac{u_m^n}{M}\right|^{n-2} \frac{u_m^n}{M}(u_m^n - \theta v) \\ &\qquad\qquad = \int_\Omega T_m(\rho_m(u_m^n)|\nabla\varphi_m^n|^2)(u_m^n - \theta v). \end{aligned} \right. \tag{5.13}$$

For the right hand side, using (5.11) again, since $T_m(\rho_m(u_m^n)|\nabla\varphi_m^n|^2)$ is bounded in $L^\infty(\Omega)$ and $|\nabla\varphi_m^n|^2 \to |\nabla\varphi_m|^2$ strongly in $L^1(\Omega)$ and a.e. in Ω (modulo a subsequence), one has

$$T_m(\rho_m(u_m^n)|\nabla\varphi_m^n|^2) \to T_m(\rho_m(u_m)|\nabla\varphi_m|^2) \text{ in } L^q(\Omega) \text{ for all } 1 \le q < \infty.$$

Thus, bearing in mind that u_m^n converges to u_m weakly in $W_0^{1,\vec{p}}(\Omega)$ yields

$$\int_\Omega T_m(\rho_m(u_m^n)|\nabla\varphi_m^n|^2)(u_m^n - \theta v) \to \int_\Omega T_m(\rho_m(u_m)|\nabla\varphi_m|^2)(u_m - \theta v) \text{ as } n \to \infty. \tag{5.14}$$

For the second term in the left hand side of (5.13), we observe that $u_m^n(u_m^n - \theta v) \ge 0$ in the set P defined by $P = \{u_m^n \ge 0 \text{ and } u_m^n \ge \theta v\} \cup \{u_m^n \le 0 \text{ and } u_m^n \le \theta v\}$. The complimentary set of P, $\bar{P}$, is given by

$$\begin{aligned} \bar{P} &= \{u_m^n < 0 \text{ and } u_m^n > \theta v\} \cup \{u_m^n > 0 \text{ and } u_m^n < \theta v\} \\ &= \{0 < u_m^n < \theta v\} \cup \{\theta v < u_m^n < 0\}. \end{aligned}$$

Thus

$$\left\{ \begin{aligned} \int_\Omega \left|\frac{u_m^n}{M}\right|^{n-2} \frac{u_m^n}{M}(u_m^n - \theta v) &\ge \int_{\{0<u_m^n<\theta v\}} \left|\frac{u_m^n}{M}\right|^{n-2} \frac{u_m^n}{M}(u_m^n - \theta v) \\ &\quad + \int_{\{\theta v<u_m^n<0\}} \left|\frac{u_m^n}{M}\right|^{n-2} \frac{u_m^n}{M}(u_m^n - \theta v) = I_{m,n}^1 + I_{m,n}^2. \end{aligned} \right. \tag{5.15}$$

Notice that $\left|\frac{u_m^n}{M}\right| \le \theta$ in both sets $\{0 < u_m^n < \theta v\}$ and $\{\theta v < u_m^n < 0\}$. Hence, bearing in mind that $0 < \theta < 1$ and $v \in \mathscr{K}_M$, one has

$$\begin{aligned} 0 \ge I_{m,n}^1 &= \int_{\{0<u_m^n<\theta v\}} M\left|\frac{u_m^n}{M}\right|^n - \int_{\{0<u_m^n<\theta v\}} \left|\frac{u_m^n}{M}\right|^{n-1} \theta v \\ &\ge -\int_{\{0<u_m^n<\theta v\}} \left|\frac{u_m^n}{M}\right|^{n-1} \theta v \ge -\int_{\{0<u_m^n<\theta v\}} \theta^n v \ge -\theta^n \int_\Omega v, \end{aligned}$$

and

$$\begin{aligned} 0 \ge I_{m,n}^2 &= \int_{\{\theta v<u_m^n<0\}} M\left|\frac{u_m^n}{M}\right|^n + \int_{\{\theta v<u_m^n<0\}} \left|\frac{u_m^n}{M}\right|^{n-1} \theta v \\ &\ge \int_{\{\theta v<u_m^n<0\}} \left|\frac{u_m^n}{M}\right|^{n-1} \theta v \ge \int_{\{\theta v<u_m^n<0\}} |\theta|^n v \ge \theta^n \int_\Omega v. \end{aligned}$$

Thus $\lim_{n\to\infty} I_{m,n}^1 = 0$ and $\lim_{n\to\infty} I_{m,n}^2 = 0$. Consequently, owing to (5.15),

$$\liminf_{n\to\infty} \int_\Omega \left|\frac{u_m^n}{M}\right|^{n-2} \frac{u_m^n}{M}(u_m^n - \theta v) \ge 0 \tag{5.16}$$

For the first term in the left hand side of (5.13), by (3.3) and (5.10), we see that there exist $\delta_\nu^m \in L^{p'_\nu}(\Omega)$ for each $\nu = 1,\dots,N$ such that

$$a_\nu(x,u_m^n,\nabla u_m^n) \to \delta_\nu^m \text{ weakly in } L^{p'_\nu}(\Omega), \text{ as } n\to\infty, \text{ for each } \nu = 1,\dots,N. \tag{5.17}$$

Let $\delta^m = (\delta_1^m,\dots,\delta_N^m)$. Then, (5.17) implies that $A(u_m^n) \to -\operatorname{div}\delta^m$ weakly in $W^{-1,\vec{p}'}(\Omega)$ as $n\to\infty$. We next prove that

$$-\operatorname{div}\delta^m = A(u_m). \tag{5.18}$$

Indeed, taking $v = u_m$ in (4.3), we see that

$$\sum_{\nu=1}^N \int_\Omega a_\nu(x,u_m^n,\nabla u_m^n)\partial_\nu(u_m^n-u_m) + \int_\Omega \left|\frac{u_m^n}{M}\right|^{n-2}\frac{u_m^n}{M}(u_m^n-u_m) = \int_\Omega T_m(\rho_m(u_m^n)|\nabla\varphi_m^n|^2)(u_m^n-u_m).$$

For the second term and the right hand side, by a similar argument as in (5.14) and (5.16), one can prove that

$$\liminf_{n\to\infty}\int_\Omega \left|\frac{u_m^n}{M}\right|^{n-2}\frac{u_m^n}{M}(u_m^n-u_m) \geq 0$$

and

$$\lim_{n\to\infty}\int_\Omega T_m(\rho_m(u_m^n)|\nabla\varphi_m^n|^2)(u_m^n-u_m) = 0.$$

Thus

$$\limsup_{n\to\infty}\langle A(u_m^n),u_m^n-u_m\rangle = \limsup_{n\to\infty}\sum_{\nu=1}^N\int_\Omega a_\nu(x,u_m^n,\nabla u_m^n)\partial_\nu(u_m^n-u_m) \leq 0.$$

Since the mapping A is pseudo-monotone, we deduce in particular that $A(u_m^n) \to A(u_m)$ weakly in $W^{-1,\vec{p}'}(\Omega)$, that is, (5.18). Returning to (5.13), in view of (5.14) and (5.16), we have, for all $v \in \mathscr{K}_M$ and $\theta \in (0,1)$,

$$\limsup_{n\to\infty}\sum_{\nu=1}^N\int_\Omega a_\nu(x,u_m^n,\nabla u_m^n)\partial_\nu(u_m^n-\theta v) \leq \int_\Omega T_m(\rho_m(u_m)|\nabla\varphi_m|^2)(u_m-\theta v). \tag{5.19}$$

Using again the pseudo-monotonicity of A, we have

$$\langle A(u_m),u_m-\theta v\rangle \leq \liminf_{n\to\infty}\langle A(u_m^n),u_m^n-\theta v\rangle \leq \limsup_{n\to\infty}\langle A(u_m^n),u_m^n-\theta v\rangle$$

Combining this last expression with (5.19), we find that

$$\begin{cases} \text{For all } v\in\mathscr{K}_M, \\ \displaystyle\sum_{\nu=1}^N\int_\Omega a_\nu(x,u_m,\nabla u_m)\partial_\nu(u_m-\theta v) \leq \int_\Omega T_m(\rho_m(u_m)|\nabla\varphi_m|^2)(u_m-\theta v). \end{cases}$$

Finally, letting θ tend to 1, we readily obtain that, for all $v\in\mathscr{K}_M$,

$$\sum_{\nu=1}^N\int_\Omega a_\nu(x,u_m,\nabla u_m)\partial_\nu(u_m-v) \leq \int_\Omega T_m(\rho_m(u_m)|\nabla\varphi_m|^2)(u_m-v). \tag{5.20}$$

This ends the proof of Lemma 4.2. □

Proof of Theorem 4.1. We begin with the derivation of some a priori estimates for (u_m) and (φ_m), and then we show that (u_m, φ_m) converges, up to a subsequence, to a bilateral solution to problem (3.1). Notice that $\rho_m(u_m) = \rho(u_m)$ for $m \geq M$. From now on, we assume that $m \geq M$. By taking $\psi = \varphi_m - \varphi_0$ as a test function in (4.6), we obtain $\int_\Omega \rho(u_m)|\nabla\varphi_m|^2 = \int_\Omega \rho(u_m)\nabla\varphi_m\nabla\varphi_0$. Since ρ is continuous, then Lemma 5.3 yields

$$\min_{|s|\leq M}\rho(s)\int_\Omega |\nabla\varphi_m|^2 \leq \int_\Omega \rho(u_m)|\nabla\varphi_m|^2 \leq \max_{|s|\leq M}\rho(s)\int_\Omega |\nabla\varphi_m||\nabla\varphi_0|.$$

Hence,

$$\int_\Omega |\nabla\varphi_m|^2 \leq C, \text{ for all } m \geq 1, \tag{5.21}$$

where C does not depend on m. Observe that $\varphi_m|_{\partial\Omega} = \varphi_0$ for all $m \geq 1$. From (5.8) and (5.21), we deduce that (φ_m) is bounded in $H^1(\Omega)$. Consequently, there exist a function $\varphi \in H^1(\Omega)$ and a subsequence, still denoted in the same way, such that $\varphi_m \to \varphi$ weakly in $H^1(\Omega)$, $\varphi_{|\partial\Omega} = \varphi_0$. In fact, once we establish the almost everywhere convergence, for some suitable subsequence, of (u_m), we obtain

$$\varphi_m \to \varphi \text{ strongly in } H^1(\Omega). \tag{5.22}$$

Taking $v = 0$ in (5.20) yields

$$\sum_{\nu=1}^N \int_\Omega a_\nu(x, u_m, \nabla u_m)\partial_\nu u_m \leq \int_\Omega T_m(\rho(u_m)|\nabla\varphi_m|^2)u_m.$$

Thus, using Lemma 5.3, one arrives at

$$\sum_{\nu=1}^N \int_\Omega a_\nu(x, u_m, \nabla u_m)\partial_\nu u_m \leq M\int_\Omega T_m(\rho(u_m)|\nabla\varphi_m|^2).$$

From (3.4) and the definition of the truncation function, we further obtain

$$\alpha\sum_{\nu=1}^N \int_\Omega |\partial_\nu u_m|^{p_\nu} \leq M\int_\Omega \rho(u_m)|\nabla\varphi_m|^2.$$

Therefore, from Corollary 2.1 and (5.21), we have $\|u_m\|_{\vec{p}} \leq C$ for all $m \geq 1$. Hence, there exist $u \in W_0^{1,\vec{p}}(\Omega)$ and a subsequence, still denoted in the same way, such that

$$u_m \to u \text{ weakly in } W_0^{1,\vec{p}}(\Omega), \text{ strongly in } L^{p_0}(\Omega) \text{ and a.e. in } \Omega. \tag{5.23}$$

Now, we are going to pass to the limit in (5.20). Using (5.23) and the fact that ρ is continuous, we have $\rho(u_m) \to \rho(u)$ a.e. in Ω. Using (3.5) and the Lebesgue theorem yields $\rho(u_m) \to \rho(u)$ in $L^q(\Omega)$ for all $q < \infty$. Hence, it follows from (5.22) that

$$\rho(u_m)|\nabla\varphi_m|^2 \to \rho(u)|\nabla\varphi|^2 \text{ strongly in } L^1(\Omega),$$

and

$$T_m(\rho(u_m)|\nabla\varphi_m|^2) \to \rho(u)|\nabla\varphi|^2 \text{ strongly in } L^1(\Omega). \tag{5.24}$$

Using (5.24), having in mind that $|u_m| \leq M$ a.e in Ω and that $u_m \to u$ a.e in Ω, we can pass to the limit in the right hand side of (5.20) to obtain

$$\int_\Omega T_m(\rho(u_m)|\nabla\varphi_m|^2)(u_m - v) \to \int_\Omega \rho(u)|\nabla\varphi|^2(u - v). \tag{5.25}$$

In particular, combining (5.20) and (5.25), for $v = u$, we have

$$\limsup_{m\to\infty} \sum_{\nu=1}^{N} \int_\Omega a_\nu(x,u_m,\nabla u_m)\partial_\nu(u_m - u) \le 0.$$

Since A is a pseudo-monotone mapping, we conclude that $A(u_m) \to A(u)$ weakly in $W^{-1,\vec{p}'}(\Omega)$ and also $\sum_{\nu=1}^{N} \int_\Omega a_\nu(x,u_m,\nabla u_m)\partial_\nu(u_m - u) \to 0$ as $m \to \infty$. In particular,

$$\lim_{n\to\infty} \sum_{\nu=1}^{N} \int_\Omega a_\nu(x,u_m,\nabla u_m)\partial_\nu u_m = \sum_{\nu=1}^{N} \int_\Omega a_\nu(x,u,\nabla u)\partial_\nu u,$$

and, for all $v \in \mathscr{K}_M$,

$$\lim_{n\to\infty} \sum_{\nu=1}^{N} \int_\Omega a_\nu(x,u_m,\nabla u_m)\partial_\nu(u_m - v) = \sum_{\nu=1}^{N} \int_\Omega a_\nu(x,u,\nabla u)\partial_\nu(u - v). \tag{5.26}$$

Consequently, from (5.25) and (5.26), we can pass to the limit in inequality (5.20). This yields

$$\sum_{\nu=1}^{N} \int_\Omega a_\nu(x,u,\nabla u)\partial_\nu(u - v) \le \int_\Omega \rho(u)|\nabla\varphi|^2(u - v) \text{ for all } v \in \mathscr{K}_M.$$

Finally, using (5.22) and (5.23), we can pass to the limit in (5.12) and obtain $\int_\Omega \rho(u)\nabla\varphi\nabla\psi = 0$ for all $\psi \in H_0^1(\Omega)$. This completes the proof of Theorem 4.1. ☐

APPENDIX

Proof of Lemma 5.2. Here, m and n are two fixed positive integers. We will write u instead of u_m^n and we put $G(\omega) = u$. First of all, we show that G has an invariant convex, closed, and bounded set. Indeed, let us consider $v = 0$ as a test function in (5.3). Note that

$$\sum_{\nu=1}^{N} \int_\Omega a_\nu(x,\omega,\nabla u)\partial_\nu u + \int_\Omega \left|\frac{u}{M}\right|^{n-2} \frac{u}{M} u = \int_\Omega T_m(\rho_m(\omega)|\nabla\varphi_m^n|^2)u.$$

Thus, by the definition of T_m, one has

$$\sum_{\nu=1}^{N} \int_\Omega a_\nu(x,\omega,\nabla u)\partial_\nu u + \int_\Omega M\left|\frac{u}{M}\right|^{n} \le m\int_\Omega |u|.$$

By repeating the same steps as in the proof of the Lemma 4.2, we deduce that

$$\|u\|_{\vec{p}} \le C_m. \tag{5.27}$$

In view of (5.5), we have $\|u\|_{L^\infty(\Omega)} \le K_m$. Let $C_m > 0$ be the constant appearing in (5.27) and consider the closed ball $B_m \subset L^{p_0}(\Omega)$ given by $B_m = \left\{v \in L^{p_0}(\Omega)/\|v\|_{p_0} \le C_m\right\}$. Since $\|u\|_{p_0} \le \|u\|_{\vec{p}}$ for all $u \in W_0^{1,\vec{p}}(\Omega)$, from (5.27), we have $G(B_m) \subset B_m$ and B_m is bounded, closed, and convex. On the other hand, due to the estimate (5.27) together with the compact embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^{p_0}(\Omega)$, we deduce that G is compact. It remains to show that G is continuous. To this end, let $(\omega_j) \subset B_m$ such that

$$\omega_j \to \omega \text{ strongly in } L^{p_0}(\Omega) \tag{5.28}$$

and consider the corresponding functions to ω and ω_j, that is, $u = G(\omega)$, $u_j = G\left(\omega_j\right)$, φ_j and φ, i.e., the couple (u_j, φ_j) verifies the following system

$$\begin{cases} -\operatorname{div} a(x, \omega_j, \nabla u_j) + \left|\dfrac{u_j}{M}\right|^{n-2}\dfrac{u_j}{M} = T_m(\rho_m(\omega_j)|\nabla\varphi_j|^2) & \text{in } \Omega, \\ \operatorname{div}(\rho_m(\omega_j)\nabla\varphi_j) = 0 & \text{in } \Omega, \\ u_j = 0, \ \varphi_j = \varphi_0, & \text{on } \partial\Omega, \end{cases} \tag{5.29}$$

whereas (u, φ) is the unique solution to

$$\begin{cases} -\operatorname{div} a(x, \omega, \nabla u) + \left|\dfrac{u}{M}\right|^{n-2}\dfrac{u}{M} = T_m(\rho_m(\omega)|\nabla\varphi|^2) & \text{in } \Omega, \\ \operatorname{div}(\rho_m(\omega)\nabla\varphi) = 0 & \text{in } \Omega, \\ u = 0, \ \varphi = \varphi_0, & \text{on } \partial\Omega, \end{cases} \tag{5.30}$$

We want to show that $u_j \to u$ strongly in $L^{p_0}(\Omega)$. Let $\mathscr{A}_j$, for each $j \geq 1$, and $\mathscr{A}: W_0^{1,\vec{p}}(\Omega) \mapsto W^{-1,\vec{p}'}(\Omega)$ be the mappings defined as follows

$$\langle \mathscr{A}_j(v), w\rangle = \sum_{\nu=1}^{N} \int_\Omega a_\nu(x, \omega_j, \nabla v)\nabla w \, \mathrm{d}x, \quad \langle \mathscr{A}(v), w\rangle = \sum_{\nu=1}^{N} \int_\Omega a_\nu(x, \omega, \nabla v)\nabla w \, \mathrm{d}x.$$

Now, we want to check that the sequences $(\mathscr{A}_j)$ and (u_j), together with $\mathscr{A}$, verify the conditions (a)-(f) of the Lemma 2.5.

(a) From (3.2), $\mathscr{A}_j$ is monotone for each $j \geq 1$.

(b) The variational formulation of the first equation of (5.29) is as follows

$$\begin{cases} \text{To find } u_j \in W^{1,\vec{p}} \cap L^\infty(\Omega) \text{ such that} \\ \langle \mathscr{A}_j(u_j), v\rangle + \displaystyle\int_\Omega \left|\frac{u_j}{M}\right|^{n-2}\frac{u_j}{M} v = \int_\Omega T_m(\rho_m(\omega_j)|\varphi_j|^2) v \quad \text{for all } v \in W_0^{1,\vec{p}}(\Omega). \end{cases} \tag{5.31}$$

According to the estimates already derived, we know that

$$\|u_j\|_{\vec{p}} \leq C_m, \text{ for all } j \geq 1, \ \|u_j\|_\infty \leq K_m, \text{ for all } j \geq 1, \tag{5.32}$$

where C_m and K_m do not depend on j. Hence, for a suitable subsequence, there exists $\tilde{u} \in W_0^{1,\vec{p}}(\Omega)$ such that

$$u_j \to \tilde{u} \text{ weakly in } W_0^{1,\vec{p}}(\Omega), \text{ strongly in } L^q(\Omega) \text{ for all } q < +\infty \text{ and a.e. in } \Omega. \tag{5.33}$$

(c) Using (5.33), we obtain

$$\left|\frac{u_j}{M}\right|^{n-2}\frac{u_j}{M} \to \left|\frac{\tilde{u}}{M}\right|^{n-2}\frac{\tilde{u}}{M} \text{ in } L^q(\Omega) \text{ for all } q < +\infty.$$

As in the proof of Theorem 4.1, we can show that, for some subsequence, still denoted in the way,

$$\rho_m(\omega_j)|\nabla\varphi_j|^2 \to \rho_m(\omega)|\nabla\varphi|^2 \text{ strongly in } L^1(\Omega), \tag{5.34}$$

where $\varphi \in H^1(\Omega)$ verifies the second equation of (5.30) together with the boundary condition $\varphi_{|\partial\Omega} = \varphi_0$. From (3.3), (5.28), and (5.32), we deduce that $(a_\nu(\cdot, \omega_j, u_j))$ is bounded in $L^{p_\nu}(\Omega)$ for every $\nu = 1, \ldots, N$. Thus, $\mathscr{A}_j(u_j) = -\operatorname{div} a(\cdot, \omega_j, u_j)$ is bounded in $W^{-1,\vec{p}'}(\Omega)$. Consequently, there exists a subsequence, still denoted in the same way, and there exists $\chi \in W^{-1,\vec{p}'}(\Omega)$ such that $\mathscr{A}_j(u_j) \to \chi$ weakly in $W^{-1,\vec{p}'}(\Omega)$.

(d) Putting $v = u_j$ in (5.31) and using (5.33)-(5.34), we obtain

$$\langle \mathscr{A}_j(u_j), u_j \rangle \to \int_\Omega T_m(\rho_m(\omega)|\nabla\varphi|^2)\tilde{u} - \int_\Omega M \left|\frac{\tilde{u}}{M}\right|^n.$$

On the other hand, passing to the limit as $j \to \infty$ in (5.31), we deduce that

$$\langle \chi, v \rangle = \int_\Omega T_m(\rho_m(\omega)|\nabla\varphi|^2)v - \int_\Omega \left|\frac{\tilde{u}}{M}\right|^{n-2} \frac{\tilde{u}}{M}v, \text{ for all } v \in W_0^{1,\vec{p}}.$$

B taking $v = \tilde{u}$, we infer that $\langle \mathscr{A}_j(u_j), u_j \rangle \to \langle \chi, \tilde{u} \rangle$.

(e) Let $v \in W_0^{1,\vec{p}}$. Then, from (3.3), there exists a subsequence, still denoted in the say, such that $a_\nu(\cdot, \omega_j, \nabla v) \to a_\nu(\cdot, \omega, \nabla v)$ strongly in $L^{p'_\nu}(\Omega)$ for all $\nu = 1, \dots, N$. Consequently,

$$\langle \mathscr{A}_j(v), u_j \rangle = \sum_{\nu=1}^N \int_\Omega a_\nu(x, \omega_j, \nabla v)\nabla u_j \,\mathrm{d}x \to \sum_{\nu=1}^N \int_\Omega a_\nu(x, \omega, \nabla v)\nabla\tilde{u}\,\mathrm{d}x = \langle \mathscr{A}(v), \tilde{u} \rangle.$$

(f) Finally, thanks to (3.3), we see that $\mathscr{A}$ is continuous and. In particular, it is hemicontinuous.

Therefore, we can apply Lemma 2.5 to deduce that $\mathscr{A}(\tilde{u}) = \chi$, which means that $(\tilde{u}, \varphi)$ is also a solution to problem (5.30). Since the solution is unique, we deduce that $\tilde{u} = u$ and it is the whole sequence $G(\omega_j) = u_j$ that converges to $u = G(\omega)$. This shows that G is continuous. □

CONCLUSIONS

In this paper, we studied a strongly coupled, nonlinear and nonuniformly elliptic problem in the framework of reflexive anisotropic Sobolev spaces, $W_0^{1,\vec{p}}(\Omega)$. In fact, in this setting, the search for weak solutions, or even capacity solutions if one of the exponents p_ν belongs to the interval $(1,2)$, is not well suited. Here we presented another approach by using the concept of bilateral solutions at a given height. The main result of this work establishes the existence of a bilateral solution to this strongly coupled nonlinear system. The problem may be regarded as a generalization of the well-known thermistor problem. The proof of this result is based on a penalization technique combined with a fixed point argument. Indeed, this kind of solution (u, φ) was obtained as the limit of solutions to certain approximate problems. The analysis relies in the theory of monotone and pseudo-monotone mappings from a reflexive Banach space onto its dual space. This work generalizes the results obtained in [9] in the isotropic case to the anisotropic case.

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