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AN EMBEDDING RESULT FOR A CLASS OF EPIGRAPHICAL SETS AND APPLICATIONS TO SET OPTIMIZATION

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Abstract. We extend to the case of a class of unbounded generalized convex sets, the famous embedding result of Rådström. The main motivation for this is to have a new look to epigraphical set-valued maps that are usually involved in set optimization problem. From this point of view, we study several aspects concerning the order of the embedding space induced by the order of the output space of objective mappings. Next, we associate to a set-valued map, an embedding single valued map, and we compare two types of efficiency related to them. Then, some new optimality conditions for set optimization problems are obtained.

Keywords. Cancellation law; Embedding; Epigraphical set-valued maps; Set optimization.

1. INTRODUCTION

The aim of this paper is to put in a wider perspective some results from [6] and [10] dealing with conic cancellation laws and generalized differentiation calculus for set-valued maps applied to set optimization problems. According to this statement, this work is twofold. On the one hand, we extend the celebrated embedding result of Rådström (see, e.g., [18, 19]), and on the other hand, we apply this extension to set optimization problems (which actually is the primary motivation of it). Notice that the extension of the Rådström's embedding result to unbounded sets was studied in many papers; see, e.g., [11, 12] and the references therein. Therefore, the proposed novelties in this work refer to one theoretical result that allows to see a set-valued map with values in a specific class of sets as a function into a normed vector space. Notice as well that taking sets with special structures in order to devise optimality conditions in set optimization is a widely used technique in the literature dedicated to this topic (see [22] and the references cited therein). Concerning the construction of the embedding space, we mention that we follow the classical procedure from the references in this papers, but using, as the main tool, an extension of the Rådström cancellation law to an epigraphical set in partially ordered normed vector spaces (see [6, 8]), while the emphasis will be on the partial order this space inherits from the original one. Our investigation of the partial order structure on the embedding

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space is instrumental for obtaining a new view on set optimization problems, since we are in position to see a set-valued map as a single-valued mapping taking values in the embedding space (see [3]) and its minima from set optimization perspective as Pareto minima of the associated vectorial function. This association seems to be fruitful, allowing to transfer Pareto optimality conditions to set optimization problems.

The paper is organized into a preliminary section, two main sections and some concluding remarks, as we briefly describe below. The second section deals with the main notation, preliminary results, and several topological facts which are needed in the sections to come. The first main section, Section 3, describes the steps that allow us, via an appropriate cancellation law, to extend the embedding result of Rådström ([18, 19]) to our new setting that considers a class of unbounded, generalized convex sets. In addition to the completion of the usual stages of this construction, we put a special emphasis on the induced partial order of the embedding space, as a main instrument to treat set optimization problems. The fourth section is divided into two subsections. The first relates a set-valued map with values having appropriate properties to a single valued map by the embedding procedure of the preceding section. Several topological properties of the two mappings are paralleled, showing interesting similarities which are illustrated by a theoretical example concerning equilibrium problems. Moreover, several differentiability concepts of the two mappings in relation to some results in literature are explored as well. The second subsection deals with set optimization problems. It is shown that an usual type of efficiency becomes the classical Pareto efficiency for a problem driven by the embedded single valued map and this gives us the possibility to transfer some primal space necessary optimality conditions for Pareto optimality to set optimization problems. Finally, the last section concludes the paper with some ideas and difficulties to overcome in a future research continuing this paper.

2. Preliminaries

In what follows, X is a normed vector space over the field of real numbers. The symbols B_X and D_X stand for the open and closed unit ball of X, respectively. For $A,B\subset X$ and $\lambda\in\mathbb{R}$, we denote by $A+B$ the Minkowski sum of A and B, that is, $A+B=\{a+b\mid a\in A, b\in B\}$ and by λA the set $\{\lambda a\mid a\in A\}$. The symbols cl, int, bd, diam, conv are for the closure, the interior, the boundary, the diameter, and the convex hull of a set, respectively.

The following useful relation is needed in the sequel

$$\operatorname{cl}(A+B)=\operatorname{cl}(\operatorname{cl}A+B). \tag{2.1}$$

Another celebrated topological fact that is needed later is

$$A\subset\operatorname{cl}B \iff \forall\varepsilon>0,\ A\subset B+\varepsilon B_X.$$

We recall that if A,B are nonempty subsets of X, the excess from A to B is $e(A,B)=\sup_{x\in A}d(x,B)$, where $d(x,B)=\inf\{\|x-b\|\mid b\in B\}$. The Pompeiu-Hausdorff distance between A and B is $h(A,B)=\max\{e(A,B),e(B,A)\}$. Notice that $h(A,B)$ is not necessarily a finite number in the absence of some boundedness assumptions on the involved sets. It is well-known that

$$e(A,B)=\inf\{\alpha>0\mid A\subset B+\alpha B_X\}=\inf\{\alpha>0\mid A\subset B+\alpha D_X\},$$

and

$$e(A,B)=e(\operatorname{cl}A,B)=e(A,\operatorname{cl}B)=e(\operatorname{cl}A,\operatorname{cl}B).$$

We also use the notation $d(A,B) = \inf_{a\in A} d(a,B)$. Let $K \subset X$ be a closed convex and pointed cone. We use the following notions for a nonempty set $A \subset X$: A is called K-bounded if there exists a bounded set $M \subset X$ such that $A \subset M + K$ and A is called K-convex if $A+K$ is convex.

Remark 2.1. Observe that if A is K-bounded, then it is not necessary to exist a bounded set N for which $A+K = N+K$. One can easily see that on the example of the epigraph (in $\mathbb{R}^2$) of the function $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^{-1}$ when $K = \mathbb{R}^2_+$.

Furthermore, set A is called K-compact (or compact with respect to the cone K) if from any cover of A with the sets of the form $U+K$, where U is open, one can extract a finite subcover of it.

The next result is a conic version of Rådström cancellation lemma from [7, Lemma 3.1].

Lemma 2.1. *Let $A,B,C \subset X$ be nonempty sets such that C is K-bounded and $A+C \subset \operatorname{cl}(C+B+K)$. Then $A \subset \operatorname{cl}\operatorname{conv}(B+K)$.*

This result is the cornerstone of the construction we propose in this work. We denote

$$\mathscr{CB}_K(X) = \{A \subset X \mid A \text{ is nonempty, } K-\text{bounded and } K-\text{convex}\}$$

and

$$\mathscr{C}_K(X) = \{\operatorname{cl}(A+K) \mid A \in \mathscr{CB}_K(X)\}.$$

In order to simplify the notation, we put $\tilde{A} := \operatorname{cl}(A+K)$ for any $A \in \mathscr{CB}_K(X)$. So, every time when we use tilde for a set, we automatically understand that the respective set is in $\mathscr{CB}_K(X)$. Some easy remarks are in order for further consideration. Observe first that $\tilde{A} = \tilde{A}+K$. Indeed, one finds from (2.1) that

$$\tilde{A}+K \supset \tilde{A} = \operatorname{cl}(A+K) = \operatorname{cl}(A+K+K) = \operatorname{cl}(\operatorname{cl}(A+K)+K) \supset \tilde{A}+K. \tag{2.2}$$

Secondly, if $\operatorname{int} K \neq \emptyset$, one has $\tilde{A} = \operatorname{cl}(A+\operatorname{int} K \cup \{0\}) = \operatorname{cl}(A+\operatorname{int} K)$, due to (2.1) and the convexity of K,

$$\operatorname{cl}(A+\operatorname{int} K) = \operatorname{cl}(A+\operatorname{cl}\operatorname{int} K) = \tilde{A}. \tag{2.3}$$

Thirdly, observe that if $\operatorname{int} K \neq \emptyset$, then

$$\operatorname{int}(A+K) = A+\operatorname{int} K. \tag{2.4}$$

The right-hand side is clearly included in the left-hand side. Take $u \in \operatorname{int}(A+K)$ and let $\varepsilon > 0$ be the radius of the open ball around u included in this set. Then $\varepsilon B_X \subset A+K-u$. Take $v \in \varepsilon B_X \cap -\operatorname{int} K$. Then there exists $a \in A$ such that $v \in a-u+K$, which implies that

$$a-u \in v-K \subset -\operatorname{int} K - K = -\operatorname{int} K.$$

We deduce that $u \in a+\operatorname{int} K \subset A+\operatorname{int} K$.

3. An Embedding Procedure for Epigraphical Sets

Now, we aim at embedding of $\mathscr{C}_K(X)$ to a normed vector space, by extending to our setting the construction from [3, 18, 19] and [4, Chapter IX] (see also [16] and [13]). Notice that the trivial case $K = \{0\}$ reduces to the situation considered in these works.

For two nonempty K-bounded and K-convex sets A and B, we introduce a sum operation by $\tilde{A} \oplus \tilde{B} := \operatorname{cl}(A+B+K)$. Then Lemma 2.1 yields the following implication:

$$\tilde{A} \oplus \tilde{C} \subset \tilde{B} \oplus \tilde{C} \Rightarrow \tilde{A} \subset \tilde{B},$$

and, consequently,

$$\tilde{A} \oplus \tilde{C} = \tilde{B} \oplus \tilde{C} \Rightarrow \tilde{A} = \tilde{B}.$$

Now, it is easy to see that $(\mathscr{C}_K(X), \oplus)$ is a commutative monoid, with cancellation law, the neutral element being $K = \operatorname{cl}(\{0\} + K) \in \mathscr{C}_K(X)$. We define, as an external operation on $\mathscr{C}_K(X)$, the multiplication with nonnegative scalars, denoted $\odot$, as follows:

$$\lambda \odot \tilde{A} := \widetilde{\lambda A} = \begin{cases} K, \text{ if } \lambda = 0 \\ \lambda \tilde{A}, \text{ if } \lambda > 0. \end{cases}$$

Moreover, for any $\tilde{A}, \tilde{B} \in \mathscr{C}_K(X)$ and any $\lambda_1, \lambda_2 \geq 0$, one readily gets $\lambda_1 \odot \left(\tilde{A} \oplus \tilde{B}\right) = \lambda_1 \odot \tilde{A} \oplus \lambda_1 \odot \tilde{B}$, $(\lambda_1 + \lambda_2) \odot \tilde{A} = \lambda_1 \odot \tilde{A} \oplus \lambda_2 \odot \tilde{A}$, $\lambda_1 \odot \left(\lambda_2 \odot \tilde{A}\right) = (\lambda_1 \lambda_2) \odot \tilde{A}$, $1 \odot \tilde{A} = \tilde{A}$.

Now, we observe that the Pompeiu-Hausdorff distance on $\mathscr{C}_K(X)$ has some useful properties.

Lemma 3.1. *Consider $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in \mathscr{C}_K(X)$ and $\lambda, \gamma \geq 0$. Then:*

(i) $h\left(\tilde{A}, \tilde{B}\right) \in \mathbb{R}$;

(ii) $h\left(\tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{C}\right) = h\left(\tilde{A}, \tilde{B}\right)$;

(iii) $h\left(\lambda \odot \tilde{A}, \lambda \odot \tilde{B}\right) = \lambda h\left(\tilde{A}, \tilde{B}\right)$;

(iv) $h\left(\tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{D}\right) \leq h\left(\tilde{A}, \tilde{B}\right) + h\left(\tilde{C}, \tilde{D}\right)$.

(v) If, additionally, there is a bounded set M for which $A + K = M + K$, then $h\left(\lambda \odot \tilde{A}, \gamma \odot \tilde{A}\right) \leq |\lambda - \gamma| \operatorname{diam} M$.

Proof. (i) We denote by N a bounded set for which $A \subset N + K$. Taking $b \in B$, we have

$$\begin{aligned} e(A+K, B+K) &= \sup\{d(a+c, B+K) \mid a \in A, c \in K\} \\ &\leq \sup\{d(a+c, B+K) \mid a \in N, c \in K\} \\ &\leq \sup\{d(a+c, b+K) \mid a \in N, c \in K\} \\ &\leq \sup\{\|a-b\| \mid a \in N\} < +\infty. \end{aligned}$$

So, the conclusion is true (see also [8, Remark 3.1]).

(ii) It is enough to prove that $e\left(\tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{C}\right) = e\left(\tilde{A}, \tilde{B}\right)$. Observe that

$$e\left(\tilde{A}, \tilde{B}\right) = e(\operatorname{cl}(A+K), \operatorname{cl}(B+K)) = e(A+K, B+K) = e(A, B+K)$$

and

$$e\left(\tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{C}\right) = e(A+C, B+C+K).$$

But, as mentioned,

$$e(A, B+K) = \inf\{\alpha > 0 \mid A \subset B + \alpha B_X + K\},$$

while

$$e(A+C, B+K+C) = \inf\{\alpha > 0 \mid A+C \subset B + \alpha B_X + K + C\}.$$

So, the inequality $e(A+C, B+K+C) \leq e(A, B+K)$ is clear. If $A + C \subset B + \alpha B_X + K + C$ then, by Lemma 2.1, $A \subset \operatorname{cl}(B + \alpha B_X + K) \subset B + (\alpha + \varepsilon) B_X + K$ for any $\varepsilon > 0$. Thus we have the reverse inequality as well.

(iii), (iv), and (v) In view of the definitions of the operations, these are obvious. □

Remark 3.1. Observe that the item (iv) from the above proposition gives the continuity of $\oplus : \mathscr{C}_K(X) \times \mathscr{C}_K(X) \to \mathscr{C}_K(X)$ when on $\mathscr{C}_K(X)$ one takes the topology given by h. Moreover (v) gives the continuity (for the obvious topologies) of $\odot : [0, \infty) \times \mathscr{C}_K^b(X) \to \mathscr{C}_K^b(X)$, where $\mathscr{C}_K^b(X)$ is the class of sets in $\mathscr{C}_K(X)$ with the property mentioned at (v).

The last preparatory step is to define a partial order on $\mathscr{C}_K(X)$. This is designed to fulfill two objectives: to extend the partial order from X given by the cone K and to be in accordance to the set optimization problems we are going to investigate later. So, the natural choice for such an order, denoted by $\preceq$, is $\tilde{A} \preceq \tilde{B} \Leftrightarrow \tilde{B} \subset \tilde{A}$. Notice that if A, B are K-closed, then this coincides with the $\preceq_K^{\ell}$ order relation between sets defined by Kuroiwa (see [13] and the next section for further details). Indeed, let us observe that the first objective is achieved: let $x, y \in X$ such that x is less than y with respect to K, that is, $y - x \in K$. This is equivalent to $y + K \subset x + K$, and, consequently, to the inclusion $\widetilde{\{y\}} \subset \widetilde{\{x\}}$ as sets in $\mathscr{C}_K(X)$. This is exactly $\widetilde{\{x\}} \preceq \widetilde{\{y\}}$.

The above discussion ensures all the necessary ingredients in order to present the embedding procedure on the steps of [18, 19] and [4, Chapter IX]. However, it is important to mention that some of the properties of constructions from [4, 18] (e.g., the continuity of the scalar multiplication or the Archimedean property) are not preserved in our more general setting, whence we are closer in our demarche to [19, Section 6].

We consider two pairs $(\tilde{A}, \tilde{B})$ and $(\tilde{C}, \tilde{D})$ of elements from $\mathscr{C}_K(X)$ and we say that they are equivalent if $\tilde{A} \oplus \tilde{D} = \tilde{B} \oplus \tilde{C}$. This is an equivalence relation and we denote by $\mathscr{G}_K(X)$ all equivalence classes. Moreover, we denote a particular class defined by a pair $(\tilde{A}, \tilde{B}) \in \mathscr{C}_K(X) \times \mathscr{C}_K(X)$ by $\langle \tilde{A}, \tilde{B} \rangle$. As usual, the addition on $\mathscr{G}_K(X)$ is defined by

$$\langle \tilde{A}, \tilde{B} \rangle + \langle \tilde{C}, \tilde{D} \rangle = \langle \tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{D} \rangle,$$

while the multiplication by a scalar $\lambda \in \mathbb{R}$ is given by

$$\lambda \langle \tilde{A}, \tilde{B} \rangle = \begin{cases} \langle \lambda \odot \tilde{A}, \lambda \odot \tilde{B} \rangle, \text{ if } \lambda \geq 0 \\ \langle -\lambda \odot \tilde{B}, -\lambda \odot \tilde{A} \rangle, \text{ if } \lambda < 0. \end{cases}$$

These operations ensure a structure of a real linear space for $\mathscr{G}_K(X)$. Notice that the zero element is the class $\langle K, K \rangle$. So, one can define the substraction as

$$\langle \tilde{A}, \tilde{B} \rangle - \langle \tilde{C}, \tilde{D} \rangle = \langle \tilde{A} \oplus \tilde{D}, \tilde{B} \oplus \tilde{C} \rangle.$$

Moreover, $\mathscr{C}_K(X)$ is embedded into this vectorial space by the mapping $\varphi : \mathscr{C}_K(X) \to \mathscr{G}_K(X)$ given by $\varphi(\tilde{A}) = \langle \tilde{A}, K \rangle$. The distance h from $\mathscr{C}_K(X)$ is extended as well on $\mathscr{G}_K(X)$. Keeping the same notation, this becomes

$$h\left(\langle \tilde{A}, \tilde{B} \rangle, \langle \tilde{C}, \tilde{D} \rangle\right) = h\left(\tilde{A} + \tilde{C}, \tilde{B} + \tilde{D}\right),$$

and it defines a norm on $\mathscr{G}_K(X)$, that is, $\|\cdot\| : \mathscr{G}_K(X) \to \mathbb{R}$, $\left\| \langle \tilde{A}, \tilde{B} \rangle \right\| = h(\tilde{A}, \tilde{B})$ is a norm. Finally, we can define a partial order on $\mathscr{G}_K(X)$, still denoted $\preceq$, by

$$\langle \tilde{A}, \tilde{B} \rangle \preceq \langle \tilde{C}, \tilde{D} \rangle \iff \tilde{A} \oplus \tilde{D} \preceq \tilde{B} \oplus \tilde{C},$$

and this extends indeed the partial order on $\mathscr{C}_K(X)$ defined above since $\tilde{A} \preceq \tilde{B}$ if and only if $\varphi(\tilde{A}) \preceq \varphi(\tilde{B})$. All these constructions do not depend on the members of the classes, the arguments for seeing this being similar to those in the original construction of Rådström in [18]. Accordingly, one can define the ordering cone (that is, the cone of positive elements) associated to this partial order as $\mathscr{K}_K(X) = \left\{ \langle \tilde{A}, \tilde{B} \rangle \in \mathscr{G}_K(X) \mid \tilde{A} \subset \tilde{B} \right\}$. The main properties of $\mathscr{K}_K(X)$ are given below.

Proposition 3.1. *The cone $\mathscr{K}_K(X)$ is convex and closed.*

Proof. The convexity of $\mathscr{K}_K(X)$ is obvious. We show that it is closed. For this, consider a sequence $\left(\left\langle\widetilde{A_n},\widetilde{B_n}\right\rangle\right)_n \subset \mathscr{K}_K(X)$ and $\langle\tilde{A},\tilde{B}\rangle \in \mathscr{G}_K(X)$ such that $\left\langle\widetilde{A_n},\widetilde{B_n}\right\rangle \overset{\|\cdot\|}{\to} \langle\tilde{A},\tilde{B}\rangle$. So, $\widetilde{A_n} \subset \widetilde{B_n}$ for all n and $\left\|\left\langle\widetilde{A_n},\widetilde{B_n}\right\rangle - \langle\tilde{A},\tilde{B}\rangle\right\| \to 0$. The last relation means that $h\left(\widetilde{A_n}\oplus\tilde{B},\widetilde{B_n}\oplus\tilde{A}\right) \to 0$. We see that, for all $\varepsilon > 0$, for n large enough,

$$A+B_n+K \subset \operatorname{cl}(A_n+B+K)+\varepsilon B_X \subset A_n+B+K+2\varepsilon B_X \subset B+B_n+K+3\varepsilon B_X.$$

Again by the cancellation rule from Lemma 2.1, we obtain that

$$A+K \subset \operatorname{cl}(B+K+3\varepsilon B_X) \subset B+K+4\varepsilon B_X.$$

Since this is true for all $\varepsilon > 0$, we obtain that $A+K \subset \tilde{B}$, whence $\tilde{A} \subset \tilde{B}$. We conclude that $\langle\tilde{A},\tilde{B}\rangle \in \mathscr{K}_K(X)$, so $\mathscr{K}_K(X)$ is closed. □

We tackle the question of the nonemptiness of the interior of $\mathscr{K}_K(X)$.

Proposition 3.2. *Let* $\operatorname{int}K \neq \emptyset$. *Then* $\operatorname{int}\mathscr{K}_K(X) \neq \emptyset$ *and, moreover,*

$$\left\{\langle\tilde{A},\tilde{B}\rangle \in \mathscr{K}_K(X) \mid \exists\varepsilon > 0 \text{ s.t. } \tilde{A}+\varepsilon B_X \subset \tilde{B}\right\} = \operatorname{int}\mathscr{K}_K(X). \tag{3.1}$$

Proof. Let $\langle\tilde{A},\tilde{B}\rangle \in \mathscr{K}_K(X)$ with the property that there exists $\varepsilon > 0$ such that $\tilde{A}+\varepsilon B_X \subset \tilde{B}$. Using (2.2), $\tilde{A}+\varepsilon B_X \subset \tilde{B}+K$, and (2.4), one has

$$\tilde{A}+2^{-1}\varepsilon B_X \subset \operatorname{int}\left(\tilde{B}+K\right) = \tilde{B}+\operatorname{int}K. \tag{3.2}$$

Let $\mu \in \left(0,2^{-1}\varepsilon\right)$. We show that the open ball with respect to the norm of $\mathscr{G}_K(X)$ with center $\langle\tilde{A},\tilde{B}\rangle$ and radius μ is included in $\mathscr{K}_K(X)$, and this will conclude the proof of the direct inclusion in (3.1). Let $\langle\tilde{C},\tilde{D}\rangle \in \mathscr{G}_K(X)$ such that $\left\|\langle\tilde{C},\tilde{D}\rangle - \langle\tilde{A},\tilde{B}\rangle\right\| < \mu$. Then, for all $\delta > 0$, $B+C+K \subset A+D+K+(\mu+\delta)B_X$. In particular, this is true for a constant $\delta \in \left(0,2^{-1}\left(2^{-1}\varepsilon-\mu\right)\right)$, whence, by (3.2),

$$\begin{aligned} B+C+\delta B_X &\subset A+D+K+(\mu+2\delta)B_X \subset A+D+K+2^{-1}\varepsilon B_X \\ &\subset D+\operatorname{cl}(B+K)+\operatorname{int}K \subset D+B+\operatorname{int}K+\delta B_X. \end{aligned}$$

Therefore, $B+C+\delta B_X \subset D+B+\operatorname{int}K+\delta B_X+K$. We apply Lemma 2.1 to see that

$$C+\delta B_X \subset \operatorname{cl}(D+\operatorname{int}K+\delta B_X).$$

Now, by a classical variant of Rådström cancellation lemma (that is, Lemma 2.1 for $K=\{0\}$) we obtain (by using (2.4) as well) that $C \subset \operatorname{cl}(D+\operatorname{int}K) = \operatorname{cl}(\operatorname{int}(D+K)) = \tilde{D}$. The proof of the direct inclusion for (3.1) is complete. In particular, $\operatorname{int}\mathscr{K}_K(X) \neq \emptyset$.

For the reverse inclusion in (3.1), we consider $\langle\tilde{A},\tilde{B}\rangle \in \operatorname{int}\mathscr{K}_K(X)$ and suppose, by way of contradiction, that, for all natural number $n > 0$, there exists $a_n \in \tilde{A}$ and $u_n \in B_X$ such that $a_n+n^{-1}u_n \notin \tilde{B}$. Then

$$\left\|\left\langle\tilde{A}\oplus\widetilde{\{n^{-1}u_n\}},\tilde{B}\right\rangle - \langle\tilde{A},\tilde{B}\rangle\right\| \to 0,$$

but $\left\langle\tilde{A}\oplus\widetilde{\{n^{-1}u_n\}},\tilde{B}\right\rangle \notin \mathscr{K}_K(X)$ for any n. This contradicts the fact that $\langle\tilde{A},\tilde{B}\rangle \in \operatorname{int}\mathscr{K}_K(X)$. □

Remark 3.2. For $\langle\tilde{A},\tilde{B}\rangle \in \mathscr{K}_K(X)$, the condition that there exists $\varepsilon > 0$ such that $\tilde{A}+\varepsilon B_X \subset \tilde{B}$ is equivalent to the condition $d\left(\operatorname{bd}\tilde{A},\operatorname{bd}\tilde{B}\right) > 0$.

We record some important consequences of the above result and its proof.

Proposition 3.3. *If* $\operatorname{int} K \neq \emptyset$, *then the following elements belong to* $\operatorname{int} \mathscr{K}_K(X)$:

(i) $\langle \tilde{U}, K \rangle$ *for which there exists* $\varepsilon > 0$ *such that* $U + \varepsilon B_X \subset K$;

(ii) $\langle \{e\} + K, K \rangle$ *with* $e \in \operatorname{int} K$;

(iii) $\langle \tilde{A}, \tilde{B} \rangle \in \mathscr{G}_K(X)$ *such that* $\tilde{A} \subset \tilde{B} + \operatorname{int} K$ *and* A *is* K*-compact.*

Proof. (i) Relation (3.2) is clearly fulfilled for $\langle \tilde{U}, K \rangle$.

(ii) This is a particular case of (i).

(iii) Since A is K-compact, $\tilde{A} = A + K$ and the inclusion from the hypothesis reads as $A \subset \operatorname{cl}(B+K) + \operatorname{int} K$. According to [9, Lemma 4.1], the K-compactness of A ensures the existence of a number $\varepsilon > 0$ such that $A + \varepsilon B_X \subset \operatorname{cl}(B+K) + \operatorname{int} K$. Thus (3.2) holds. □

Proposition 3.4. *For* $A \in \mathscr{CB}_K(X)$, *it holds the equivalence* $\langle K, \tilde{A} \rangle \notin \operatorname{int} \mathscr{K}_K(X) \iff A \cap -\operatorname{int} K = \emptyset$.

Proof. Suppose that $\langle K, \tilde{A} \rangle \notin \operatorname{int} \mathscr{K}_K(X)$. So, for all $\varepsilon > 0$, $K + \varepsilon B_X \not\subset \operatorname{cl}(A+K)$. If there exists $e \in A$ and $\delta > 0$ such that $-e + \delta B_X \subset K$, then $-e + (K+e) + \delta B_X \subset A + K$, whence $K + \delta B_X \subset \operatorname{cl}(A+K)$, which is a contradiction.

Conversely, we consider that $A \cap -\operatorname{int} K = \emptyset$ and suppose that there exists $\varepsilon > 0$ such that $\varepsilon B_X \subset \operatorname{cl}(A+K)$. Then we can find $e \in \operatorname{int} K \cap \varepsilon B_X$, which gives a positive δ for which $-e + \delta B_X \subset -\operatorname{int} K \cap \varepsilon B_X$. Now, since $-e \in \operatorname{cl}(A+K)$, we see that there exist some sequences $(a_n) \subset A$ and $(k_n) \subset K$ such that $a_n + k_n \to -e$. But, for n large enough, $a_n + k_n \in -e + \delta B_X \subset -\operatorname{int} K$, which means that $a_n \in -\operatorname{int} K \cap A$, which is a contradiction. □

We consider now the question of the representation of an element of positive dual of $\mathscr{K}_K(X)$. As usual, we denote by X^* the topological dual of X and by K^+ the positive polar of K, that is,

$$K^+ := \{x^* \in X^* \mid x^*(x) \geq 0,\ \forall x \in K\}.$$

Proposition 3.5. *Let* $x^* \in K^+$. *Define* $T_{x^*} : \mathscr{G}_K(X) \to \mathbb{R}$ *by* $T_{x^*}\left(\langle \tilde{A}, \tilde{B} \rangle\right) = \inf x^*\left(\tilde{A}\right) - \inf x^*\left(\tilde{B}\right)$. *Then following properties hold:*

(i) $T_{x^*} \in (\mathscr{K}_K(X))^+$ *and* $\|T_{x^*}\| = \|x^*\|$;

(ii) For $x^*, y^* \in K^+ \setminus \{0\}$ *there exists* $z^* \in K^+$ *such that* $T_{x^*} + T_{y^*} = T_{z^*}$ *if and only if there exists* $\alpha \in \mathbb{R}_+ \setminus \{0\}$ *such that* $x^* = \alpha y^*$;

(iii) If $K \cup -K \neq X$, *then there exists* $T \in (\mathscr{K}_K(X))^+$ *such that* $T \neq T_{x^*}$ *for all* $x^* \in K^+$.

(iv) For every $T \in (\mathscr{K}_K(X))^+$, *there exists* $x_T^* \in K^+$ *such that*

$$T(\langle \{a\} + K, K \rangle) = T_{x_T^*}(\langle \{a\} + K, K \rangle),\ \forall a \in X.$$

and

$$T\left(\langle \tilde{A}, K \rangle\right) \leq T_{x_T^*}\left(\langle \tilde{A}, K \rangle\right),\ \forall A \in \mathscr{CB}_K(X).$$

Moreover, $\|T\| \geq \|x_T^*\|$ *and, for* $A \in \mathscr{CB}_K(X)$,

$$\sup_{\substack{\|T\| \leq 1 \\ T \in (\mathscr{K}_K(X))^+}} T\left(\langle \tilde{A}, K \rangle\right) \leq \sup_{\|x_T^*\| \leq 1} T_{x_T^*}\left(\langle \tilde{A}, K \rangle\right).$$

Proof. (i) First of all, we observe that, for $A \in \mathscr{CB}_K(X)$,

$$\inf x^*\left(\tilde{A}\right) = \inf x^*(A+K) = \inf\left(x^*(A) + [0, \infty)\right) = \inf x^*(A) \in \mathbb{R},$$

where the last relation is due to the K-boundedness of A. Therefore, T_{x^*} is well-defined. Clearly, T_{x^*} is linear and if $\langle \tilde{A}, \tilde{B} \rangle \in \mathscr{K}_K(X)$, then $T_{x^*}(\langle \tilde{A}, \tilde{B} \rangle) \geq 0$. We have to show that T_{x^*} is also continuous. Consider, as usual, a sequence and its limit in $\mathscr{G}_K(X)$, that is, $\left\langle \widetilde{A_n}, \widetilde{B_n} \right\rangle \overset{\|\cdot\|}{\to} \langle \tilde{A}, \tilde{B} \rangle$. Then, as in the proof of Proposition 3.1, we have, for all $\varepsilon > 0$, a natural number n_ε such that for all $n \geq n_\varepsilon$, $A_n + B \subset A + B_n + K + \varepsilon B_X$ and $A + B_n \subset A_n + B + K + \varepsilon B_X$. These relations imply

$$\inf x^*(A_n) + \inf x^*(B) \geq \inf x^*(A) + \inf x^*(B_n) - \varepsilon \|x^*\|$$

and

$$\inf x^*(A) + \inf x^*(B_n) \geq \inf x^*(A_n) + \inf x^*(B) - \varepsilon \|x^*\|.$$

Consequently, for all $n \geq n_\varepsilon$,

$$|(\inf x^*(A_n) - \inf x^*(B_n)) - (\inf x^*(A) - \inf x^*(B))| \leq \varepsilon \|x^*\|,$$

whence $\left| T_{x^*}(\langle \tilde{A}_n, \tilde{B}_n \rangle) - T_{x^*}(\langle \tilde{A}, \tilde{B} \rangle) \right| \leq \varepsilon \|x^*\|$. We conclude that T_{x^*} is continuous. Concerning its norm, we have $\|T_{x^*}\| = \sup_{\|\langle \tilde{A}, \tilde{B} \rangle\| \leq 1} \left\| T_{x^*}(\langle \tilde{A}, \tilde{B} \rangle) \right\|$. Since, for all $x \in X$, $\|\langle x + K, K \rangle\| \leq \|x\|$, we have

$$\|T_{x^*}\| \geq \|\langle x + K, K \rangle\| = x^*(x), \ \forall x \in D_X,$$

so $\|T_{x^*}\| \geq \|x^*\|$.

On the other hand, take $\langle \tilde{A}, \tilde{B} \rangle \in \mathscr{G}_K(X)$ with $\|\langle \tilde{A}, \tilde{B} \rangle\| = \rho \geq 0$. This is to say that, for all $\varepsilon > 0$, $\tilde{A} \subset \tilde{B} + (\rho + \varepsilon) D_X$ and $\tilde{B} \subset \tilde{A} + (\rho + \varepsilon) D_X$. It follows that $\inf x^*(\tilde{A}) \geq \inf x^*(\tilde{B}) - (\rho + \varepsilon)\|x^*\|$ and $\inf x^*(\tilde{B}) \geq \inf x^*(\tilde{A}) - (\rho + \varepsilon)\|x^*\|$, whence

$$T_{x^*}(\langle \tilde{A}, \tilde{B} \rangle) = \inf x^*(\tilde{A}) - \inf x^*(\tilde{B}) \in [-(\rho + \varepsilon)\|x^*\|, (\rho + \varepsilon)\|x^*\|].$$

We deduce that $\left| T_{x^*}(\langle \tilde{A}, \tilde{B} \rangle) \right| \leq \|x^*\| \left\| \langle \tilde{A}, \tilde{B} \rangle \right\|$ for any $\langle \tilde{A}, \tilde{B} \rangle \in \mathscr{G}_K(X)$, so $\|T_{x^*}\| \leq \|x^*\|$. We conclude that $\|T_{x^*}\| = \|x^*\|$.

(ii) Let $x^*, y^* \in K^+$. If there exists $z^* \in K^+$ such that $T_{z^*}(\langle \tilde{A}, K \rangle) = T_{x^*}(\langle \tilde{A}, K \rangle) + T_{y^*}(\langle \tilde{A}, K \rangle)$ for all $A \in \mathscr{CB}_K(X)$, then, in particular, for $A = \{x\}$ with $x \in X$, we have $z^*(x) = x^*(x) + y^*(x)$. If there exist $a, b \in X$ such that $x^*(a) < x^*(b)$ and $y^*(a) > y^*(b)$, then, for $A = \operatorname{conv}\{a, b\} = [a, b] \in \mathscr{CB}_K(X)$, we have $T_{z^*}(\langle \tilde{A}, K \rangle) = T_{x^*}(\langle \tilde{A}, K \rangle) + T_{y^*}(\langle \tilde{A}, K \rangle) = x^*(a) + y^*(b)$. Since $T_{z^*}(\langle \tilde{A}, K \rangle) = \inf z^*([a, b])$ and $[a, b]$ is compact, there exists $\lambda \in [0, 1]$ such that $T_{z^*}(\langle \tilde{A}, K \rangle) = z^*(\lambda a + (1 - \lambda) b)$. We deduce that

$$\lambda z^*(a) + (1 - \lambda) z^*(b) = x^*(a) + y^*(b),$$

whence $\lambda(x^*(a) + y^*(a)) + (1 - \lambda)(x^*(b) + y^*(b)) = x^*(a) + y^*(b)$, which gives

$$\lambda(y^*(a) - y^*(b)) = (1 - \lambda)(x^*(a) - x^*(b)),$$

which is a contradiction. We obtain that, in particular, $\{x \in X \mid x^*(x) < 0\} \subset \{x \in X \mid y^*(x) \leq 0\}$, so

$$\operatorname{cl}\{x \in X \mid x^*(x) < 0\} = \{x \in X \mid x^*(x) \leq 0\} \subset \{x \in X \mid y^*(x) \leq 0\}.$$

According to the classical Farkas Lemma, this is equivalent with the existence of $\alpha \in \mathbb{R}_+$ such that $x^* = \alpha y^*$. Of course, $\alpha \neq 0$. The other implication is obvious.

(iii) Since $K \cup -K \neq X$, then there exist $a, b \in X$ such that $a \notin b + K$ and $b \notin a + K$. Thus there exists $x^*, y^* \in K^+$ such that $x^*(a) < x^*(b)$ and $y^*(b) < y^*(a)$. Following (i), $T_{x^*} + T_{y^*} \in (\mathscr{K}_K(X))^+$ but, according to the proof of (ii), $T_{x^*} + T_{y^*} \neq T_{z^*}$ for all $z^* \in K^+$.

(iv) Consider now $T \in (\mathscr{K}_K(X))^+$. Then the functional $x_T^* : X \to \mathbb{R}$ given by $x_T^*(x) = T(\langle \{x\} + K, K \rangle)$ belongs to K^+ (all the properties ensuring the validity of this assertion are easy to check). Observe that, for any $\langle \tilde{A}, \tilde{B} \rangle \in \mathscr{G}_K(X)$,

$$T\left(\left\langle \tilde{A}, \tilde{B} \right\rangle\right) = T\left(\left\langle \tilde{A}, K \right\rangle - \left\langle \tilde{B}, K \right\rangle\right) = T\left(\left\langle \tilde{A}, K \right\rangle\right) - T\left(\left\langle \tilde{B}, K \right\rangle\right).$$

Take $A \in \mathscr{C}\mathscr{B}_K(X)$. Then, for any $a \in A$, $\left\langle \{a\} + K, \tilde{A} \right\rangle \in \mathscr{K}_K(X)$, so, $T\left(\left\langle \{a\} + K, \tilde{A} \right\rangle\right) \geq 0$, whence $x_T^*(a) \geq T\left(\left\langle \tilde{A}, K \right\rangle\right)$. We deduce that $T\left(\left\langle \tilde{A}, K \right\rangle\right) \leq \inf x_T^*(A)$, which means

$$T\left(\left\langle \tilde{A}, K \right\rangle\right) \leq T_{x_T^*}\left(\left\langle \tilde{A}, K \right\rangle\right), \ \forall A \in \mathscr{C}\mathscr{B}_K(X).$$

We also have $|x_T^*(x)| = |T(\langle \{x\} + K, K \rangle)| \leq \|T\| \|x\|$, whence $\|x_T^*\| \leq \|T\|$. Now, for $A \in \mathscr{C}\mathscr{B}_K(X)$, we easily see that

$$\sup_{\substack{\|T\| \leq 1 \\ T \in (\mathscr{K}_K(X))^+}} T\left(\left\langle \tilde{A}, K \right\rangle\right) \leq \sup_{\|x_T^*\| \leq 1} T_{x_T^*}\left(\left\langle \tilde{A}, K \right\rangle\right).$$

The proof is complete. □

The next result concerns a density property, with respect to the weak star topology of $(\mathscr{G}_K(X))^*$ (denoted by w^*), of the set of positive operators on $\mathscr{G}_K(X)$ generated by the elements of K^+.

Proposition 3.6. *In the notation from Proposition 3.5, $\mathscr{D} = \{T_{x^*} \mid x^* \in K^+\}$ is a cone in $(\mathscr{K}_K(X))^+$ and $w^* - \mathrm{cl}(\mathrm{conv}\,\mathscr{D}) = (\mathscr{K}_K(X))^+$.*

Proof. The fact that $\mathscr{D}$ is a cone is obvious since $\lambda T_{x^*} = T_{\lambda x^*}$ for all $\lambda \geq 0$ and $x^* \in K^+$. Moreover, the inclusion $w^* - \mathrm{cl}(\mathrm{conv}\,\mathscr{D}) \subset (\mathscr{K}_K(X))^+$ is also clear. Another important thing to take into account is from the application of a classical separation result for convex sets, and reads as follows

$$\begin{aligned} \left\langle \tilde{A}, \tilde{B} \right\rangle \in \mathscr{K}_K(X) \Leftrightarrow \tilde{A} \subset \tilde{B} \Leftrightarrow \inf x^*\left(\tilde{A}\right) \geq \inf x^*\left(\tilde{B}\right), \ \forall x^* \in K^+ \\ \Leftrightarrow T_{x^*}\left(\left\langle \tilde{A}, \tilde{B} \right\rangle\right) \geq 0, \ \forall x^* \in K^+. \end{aligned}$$

Now, we suppose that the equality from the conclusion does not hold. Thus there exists $T \in (\mathscr{K}_K(X))^+$ such that $T \notin w^* - \mathrm{cl}(\mathrm{conv}\,\mathscr{D})$. Again by a separation result, there exists $\left\langle \tilde{A}, \tilde{B} \right\rangle \in \mathscr{G}_K(X)$ such that

$$T\left(\left\langle \tilde{A}, \tilde{B} \right\rangle\right) < \inf\left\{T_{x^*}\left(\left\langle \tilde{A}, \tilde{B} \right\rangle\right) \mid x^* \in K^+\right\}.$$

Taking into account that $\mathscr{D}$ is a cone, we easily see, by a standard argument, that $T_{x^*}\left(\left\langle \tilde{A}, \tilde{B} \right\rangle\right) \geq 0$ for all $x^* \in K^+$. The above chain of equivalences ensures that $\left\langle \tilde{A}, \tilde{B} \right\rangle \in \mathscr{K}_K(X)$, whence $\inf\left\{T_{x^*}\left(\left\langle \tilde{A}, \tilde{B} \right\rangle\right) \mid x^* \in K^+\right\} = 0$. Consequently, $T\left(\left\langle \tilde{A}, \tilde{B} \right\rangle\right) < 0$. This is a contradiction, so the equality holds. □

4. An Embedding of a Set-valued Map and Applications

4.1. **An embedding of a set-valued map and related properties.** In this subsection, using an idea from [3], we associate to a set-valued map having values in $\mathscr{C}\mathscr{B}_K(X)$ a function with values in $\mathscr{G}_K(X)$ by using the embedding procedure described before and then we explore the way several classical properties of these objects (convexity, continuity, generalized differentiation, etc.) link each other.

Let Z be a real normed vector space and $F : Z \rightrightarrows X$ be a set-valued map with nonempty, K-bounded and K-convex values. This will be a standing assumption in the sequel. Then its epigraphical mapping $\operatorname{Epi} F : Z \rightrightarrows X$ by $\operatorname{Epi} F(z) = F(z) + K$ has values whose closedness are in $\mathscr{C}_K(X)$. Using the embedding of $\mathscr{C}_K(X)$ into the normed vector space $\mathscr{G}_K(X)$ given in the preceding section, we associate to F the single-valued function $f : Z \to \mathscr{G}_K(X)$ defined as $f(z) = \left\langle \widetilde{F(z)}, K \right\rangle$.

Firstly, we put into relation the generalized convexities for F and f. We recall (see, e.g., [1]) that if $A \subset Z$ is a nonempty convex set, then one says that F is K-convex on A if, for every $x, y \in A$ and every $\lambda \in (0,1)$, $\lambda F(x) + (1-\lambda) F(y) \subset F(\lambda x + (1-\lambda) y) + K$. For a single-valued map, this notion is similar and it requires only the replacement of $\subset$ by $\in$.

Proposition 4.1. *In the above notation, suppose that F has K-closed valued. Then F is K-convex on A if and only if f is $\mathscr{K}_K(X)$-convex on A.*

Proof. The K-convexity of F on A can be equivalently written as

$$\operatorname{cl}(\lambda F(x) + (1-\lambda) F(y) + K) \subset \operatorname{cl}(F(\lambda x + (1-\lambda) y) + K),\ \forall x, y \in A, \forall \lambda \in (0,1),$$

that is,

$$\lambda \odot \widetilde{F(x)} \oplus (1-\lambda) \odot \widetilde{F(y)} \subset F(\lambda x \widetilde{+ (1-} \lambda) y), \forall x, y \in A, \forall \lambda \in (0,1).$$

Furthermore, the above relation means that

$$\left\langle \lambda \odot \widetilde{F(x)} \oplus (1-\lambda) \odot \widetilde{F(y)}, F(\lambda x \widetilde{+ (1-} \lambda) y) \right\rangle \in \mathscr{K}_K(X), \forall x, y \in A, \forall \lambda \in (0,1),$$

and this is to say that

$$\lambda \left\langle \widetilde{F(x)}, K \right\rangle + (1-\lambda) \left\langle \widetilde{F(y)}, K \right\rangle - \left\langle F(\lambda x \widetilde{+ (1-} \lambda) y), K \right\rangle \in \mathscr{K}_K(X), \forall x, y \in A, \forall \lambda \in (0,1).$$

The latter relation is exactly the $\mathscr{K}_K(X)$-convexity of f on A. □

Concerning another useful property, that is, the Lipschitz continuity, it is straightforward to see that the next assertion holds.

Proposition 4.2. *The single-valued map f is Lipschitz around $\bar{z}$ (with respect to the norm on $\mathscr{G}_K(X)$) if and only if the set-valued map $z \rightrightarrows \widetilde{F(z)}$ is Lipschitz around $\bar{z}$ (with respect to h).*

A useful parenthesis here is the immediate possible use of these transfer properties from a set-valued map to its embedding single-valued map (and vice-versa) in discussing some vector equilibrium problems. Therefore, for a short look, we turn our attention to vector equilibrium problems and we follow, as before, the way the behavior of the embedding function impacts the behavior of the embedded set-valued map from the perspective of the existence of solutions for some equilibrium problems they define. Actually, we consider some usual vector equilibrium problems as presented, for instance, in [1, Chapter 10]. Only for our purpose in this small discussion, we consider a (bi)set-valued map $G : Z \times Z \rightrightarrows X$ with nonempty, K-bounded and K-convex values. Again, we associate the (bi)function $g : Z \times Z \to \mathscr{G}_K(X)$ defined as $g(u,v) = \left\langle \widetilde{G(u,v)}, K \right\rangle$.

Proposition 4.3. *Let $S \subset Z$ be a convex compact set and* $\operatorname{int} K \neq \emptyset$. *Assume the following properties:*

(i) $G(u,u) \cap -\operatorname{int} K = \emptyset$, for all $u \in S$;

(ii) $u \rightrightarrows G(u,v)$ is K-convex on S for every $v \in S$;

(iii) the set $\{u \in S \mid G(u,v) \subset X \setminus -\operatorname{int} K\}$ *is closed for all* $v \in S$.
Then there exists $\overline{u} \in S$ *such that, for all* $v \in S$, $G(\overline{u},v) \cap -\operatorname{int} K = \emptyset$.

Proof. According to Proposition 3.4, for $(u,v) \in Z \times Z$, the relation $G(u,v) \cap -\operatorname{int} K = \emptyset$ is equivalent to $\left\langle \widetilde{G(u,v)}, K \right\rangle \notin -\operatorname{int} \mathscr{K}_K(X)$. Now, it is easy to see that assumptions (i), (ii), and (iii) mean, respectively, that $g(u,u) \notin -\operatorname{int} \mathscr{K}_K(X)$, $u \rightrightarrows g(u,v)$ is $\mathscr{K}_K(X)$-convex on S for every $v \in S$, and the set $\{u \in S \mid g(u,v) \in (\mathscr{G}_K(X) \setminus -\operatorname{int} \mathscr{K}_K(X))\}$ is closed for all $v \in S$. These properties on g ensure that the equilibrium problem

$$\exists \overline{u} \in S \text{ s.t. } g(\overline{u},v) \notin -\operatorname{int} \mathscr{K}_K(X), \ \forall v \in S,$$

has a solution by the use of Ky Fan Lemma (see, e.g., [1]). The conclusion follows. □

At this point, after this new argument on the usefulness of the embedding procedure, we follow here, the next step to take is to look at some classical generalized differentiation objects associated to the function f and to devise generalized differentiation for the initial set-valued map F. We consider again the setting fixed in the beginning of this section. Recall that the graph of F is the set

$$\operatorname{Gr} F := \{(z,x) \mid z \in Z, x \in F(z)\}.$$

Following [2], one defines the Bouligand derivative of F at $(\overline{z},\overline{x}) \in \operatorname{Gr} F$ as the set-valued map $D_B F(\overline{z},\overline{x}) : Z \rightrightarrows X$ given by

$$v \in D_B F(\overline{z},\overline{x})(u) \iff \liminf_{u' \to u, t \downarrow 0} d\left(v, \frac{F(\overline{z}+tu') - \overline{x}}{t}\right) = 0. \tag{4.1}$$

Accordingly, for the function f, using a slightly simplified notation due to the fact that f is single-valued, one has

$$v_f \in D_B f(\overline{z})(u) \iff \liminf_{u' \to u, t \downarrow 0} \left\| \frac{f(\overline{z}+tu') - f(\overline{z})}{t} - v_f \right\| = 0.$$

Taking $v_f = \langle \tilde{A}, \tilde{B} \rangle \in \mathscr{G}_K(X)$, one has that

$$\liminf_{u' \to u, t \downarrow 0} h\left(\tilde{A} + \frac{\widetilde{F(\overline{z})}}{t}, \frac{\widetilde{F(\overline{z}+tu')}}{t} + \tilde{B}\right) = 0.$$

Consequently,

$$\liminf_{u' \to u, t \downarrow 0} e\left(\tilde{A} + \frac{\widetilde{F(\overline{z})}}{t}, \frac{\widetilde{F(\overline{z}+tu')}}{t} + \tilde{B}\right) = 0$$

and

$$\liminf_{u' \to u, t \downarrow 0} e\left(\tilde{B} + \frac{\widetilde{F(\overline{z}+tu')}}{t}, \tilde{A} + \frac{\widetilde{F(\overline{z})}}{t}\right) = 0.$$

Proposition 4.4. *In this notation, the following assertions hold:*
(i) for all $v \in \tilde{A}$ *and* $\overline{x} \in \widetilde{F(\overline{z})}$,

$$\liminf_{u' \to u, t \downarrow 0} d\left(v + \frac{\overline{x}}{t}, \frac{\widetilde{F(\overline{z}+tu')}}{t} + \tilde{B}\right) = 0;$$

(ii) for all $v \in \tilde{A}$,

$$\liminf_{u' \to u, t \downarrow 0} e\left(v + \frac{\widetilde{F(\bar{z})}}{t}, \frac{\widetilde{F(\bar{z}+tu')}}{t} + \tilde{B}\right) = 0;$$

(iii) if $\tilde{B} = K$*, then*

$$\tilde{A} \subset \bigcap_{\bar{x} \in F(\bar{z})} D_B \widetilde{F}(\bar{z}, \bar{x})(u),$$

where $\widetilde{F}$ *is the set-valued map* $z \rightrightarrows \widetilde{F(z)}$*. In addition, if* $F(\bar{z})$ *has ideal minimum (i.e., there exists* $\bar{y} \in F(\bar{z})$*,* $\bar{x} - \bar{y} \in K$ *for each* $\bar{x} \in F(\bar{z})$*), then*

$$v \in \bigcap_{\bar{x} \in \widetilde{F(\bar{z})}} D_B \widetilde{F}(\bar{z}, \bar{x})(u) \Longleftrightarrow \liminf_{u' \to u, t \downarrow 0} e\left(v + \frac{\widetilde{F(\bar{z})}}{t}, \frac{\widetilde{F(\bar{z}+tu')}}{t}\right) = 0.$$

Proof. We remark that, for all $v \in \tilde{A}$, $\bar{x} \in \widetilde{F(\bar{z})}$, and $t > 0$ the following inequalities hold:

$$d\left(v + \frac{\bar{x}}{t}, \frac{\widetilde{F(\bar{z}+tu')}}{t} + \tilde{B}\right) \le h\left(\tilde{A} + \frac{\widetilde{F(\bar{z})}}{t}, \frac{\widetilde{F(\bar{z}+tu')}}{t} + \tilde{B}\right),$$

$$e\left(v + \frac{\widetilde{F(\bar{z})}}{t}, \frac{\widetilde{F(\bar{z}+tu')}}{t} + \tilde{B}\right) \le h\left(\tilde{A} + \frac{\widetilde{F(\bar{z})}}{t}, \frac{\widetilde{F(\bar{z}+tu')}}{t} + \tilde{B}\right),$$

and

$$e\left(\tilde{A} + \frac{\bar{x}}{t}, \frac{\widetilde{F(\bar{z}+tu')}}{t} + \tilde{B}\right) \le h\left(\tilde{A} + \frac{\widetilde{F(\bar{z})}}{t}, \frac{\widetilde{F(\bar{z}+tu')}}{t} + \tilde{B}\right).$$

Thus (i) and (ii) are consequences of these inequalities. Also, if $\tilde{B} = K$, (i) implies the inclusion from (iii) by taking into account the definition of $D_B \widetilde{F}(\bar{z}, \bar{x})(u)$.

Now, it remains to prove the direct implication from (iii) for the case when $F(\bar{z})$ has ideal minimum denoted $\bar{y}$. Let $v \in \bigcap_{\bar{x} \in F(\bar{z})} D_B \widetilde{F}(\bar{z}, \bar{x})(u)$. This implies

$$\liminf_{u' \to u, t \downarrow 0} d\left(v + \frac{\bar{x}}{t}, \frac{\widetilde{F(\bar{z}+tu')}}{t}\right) = 0, \ \forall \bar{x} \in F(\bar{z}).$$

Since, for each $k \in K$,

$$d\left(v + \frac{\bar{y}}{t}, \frac{\widetilde{F(\bar{z}+tu')}}{t}\right) \ge d\left(v + \frac{\bar{y}+k}{t}, \frac{\widetilde{F(\bar{z}+tu')}}{t} + K\right),$$

we obtain

$$d\left(v + \frac{\bar{y}}{t}, \frac{\widetilde{F(\bar{z}+tu')}}{t}\right) \ge e\left(v + \frac{\widetilde{F(\bar{z})}}{t}, \frac{\widetilde{F(\bar{z}+tu')}}{t} + K\right)$$

and the conclusion follows. □

We discuss now the Fréchet and the Gâteaux differentiability of the embedding map $f : Z \to \mathscr{G}_K(X)$ and we compare their consequences for F with other generalized differentiability concepts for set-valued maps that one can find in literature.

First, we consider the Fréchet differentiability. Denote by $L(Z,\mathscr{G}_K(X))$ the space of linear and continuous operators from Z to $\mathscr{G}_K(X)$. Recall that f is Fréchet differentiable at $\bar{z} \in Z$ if and only if there exists $T \in L(Z,\mathscr{G}_K(X))$ (denoted $\nabla f(\bar{z})$) such that

$$\lim_{z\to\bar{z}} \frac{f(z)-f(\bar{z})-T(z-\bar{z})}{\|z-\bar{z}\|} = 0.$$

In our setting, this is to say that there exists$T \in L(Z,\mathscr{G}_K(X))$, $T(z) = \left\langle \widetilde{T_1(z)}, \widetilde{T_2(z)} \right\rangle$ such that

$$\lim_{z\to\bar{z}} \frac{h(F(z)+\widetilde{T_2(z-\bar{z})}, F(\bar{z})+\widetilde{T_1(z-\bar{z})})}{\|z-\bar{z}\|} = 0.$$

This means that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $z \in B(\bar{z},\delta)$,

$$F(z)+\widetilde{T_2(z-\bar{z})} \subset F(\bar{z})+\widetilde{T_1(z-\bar{z})}+\varepsilon\|z-\bar{z}\|D_X$$

and

$$F(\bar{z})+\widetilde{T_1(z-\bar{z})} \subset F(z)+\widetilde{T_2(z-\bar{z})}+\varepsilon\|z-\bar{z}\|D_X.$$

Similarly to [3], we say that a multifunction $F : Z \to \mathscr{C}_K(X)$ satisfying the precedent conditions, denoted (C), with $\tilde{T}_1, \tilde{T}_2 : Z \to \mathscr{C}_K(X)$ (where $\tilde{T}_i(z) = \widetilde{T_i(z)}$, for all $z \in Z$ and $i \in 1,2$) and $T = \left\langle \tilde{T}_1, \tilde{T}_2 \right\rangle \in L(Z,\mathscr{G}_K(X))$ is π-differentiable. If there exists $U : Z \to \mathscr{C}_K(X)$ such that $\nabla f(\bar{z})(\cdot) = \left\langle \widetilde{U(\cdot)}, K \right\rangle$, then conditions (C) implies the following conditions, denoted (C') : for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $z \in B(\bar{z},\delta)$,

$$F(z) \subset F(\bar{z})+\widetilde{U(z-\bar{z})}+\varepsilon\|z-\bar{z}\|D_X,\ F(\bar{z}) \subset F(z)+\widetilde{U(z-\bar{z})}+\varepsilon\|z-\bar{z}\|D_X.$$

Following [17], a multifunction F which satisfies (C'), where $\tilde{U} : Z \to \mathscr{C}_K(X)$ (again, in the notation $\widetilde{U}(z) = \widetilde{U(z)}$, for all $z \in Z$) is a positive homogeneous map, is called $\tilde{U}-$differentiable at $\bar{z}$. If f is Fréchet differentiable and $\left\langle \widetilde{U(\cdot)}, K \right\rangle = \nabla f(\bar{z})(\cdot)$, then F is $\tilde{U}-$differentiable at $\bar{z}$. Conversely, if F is $\tilde{U}-$differentiable at $\bar{z}$ and U is single valued, linear and continuous, then $\langle U(\cdot)+K, K\rangle = \nabla f(\bar{z})(\cdot)$. We remark that if $T(\cdot) = \langle \{T_1(\cdot)\}+K, \{T_2(\cdot)\}+K\rangle$, where T_1 and T_2 are single-valued continuous operators, then the linearity of T is equivalent with the linearity of $T_1 - T_2$ and $T = \nabla f(\bar{z})$ if and only if $T_1 - T_2 \in \widehat{\partial} F(\bar{z}) \cap \widehat{\partial}_u F(\bar{z})$, where $\widehat{\partial} F(\bar{z})$ and $\widehat{\partial}_u F(\bar{z})$ are given in [7]. If $\widehat{\partial} F(\bar{z}) \cap \widehat{\partial}_u F(\bar{z}) \neq \emptyset$, then it contains only one element T, F is $\pi-$differentiable, $T-$differentiable, f is Fréchet differentiable at $\bar{z}$ and $\nabla f(\bar{z})(u) = \langle T(u)+K, K\rangle$.

Now, we turn our attention on Gâteaux differentiability of f. Recall that f is Gâteaux differentiable at $\bar{z} \in Z$ if, for each $u \in X$, there exists $\lim_{t\to 0} \frac{f(\bar{z}+tu)-f(\bar{z})}{t}$, denoted by $D'f(\bar{z})(u)$, and the map $u \mapsto D'f(\bar{z})(u)$ is linear and continuous. We recall (see [15]) that for two sets $A,B,C \in \mathscr{C}_K(X)$, $A \ominus B = C$ if and only if $A = B+C$ or, equivalently in this case, with $A = B \oplus C$. Similarly to [15], we may consider the Hukuhara differential for a $\mathscr{C}_K(X)$ valued map as follows: if, for each $u \in X$, the following limits

$$\lim_{t\downarrow 0} \frac{F(\bar{z}+tu) \ominus F(\bar{z})}{t} \in \mathscr{C}_K(X),$$

$$\lim_{t\downarrow 0}\frac{F(\bar z)\ominus F(\bar z-tu)}{t}\in\mathscr{C}_K(X)$$

exist and are equal, denoted by $D'F(\bar z)(u)$, and $u\rightrightarrows D'F(\bar z)(u)$ is linear and continuous, then F is called Hukuhara differentiable at $\bar z$ and $D'F(\bar z)$ is the Hukuhara differential of F at $\bar z$.

It is not difficult to verify that if F is Hukuhara differentiable at $\bar z$, then $\langle D'F(\bar z)(\cdot),K\rangle = D'f(\bar z)(\cdot)$. Conversely, if $D'f(\bar z)(\cdot)=\langle U(\cdot),K\rangle$, where $U:Z\to\mathscr{C}_K(X)$ is a linear continuous map, and if, for each $t>0$ and $u\in X$, there exist $F(\bar z+tu)\ominus F(\bar z)$ and $F(\bar z)\ominus F(\bar z-tu)$, then F is Hukuhara differentiable at $\bar z$ and $U=D'F(\bar z)$.

Example 4.1. Let $g:Z\to X$, $h:Z\to\mathbb{R}_+$, $M\in\mathscr{C}_K(X)$, and $F_1(z)=\{g(z)\}+K$, $F_2(z)=\{g(z)\}+M$ and $F_3(z)=h(z)\cdot M$. Accordingly, we have the embedding maps, denoted by f_1, f_2, and f_3, respectively.

(i) If g is a Fréchet differentiable map at $\bar z$, then, for each $\varepsilon>0$, there exists $\delta>0$ such that, for each $z\in B(\bar z,\delta)$,

$$g(z)\subset g(\bar z)+\nabla g(\bar z)(z-\bar z)+\varepsilon\|z-\bar z\|D_X,$$
$$g(\bar z)+\nabla g(\bar z)(z-\bar z)\subset g(z)+\varepsilon\|z-\bar z\|D_X.$$

Thus

$$F_1(z)\subset F_1(\bar z)+\nabla g(\bar z)(z-\bar z)+K+\varepsilon\|z-\bar z\|D_X,$$
$$F_1(\bar z)+\nabla g(\bar z)(z-\bar z)+K\subset F_1(z)+\varepsilon\|z-\bar z\|D_X.$$

whence $\langle\nabla g(\bar z)(\cdot)+K,K\rangle=\nabla f_1(\bar z)(\cdot)$. Similarly we get $\langle\nabla g(\bar z)(\cdot)+K,K\rangle=\nabla f_2(\bar z)(\cdot)$. If h is Fréchet differentiable at $\bar z$, for $u\in Z$ such that $\nabla h(\bar z)(u)>0$, then $\langle\nabla h(\bar z)(u)\cdot M+K,K\rangle=\nabla f_3(\bar z)(u)$.

(ii) If g is Gâteaux differentiable at $\bar z$, then $F_1(\bar z+tu)\ominus F_1(\bar z)=F_2(\bar z+tu)\ominus F_2(\bar z)=g(\bar z+tu)-g(\bar z)+K$ and $F_3(\bar z+tu)\ominus F_3(\bar z)=(h(\bar z+tu)-h(\bar z))\cdot M+K$ for each t and $u\in Z$. Thus, for $i\in\{1,2\}$, $D'f_i(\bar z)(\cdot)=D'F_i(\bar z)(\cdot)=\langle D'g(\bar z)(\cdot)+K,K\rangle$. If h is Gâteaux differentiable at $\bar z$, then $D'f_3(\bar z)(u)=D'F_3(\bar z)(u)=\langle D'h(\bar z(u)\cdot M+K,K\rangle$ for each $u\in X$ such that $D'h(\bar z)(u)>0$.

(iii) In the same hypothesis as in Proposition 4.4, one has

$$v\in D_Bg(\bar z)(u)\implies\langle v+K,K\rangle\in D_Bf_1(\bar z)(u)\implies v+K\subset\bigcap_{\bar x\in F_1(\bar z)}D_BF_1(\bar z,\bar x)(u).$$

Indeed,

$$v\in D_Bg(\bar z)(u)\iff\liminf_{u'\to u,t\downarrow 0}d\left(v+\frac{g(\bar z)}{t},\frac{g(\bar z+tu')}{t}\right)=0,$$

which implies

$$\liminf_{u'\to u,t\downarrow 0}h\left(v+\frac{g(\bar z)}{t}+K,\frac{g(\bar z+tu')}{t}+K\right)=0,$$

which is equivalent to $\langle v+K,K\rangle\in D_Bf_1(\bar z)(u)$. Finally, the latter relation implies, according to Proposition 4.4 (iii),

$$v+K\subset\bigcap_{\bar x\in F_1(\bar z)}D_BF_1(\bar z,\bar x)(u).$$

We refrain ourselves from exploring possible calculus associated to the above objects (see the last section for more comments on this), concentrating instead on some links associated to generalized differentiation objects that have the potential to generate optimality conditions for optimization problems we envisage, which is set optimization problems.

4.2. **Set optimization problems and primal optimality conditions.** In this subsection we consider set optimization problems and we link the objective set-valued map that drives these problems with the associated embedding function. The aim is to see the efficiencies of set optimization as classical Pareto efficiencies for this associated function, and this view is instrumental in order to import in set optimization some techniques from vector optimization. The final aim is to build up optimality conditions for set optimization problems using known optimality conditions for Pareto minimality.

For some nonempty sets $A,B \subset X$, one defines the $\preceq_K^{\ell}$ order (see, e.g., [14, 16]) by $A \preceq_K^{\ell} B \Leftrightarrow B \subset A+K$. On this basis, one defines an efficiency concept for constrained set optimization problems. Let $M \subset Z$ be a nonempty closed set. An element $\overline{z} \in M$ is said to be ℓ-minimum for F on M if

$$z \in M, F(z) \preceq_K^{\ell} F(\overline{z}) \Longrightarrow F(\overline{z}) \preceq_K^{\ell} F(z).$$

So, the ℓ-minimality of $\overline{z}$ means that when we consider an element $z \in M$, there exist two possibilities: either $F(\overline{z}) \not\subset F(z)+K$, or $F(\overline{z})+K = F(z)+K$.

Suppose that $\operatorname{int} K \neq \emptyset$ and consider now the weak notion of ℓ-efficiency. More precisely, $\overline{z} \in M$ is said to be weak ℓ-minimum for F on M if, for all $z \in M$, either $F(\overline{z}) \not\subset F(z)+\operatorname{int} K$, or $F(\overline{z})+\operatorname{int} K = F(z)+\operatorname{int} K$. It is known (see, e.g., [10]) that if the set of weak Pareto minimal points with respect to K of $F(\overline{z})$ is nonempty, then the weak ℓ-minimality of $\overline{z}$ for F on M means that $F(\overline{z}) \not\subset F(z)+\operatorname{int} K$, for all $z \in M$.

We study the links between these efficiency notions for F and the Pareto efficiencies for f, the latter considered with respect to the ordering cone $\mathscr{K}_K(X)$.

Proposition 4.5. *Let F be with nonempty, K-closed values, and $\overline{z}, z \in M$. Then,*
(i) $F(\overline{z}) \preceq_K^{\ell} F(z)$ if and only if $f(z) - f(\overline{z}) \in \mathscr{K}_K(X)$;
(ii) $\overline{z}$ is ℓ-minimum for F on M if and only if $\overline{z}$ is a Pareto minimum point for f on M.

Proof. Indeed, if $F(\overline{z}) \preceq_K^{\ell} F(z)$, then $F(z) \subset F(\overline{z})+K$, whence $\widetilde{F(z)} \subset \widetilde{F(\overline{z})}$, which is equivalent to $f(z) - f(\overline{z}) \in \mathscr{K}_K(X)$. So, this implication holds without any supplementary conditions. For the converse, since F has K-closed values, $\widetilde{F(z)} = F(z)+K$, and $f(z) - f(\overline{z}) \in \mathscr{K}_K(X)$ actually means that $F(z)+K \subset F(\overline{z})+K$, so, $F(\overline{z}) \preceq_K^{\ell} F(z)$.

(ii) The fact that $\overline{z}$ is a Pareto minimum for f on M means that $f(z) - f(\overline{z}) \notin -\mathscr{K}_K(X) \setminus \{\langle K,K\rangle\}$, which can be written as $\left\langle \widetilde{F(\overline{z})}, K\right\rangle - \left\langle \widetilde{F(z)}, K\right\rangle \notin \mathscr{K}_K(X) \setminus \{\langle K,K\rangle\}$. This is equivalently to say that either

$$\left\langle \widetilde{F(\overline{z})}, K\right\rangle - \left\langle \widetilde{F(z)}, K\right\rangle \notin \mathscr{K}_K(X)$$

or

$$\left\langle \widetilde{F(\overline{z})}, K\right\rangle - \left\langle \widetilde{F(z)}, K\right\rangle = \langle K,K\rangle .$$

Under our assumptions on the values of F (that is, they are K-bounded, K-convex and K-closed) the first is to say that $F(\overline{z})+K \not\subset F(z)+K$, while the latter means that $F(\overline{z})+K = F(z)+K$. The claim is done. □

Proposition 4.6. *Suppose that* $\operatorname{int} K \neq \emptyset$ *and* F *has* K*-closed values. If* $\bar{z} \in M$ *is weak* ℓ*-minimum for* F *on* M*, then it is weak Pareto minimum for* f *on* M*. If* $F(\bar{z})$ *is* K*-compact, then the converse also holds.*

Proof. Suppose that $\bar{z} \in M$ is weak ℓ-minimum for F on M. Then, for any $z \in M$, $F(\bar{z}) \not\subset F(z) + \operatorname{int} K$, or $F(\bar{z}) + \operatorname{int} K = F(z) + \operatorname{int} K$. In the first case, since $f(\bar{z}) - f(z) = \left\langle \widetilde{F(\bar{z})}, \widetilde{F(z)} \right\rangle = \langle F(\bar{z}) + K, F(z) + K \rangle$, by Proposition 3.2, we have $f(\bar{z}) - f(z) \notin \operatorname{int} \mathscr{K}_K(X)$. In the second case, $\operatorname{cl}(F(\bar{z}) + \operatorname{int} K) = \operatorname{cl}(F(z) + \operatorname{int} K)$, that is, $\widetilde{F(\bar{z})} = \widetilde{F(z)}$. Thus we deduce that

$$d\left(\operatorname{bd}\left(\widetilde{F(\bar{z})}\right), \operatorname{bd}\widetilde{F(z)}\right) = 0.$$

According to Remark 3.2, $f(\bar{z}) - f(z) \notin \operatorname{int} \mathscr{K}_K(X)$.

For the converse, if $\bar{z}$ is a weak Pareto minimum for f on M with respect to $\mathscr{K}_K(X)$, then, for all $z \in M$, $f(z) - f(\bar{z}) \notin -\operatorname{int} \mathscr{K}_K(X)$, and this amounts to say that $\left\langle \widetilde{F(\bar{z})}, \widetilde{F(z)} \right\rangle \notin \operatorname{int} \mathscr{K}_K(X)$. In particular, according to Proposition 3.3 (iii), since $F(\bar{z})$ is K-compact, one has that $F(\bar{z}) \not\subset \widetilde{F(z)} + \operatorname{int} K$. Under the K-closedness of $F(z)$, that is, $F(\bar{z}) \not\subset F(z) + \operatorname{int} K$, which proves the assertion. □

Example 4.2. Without the K-compactness assumption on $F(\bar{z})$, the converse in the above result is not true. To see this, let us consider $Z := \mathbb{R}$, $X := \mathbb{R}^2$, $K := \mathbb{R}^2_+$, $F : Z \rightrightarrows X$,

$$F(z) = \begin{cases} \{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0\}, & \text{if } z \neq 0 \\ \left\{(x,y) \in \mathbb{R}^2 \mid x > 0, y \geq \exp\left(\frac{1}{x}\right)\right\}, & \text{if } z = 0. \end{cases}$$

Then $\bar{z} := 0$ is a weak Pareto minimum for f on Z (see Proposition 3.2), but it is not a weak ℓ-minimum for F on Z, since, for all z one has $F(\bar{z}) \subset F(z) + \operatorname{int} K$, but the converse inclusion does not hold. Clearly, $F(\bar{z})$ is not K-compact.

Remark 4.1. Notice that the corresponding concepts of local efficiencies and the associated results are easy to write.

To present optimality conditions, we firstly recall some well-known definitions (see, e.g., [2]). If in the preceding section we gave the definition of the Bouligand derivative by inferior limit, we prefer here a sequential approach for several related concepts.

Definition 4.1. Let D be a nonempty subset of Z and $\bar{z} \in X$.

(i) the Bouligand tangent cone to D at $\bar{z}$ (the contingent cone in [2]) is the set

$$T_B(D, \bar{z}) = \{u \in X \mid \exists (t_n) \downarrow 0, \exists (u_n) \to u, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \bar{z} + t_n u_n \in D\}$$

where $(t_n) \downarrow 0$ means $(t_n) \subset (0, \infty)$ and $t_n \to 0$;

(ii) the Ursescu tangent cone to D at $\bar{z}$ (the intermediate cone in [2]) is the set

$$T_U(D, \bar{z}) = \{u \in X \mid \forall (t_n) \downarrow 0, \exists (u_n) \to u, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \bar{z} + t_n u_n \in D\}.$$

Definition 4.2. Let $(\bar{z}, \bar{x}) \in \operatorname{Gr} F$. The Bouligand derivative of F at $(\bar{z}, \bar{x})$ is the set valued map $D_B F(\bar{z}, \bar{x})$ from Z into X defined by

$$\operatorname{Gr} D_B F(\bar{z}, \bar{x}) = T_B(\operatorname{Gr} F, (\bar{z}, \bar{x})).$$

(Notice that this coincides with the definition from relation (4.1).) The Ursescu derivative has a similar definition.

In addition to these classical objects, we introduce a different kind of approximation for a set-valued map.

Definition 4.3. We call an element $\left\langle \tilde{A}, \tilde{B} \right\rangle \in \mathscr{G}_K(X)$ an approximating pair of F at $\bar{z}$ in direction $u \in Z$ if there exists $(u_n) \to u$, $(t_n) \subset (0, \infty)$, $t_n \to 0$ such that, for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that, for all $n \geq n_\varepsilon$,

$$F(\bar{z}) + t_n A \subset t_n \odot \tilde{B} \oplus \widetilde{F(\bar{z} + t_n u_n)} + \varepsilon t_n B_X$$

and

$$F(\bar{z} + t_n u_n) + t_n B \subset t_n \odot \tilde{A} \oplus \widetilde{F(\bar{z})} + \varepsilon t_n B_X.$$

Now we are able to present our necessary optimality conditions for weak ℓ-minimality by using the above primal space approximation object.

Proposition 4.7. *(i) If there exists a neighborhood V of $\bar{z}$ such that $F(\bar{z}) \preceq_K^\ell F(z)$ for every $z \in V$, then, for every approximating pair $\left\langle \tilde{A}, \tilde{B} \right\rangle \in \mathscr{G}_K(X)$ of F at $\bar{z}$ in a direction $u \in X$, $\tilde{A} \subset \tilde{B}$.*

(ii) Suppose that $\operatorname{int} K \neq \emptyset$, F has K-closed values, and $F(\bar{z})$ is K-compact. Moreover, assume that F is Lipschitz around $\bar{z}$. Let $\bar{z}$ be a local weak ℓ-minimum for F on M. Then, for $u \in T_U(M, \bar{z})$ and every approximating pair $\left\langle \tilde{A}, \tilde{B} \right\rangle \in \mathscr{G}_K(X)$ of F at $\bar{z}$ in direction u,

$$\tilde{B} + \varepsilon B_X \not\subset \tilde{A},\ \forall \varepsilon > 0. \tag{4.2}$$

If $\tilde{B} = K$, then relation (4.2) means that $A \cap -\operatorname{int} K = \emptyset$.

Proof. (i) Let $\left\langle \tilde{A}, \tilde{B} \right\rangle \in \mathscr{G}_K(X)$ be an approximating pair of F at $\bar{z}$ in a direction $u \in Z$. It is easy to see that this means that $\left\langle \tilde{A}, \tilde{B} \right\rangle \in D_B f(\bar{z})(u)$. According to Proposition 4.5 (i), $f(z) - f(\bar{z}) \in \mathscr{K}_K(X)$ for all $z \in V$. Taking into account that $\mathscr{K}_K(X)$ is a closed cone, we see that $D_B f(\bar{z})(u) \subset \mathscr{K}_K(X)$ whence $\tilde{A} \subset \tilde{B}$.

(ii) Following Proposition 4.6, hypothesis (ii) ensures that $\bar{z}$ is a local weak Pareto minimum for f on M and, moreover, f is Lipschitz around $\bar{z}$ (see Proposition 4.2). Let $\left\langle \tilde{A}, \tilde{B} \right\rangle \in \mathscr{G}_K(X)$ be an approximating pair of F at $\bar{z}$ in direction $u \in T_U(M, \bar{z})$, meaning, as before, $\left\langle \tilde{A}, \tilde{B} \right\rangle \in D_B f(\bar{z})(u)$. Using an well-known necessary optimality condition for Pareto minimality (see, e.g., [5, Corollary 3.2]), we see that $\left\langle \tilde{A}, \tilde{B} \right\rangle \notin -\operatorname{int} \mathscr{K}_K(X)$. According to Proposition 3.2, this is the conclusion given by relation (4.2).

Finally, takeing $\tilde{B} = K$ and using Proposition 3.4, we see that $A \cap -\operatorname{int} K = \emptyset$, which is the last conclusion. □

Remark 4.2. If $\tilde{B} = K$, Proposition 4.7 (i) and Proposition 4.4 (iii) ensure that if there exists a neighborhood V of $\bar{z}$ such that $F(\bar{z}) \preceq_K^\ell F(z)$ for every $z \in V$, then, for all $u \in Z$,

$$\bigcap_{\bar{x} \in F(\bar{z})} D_B \widetilde{F}(\bar{z}, \bar{x})(u) \cap K \neq \emptyset.$$

Following Example 4.1 (iii) and the notation therein, if the set-valued map is F_1, then ,for each $u \in X$ and $v \in D_B g(\bar{z})(u)$,t $\langle v + K, K \rangle \in D_B f_1(\bar{z})(u)$. Thus $v \in K$, which implies

$$v + K \subset \bigcap_{\bar{x} \in F_1(\bar{z})} D_B F_1(\bar{z}, \bar{x})(u).$$

For the next result, which aims to obtain a scalarization of ℓ-minimality, we recall (see [2]) that a set-valued map $F : Z \rightrightarrows X$ is said to be continuous at $z \in Z$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $v \in B(z,\delta)$, $h(F(v), F(z)) < \varepsilon$. As usual, F is said to be continuous if it is continuous at all points in Z.

Proposition 4.8. *Let $F : Z \rightrightarrows X$ be a K-convex set valued map with values in $\mathscr{C}_K(X)$ and $M \subset Z$ be a nonempty convex and compact set. Suppose that F is continuous and $\bar{z} \in M$ is a ℓ-minimum for F on M. Then there exists $U : \mathscr{C}_K(X) \to \mathbb{R}$, a positive homogeneous, additive and monotone (with respect to the order of $\mathscr{C}_K(X)$, that is, $\preceq$) operator such that $U(F(\bar{z})) \le U(F(z))$ for all $z \in M$. Moreover, for all $z \in M$ and all nonzero natural number n, there exists a finite subset $I_n \subset \mathbb{N}$ and a set $\{x_i^* \mid i \in I_n\} \subset K^+$ such that the operator $T_n : \mathscr{C}_K(X) \to \mathbb{R}$ defined by $T_n(A) = \sum_{i\in I_n} \inf x_i^*(A)$ satisfies $T_n(F(z)) - T_n(F(\bar{z})) > -n^{-1}$ and $\lim T_n(F(z)) = U(F(z))$.*

Proof. Taking into account Propositions 4.1 and 4.5, we deduce that $f(M) + \mathscr{K}_K(X)$ is a convex set and $\bar{z}$ is a Pareto minimum for f on M. Moreover, it is easy to see that the continuity of F is equivalent to the continuity of f (as a single valued function), whence the set $f(M) + \mathscr{K}_K(X)$ is also closed. Therefore, by a well-known separation argument, there exists $T \in (\mathscr{K}_K(X))^+$ such that

$$T(\langle F(\bar{z}), K\rangle) \le T(\langle F(z), K\rangle), \forall z \in M.$$

Define $U : \mathscr{C}_K(X) \to \mathbb{R}$, by $U(A) = T(\langle A, K\rangle)$. In view of the properties of T, U satisfies the first part of the conclusion.

For the second part of the conclusion, we use Proposition 3.6. Fix $z \in M$ and take $v := \langle F(z), F(\bar{z})\rangle \in \mathscr{G}_K(X)$. Therefore, for all $n \in \mathbb{N} \setminus \{0\}$, there exist a finite subset I_n of $\mathbb{N}$ and a set $\{x_i^* \mid i \in I_n\} \subset K^+$ such that

$$\left| T(v) - \sum_{i\in I_n} T_{x_i^*}(v) \right| < \frac{1}{n}.$$

Putting $\widetilde{T}_n = \sum_{i\in I_n} T_{x_i^*}$, we have

$$T(\langle F(z), K\rangle) - T(\langle F(\bar{z}), K\rangle) - \widetilde{T}_n(\langle F(z), K\rangle) + \widetilde{T}_n(\langle F(\bar{z}), K\rangle) < n^{-1}$$

since $T \in (\mathscr{K}_K(X))^+$, $\widetilde{T}_n(\langle F(z), K\rangle) - \widetilde{T}_n(\langle F(\bar{z}), K\rangle) > -n^{-1}$. Take now $T_n : \mathscr{C}_K(X) \to \mathbb{R}$ given by $T_n(A) = \widetilde{T}_n(\langle A, K\rangle)$, which satisfies the requirements. Indeed,

$$\lim T_n(F(z)) = \lim \widetilde{T}_n(\langle F(z), K\rangle) = T(\langle F(z), K\rangle) = U(F(z)).$$

□

Remark 4.3. Let $F : Z \rightrightarrows X$ be as a set valued map with values in $\mathscr{C}_K(X)$, $M \subset Z$ be a nonempty set, and $\bar{z} \in M$. It is easy to see that if there exists $U : \mathscr{C}_K(X) \to \mathbb{R}$ a strictly monotone operator (that is $A \subset B$ and $A \ne B$, then $U(A) > U(B)$) such that $U(F(\bar{z})) \le U(F(z))$ for all $z \in M$, then $\bar{z}$ is a ℓ-minimum for F on M.

5. Concluding Remarks

In this paper, we devised an embedding result for the class of nonempty, bounded, and convex sets with respect to a closed convex and pointed cone. This allows us to extend, in an nontrivial manner, a classical result of Rådström. Along the main discussion, we presented several applications and we investigated optimality conditions for set optimization that can be obtained by simply seeing, via the embedding result, a set-valued maps as a single valued one. Several other possible applications can be envisaged by extending or offering alternatives to some results in [3, 15, 17, 20, 21], for instance. In a future work, we intend to obtain suitable calculus for some generalized differentiation objects, a task that is not straightforward to have in mind that several of the main analytical properties usually used for such calculus (completeness, Asplundness, etc.) of the embedding space are not available.

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