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SUFFICIENT WEIGHTED SECOND-ORDER CONE COMPLEMENTARITY PROBLEMS

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Abstract. In this paper, we investigate the properties of a weighted second-order cone complementarity problem (wSOCCP), which covers a wide range of complementarity problems with applications in equilibrium problems from engineering and economics. We show that the column sufficient wSOCCP with cross commutative property has a convex (perhaps empty) solution set. In particular, when the weight vector is zero, the reverse implication is also true.

Keywords. Complementarity problem; Column sufficient; Maximal complementarity; Second-order cone.

1. Introduction

The *weighted second-order cone complementarity problem* (wSOCCP) is to find vectors $(x,s,y) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ such that

$$\begin{array}{l} x \circ s = w, \\ Ax + Bs + Cy = d, \\ x \in \mathscr{K},\ s \in \mathscr{K}, \end{array} \tag{1.1}$$

where $\circ$ represents the Jordan product, $A \in \mathbb{R}^{(n+m)\times n}$, $B \in \mathbb{R}^{(n+m)\times n}$, and $C \in \mathbb{R}^{(n+m)\times m}$ are given matrices, $d \in \mathbb{R}^{n+m}$ is a given vector, $w \in \mathscr{K}$ is a given weight vector, and $\mathscr{K} = \mathscr{K}^{n_1} \times \mathscr{K}^{n_2} \times \cdots \times \mathscr{K}^{n_r}$ with $n = n_1 + n_2 + \cdots + n_r$ is the Cartesian product of second-order cones (SOC) [1, 2]. Here the n_i-dimensional SOC $\mathscr{K}^{n_i}(i = 1, \ldots, r)$ is defined by

$$\mathscr{K}^{n_i} = \{x_i = (x_{i0}, \overline{x_i}) \in \mathbb{R} \times \mathbb{R}^{n_i-1} : x_{i0} - \|\overline{x_i}\| \geq 0\},$$

where $\|\cdot\|$ refers to the Euclidean norm. It is well known that the interior of the SOC $\mathscr{K}$ can be characterized by

$$\text{int}\mathscr{K} = \text{int}\mathscr{K}^{n_1} \times \text{int}\mathscr{K}^{n_2} \times \cdots \times \text{int}\mathscr{K}^{n_r},$$

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where $\text{int}\mathscr{K}^{n_i} = \{x_i = (x_{i0}, \overline{x_i}) \in \mathbb{R} \times \mathbb{R}^{n_i-1} : x_{i0} - \|\overline{x_i}\| > 0\}$ for $i = 1, \ldots, r$. It is easy to verify that the SOC $\mathscr{K}$ is self-dual, i.e., $\mathscr{K} = \mathscr{K}^* := \{s \in \mathbb{R}^n : s^T x \geq 0, \forall x \in \mathscr{K}\}$. We recall that the wSOCCP (1.1) is said to be monotone if $A\Delta x + B\Delta s + C\Delta y = 0$ implies $\Delta x^T \Delta s \geq 0$, and it is said to be skew-symmetric if $A\Delta x + B\Delta s + C\Delta y = 0$ implies $\Delta x^T \Delta s = 0$.

For any matrix $D \in \mathbb{R}^{(n+m)\times n}$ whose columns form a basis of $\text{Ker}C^T$, the wSOCCP (1.1) is equivalent to finding (x, s) such that

$$\begin{aligned} & x \circ s = w, \\ & Qx + Rs = a, \\ & x \in \mathscr{K},\ s \in \mathscr{K}, \end{aligned} \tag{1.2}$$

where

$$Q = D^T A \in \mathbb{R}^{n\times n}, \quad R = D^T B \in \mathbb{R}^{n\times n}, \quad a = D^T d \in \mathbb{R}^n.$$

If $w = 0$, then wSOCCP (1.2) reduces to the *second-order cone complementarity problem*, denoted by SOCCP(Q, R, a). If $w = 0$, $Q = I$, $R = -M$, and $a = q$, then SOCCP(Q, R, a) reduces to the *second-order cone complementarity problem* SOCCP(M, q). In particular, when $\mathscr{K} = \mathbb{R}^n_+$, this reduces to the *standard linear complementarity problem* LCP(M, q).

The set of all feasible points and the set of all strictly feasible points for the wSOCCP (1.1) are defined by

$$\mathscr{F} = \{z = (x, s, y) \in \mathscr{K} \times \mathscr{K} \times \mathbb{R}^m : Ax + Bs + Cy = d\},$$

$$\mathscr{F}^0 = \{z = (x, s, y) \in \text{int}\mathscr{K} \times \text{int}\mathscr{K} \times \mathbb{R}^m : Ax + Bs + Cy = d\},$$

respectively. In addition, the solution set of the wSOCCP (1.1) is given by

$$\mathscr{F}^* = \{z = (x, s, y) \in \mathscr{F} : x \circ s = w\}.$$

The weighted complementarity problem (wCP) originates from the monograph by Kojima et al. [3], while the weighted linear complementarity problem (wLCP) with $P_*(\kappa)$ (sufficient) matrices was studied in [4, 5]. In 2012, the notion of a wCP was introduced by Potra in [6], which consists of finding a pair of vectors (x, s) belonging to the intersection of a manifold with a cone such that their product in a certain algebra, $x \circ s$, equals a given weight vector w. If w is the zero vector, then the wCP reduces to a complementarity problem (CP) [6] which has been extensively studied. Compared to CP [7, 8], the wCP can be used to modele a larger class of equilibrium problems from engineering and economics. For example, the Fisher market equilibrium problem with linear utilities reduces to a monotone linear wCP [9, 10] by using the Eisenberg-Gale formulation of the Fisher problem [11]. Moreover, the linear programming and weighted centering (LPWC) problem introduced by Anstreicher [12] also can be formulated as a monotone linear wCP [9]. Potra [6] presented and analyzed two interior-point methods for solving the monotone linear wCP over the nonnegative orthant. In 2016, Potra [9] gave some fundamental results about sufficient linear wCP over the nonnegative orthant, and proposed a corrector-predictor interior-point method for its numerical solution. Potra [9] associated with each sufficient linear wCP an appropriate optimization problem, and showed that a linear wCP is row sufficient if and only if every KKT point of that optimization problem is a solution of the wCP. Every column sufficient linear wCP was proved to have a convex (perhaps empty) solution set [9]. Jian [13] presented a smoothing Newton method for solving monotone linear wCP over the nonnegative orthant. Chi, Gowda and Tao [14] studied the weighted horizontal linear complementarity problem in the setting of Euclidean Jordan algebras and established some existence and uniqueness results. Asadi, Darvay, Lesaja et al. [15] presented a full-Newton step interior-point method for monotone weighted linear complementarity problems. Tang and Zhang [16] developed a nonmonotone smoothing Newton algorithm for wCP with good numerical performance. Based on a kernel function, Chi, Wang and Lesaja [17] proposed a full-Newton step feasible interior-point algorithm for $P_*(\kappa)$-weighted linear complementarity problem.

In this paper, we aim to study the properties of the wSOCCP (1.1). We show that the column sufficient wSOCCP with cross commutative property has a convex (perhaps empty) solution set. If the weight vector is zero, the reverse implication is also true. The analysis is based on properties associated with SOC, which differ from the traditional techniques for the wCP [9]. To sum up, we compare the conditions in the results of the wCP [9] and our results of the wSOCCP in Table 1.

The organization of this paper is as follows. In Section 2, we review some preliminaries including the Euclidean Jordan algebra associated with SOC, and the Cartesian mixed P_0-property. In Section 3 and Section 4, we discuss some properties of wSOCCP, such as the convexity of the solution set, globally uniquely solvable property, etc. Section 5, the last section, is devoted to the maximal complementarity for the wSOCCP.

TABLE 1. Properties of the wCP and the wSOCCP

property	wCP [9]	wSOCCP
convex solution set	column sufficient	column sufficient, cross commutative
nonempty convex set of maximal complementarity solutions	column sufficient, solvable	column sufficient, cross commutative, solvable, $\mathscr{O}=\emptyset$
a unique solution	sufficient, strictly feasible	cross commutative, solvable, $P-$property

The following notations are used throughout this paper. Let $\mathbb{R}^n$ (respectively, $\mathbb{R}$) denote the space of n-dimensional real column vectors (respectively, real numbers). The cone of symmetric, positive definite matrices of order n is denoted by $\mathbb{S}^n_{++}$. For convenience, we often use $x=(x_0,\bar{x})$ for the column vector $x=(x_0,\bar{x}^T)^T\in\mathbb{R}\times\mathbb{R}^{n-1}$, and use (x,y,z) for adjoining vectors x,y,z in a column $(x^T,y^T,z^T)^T$. For any $x\in\mathbb{R}^n$, x^+ denotes the nearest-point (in the Euclidean norm) projection of x onto the SOC $\mathscr{K}$. For any $x,y\in\mathbb{R}^n$, we write $x\succeq_{\mathscr{K}} y$ if $x-y$ is in $\mathscr{K}$. For convenience, we define $x\sqcap y := x-(x-y)^+$. The symbol $\|\cdot\|$ denotes the Euclidean norm defined by $\|x\| := \sqrt{x^Tx}$ for a vector x. If $\mathscr{A}\subseteq\mathbb{R}^k$ and $\mathscr{B}\subseteq\mathbb{R}^l$, then $\mathscr{A}\times\mathscr{B}=\{(x,y): x\in\mathscr{A} \text{ and } y\in\mathscr{B}\}$ is their Cartesian product.

2. PRELIMINARIES

In this section, we recall some concepts and backgrounds, which include the Euclidean Jordan algebra associated with SOC [18, 19, 20], and the Cartesian mixed P_0-property [21].

First, we recall some concepts and the Euclidean Jordan algebra associated with the SOC [18, 19]. The SOC (also called Lorentz cone or ice-cream cone) is defined by

$$\mathcal{K}^n = \{x = (x_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : \|\bar{x}\| \le x_0\},$$

where $\bar{x} \in \mathbb{R}^{n-1}$, and $\|\cdot\|$ refers to the Euclidean norm. The Euclidean Jordan algebra for the SOC $\mathcal{K}^n$ is the algebra defined by $x \circ s = (x^T s, x_0\bar{s} + s_0\bar{x})$, $\forall x, s \in \mathbb{R}^n$, with $e = (1, 0, \cdots, 0)^T \in \mathbb{R}^n$ being its unit element. Given an element $x = (x_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define

$$L(x) = \begin{pmatrix} x_0 & \bar{x}^T \\ \bar{x} & x_0 I_{n-1} \end{pmatrix},$$

where I_{n-1} represents the $(n-1) \times (n-1)$ identity matrix. Moreover, $L(x)$ is symmetric positive definite (and hence invertible) if and only if $x \in \text{int}\mathcal{K}^n$. It is easy to verify that $x \circ s = L(x)s$ for any $s \in \mathbb{R}^n$. It should be noted that x and s do not operator commute in general, i.e., $L(x)L(s) \neq L(s)L(x)$.

We now introduce the spectral factorization of vectors in $\mathbb{R}^n$ associated with the SOC $\mathcal{K}^n$, which is an important character of Jordan algebra. For any $x = (x_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, its spectral factorization relative to the SOC $\mathcal{K}^n$ is defined as $x = \lambda_1(x)c^{(1)}(x) + \lambda_2(x)c^{(2)}(x)$. Here $\lambda_1(x)$ and $\lambda_2(x)$ are the spectral values given by $\lambda_i(x) = x_0 + (-1)^i\|\bar{x}\|$, $i = 1, 2$, and $c^{(1)}(x)$ and $c^{(2)}(x)$ are the associated spectral vectors given by $c^{(i)}(x) = \frac{1}{2}\left(1, (-1)^i\tilde{x}\right)$, $i = 1, 2$, with $\tilde{x} = \frac{\bar{x}}{\|\bar{x}\|}$ if $\bar{x} \neq 0$, and any vector $\tilde{x} \in \mathbb{R}^{n-1}$ satisfying $\|\tilde{x}\| = 1$ if $\bar{x} = 0$. Observe that

$$c^{(1)}(x) \circ c^{(2)}(x) = 0, \quad c^{(i)}(x) \circ c^{(i)}(x) = c^{(i)}(x), \quad c^{(i)}(x) \in \text{bd}\mathcal{K}^n$$

for $i = 1, 2$. By using the spectral factorization, we may extend scalar functions to the SOC functions. For example, we define $x^2 = \lambda_1^2(x)c^{(1)}(x) + \lambda_2^2(x)c^{(2)}(x)$ for all $x \in \mathbb{R}^n$. Since both λ_1 and λ_2 are nonnegative for any $x \in \mathcal{K}^n$, we define

$$\sqrt{x} = \sqrt{\lambda_1(x)}c^{(1)}(x) + \sqrt{\lambda_2(x)}c^{(2)}(x), \ \ \forall x \in \mathcal{K}^n.$$

Now let us introduce the concept of the Cartesian mixed P_0-property.

Definition 2.1. [21] Let the matrix $D = (A, B, C)$, where $A, B \in \mathbb{R}^{(n+m)\times n}$ and $C \in \mathbb{R}^{(n+m)\times m}$. Matrix D is said to have the Cartesian mixed P_0-property if and only if C has full column rank and

$$\left.\begin{array}{l} A\Delta x + B\Delta s + C\Delta y = 0, \ (\Delta x, \Delta s) \neq 0, \ \Delta y \in \mathbb{R}^m \\ \Delta x = (\Delta x_1, \cdots, \Delta x_r) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r} \\ \Delta s = (\Delta s_1, \cdots, \Delta s_r) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r} \end{array}\right\} \Rightarrow$$

there exists an index i such that $(\Delta x_i, \Delta s_i) \neq 0$ and $\langle \Delta x_i, \Delta s_i \rangle \ge 0$.

Clearly, when $r = n$ and $n_1 = \cdots = n_r = 1$, matrix Q having the Cartesian mixed P_0-property coincides with Q having the mixed P_0-property [22]. Therefore, (A, B, C) having the Cartesian mixed P_0-property, is a weaker assumption than the monotonicity assumption usually used in SOCCPs (see, e.g., [23]).

3. Weighted Second-order Cone Complementarity Problem

For subsequent needs, we introduce some definitions.

Definition 3.1. The triplet (A,B,C) is said to have the column sufficient property if

$$\left.\begin{array}{l}(\Delta x,\Delta s,\Delta y)\in \mathrm{Ker}(A,B,C)\\ \Delta x \text{ and } \Delta s \text{ operate commute}\\ \Delta x\circ\Delta s\in -\mathscr{K}\end{array}\right\}\Rightarrow \Delta x\circ\Delta s=0.$$

Definition 3.2. [14] The triplet (A,B,C) is said to have the P-property over $\mathbb{R}^{2n+m}$ if

$$\left.\begin{array}{l}(\Delta x,\Delta s,\Delta y)\in \mathrm{Ker}(A,B,C)\\ \Delta x \text{ and } \Delta s \text{ operate commute}\\ \Delta x\circ\Delta s\in -\mathscr{K}\end{array}\right\}\Rightarrow \Delta x=0 \text{ and } \Delta s=0.$$

Definition 3.3. The triplet (A,B,C) is said to have the cross commutative property if, for any two distinct solutions (x,s,y) and (u,v,h) of the wSOCCP (1.1), x operator commutes with v and u operator commutes with s, where $Ax+Bs+Cy=d$, $Au+Bv+Ch=d$ for any given vector $d\in\mathbb{R}^{n+m}$, and $x\circ s=w$, $u\circ v=w$ for any weight vector $w\in\mathscr{K}$.

Example 3.1. Consider the wSOCCP (1.2) associated with the SOC $\mathscr{K}^3$, where

$$Q=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad R=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$a=\begin{pmatrix}2\\1\\1\end{pmatrix},\quad w=\begin{pmatrix}4\\2\\2\end{pmatrix}\in \mathrm{int}\mathscr{K}^3,\quad u=\begin{pmatrix}0\\0\\-1\end{pmatrix},\quad v=\begin{pmatrix}1\\0\\1\end{pmatrix}.$$

The solution set of the wSOCCP (1.2) with (Q,R,a,w), described by

$$\left\{x=\begin{pmatrix}2\\1\\1\end{pmatrix}\text{ and } s=\begin{pmatrix}2\\0\\0\end{pmatrix},\quad x'=\begin{pmatrix}2\\1\\0\end{pmatrix}\text{ and } s'=\begin{pmatrix}2\\0\\1\end{pmatrix}\right\},$$

is not convex. Here x and s operator commute, and x' and s' do not operator commute. Since x and s' do not operator commute, and x' and s operator commute, the pair (Q,R) is not cross commutative. Here $(u,v)\in\mathrm{Ker}(Q,R)$, $u\circ v\in-\mathscr{K}^3$ and $u\circ v=(-1,0,-1)^T\neq 0$, which implies that the pair (Q,R) with full of row rank is not column sufficient. Moreover, for any $(u',v')\in\mathrm{Ker}(Q,R)$, we have $u'=(0,0,-v'_3)^T$. Thus $u'^Tv'=-v_3'^2\le 0$, which implies that the wSOCCP (1.2) with (Q,R,a,w) is not monotone.

From Example 3.1, we summarize couple observations as below.

- **(a):** Although some pair (Q,R) is full of row rank, it may be neither column sufficient nor monotone, and perhaps the solution set of the corresponding wSOCCP (1.2) is not convex.
- **(b):** Even for the same pair (Q,R) in the wSOCCP (1.2), some solutions x and s operator commute, and some solutions x' and s' do not operator commute. Since x and s' do not operator commute, and x' and s operator commute, the pair (Q,R) is not necessarily cross commutative.

Definition 3.4. Let the matrix $D = (A, B, C)$, where $A, B \in \mathbb{R}^{(n+m)\times n}$ and $C \in \mathbb{R}^{(n+m)\times m}$. Matrix D is said to be Cartesian monotone if

$$\left.\begin{array}{l} A\Delta x + B\Delta s + C\Delta y = 0,\ \Delta y \in \mathbb{R}^m \\ \Delta x = (\Delta x_1, \cdots, \Delta x_r) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r} \\ \Delta s = (\Delta s_1, \cdots, \Delta s_r) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r} \end{array}\right\} \Rightarrow$$

$\langle \Delta x_i, \Delta s_i \rangle \geq 0$ for any index $i = 1, \cdots, r$.

4. Properties of the wSOCCP

In this section, we discuss some properties of the proposed wSOCCPs (1.1) and (1.2).

Definition 4.1. [14] The pair $\{Q, R\}$ is said to be an $\mathbf{R}_0$-pair if zero is the only solution of the LCP$(Q, R, 0)$. This means that $\Theta(z) = 0 \Leftrightarrow z = 0$, where

$$\Theta(z) := \left[\begin{array}{c} x \sqcap y \\ Qx + Ry \end{array}\right]$$

with $z = (x, y)$.

Here LCP(Q, R, a) in Definition 4.1 denotes wLCP (Q, R, a, w) with $w = 0$. Let SOL(a, w) denote the solution set of wLCP(Q, R, a, w) on a Euclidean Jordan algebra.

Theorem 4.1. [14] *Let $\{Q, R\}$ be an R_0-pair with deg(Q, R) nonzero.*

- **(i):** *For any $(a, w) \in \mathbb{R}^n \times \mathcal{K}$, the wLCP$(Q, R, a, w)$ has a nonempty compact solution set.*
- **(ii):** *Let $w > 0$, $t_k \downarrow 0$, and $(x_k, s_k) \in SOL(t_k a, w)$ for all k. Then, $x_k > 0$ and $s_k > 0$ for all k, the sequence $\{(x_k, s_k)\}$ is bounded, and any accumulation point of this sequence solves LCP(Q, R, a).*

Theorem 4.1 (i) shows that wLCP(Q, R, a, w) is solvable in a general Euclidean Jordan algebra with a symmetric cone, which is just wSOCCP(Q, R, a, w) (1.2) if the symmetric cone is the SOC $\mathcal{K}$. Theorem 4.1 (ii) is very useful in designing interior-point algorithms for LCP on a Euclidean Jordan algebra. Moreover, wLCP over symmetric cone together with Theorem 4.1 can be applied to *weighted centers* for *linear programming* (LP), *semidefinite programming* (SDP), and *second-order cone programming* (SOCP), which plays an important role in interior-point algorithms (see [3]).

- **(i):** One notation of weighted centers for LP [24] may be characterized by the following wLCP on the nonnegative orthant [25]

$$\begin{array}{lll} Ax = b, & s = c - A^T y & \text{for some } y \in \mathbb{R}^m, \\ xs = w, & x > 0, & s > 0, \end{array}$$

  where xs denotes the componentwise product of x and s.
- **(ii):** One of the available notions of weighted centers for SDP [26] is the weighted centers defined as solutions of the wLCP over the cone of symmetric and positive semidefinite matrices

$$\begin{array}{lll} A(X) = b, & S = C - A^* y & \text{for some } y \in \mathbb{R}^m, \\ XS + SX = 2W, & X \in \mathbb{S}^n_{++}, & S \in \mathbb{S}^n_{++}, \end{array}$$

  where the symmetric matrix W is the weight.

(iii): Weighted centers for SOCP may be given by solutions of the following system of equations

$$\begin{aligned} &Ax = b, \quad s = c - A^T y \quad \text{for some } y \in \mathbb{R}^m, \\ &x \circ s = w, \quad x \in \mathscr{K}, \quad s \in \mathscr{K}, \end{aligned}$$

which is a skew-symmetric wSOCCP.

Lemma 4.1. *For wSOCCP (1.1) with $A, B \in \mathbb{R}^{(n+m)\times n}$ and $C \in \mathbb{R}^{(n+m)\times m}$, consider the following statements*

(i): *wSOCCP (1.1) is Cartesian monotone, and C has full column rank.*
(ii): *wSOCCP (1.1) is monotone, and C has full column rank.*
(iii): *(A,B,C) has the Cartesian mixed P_0-property.*

Then, there holds (i)⇒(ii)⇒(iii).

Proof. The implications follow from Definition 2.1, Definition 3.4, and the concept of monotonicity. □

Lemma 4.2. *For the wSOCCP (1.2) with $(Q,R) \in \mathbb{R}^{n\times(n+n)}$, consider the following statement*

(i): *wSOCCP (1.2) is Cartesian monotone.*
(ii): *(Q,R) is column sufficient.*
(iii): *(Q,R) is full of row rank.*

Then, there holds (i)⇒(ii)⇒(iii).

Proof. (i) ⇒ (ii). Consider $(\Delta x, \Delta s) \in \text{Ker}(Q,R)$ satisfying that Δx and Δs operate commute, and $\Delta x \circ \Delta s \in -\mathscr{K}$. Then, $Q\Delta x + R\Delta s = 0$, and consequently $\Delta x_i^T \Delta s_i \geq 0$ for any $i = 1, \cdots, r$, since wSOCCP (1.2) is Cartesian monotone. Moreover, due to

$$\Delta x_i \circ \Delta s_i = \begin{pmatrix} \Delta x_i^T \Delta s_i \\ \Delta x_{i0}\overline{\Delta s_i} + \Delta s_{i0}\overline{\Delta x_i} \end{pmatrix} \in -\mathscr{K}^{n_i},$$

we obtain

$$\Delta x_i^T \Delta s_i \leq 0 \text{ and } \|\Delta x_{i0}\overline{\Delta s_i} + \Delta s_{i0}\overline{\Delta x_i}\| \leq -\Delta x_i^T \Delta s_i.$$

Thus $\Delta x_i^T \Delta s_i = 0$ and $\|\Delta x_{i0}\overline{\Delta s_i} + \Delta s_{i0}\overline{\Delta x_i}\| = 0$, which implies $\Delta x_i \circ \Delta s_i = 0$ for any $i = 1, \cdots, r$, i.e., $\Delta x \circ \Delta s = 0$.
(ii) ⇒ (iii). By following the proof of Theorem 2.1 [27] for sufficient LCPs, we obtain the desired result. For simplicity, we omit the details. □

Remark 4.1. Consider the following statements

(i'): (Q,R) is not full of row rank.
(ii'): (Q,R) is not column sufficient.
(iii'): wSOCCP (1.2) is not Cartesian monotone.

It follows from Lemma 4.2 that (i') ⇒ (ii') ⇒ (iii').

Lemma 4.3. *For wSOCCP (1.1), the following statements are equivalent*

(i): *The solution set of wSOCCP (1.1) is convex.*
(ii): *For any two solutions (x,s,y) and (u,v,h) of wSOCCP (1.1), $x \circ v + u \circ s = 2w$. Especially, if $w = 0$, then $x \circ v = u \circ s = 0$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that the solution set of wSOCCP (1.1) is convex. Then, for any $\xi \in [0,1]$ and any two solutions (x,s,y) and (u,v,h) of wSOCCP (1.1), $\xi(x,s,y)+(1-\xi)(u,v,h)$ is a solution to wSOCCP (1.1). Consequently, we obtain

$$\begin{aligned} w &= (\xi x+(1-\xi)u)\circ(\xi s+(1-\xi)v) \\ &= \xi^2 x\circ s+\xi(1-\xi)(x\circ v+u\circ s)+(1-\xi)^2 u\circ v \\ &= \xi^2 w+\xi(1-\xi)(x\circ v+u\circ s)+(1-\xi)^2 w, \end{aligned}$$

and therefore $x\circ v+u\circ s=2w$.
(ii) $\Rightarrow$ (i) Suppose that, for any two solutions (x,s,y) and (u,v,h) of wSOCCP (1.1), $x\circ v+u\circ s=2w$. Then, for any $\xi\in\mathbb{R}$,

$$\begin{aligned} &(\xi x+(1-\xi)u)\circ(\xi s+(1-\xi)v) \\ &= \xi^2 x\circ s+\xi(1-\xi)(x\circ v+u\circ s)+(1-\xi)^2 u\circ v \\ &= \xi^2 w+2\xi(1-\xi)w+(1-\xi)^2 w \\ &= w, \end{aligned}$$

and

$$\begin{aligned} &A(\xi x+(1-\xi)u)+B(\xi s+(1-\xi)v)+C(\xi y+(1-\xi)h) \\ &= \xi(Ax+Bs+Cy)+(1-\xi)(Au+Bv+Ch) \\ &= \xi d+(1-\xi)d \\ &= d. \end{aligned}$$

If $\xi\in[0,1]$, it yields that $\xi(x,s,y)+(1-\xi)(u,v,h)$ is a solution to wSOCCP (1.1), which implies that the solution set of wSOCCP (1.1) is convex. $\square$

Theorem 4.2. *For wSOCCP (1.1), if (A,B,C) has the column sufficient and cross commutative properties, then the following results hold*

- **(a):** *For any weight vector $w\in\mathscr{K}$ and any given vector $d\in\mathbb{R}^{n+m}$, the solution set $\mathscr{F}^*$ of wSOCCP (1.1) is convex (possibly empty).*
- **(b):** *For any $(x,s,y)\in\mathscr{F}^*$, x and s operator commute.*
- **(c):** *For any $i=1,2,\ldots,r$ such that $w_i\in \mathrm{int}\mathscr{K}^{n_i}$, $x_i\in \mathrm{int}\mathscr{K}^{n_i}$ and $s_i\in \mathrm{int}\mathscr{K}^{n_i}$ are uniquely defined for any $(x,s,y)\in\mathscr{F}^*$; For any $i=1,2,\ldots,r$ such that $w_i\in \mathrm{bd}\mathscr{K}^{n_i}$, we have either $x_i\in \mathrm{int}\mathscr{K}^{n_i}$, $s_i\in \mathrm{bd}\mathscr{K}^{n_i}$, or $x_i\in \mathrm{bd}\mathscr{K}^{n_i}$, $s_i\in \mathrm{int}\mathscr{K}^{n_i}$, or $x_i\in \mathrm{bd}\mathscr{K}^{n_i}$, $s_i\in \mathrm{bd}\mathscr{K}^{n_i}$.*

Proof. For any given weight vector $w\in\mathscr{K}$ and any given vector $d\in\mathbb{R}^{n+m}$, we only consider the case $\mathscr{K}=\mathscr{K}^n$ without loss of generality.

(a) If the wSOCCP (1.1) has less than two solutions, the argument is obvious. Suppose that (x,s,y) and (u,v,h) are arbitrary two distinct solutions of the wSOCCP (1.1), i.e.,

$$\begin{aligned} Ax+Bs+Cy=d, \\ Au+Bv+Ch=d, \end{aligned}$$

and

$$\begin{aligned} x\circ s=w,\ x\in\mathscr{K}^n,\ s\in\mathscr{K}^n, \\ u\circ v=w,\ u\in\mathscr{K}^n,\ v\in\mathscr{K}^n. \end{aligned} \tag{4.1}$$

By the definition of Jordan product (4.1) associated with SOC, we have

$$\begin{aligned} x_0s_0+\bar{x}^T\bar{s}=u_0v_0+\bar{u}^T\bar{v}=w_0, \\ x_0\bar{s}+s_0\bar{x}=u_0\bar{v}+v_0\bar{u}=\bar{w}. \end{aligned} \tag{4.2}$$

According to the cross commutative property, x operator commutes with v and u operator commutes with s. Then, it follows from [18, Corollary 7] that $\bar{x}=0$ or $\bar{v}=0$ or $\bar{x}$ and $\bar{v}$ are proportional, and $\bar{u}=0$ or $\bar{s}=0$ or $\bar{u}$ and $\bar{s}$ are proportional.

Next, we show $(x-u)\circ(s-v)\in -\mathscr{K}^n$ by considering the following cases.

(i) For $\bar{x}=0$, $\bar{s}=0$, $\bar{u}=0$, and $\bar{v}=0$, we obtain from (4.2) that $x_0s_0=u_0v_0=w_0$. For $w_0=0$, it is not difficult to see that $(x-u)\circ(s-v)\in -\mathscr{K}^n$. Therefore, we only consider the case $w_0>0$. To this end, using (4.1) yields

$$\begin{aligned}(x-u)\circ(s-v) &= 2w-u\circ s-x\circ v\\ &= 2\begin{pmatrix} w_0\\ 0\end{pmatrix}-\begin{pmatrix} u_0s_0\\ 0\end{pmatrix}-\begin{pmatrix} x_0v_0\\ 0\end{pmatrix}\\ &= 2\begin{pmatrix} w_0\\ 0\end{pmatrix}-\begin{pmatrix} \frac{w_0u_0}{x_0}\\ 0\end{pmatrix}-\begin{pmatrix} \frac{w_0x_0}{u_0}\\ 0\end{pmatrix}\\ &= -w_0\begin{pmatrix} \frac{x_0}{u_0}+\frac{u_0}{x_0}-2\\ 0\end{pmatrix}\\ &= -w_0\begin{pmatrix} \left(\sqrt{\frac{x_0}{u_0}}-\sqrt{\frac{u_0}{x_0}}\right)^2\\ 0\end{pmatrix}\in -\mathscr{K}^n.\end{aligned}$$

(ii) For $\bar{u}=0$ and $\bar{v}=0$, it follows from (4.2) that $\bar{w}=0$ and hence $\bar{x}$ and $\bar{s}$ are proportional. Then, we see that

$$\lambda_1(u)=\lambda_2(u)=u_0,\quad \lambda_1(v)=\lambda_2(v)=v_0,$$

$$c^{(1)}(s)=c^{(2)}(x),\quad c^{(2)}(s)=c^{(1)}(x).$$

By the spectral factorization, we compute

$$\begin{aligned}w=x\circ s\ &=[\lambda_1(x)c^{(1)}(x)+\lambda_2(x)c^{(2)}(x)]\circ[\lambda_1(s)c^{(2)}(x)+\lambda_2(s)c^{(1)}(x)]\\ &=\lambda_1(x)\lambda_2(s)c^{(1)}(x)+\lambda_2(x)\lambda_1(s)c^{(2)}(x),\end{aligned}$$

$$\begin{aligned}w=u\circ v\ &=[\lambda_1(u)c^{(1)}(x)+\lambda_2(u)c^{(2)}(x)]\circ[\lambda_1(v)c^{(1)}(x)+\lambda_2(v)c^{(2)}(x)]\\ &=u_0v_0c^{(1)}(x)+u_0v_0c^{(2)}(x)\\ &=w_0c^{(1)}(x)+w_0c^{(2)}(x),\end{aligned}$$

$$\begin{aligned}u\circ s\ &=[\lambda_1(u)c^{(1)}(x)+\lambda_2(u)c^{(2)}(x)]\circ[\lambda_1(s)c^{(2)}(x)+\lambda_2(s)c^{(1)}(x)]\\ &=u_0\lambda_2(s)c^{(1)}(x)+u_0\lambda_1(s)c^{(2)}(x),\end{aligned}$$

and

$$\begin{aligned}x\circ v\ &=[\lambda_1(x)c^{(1)}(x)+\lambda_2(x)c^{(2)}(x)]\circ[\lambda_1(v)c^{(1)}(x)+\lambda_2(v)c^{(2)}(x)]\\ &=v_0\lambda_1(x)c^{(1)}(x)+v_0\lambda_2(x)c^{(2)}(x).\end{aligned}$$

Consequently, they lead to

$$\lambda_1(x)\lambda_2(s)=u_0v_0=w_0>0,\quad \lambda_2(x)\lambda_1(s)=u_0v_0=w_0>0.$$

It indicates that $\lambda_1(x) \geq u_0$ if and only if $\lambda_2(s) \leq v_0$ and $\lambda_2(x) \geq u_0$ if and only if $\lambda_1(s) \leq v_0$. This further shows $(\lambda_1(x) - u_0)(\lambda_2(s) - v_0) \leq 0$ and $(\lambda_2(x) - u_0)(\lambda_1(s) - v_0) \leq 0$. Therefore, we conclude that

$$\begin{array}{rcl} (x-u)\circ(s-v) & = & x\circ s + u\circ v - u\circ s - x\circ v \\ & = & [\lambda_1(x)\lambda_2(s) + u_0v_0 - u_0\lambda_2(s) - v_0\lambda_1(x)]c^{(1)}(x) \\ & & +[\lambda_2(x)\lambda_1(s) + u_0v_0 - u_0\lambda_1(s) - v_0\lambda_2(x)]c^{(2)}(x) \\ & = & (\lambda_1(x) - u_0)(\lambda_2(s) - v_0)c^{(1)}(x) \\ & & +(\lambda_2(x) - u_0)(\lambda_1(s) - v_0)c^{(2)}(x) \\ & \in & -\mathscr{K}^n. \end{array}$$

(iii) For $\bar{x} = 0$ and $\bar{u} = 0$, it follows from (4.1) and (4.2) that

$$\lambda_1(x) = \lambda_2(x) = x_0 > 0, \quad \lambda_1(u) = \lambda_2(u) = u_0 > 0,$$

$$(x_0s_0, x_0\bar{s}) = x\circ s = w = u\circ v = (u_0v_0, u_0\bar{v}).$$

For notational convenience, we denote

$$c^{(1)}(x) = c^{(1)}(s) = c^{(1)}(v) = c^{(1)}(u), \quad c^{(2)}(x) = c^{(2)}(s) = c^{(2)}(v) = c^{(2)}(u).$$

By the spectral factorization, we have

$$\begin{aligned} w = x\circ s &= [\lambda_1(x)c^{(1)}(x) + \lambda_2(x)c^{(2)}(x)]\circ[\lambda_1(s)c^{(1)}(x) + \lambda_2(s)c^{(2)}(x)] \\ &= x_0\lambda_1(s)c^{(1)}(x) + x_0\lambda_2(s)c^{(2)}(x) \\ &= \lambda_1(w)c^{(1)}(x) + \lambda_2(w)c^{(2)}(x), \end{aligned}$$

$$\begin{aligned} w = u\circ v &= [\lambda_1(u)c^{(1)}(x) + \lambda_2(u)c^{(2)}(x)]\circ[\lambda_1(v)c^{(1)}(x) + \lambda_2(v)c^{(2)}(x)] \\ &= u_0\lambda_1(v)c^{(1)}(x) + u_0\lambda_2(v)c^{(2)}(x), \end{aligned}$$

$$\begin{aligned} u\circ s &= [\lambda_1(u)c^{(1)}(x) + \lambda_2(u)c^{(2)}(x)]\circ[\lambda_1(s)c^{(1)}(x) + \lambda_2(s)c^{(2)}(x)] \\ &= u_0\lambda_1(s)c^{(1)}(x) + u_0\lambda_2(s)c^{(2)}(x), \end{aligned}$$

and

$$\begin{aligned} x\circ v &= [\lambda_1(x)c^{(1)}(x) + \lambda_2(x)c^{(2)}(x)]\circ[\lambda_1(v)c^{(1)}(x) + \lambda_2(v)c^{(2)}(x)] \\ &= x_0\lambda_1(v)c^{(1)}(x) + x_0\lambda_2(v)c^{(2)}(x). \end{aligned}$$

Then, we verify that

$$x_0\lambda_1(s) = u_0\lambda_1(v) = \lambda_1(w) \geq 0, \quad x_0\lambda_2(s) = u_0\lambda_2(v) = \lambda_2(w) > 0.$$

Consequently, it indicates that $x_0 \geq u_0$ if and only if $\lambda_1(s) \leq \lambda_1(v)$ and $x_0 \geq u_0$ if and only if $\lambda_2(s) \leq \lambda_2(v)$. This further implies $(x_0 - u_0)(\lambda_1(s) - \lambda_1(v)) \leq 0$ and $(x_0 - u_0)(\lambda_2(s) - \lambda_2(v)) \leq 0$. Therefore, we conclude that

$$\begin{array}{rcl} (x-u)\circ(s-v) & = & x\circ s + u\circ v - u\circ s - x\circ v \\ & = & [x_0\lambda_1(s) + u_0\lambda_1(v) - u_0\lambda_1(s) - x_0\lambda_1(v)]c^{(1)}(x) \\ & & +[x_0\lambda_2(s) + u_0\lambda_2(v) - u_0\lambda_2(s) - x_0\lambda_2(v)]c^{(2)}(x) \\ & = & (x_0 - u_0)(\lambda_1(s) - \lambda_1(v))c^{(1)}(x) \\ & & +(x_0 - u_0)(\lambda_2(s) - \lambda_2(v))c^{(2)}(x) \\ & \in & -\mathscr{K}^n. \end{array}$$

(iv) For $\overline{u} = 0$, $\overline{s} \neq 0$, and $\overline{x}$ and $\overline{v}$ are proportional, it follows from (4.1) that

$$\lambda_1(u) = \lambda_2(u) = u_0 > 0,$$

$$\begin{pmatrix} u_0 v_0 \\ u_0 \overline{v} \end{pmatrix} = u \circ v = w = \begin{pmatrix} w_0 \\ \overline{w} \end{pmatrix},$$

$$\begin{pmatrix} x_0 s_0 + \overline{x}^T \overline{s} \\ x_0 \overline{s} + s_0 \overline{x} \end{pmatrix} = x \circ s = w = \begin{pmatrix} w_0 \\ \overline{w} \end{pmatrix}.$$

Then, $\overline{x}$, $\overline{v}$, $\overline{s}$ and $\overline{w}$ are all proportional. Without loss of generality, we assume

$$\begin{aligned} c^{(1)}(x) = c^{(1)}(v) = c^{(1)}(u) = c^{(2)}(s), \\ c^{(2)}(x) = c^{(2)}(v) = c^{(2)}(u) = c^{(1)}(s). \end{aligned}$$

By the spectral factorization, we obtain

$$\begin{aligned} w = x \circ s &= [\lambda_1(x)c^{(1)}(x) + \lambda_2(x)c^{(2)}(x)] \circ [\lambda_1(s)c^{(2)}(x) + \lambda_2(s)c^{(1)}(x)] \\ &= \lambda_1(x)\lambda_2(s)c^{(1)}(x) + \lambda_2(x)\lambda_1(s)c^{(2)}(x) \\ &= \lambda_1(w)c^{(1)}(x) + \lambda_2(w)c^{(2)}(x), \end{aligned}$$

$$\begin{aligned} w = u \circ v &= [\lambda_1(u)c^{(1)}(x) + \lambda_2(u)c^{(2)}(x)] \circ [\lambda_1(v)c^{(1)}(x) + \lambda_2(v)c^{(2)}(x)] \\ &= u_0\lambda_1(v)c^{(1)}(x) + u_0\lambda_2(v)c^{(2)}(x), \end{aligned}$$

$$\begin{aligned} u \circ s &= [\lambda_1(u)c^{(1)}(x) + \lambda_2(u)c^{(2)}(x)] \circ [\lambda_1(s)c^{(2)}(x) + \lambda_2(s)c^{(1)}(x)] \\ &= u_0\lambda_2(s)c^{(1)}(x) + u_0\lambda_1(s)c^{(2)}(x), \end{aligned}$$

and

$$\begin{aligned} x \circ v &= [\lambda_1(x)c^{(1)}(x) + \lambda_2(x)c^{(2)}(x)] \circ [\lambda_1(v)c^{(1)}(x) + \lambda_2(v)c^{(2)}(x)] \\ &= \lambda_1(x)\lambda_1(v)c^{(1)}(x) + \lambda_2(x)\lambda_2(v)c^{(2)}(x). \end{aligned}$$

They yield

$$\lambda_1(x)\lambda_2(s) = u_0\lambda_1(v) = \lambda_1(w) \geq 0, \quad \lambda_2(x)\lambda_1(s) = u_0\lambda_2(v) = \lambda_2(w) > 0,$$

which says that $\lambda_1(x) \geq u_0$ if and only if $\lambda_2(s) \leq \lambda_1(v)$ and $\lambda_2(x) \geq u_0$ if and only if $\lambda_1(s) \leq \lambda_2(v)$. This implies $(\lambda_1(x) - u_0)(\lambda_2(s) - \lambda_1(v)) \leq 0$ and $(\lambda_2(x) - u_0)(\lambda_1(s) - \lambda_2(v)) \leq 0$. Therefore, we conclude that

$$\begin{aligned} (x-u)\circ(s-v) &= x \circ s + u \circ v - u \circ s - x \circ v \\ &= [\lambda_1(x)\lambda_2(s) + u_0\lambda_1(v) - u_0\lambda_2(s) - \lambda_1(x)\lambda_1(v)]c^{(1)}(x) \\ &\quad + [\lambda_2(x)\lambda_1(s) + u_0\lambda_2(v) - u_0\lambda_1(s) - \lambda_2(x)\lambda_2(v)]c^{(2)}(x) \\ &= (\lambda_1(x) - u_0)(\lambda_2(s) - \lambda_1(v))c^{(1)}(x) \\ &\quad + (\lambda_2(x) - u_0)(\lambda_1(s) - \lambda_2(v))c^{(2)}(x) \\ &\in -\mathcal{K}^n. \end{aligned}$$

(v) If $\overline{x}$ and $\overline{v}$ are proportional, $\overline{u}$ and $\overline{s}$ are proportional, and $\overline{x}$ and $\overline{s}$ are also proportional, we obtain that x, v, u and s all share the same spectral vectors. Without loss of generality, we assume

$$\begin{aligned} c^{(1)}(x) = c^{(2)}(v) = c^{(2)}(u) = c^{(1)}(s), \\ c^{(2)}(x) = c^{(1)}(v) = c^{(1)}(u) = c^{(2)}(s). \end{aligned}$$

By the spectral factorization, we have

$$\begin{aligned} w = x \circ s &= [\lambda_1(x)c^{(1)}(x) + \lambda_2(x)c^{(2)}(x)] \circ [\lambda_1(s)c^{(1)}(x) + \lambda_2(s)c^{(2)}(x)] \\ &= \lambda_1(x)\lambda_1(s)c^{(1)}(x) + \lambda_2(x)\lambda_2(s)c^{(2)}(x), \end{aligned}$$

$$\begin{aligned} w = u \circ v &= [\lambda_1(u)c^{(2)}(x) + \lambda_2(u)c^{(1)}(x)] \circ [\lambda_1(v)c^{(2)}(x) + \lambda_2(v)c^{(1)}(x)] \\ &= \lambda_2(u)\lambda_2(v)c^{(1)}(x) + \lambda_1(u)\lambda_1(v)c^{(2)}(x), \end{aligned}$$

$$\begin{aligned} u \circ s &= [\lambda_1(u)c^{(2)}(x) + \lambda_2(u)c^{(1)}(x)] \circ [\lambda_1(s)c^{(1)}(x) + \lambda_2(s)c^{(2)}(x)] \\ &= \lambda_2(u)\lambda_1(s)c^{(1)}(x) + \lambda_1(u)\lambda_2(s)c^{(2)}(x), \end{aligned}$$

and

$$\begin{aligned} x \circ v &= [\lambda_1(x)c^{(1)}(x) + \lambda_2(x)c^{(2)}(x)] \circ [\lambda_1(v)c^{(2)}(x) + \lambda_2(v)c^{(1)}(x)] \\ &= \lambda_1(x)\lambda_2(v)c^{(1)}(x) + \lambda_2(x)\lambda_1(v)c^{(2)}(x). \end{aligned}$$

They lead to

$$\begin{aligned} \lambda_1(x)\lambda_1(s) = \lambda_2(u)\lambda_2(v) = \lambda_1(w) \geq 0, \\ \lambda_2(x)\lambda_2(s) = \lambda_1(u)\lambda_1(v) = \lambda_2(w) > 0, \end{aligned}$$

which indicate that $\lambda_1(x) \geq \lambda_2(u)$ if and only if $\lambda_1(s) \leq \lambda_2(v)$ and $\lambda_2(x) \geq \lambda_1(u)$ if and only if $\lambda_2(s) \leq \lambda_1(v)$. This implies $(\lambda_1(x) - \lambda_2(u))(\lambda_1(s) - \lambda_2(v)) \leq 0$ and $(\lambda_2(x) - \lambda_1(u))(\lambda_2(s) - \lambda_1(v)) \leq 0$. Therefore, we conclude that

$$\begin{aligned} &(x-u) \circ (s-v) \\ &= x \circ s + u \circ v - u \circ s - x \circ v \\ &= [\lambda_1(x)\lambda_1(s) + \lambda_2(u)\lambda_2(v) - \lambda_2(u)\lambda_1(s) - \lambda_1(x)\lambda_2(v)]c^{(1)}(x) \\ &\quad + [\lambda_2(x)\lambda_2(s) + \lambda_1(u)\lambda_1(v) - \lambda_1(u)\lambda_2(s) - \lambda_2(x)\lambda_1(v)]c^{(2)}(x) \\ &= (\lambda_1(x) - \lambda_2(u))(\lambda_1(s) - \lambda_2(v))c^{(1)}(x) \\ &\quad + (\lambda_2(x) - \lambda_1(u))(\lambda_2(s) - \lambda_1(v))c^{(2)}(x) \\ &\in -\mathcal{K}^n. \end{aligned}$$

(vi) If $\bar{x}$ and $\bar{v}$ are proportional, and $\bar{u}$ and $\bar{s}$ are proportional, there exist real numbers $\alpha, \beta \neq 0$ such that

$$\bar{u} = \alpha \bar{s}, \quad \bar{v} = \beta \bar{x}. \tag{4.3}$$

By (4.2), one has $x_0 \bar{s} + s_0 \bar{x} = u_0 \beta \bar{x} + v_0 \alpha \bar{s}$. If $\bar{x}$ and $\bar{s}$ are not proportional, then

$$x_0 = v_0 \alpha, \quad s_0 = u_0 \beta. \tag{4.4}$$

From (4.1), (4.3), and (4.4), we obtain

$$\begin{aligned} u = (u_0, \bar{u}) = (\frac{s_0}{\beta}, \alpha \bar{s}) \in \mathcal{K}^n \Rightarrow \frac{s_0}{\beta} \geq \alpha \|\bar{s}\| \Rightarrow \alpha\beta \leq \frac{s_0}{\|\bar{s}\|}, \\ v = (v_0, \bar{v}) = (\frac{x_0}{\alpha}, \beta \bar{x}) \in \mathcal{K}^n \Rightarrow \frac{x_0}{\alpha} \geq \beta \|\bar{x}\| \Rightarrow \alpha\beta \leq \frac{x_0}{\|\bar{x}\|}. \end{aligned} \tag{4.5}$$

Combining (4.2) and (4.5) yields

$$\begin{aligned}
&x_0s_0+\bar{x}^T\bar{s}=\frac{x_0s_0}{\alpha\beta}+\alpha\beta\bar{x}^T\bar{s}\\
&\Leftrightarrow \alpha\beta x_0s_0+\alpha\beta\bar{x}^T\bar{s}=x_0s_0+\alpha^2\beta^2\bar{x}^T\bar{s}\\
&\Leftrightarrow (\alpha\beta-1)x_0s_0+\alpha\beta(1-\alpha\beta)\bar{x}^T\bar{s}=0\\
&\Leftrightarrow (\alpha\beta-1)(x_0s_0-\alpha\beta\bar{x}^T\bar{s})=0\\
&\Leftrightarrow \alpha\beta=1 \text{ or } \alpha\beta=\frac{x_0s_0}{\bar{x}^T\bar{s}}.
\end{aligned} \tag{4.6}$$

In the following, we show $\alpha\beta=1$ by contradiction. If $\alpha\beta=\frac{x_0s_0}{\bar{x}^T\bar{s}}$, we obtain from (4.5)

$$\begin{aligned}
\alpha\beta=\frac{x_0s_0}{\bar{x}^T\bar{s}}\le\frac{s_0}{\|\bar{s}\|}\Rightarrow x_0\le\frac{\bar{x}^T\bar{s}}{\|\bar{s}\|}\le\frac{\|\bar{x}\|\cdot\|\bar{s}\|}{\|\bar{s}\|}=\|\bar{x}\|,\\
\alpha\beta=\frac{x_0s_0}{\bar{x}^T\bar{s}}\le\frac{x_0}{\|\bar{x}\|}\Rightarrow s_0\le\frac{\bar{x}^T\bar{s}}{\|\bar{x}\|}\le\frac{\|\bar{x}\|\cdot\|\bar{s}\|}{\|\bar{x}\|}=\|\bar{s}\|.
\end{aligned}$$

However, it follows from (4.1) that $x\in\mathscr{K}^n$ and $s\in\mathscr{K}^n$, i.e., $x_0\ge\|\bar{x}\|$ and $s_0\ge\|\bar{s}\|$. Thus,

$$\begin{aligned}
x_0=\frac{\bar{x}^T\bar{s}}{\|\bar{s}\|}=\frac{\|\bar{x}\|\cdot\|\bar{s}\|}{\|\bar{s}\|}=\|\bar{x}\|,\\
s_0=\frac{\bar{x}^T\bar{s}}{\|\bar{x}\|}=\frac{\|\bar{x}\|\cdot\|\bar{s}\|}{\|\bar{x}\|}=\|\bar{s}\|,
\end{aligned}$$

which contradict the fact that $\bar{x}$ and $\bar{s}$ are not proportional. Then by (4.5) and (4.6), we have $\alpha\beta=1$, and

$$u=\alpha(s_0,\bar{s})\in\mathscr{K}^n,\quad v=\frac{1}{\alpha}(x_0,\bar{x})\in\mathscr{K}^n. \tag{4.7}$$

Using (4.7), it further yields

$$\begin{aligned}
&(x-u)\circ(s-v)\\
&=\begin{pmatrix} x_0-\alpha s_0\\ \bar{x}-\alpha\bar{s}\end{pmatrix}\circ\begin{pmatrix} s_0-\frac{1}{\alpha}x_0\\ \bar{s}-\frac{1}{\alpha}\bar{x}\end{pmatrix}\\
&=-\frac{1}{\alpha}(x-\alpha s)^2\in-\mathscr{K}^n.
\end{aligned}$$

Therefore, $x-u$ and $s-v$ operator commute, and

$$(x-u)\circ(s-v)\in-\mathscr{K}^n \tag{4.8}$$

holds in all the cases. Applying (4.8), the column sufficient property and the fact $(x-u,s-v,y-h)\in\mathrm{Ker}(A,B,C)$, we deduce that

$$(x-u)\circ(s-v)=0, \tag{4.9}$$

and $x\circ v+u\circ s=2w$. By Lemma 4.3, the solution set $\mathscr{F}^*$ of wSOCCP (1.1) is convex.

(b) For any $(x,s,y),(u,v,h)\in\mathscr{F}^*$, it follows from [18, Lemma 15], the proof of (a) and (4.9) that $x-u=0$, or $s-v=0$, or $\bar{x}-\bar{u}$ and $\bar{s}-\bar{v}$ are proportional. Then, there exists a real number $\delta\le0$ such that $\bar{x}-\bar{u}=\delta(\bar{s}-\bar{v})$. By the cross commutative property, x operator commutes with v and u operator commutes with s. Then it follows from [18, Corollary 7] that $\bar{x}=0$ or $\bar{v}=0$

or $\bar{x}$ and $\bar{v}$ are proportional, and $\bar{u}=0$ or $\bar{s}=0$ or $\bar{u}$ and $\bar{s}$ are proportional. Without loss of generality, there exist real numbers α,β (possibly zero) such that $\bar{v}=\alpha\bar{x}$ and $\bar{u}=\beta\bar{s}$. Therefore

$$(1+\delta\alpha)\bar{x}=(\delta+\beta)\bar{s},$$

which, together with the proof of (a), implies that $\bar{x}$ and $\bar{s}$ are proportional (possibly zero) and hence operator commute.

(c) Without loss of generality, we assume $w\in\mathrm{int}\mathscr{K}^n$. From the proof of (b), $\bar{x}$, $\bar{s}$, $\bar{u}$, $\bar{v}$ and $\bar{w}$ are all proportional (possibly zero). Again, without loss of generality, we assume

$$\begin{array}{c} c^{(1)}(x)=c^{(1)}(v)=c^{(1)}(u)=c^{(1)}(s),\\ c^{(2)}(x)=c^{(2)}(v)=c^{(2)}(u)=c^{(2)}(s). \end{array}$$

Then, it follows from (4.9) that

$$\begin{array}{rcl} 0 &=& (x-u)\circ(s-v)\\ &=& (\lambda_1(x)-\lambda_1(u))(\lambda_1(s)-\lambda_1(v))c^{(1)}(x)\\ && +(\lambda_2(x)-\lambda_2(u))(\lambda_2(s)-\lambda_2(v))c^{(2)}(x), \end{array}$$

and hence

$$\begin{array}{c} (\lambda_1(x)-\lambda_1(u))(\lambda_1(s)-\lambda_1(v))=0,\\ (\lambda_2(x)-\lambda_2(u))(\lambda_2(s)-\lambda_2(v))=0. \end{array} \tag{4.10}$$

By the spectral factorization, we compute that

$$\begin{array}{rl} w=x\circ s &= [\lambda_1(x)c^{(1)}(x)+\lambda_2(x)c^{(2)}(x)]\circ[\lambda_1(s)c^{(1)}(x)+\lambda_2(s)c^{(2)}(x)]\\ &= \lambda_1(x)\lambda_1(s)c^{(1)}(x)+\lambda_2(x)\lambda_2(s)c^{(2)}(x), \end{array}$$

$$\begin{array}{rl} w=u\circ v &= [\lambda_1(u)c^{(1)}(x)+\lambda_2(u)c^{(2)}(x)]\circ[\lambda_1(v)c^{(1)}(x)+\lambda_2(v)c^{(2)}(x)]\\ &= \lambda_1(u)\lambda_1(v)c^{(1)}(x)+\lambda_2(u)\lambda_2(v)c^{(2)}(x), \end{array}$$

Since $w\in\mathrm{int}\mathscr{K}^n$, we have

$$\begin{array}{c} \lambda_1(x)\lambda_1(s)=\lambda_1(u)\lambda_1(v)=\lambda_1(w)>0,\\ \lambda_2(x)\lambda_2(s)=\lambda_2(u)\lambda_2(v)=\lambda_2(w)>0. \end{array}$$

Therefore, there hold $\lambda_1(x)\neq\lambda_1(u)$ if and only if $\lambda_1(s)\neq\lambda_1(v)$ and $\lambda_2(x)\neq\lambda_2(u)$ if and only if $\lambda_2(s)\neq\lambda_2(v)$. Then by (4.10), we conclude that

$$\begin{array}{ll} \lambda_1(x)=\lambda_1(u)>0, & \lambda_1(s)=\lambda_1(v)>0,\\ \lambda_2(x)=\lambda_2(u)>0, & \lambda_2(s)=\lambda_2(v)>0, \end{array}$$

i.e., $x=u\in\mathrm{int}\mathscr{K}^n$ and $s=v\in\mathrm{int}\mathscr{K}^n$. If $w\in\mathrm{bd}\mathscr{K}^n$, by following the above proof, we can achieve

$$\begin{array}{c} \lambda_1(x)\lambda_1(s)=\lambda_1(w)=0,\\ \lambda_2(x)\lambda_2(s)=\lambda_2(w)\geq 0, \end{array}$$

which imply that at least one of $\lambda_1(x)$ and $\lambda_1(s)$ is zero. Therefore, either $x\in\mathrm{int}\mathscr{K}^n$, $s\in\mathrm{bd}\mathscr{K}^n$, or $x\in\mathrm{bd}\mathscr{K}^n$, $s\in\mathrm{int}\mathscr{K}^n$, or $x\in\mathrm{bd}\mathscr{K}^n$, $s\in\mathrm{bd}\mathscr{K}^n$. □

Example 4.1. Consider the wSOCCP (1.2) associated with $\mathscr{K}^3$, where

$$Q=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad R=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad R'=\begin{pmatrix}0&0&0\\0&-1&0\\0&0&0\end{pmatrix},$$

$$a=\begin{pmatrix}1\\0\\1\end{pmatrix},\quad w=\begin{pmatrix}4\\0\\4\end{pmatrix}\in \mathrm{bd}\mathscr{K}^3,\quad u=\begin{pmatrix}0\\-1\\0\end{pmatrix},\quad v=\begin{pmatrix}1\\1\\0\end{pmatrix}.$$

The solution set of the wSOCCP (1.2) with both (Q,R,a,w) and (Q,R',a,w), given by

$$\left\{x=\begin{pmatrix}1\\0\\1\end{pmatrix}, s=\begin{pmatrix}s_1\\0\\4-s_1\end{pmatrix} \text{ with } s_1\geq 2\right\},$$

is convex, where x and s are both cross and operator commutative. Here $(u,v)\in \mathrm{Ker}(Q,R)$, $u\circ v\in -\mathscr{K}^3$, $u\circ v=(-1,-1,0)^T\neq 0$, which implies that the pair (Q,R) is not column sufficient. Moreover, for any $(u',v')\in \mathrm{Ker}(Q,R')$, we have $u'=(0,v'_2,0)^T$ and $u'^Tv'=v'^2_2\geq 0$. Then, the wSOCCP (1.2) with (Q,R',a,w) is Cartesian monotone and from Lemma 4.2, it is column sufficient.

In light of Example 4.1, we sum up the following observations.

(i): The existence of a weight vector $w\in\mathscr{K}\backslash\{0\}$ and a vector $d\in\mathbb{R}^{n+m}$ such that the solution set of the wSOCCP (1.1) is convex does not imply that the triplet (A,B,C) is column sufficient.

(ii): Even if the triplet (A,B,C) for the wSOCCP (1.1) has the column sufficient and cross commutative properties, x_i and s_i are not necessarily uniquely defined for some $i=1,2,\dots,r$ such that $w_i\notin \mathrm{int}\mathscr{K}^{n_i}$ and any $(x,s,y)\in\mathscr{F}^*$.

(iii): Moreover, if the triplet (A,B,C) for the wSOCCP (1.1) is column sufficient and cross commutative, we have either

$$x_i\in \mathrm{int}\mathscr{K}^{n_i},\quad s_i\in \mathrm{bd}\mathscr{K}^{n_i},$$

or

$$x_i\in \mathrm{bd}\mathscr{K}^{n_i},\quad s_i\in \mathrm{int}\mathscr{K}^{n_i},$$

or

$$x_i\in \mathrm{bd}\mathscr{K}^{n_i},\quad s_i\in \mathrm{bd}\mathscr{K}^{n_i},$$

for any $i=1,2,\dots,r$ such that $w_i\in \mathrm{bd}\mathscr{K}^{n_i}$.

Theorem 4.3. *For the wSOCCP (1.1) with $w=0$, the following statements are equivalent.*

(i): *The triplet (A,B,C) has the column sufficient and cross commutative properties.*

(ii): *For any $d\in\mathbb{R}^{n+m}$, the wSOCCP (1.1) with $w=0$ has a convex (possibly empty) solution set.*

Proof. (i) $\Rightarrow$ (ii) It follows immediately from Theorem 4.2.
(ii) $\Rightarrow$ (i) On the contrary, we assume that the pair (Q,R) in the wSOCCP (1.2) is not column sufficient. Then, there exist vectors $u,v\in\mathbb{R}^n$ such that

$$\left.\begin{array}{l}(u,v)\in \mathrm{Ker}(Q,R)\\ u \text{ and } v \text{ operate commute}\\ u\circ v\in -\mathscr{K}\end{array}\right\}\Rightarrow u\circ v\neq 0.$$

Thus, it follows from Definition 5 in [18] that u and v have the same spectral vectors. Without loss of generality, we assume $c^{(1)}(u)=c^{(1)}(v)$, $c^{(2)}(u)=c^{(2)}(v)$, $\lambda_1(u)\lambda_1(v)\leq 0$ and $\lambda_2(u)\lambda_2(v)\leq 0$. Consequently, by the spectral factorization, we have $u^+\circ v^+=0$ and $u^-\circ v^-=$

0. Note that $u = u^+ - u^-, v = v^+ - v^-$. We define $a = Qu^+ + Rv^+ = Qu^- + Rv^-$. Clearly, there holds $\langle u, v\rangle = \langle e, u \circ v\rangle < 0$. Therefore, (u^+, v^+) and (u^-, v^-) are two distinct solutions of the wSOCCP (1.2) with $w = 0$. Since the solution set of the wSOCCP (1.2) with $w = 0$ is convex, applying Lemma 4.3 yields that $u^+ \circ v^- = u^- \circ v^+ = 0$. This implies $u \circ v = (u^+ - u^-) \circ (v^+ - v^-) = 0$, which is a contradiction with $\langle u, v\rangle < 0$. In addition, since the solution set of the wSOCCP (1.2) with $w = 0$ is convex, it follows from Lemma 4.3 that $x \circ v = u \circ s = 0$ for any two distinct solutions (x, s) and (u, v) of the wSOCCP (1.2) with $w = 0$. Thus the cross commutative property holds. □

Remark 4.2. Theorem 4.2 shows if the triplet (A, B, C) has the column sufficient and cross commutative properties, the solution set of the wSOCCP is convex. Under the same condition, Qin, Kong and Han [28] studied the convexity of solution set for the *symmetric cone linear complementarity problem* (SCLCP), which is to find $x \in \mathscr{V}$ such that

$$x \in K, \quad \mathscr{L}(x) + q \in K, \quad \langle x, \mathscr{L}(x) + q\rangle = 0,$$

where $(\mathscr{V}, \circ, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra, K is the corresponding symmetric cone, $\mathscr{L} : \mathscr{V} \to \mathscr{V}$ is a linear transformation and $q \in \mathscr{V}$. When a symmetric cone reduces to a SOC, a SCLCP becomes a SOCCP, which is the special case of wSOCCP with $w = 0$.

Theorem 4.4. *Suppose the triplet (A, B, C) has the cross commutative, solvable and P-property. Then, wSOCCP (1.1) has the globally uniquely solvable property.*

Proof. If wSOCCP (1.1) has less than two solutions, the result obviously holds. Suppose that (x, s, y) and (u, v, h) are arbitrary two solutions of wSOCCP (1.1). Using the triplet (A, B, C) with the cross commutative property and following the proof of Theorem 4.2(a) yield that $x - u$ and $s - v$ operator commute, and (4.8) holds, i.e., $(x - u) \circ (s - v) \in -\mathscr{K}^n$. Sicne (A, B, C) has the P-property, it follows from Definition 3.2 that $x - u = 0$ and $s - v = 0$, which implies that wSOCCP (1.1) has the globally uniquely solvable property. □

5. Maximal Complementarity

To present the maximal complementarity results, let us define the following index sets

$$\begin{aligned} \mathscr{W} &= \{i \in \{1, 2, \ldots, r\} : w_i = 0\}, \\ \overline{\mathscr{W}} &= \{i \in \{1, 2, \ldots, r\} : w_i \in \mathrm{bd}\mathscr{K}^{n_i}\}, \\ \widehat{\mathscr{W}} &= \{i \in \{1, 2, \ldots, r\} : w_i \in \mathrm{int}\mathscr{K}^{n_i}\}. \end{aligned}$$

Then, by [18, Corollary 24], we consider the following subsets of $\mathscr{W}$:

$$\begin{aligned} \mathscr{I} &= \{i \in \{1, 2, \ldots, r\} : \exists (x, s, y) \in \mathscr{F}^*, x_i \in \mathrm{int}\mathscr{K}^{n_i}, s_i = 0\}, \\ \mathscr{J} &= \{i \in \{1, 2, \ldots, r\} : \exists (x, s, y) \in \mathscr{F}^*, x_i = 0, s_i \in \mathrm{int}\mathscr{K}^{n_i}\}, \\ \mathscr{B} &= \{i \in \{1, 2, \ldots, r\} : \exists (x, s, y) \in \mathscr{F}^*, x_i \in \mathrm{bd}\mathscr{K}^{n_i}, s_i \in \mathrm{bd}\mathscr{K}^{n_i}\}, \\ \mathscr{O} &= \{i \in \{1, 2, \ldots, r\} : \forall (x, s, y) \in \mathscr{F}^*, x_i + s_i \notin \mathrm{int}\mathscr{K}^{n_i}\}. \end{aligned}$$

Further, according to Theorem 4.2, we define the following subsets of $\overline{\mathscr{W}}$:

$$\begin{aligned} \overline{\mathscr{I}} &= \{i \in \{1, 2, \ldots, r\} : \exists (x, s, y) \in \mathscr{F}^*, x_i \in \mathrm{int}\mathscr{K}^{n_i}, s_i \in \mathrm{bd}\mathscr{K}^{n_i}\}, \\ \overline{\mathscr{J}} &= \{i \in \{1, 2, \ldots, r\} : \exists (x, s, y) \in \mathscr{F}^*, x_i \in \mathrm{bd}\mathscr{K}^{n_i}, s_i \in \mathrm{int}\mathscr{K}^{n_i}\}, \\ \overline{\mathscr{B}} &= \{i \in \{1, 2, \ldots, r\} : \forall (x, s, y) \in \mathscr{F}^*, x_i \in \mathrm{bd}\mathscr{K}^{n_i}, s_i \in \mathrm{bd}\mathscr{K}^{n_i}\}. \end{aligned}$$

Theorem 5.1. *Suppose that wSOCCP (1.1) is column sufficient and cross commutative. Then, the following results hold.*

- **(a):** *The index sets $\mathscr{I}$, $\mathscr{J}$, $\mathscr{B}$, and $\mathscr{O}$ are disjoint and they form a partition of $\mathscr{W}$.*
- **(b):** *The index sets $\overline{\mathscr{I}}$, $\overline{\mathscr{J}}$, and $\overline{\mathscr{B}}$ are disjoint and they form a partition of $\overline{\mathscr{W}}$.*
- **(c):** *Moreover, for any solution $(x,s,y) \in \mathscr{F}^*$, there must have $x_{\mathscr{J}} = 0$, $s_{\mathscr{I}} = 0$, $x_{\overline{\mathscr{B}}} \in bd\mathscr{K}^{n_{\overline{\mathscr{B}}}}$, $s_{\overline{\mathscr{B}}} \in bd\mathscr{K}^{n_{\overline{\mathscr{B}}}}$, and $x_{\widehat{\mathscr{W}}} \in int\mathscr{K}^{n_{\widehat{\mathscr{W}}}}$, $s_{\widehat{\mathscr{W}}} \in int\mathscr{K}^{n_{\widehat{\mathscr{W}}}}$.*

Proof. Since wSOCCP (1.1) is column sufficient and cross commutative, it follows from Theorem 4.2 that the solution set of (1.1) is convex.

(a) It is obvious that $\mathscr{I} \cap \mathscr{O} = \mathscr{J} \cap \mathscr{O} = \mathscr{B} \cap \mathscr{O} = \emptyset$. To verify the desired result, we discuss three steps.

(i) Firstly, we show $\mathscr{I} \cap \mathscr{J} = \emptyset$. For $i \in \mathscr{I} \cap \mathscr{J}$, there exist $(x,s,y),(u,v,h) \in \mathscr{F}^*$ such that $x_i \in \mathrm{int}\mathscr{K}^{n_i}$, $s_i = 0$, $u_i = 0$, $v_i \in \mathrm{int}\mathscr{K}^{n_i}$. Therefore, for any $\xi \in (0,1)$, we have $\xi x_i + (1-\xi)u_i \in \mathrm{int}\mathscr{K}^{n_i}$ and $\xi s_i + (1-\xi)v_i \in \mathrm{int}\mathscr{K}^{n_i}$, which contradicts the fact that the solution set of (1.1) is convex.

(ii) Secondly, we show $\mathscr{I} \cap \mathscr{B} = \emptyset$. For $i \in \mathscr{I} \cap \mathscr{B}$, there exist $(x,s,y),(u,v,h) \in \mathscr{F}^*$ such that $x_i \in \mathrm{int}\mathscr{K}^{n_i}$, $s_i = 0$, $u_i \in \mathrm{bd}\mathscr{K}^{n_i}$, $v_i \in \mathrm{bd}\mathscr{K}^{n_i}$. Thus, for any $\xi \in (0,1)$, $\xi x_i + (1-\xi)u_i \in \mathrm{int}\mathscr{K}^{n_i}$ and $\xi s_i + (1-\xi)v_i \in \mathrm{bd}\mathscr{K}^{n_i}$, which contradicts the fact that the solution set of (1.1) is convex.

(iii) By following the same arguments of (ii), it can be verified that $\mathscr{J} \cap \mathscr{B} = \emptyset$.

To conclude, the index sets $\mathscr{I}, \mathscr{J}, \mathscr{B}, \mathscr{O}$ are disjoint and $\mathscr{W} = \mathscr{I} \cup \mathscr{J} \cup \mathscr{B} \cup \mathscr{O}$.

(b) From Theorem 4.2(c), we have $\overline{\mathscr{W}} = \overline{\mathscr{I}} \cup \overline{\mathscr{J}} \cup \overline{\mathscr{B}}$. Then, it is clear that $\overline{\mathscr{I}} \cap \overline{\mathscr{B}} = \overline{\mathscr{J}} \cap \overline{\mathscr{B}} = \emptyset$.

Now, we show $\overline{\mathscr{I}} \cap \overline{\mathscr{J}} = \emptyset$. For $i \in \overline{\mathscr{I}} \cap \overline{\mathscr{J}}$, there exist $(x,s,y),(u,v,h) \in \mathscr{F}^*$ such that $x_i \in \mathrm{int}\mathscr{K}^{n_i}$, $s_i \in \mathrm{bd}\mathscr{K}^{n_i}$, $u_i \in \mathrm{bd}\mathscr{K}^{n_i}$, $v_i \in \mathrm{int}\mathscr{K}^{n_i}$. Then, for any $\xi \in (0,1)$, we have $\xi x_i + (1-\xi)u_i \in \mathrm{int}\mathscr{K}^{n_i}$ and $\xi s_i + (1-\xi)v_i \in \mathrm{int}\mathscr{K}^{n_i}$, which contradicts the fact that the solution set of (1.1) is convex. Thus, $\overline{\mathscr{I}}$, $\overline{\mathscr{J}}$, $\overline{\mathscr{B}}$ are disjoint and $\overline{\mathscr{W}} = \overline{\mathscr{I}} \cup \overline{\mathscr{J}} \cup \overline{\mathscr{B}}$.

(c) Consider any solution $(x,s,y) \in \mathscr{F}^*$ and an index $i \in \mathscr{I}$. For $s_i \in \mathrm{int}\mathscr{K}^{n_i}$, we have $x_i = 0$ and therefore $i \in \mathscr{J}$, which contradicts $\mathscr{I} \cap \mathscr{J} = \emptyset$. For $s_i \in \mathrm{bd}\mathscr{K}^{n_i}$, we obtain $x_i \in \mathrm{bd}\mathscr{K}^{n_i}$ or $x_i = 0$; and consequently $i \in \mathscr{B}$ or $i \in \mathscr{O}$, which contradicts $\mathscr{I} \cap \mathscr{B} = \emptyset$ or $\mathscr{I} \cap \mathscr{O} = \emptyset$. Therefore $s_{\mathscr{I}} = 0$. Similarly, we can conclude that $x_{\mathscr{J}} = 0$. By the definition of $\overline{\mathscr{B}}$, we have, for any $i \in \overline{\mathscr{B}}$, $x_i \in \mathrm{bd}\mathscr{K}^{n_i}$ and $s_i \in \mathrm{bd}\mathscr{K}^{n_i}$, i.e., $x_{\overline{\mathscr{B}}} \in \mathrm{bd}\mathscr{K}^{n_{\overline{\mathscr{B}}}}$, $s_{\overline{\mathscr{B}}} \in \mathrm{bd}\mathscr{K}^{n_{\overline{\mathscr{B}}}}$. In view of the proof of Theorem 4.2(c), we obtain that, for any $i \in \widehat{\mathscr{W}}$, $x_i \in \mathrm{int}\mathscr{K}^{n_i}$ and $s_i \in \mathrm{int}\mathscr{K}^{n_i}$, i.e., $x_{\widehat{\mathscr{W}}} \in \mathrm{int}\mathscr{K}^{n_{\widehat{\mathscr{W}}}}$, $s_{\widehat{\mathscr{W}}} \in \mathrm{int}\mathscr{K}^{n_{\widehat{\mathscr{W}}}}$. ☐

Definition 5.1. A solution $(x^*,s^*,y^*) \in \mathscr{F}^*$ is said to be a maximal complementarity solution if

$$\begin{aligned}&\mathrm{Cardinal}\{i : \text{for any } i = 1,2,\ldots,r \text{ such that } x_i^* + s_i^* \in \mathrm{int}\mathscr{K}^{n_i}\}\\ &= \max_{(x,s,y)\in\mathscr{F}^*} \mathrm{Cardinal}\{i : \text{for any } i = 1,2,\ldots,r \text{ such that } x_i + s_i \in \mathrm{int}\mathscr{K}^{n_i}\}.\end{aligned}$$

Definition 5.2. A solution $(x,s,y) \in \mathscr{F}^*$ is said to be a strict complementarity solution if $x+s \in \mathrm{int}\mathscr{K}$.

For any $(x,s,y) \in \mathscr{F}^*$, it follows from [18, Corollary 24], the proof of Theorem 4.2(c), Definition 5.1, and Definition 5.2 that $x_i + s_i \in \mathrm{int}\mathscr{K}^{n_i}$ for any $i \in \mathscr{I} \cup \mathscr{J} \cup \mathscr{B} \cup \overline{\mathscr{I}} \cup \overline{\mathscr{J}} \cup \widehat{\mathscr{W}}$. In other words, a maximal complementarity solution is the strict complementarity solution if and only if $\mathscr{O} \cup \overline{\mathscr{B}} = \emptyset$.

Theorem 5.2. *Suppose that wSOCCP (1.1) is column sufficient, cross commutative, and solvable. Then, the following hold.*

- **(a):** *If $\mathscr{O}=\emptyset$, the set of its maximal complementarity solutions is a nonempty convex set, and any maximal complementarity solution must be in ri$(\mathscr{F}^*)$ (i.e., the relative interior of $\mathscr{F}^*$).*
- **(b):** *Any solution in ri$(\mathscr{F}^*)$ must be a maximal complementarity solution of the wSOCCP (1.1).*

Proof. (a) From Theorem 4.2, $\mathscr{F}^*$ is convex. For any $i\in\mathscr{I}$, we consider a solution $(x^i,s^i,y^i)\in\mathscr{F}^*$ with $x^i_i\in\mathrm{int}\mathscr{K}^{n_i}, s^i_i=0$; for any $j\in\mathscr{J}$, we consider a solution $(x^j,s^j,y^j)\in\mathscr{F}^*$ with $x^j_j=0, s^j_j\in\mathrm{int}\mathscr{K}^{n_j}$; for any $k\in\mathscr{B}$, we consider a solution $(x^k,s^k,y^k)\in\mathscr{F}^*$ with $x^k_k\in\mathrm{bd}\mathscr{K}^{n_k}, s^k_k\in\mathrm{bd}\mathscr{K}^{n_k}$; for any $i'\in\overline{\mathscr{I}}$, we consider a solution $(x^{i'},s^{i'},y^{i'})\in\mathscr{F}^*$ with $x^{i'}_{i'}\in\mathrm{int}\mathscr{K}^{n_{i'}}$, $s^{i'}_{i'}\in\mathrm{bd}\mathscr{K}^{n_{i'}}$; for any $j'\in\overline{\mathscr{J}}$, we consider a solution $(x^{j'},s^{j'},y^{j'})\in\mathscr{F}^*$ with $x^{j'}_{j'}\in\mathrm{bd}\mathscr{K}^{n_{j'}}$, $s^{j'}_{j'}\in\mathrm{int}\mathscr{K}^{n_{j'}}$. Then, we observe that

$$\begin{aligned}(x,s,y)= \quad & (\mathrm{Cardinal}(\mathscr{I}\cup\mathscr{J}\cup\mathscr{B}\cup\overline{\mathscr{I}}\cup\overline{\mathscr{J}}))^{-1}\Bigg(\sum_{i\in\mathscr{I}}(x^i,s^i,y^i)\\ &+\sum_{j\in\mathscr{J}}(x^j,s^j,y^j)+\sum_{k\in\mathscr{B}}(x^k,s^k,y^k)+\sum_{i'\in\overline{\mathscr{I}}}(x^{i'},s^{i'},y^{i'})\\ &+\sum_{j'\in\overline{\mathscr{J}}}(x^{j'},s^{j'},y^{j'})\Bigg)\end{aligned}$$

is a maximal complementarity solution of the wSOCCP (1.1). Since $\mathscr{F}^*$ is convex, its relative interior is given by (see [29])

$$\mathrm{ri}(\mathscr{F}^*)=\left\{z\in\mathscr{F}^*:\forall z'\in\mathscr{F}^*,\exists\lambda>1 \text{ s.t. } \lambda z+(1-\lambda)z'\in\mathscr{F}^*\right\}. \tag{5.1}$$

For any two solutions (x,s,y) and (u,v,h) of the wSOCCP (1.1), applying Lemma 4.3 yields that, for any $\xi\in\mathbb{R}$,

$$(A,B,C)(\xi(x,s,y)+(1-\xi)(u,v,h))=d, \tag{5.2}$$

$$(\xi x+(1-\xi)u)\circ(\xi s+(1-\xi)v)=w. \tag{5.3}$$

Let (x,s,y) be a maximal complementarity solution and (u,v,h) be an arbitrary solution to wSOCCP (1.1). It follows from Theorem 4.2(c) that x and s operator commute, u and v operator commute, and hence x,s,u,v and w all operator commute. Consequently, the following implications hold.

(i) For any $i\in\mathscr{W}$,

$$\begin{aligned}&u_i\in\mathrm{int}\mathscr{K}^{n_i}\Rightarrow x_i\in\mathrm{int}\mathscr{K}^{n_i}, \text{ for } i\in\mathscr{I},\\ &v_i\in\mathrm{int}\mathscr{K}^{n_i}\Rightarrow s_i\in\mathrm{int}\mathscr{K}^{n_i}, \text{ for } i\in\mathscr{J},\\ &u_i\in\mathrm{bd}\mathscr{K}^{n_i},v_i\in\mathrm{bd}\mathscr{K}^{n_i}\Rightarrow x_i\in\mathrm{bd}\mathscr{K}^{n_i},s_i\in\mathrm{bd}\mathscr{K}^{n_i}, \text{ for } i\in\mathscr{B};\end{aligned}$$

(ii) For any $i\in\overline{\mathscr{W}}$,

$$\begin{aligned}&u_i\in\mathrm{int}\mathscr{K}^{n_i}\Rightarrow x_i\in\mathrm{int}\mathscr{K}^{n_i} \text{ for } i\in\overline{\mathscr{I}},\\ &v_i\in\mathrm{int}\mathscr{K}^{n_i}\Rightarrow s_i\in\mathrm{int}\mathscr{K}^{n_i} \text{ for } i\in\overline{\mathscr{J}},\\ &u_i\in\mathrm{bd}\mathscr{K}^{n_i},v_i\in\mathrm{bd}\mathscr{K}^{n_i}\Rightarrow x_i\in\mathscr{K}^{n_i},s_i\in\mathscr{K}^{n_i} \text{ for } i\in\overline{\mathscr{I}}\cup\overline{\mathscr{J}}\cup\overline{\mathscr{B}};\end{aligned}$$

(iii) For any $i \in \widehat{\mathscr{W}}$,

$$u_i \in \mathrm{int}\mathscr{K}^{n_i}, v_i \in \mathrm{int}\mathscr{K}^{n_i} \Rightarrow x_i \in \mathrm{int}\mathscr{K}^{n_i}, s_i \in \mathrm{int}\mathscr{K}^{n_i}.$$

Now, for any $\lambda > 1$ and any $i = 1, 2, \cdots, r$, we obtain

$$\lambda x_i + (1-\lambda)u_i \succeq_{\mathscr{K}} x_i + (1-\lambda)u_i,$$
$$\lambda s_i + (1-\lambda)v_i \succeq_{\mathscr{K}} s_i + (1-\lambda)v_i.$$

For convenience, we denote

$$\lambda = 1 + \varsigma,$$

$$\varsigma = \min\left\{ \begin{array}{l} \min\limits_{i,j,k}\left\{\frac{\lambda_j(x_i)}{\lambda_k(u_i)} : \lambda_j(x_i) > 0, \lambda_k(u_i) > 0, \quad j,k = 1,2\right\}, \\ \min\limits_{i,j,k}\left\{\frac{\lambda_j(s_i)}{\lambda_k(v_i)} : \lambda_j(s_i) > 0, \lambda_k(v_i) > 0, \quad j,k = 1,2\right\} \end{array} \right\}.$$

If (x, s, y) is a maximal complementarity solution and $\mathscr{O} = \emptyset$, then

$$\lambda(x, s) + (1-\lambda)(u, v) \succeq_{\mathscr{K}} 0.$$

Thus it follows from (5.2) and (5.3) that

$$\lambda(x, s, y) + (1-\lambda)(u, v, h) \in \mathscr{F}^*,$$

which together with (5.1) implies $(x, s, y) \in \mathrm{ri}(\mathscr{F}^*)$.

(b) Assume that $(x, s, y) \in \mathrm{ri}(\mathscr{F}^*)$ and any $(u, v, h) \in \mathscr{F}^*$. In light of (5.1), there exists $\lambda > 1$ such that, for any $i = 1, 2, \cdots, r$,

$$\lambda x_i + (1-\lambda)u_i \succeq_{\mathscr{K}} 0,$$
$$\lambda s_i + (1-\lambda)v_i \succeq_{\mathscr{K}} 0.$$

Since x, s, u, v and w all operator commute, implications (i), (ii), and (iii) hold for any $(u, v, h) \in \mathscr{F}^*$. Hence, (x, s, y) is a maximal complementarity solution to wSOCCP (1.1). □

Remark 5.1. If the wSOCCP (1.1) is column sufficient, cross commutative, solvable, and $\mathscr{O} = \emptyset$, then the set of its maximal complementarity solutions is a nonempty convex set that coincides with the relative interior of $\mathscr{F}^*$. In contrast, in Proposition 2 [9], if the wLCP over the nonnegative orthant is column sufficient and solvable, then the set of its maximal complementarity solutions is a nonempty convex set that coincides with the relative interior of its solution set. There are more required conditions in Theorem 5.2, due to the structure of SOC. It is worthy of to relax some conditions in the future directions.

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