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HIGHER-ORDER DIFFERENTIAL SENSITIVITY FOR WEAK VECTOR VARIATIONAL INEQUALITIES IN TERMS OF TANGENT DERIVATIVES

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Abstract. This paper is devoted to the study of differential sensitivity analysis for weak vector variational inequalities in terms of the higher-order tangent derivatives, which were introduced by J.P. Penot in 2017. We first provide some calculus rules for these derivatives such as composition rule and sum rule, and we obtain some formulas for the tangent derivative of the profile mapping. Then, they are employed to investigate the differential sensitivity for the vector variational inequality problem with the help of gap functions. Several examples are given to illustrate the obtained results.

Keywords. Differential sensitivity analysis; Gap function; Tangent derivatives; Vector variational inequality.

1. INTRODUCTION

One of the main topics in sensitivity analysis for optimization-related problems is the study of derivatives/generalized derivatives of the solution mappings and the optimal-value mappings of perturbed problems. For nonlinear programming, Fiacco and Ishizuka [8] studied sensitivity results in terms of classical derivatives. However, practical optimization-related problems are often nonsmooth. To cope with this crucial difficulty, most of approaches for sensitivity analysis are based on generalized derivatives. For nonsmooth multiobjective optimization, Tanino [27] first studied the first-order contingent derivative of perturbation maps. Some related results were developed in [13, 14, 25]. By using the concepts in dual space approaches, such as sub-gradients and co-derivatives of set-valued mappings, Levy and Mordukhovich [15] investigated sensitivity analysis for parameterized vector optimization problems. Following this direction, the reader is referred to [6, 7, 21, 22] and the references therein. In primal space approaches, Luc et al. [20] presented some sufficient conditions for semi-differentiability of efficient solutions and marginal maps for parameterized multiobjective optimization under some relaxed assumptions. For second-order considerations, Wang et al. [32] provided the second-order contingent derivative for the perturbation map and the proper perturbation map of vector

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optimization problem; Tung [29] gave some sufficient conditions for inner and outer estimations of the second-order proto-derivative of the efficient solution map in vector optimization problems under some qualification conditions. For equilibrium problems, Anh [3] obtained differential sensitivity for these problems in terms of variational sets. Tung et al. [31] introduced a notion of a higher-order Hadamard-type directional derivative and obtained an implicit set-valued map theorem; then it is employed to examine the higher-order sensitivity analysis for solution maps of a parametric vector equilibrium problem. All the aforementioned works are for sensitivity analysis to optimization and equilibrium problems.

However, the sensitivity analysis for vector variational inequalities still needs to be addressed. The concept of gap functions plays an essential role in studying the sensitivity of vector variational inequalities (VVI) and weak vector variational inequalities (WVVI). Chen et al. [5] defined some gap functions for these problems and discussed some related properties. Later, Li et al. [16] studied the differential and sensitivity properties of the set-valued gap function for WVVI under some suitable conditions. Subsequently, Li and Li [19] examined the differential sensitivity for the WVVI by using the second-order contingent derivative. Li and Zhai [18] introduced an asymptotic second-order Φ-contingent cone and discussed an explicit expression of the asymptotic second-order contingent derivative for the class of set-valued maps and obtained them to investigate sensitivity analysis and optimality conditions for vector variational inequalities. Besides, to the best of our knowledge, there are only few results for the higher-order differential sensitivity of vector variational inequalities.

Motivated by the above observations, we, in this paper, aim to discuss the higher-order differential sensitivity for vector variational inequalities in terms of tangent derivatives, introduced by Penot [24]. These tangent derivatives have many nice properties and are useful in applications (see [4, 12, 23]). We first give some calculus rules such as chain rule and sum rule for these derivatives. The main idea for these rules is inspired by [17] and based on a transitivity, that is to insert an intermediate element. Our approach is different from [4], in which the assumption of metric regularity plays a crucial role. We also provide the formula for the tangent derivative of the profile mapping. We then apply them to obtain the differential sensitivity for vector variational inequality with the help of the corresponding gap function.

The rest of the paper is organized as follows. In Section 2, we collect definitions and preliminary facts for use later. Section 3 gives some calculus rules for the higher-order tangent derivatives. We discuss the differential properties of a class of set-valued maps and derive an explicit expression for them in Section 4. Finally, the higher-order sensitivity properties for vector weak variational inequality are investigated in Section 5.

2. Preliminaries

Throughout the paper, if not otherwise stated, $\mathbb{N}, \mathbb{R}^n$, and $\mathbb{R}_+$ stand for the set of natural numbers, an n-dimensional space, and the set of positive real numbers, respectively (resp). Let $\mathbb{B}^n$ denote the open unit ball of $\mathbb{R}^n$ and $\mathbb{B}^n(x,r)$ be the open ball with center x and radius r. For a set $M \subseteq \mathbb{R}^n$, $\text{int}M$, $\text{cl}M$, $\text{co}M$, and $\text{cone}M$ stand for its topological interior, closure, convex hull, and cone hull, resp. $L(\mathbb{R}^n, \mathbb{R}^m)$ denotes the space of the linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$. For any $l \in L(\mathbb{R}^n, \mathbb{R}^m)$, we recall the norm $\|l\|_L := \sup\{\|l(x)\| \mid \|x\| \leq 1\}$.

Let $C \subset \mathbb{R}^n$ be a closed and convex cone with nonempty interior. A nonempty convex set Q (see [9]) is a base of C if $0 \notin \text{cl}Q$ and $\text{cone}Q = C$. The cone C is called sequentially regular (see [26]) if each C-increasing and C-upper bounded sequence converges to an element of C. A set Ω of $\mathbb{R}^n$ is called C-upper bounded (see [28]) if there exists a point $a \in \mathbb{R}^n$ such that $\Omega \subset a - C$. The maximal and weak maximal point sets of Ω with respect to C are defined (see [10]),

$$\mathrm{Max}(\Omega, C) := \left\{ \bar{y} \in \Omega \mid (\bar{y} + C \setminus \{0\}) \cap \Omega = \emptyset \right\},$$
$$\mathrm{WMax}(\Omega, C) := \left\{ \bar{y} \in \Omega \mid (\bar{y} + \mathrm{int} C) \cap \Omega = \emptyset \right\}.$$

Let $G \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping. The domain and graph of G are

$$\mathrm{dom}\, G := \{x \in \mathbb{R}^n \mid G(x) \neq \emptyset\}, \quad \mathrm{gph}\, G := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in G(x)\}.$$

The profile map $G_- : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is defined by $G_-(x) := G(x) - C$, for every $x \in \mathbb{R}^n$. The hypograph of G with respect to C is

$$\mathrm{hypo}\, G := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in G(x) - C\}.$$

The closure mapping related to G is a set-valued mapping $\mathrm{Cl}\, G \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by

$$\mathrm{gph}(\mathrm{Cl}\, G) := \mathrm{cl}(\mathrm{gph}\, G).$$

G is said to be uniformly compact near $\bar{x} \in \mathbb{R}^n$ (see [2]) if there exists a neighborhood U of $\bar{x}$ such that the set $\bigcup_{x \in U} G(x)$ is compact.

Let us recall the higher-order tangent sets, which were introduced by Penot (see [24]).

Definition 2.1. (see [24]) Let $M \subseteq \mathbb{R}^n$, $\bar{x} \in \mathrm{cl}\, M$, $v \in \mathbb{R}^n$, and $r \in \overline{\mathbb{R}}_+ := [0, +\infty]$.

(i) The higher-order tangent set of M at $\bar{x}$ with index r is

$$T_r^h(M, \bar{x}, v) := \left\{ w \in \mathbb{R}^n \mid \exists t_k \searrow 0, \exists s_k \searrow 0, t_k s_k^{-1} \to r, \exists w_k \to w : \bar{x} + t_k v + \frac{1}{2} t_k s_k w_k \in M, \ \forall k \in \mathbb{N} \right\}.$$

(ii) The incident higher-order tangent set of MS at $\bar{x}$ with index r is

$$T_r^{hi}(M, \bar{x}, v) := \left\{ w \in \mathbb{R}^n \mid \forall t_k \searrow 0, \forall s_k \searrow 0, t_k s_k^{-1} \to r, \exists w_k \to w : \bar{x} + t_k v + \frac{1}{2} t_k s_k w_k \in M, \ \forall k \in \mathbb{N} \right\}.$$

$T_1^h(M, \bar{x}, v)$ is the well-known second-order tangent set (see [1]) and $T_0^h(M, \bar{x}, v)$ is the second-order asymptotic tangent cone (see [23]). Some nice properties of these tangent sets were studied in [11, 23, 24].

Definition 2.2. (see [24]) Let $(\bar{x}, \bar{y}) \in \mathrm{gph}\, G$, $(\bar{u}, \bar{v}) \in \mathbb{R}^n \times \mathbb{R}^m$, and $r \in \overline{\mathbb{R}}_+$.

(i) The higher-order tangent derivative of G at $(\bar{x}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$ with index r is

$$\mathrm{gph}(D_r^h G(\bar{x}, \bar{y}, \bar{u}, \bar{v})) := T_r^h(\mathrm{gph}\, G, (\bar{x}, \bar{y}), (\bar{u}, \bar{v})).$$

Equivalently, for all $x \in \mathbb{R}^n$,

$$D_r^h G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = \Big\{ y \in \mathbb{R}^m \mid \exists t_k \searrow 0, \exists s_k \searrow 0, t_k s_k^{-1} \to r, \ \exists (x_k, y_k) \to (x, y) :$$
$$\bar{y} + t_k \bar{v} + \frac{1}{2} t_k s_k y_k \in G\left(\bar{x} + t_k \bar{u} + \frac{1}{2} t_k s_k x_k \right), \forall k \in \mathbb{N} \Big\}.$$

(ii) The incident higher-order tangent derivative of G at $(\bar{x},\bar{y})$ in the direction $(\bar{u},\bar{v})$ with index r is

$$\mathrm{gph}(D_r^{hi}G(\bar{x},\bar{y},\bar{u},\bar{v})) := T_r^{hi}(\mathrm{gph}\, G,(\bar{x},\bar{y}),(\bar{u},\bar{v})).$$

Equivalently, for all $x \in \mathbb{R}^n$,

$$D_r^{hi}G\,(\bar{x},\bar{y},\bar{u},\bar{v})\,(x) = \Big\{y \in \mathbb{R}^m \mid \forall t_k \searrow 0, \forall s_k \searrow 0, t_k s_k^{-1} \to r,\ \exists (x_k,y_k) \to (x,y) : \\ \bar{y}+t_k\bar{v}+\frac{1}{2}t_ks_ky_k \in G\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right), \forall k \in \mathbb{N}\Big\}.$$

$D_0^hG\,(\bar{x},\bar{y},\bar{u},\bar{v})\,(x)$ is the second-order asymptotic tangent derivative, introduced by Penot (see [23]). While $D_1^hG\,(\bar{x},\bar{y},\bar{u},\bar{v})\,(x)$ is the second-order tangent derivative (see [1]).

We give some examples to illustrate the higher-order tangent derivatives.

Example 2.1. (i) Let $G : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined by

$$G(x) := \begin{cases} \{y \in \mathbb{R} \mid y = x^{(2p+1)/2p}, p \in \mathbb{N}\}, & \text{if } x \geq 0, \\ \emptyset, & \text{if } x < 0. \end{cases}$$

Take $(\bar{x},\bar{y}) = (0,0)$ and $(\bar{u},\bar{v}) = (1,0)$. By directly calculating, we have

$$D_r^hG\,(\bar{x},\bar{y},\bar{u},\bar{v})\,(x) = \begin{cases} \mathbb{R}_+, & \text{if } r = 0, \\ \emptyset, & \text{if } r = (0,+\infty]. \end{cases}$$

(ii) Let $G_q : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined by $G_q(x) := \{(y_1,y_2) \in \mathbb{R}^2 \mid y_1 = y_2 = x^{2q}\}$ for $q \in \mathbb{N}$. Take $(\bar{x},\bar{y}) = (0,(0,0))$ and $(\bar{u},\bar{v}) = (1,(0,0))$. After some computations, we see that

- for any $x \in \mathbb{R}$,

$$D_r^hG_1\,(\bar{x},\bar{y},\bar{u},\bar{v})\,(x) = \begin{cases} \{(2r,2r)\}, & \text{if } r = [0,+\infty), \\ \emptyset, & \text{if } r = +\infty. \end{cases}$$

- for $q > 1$ and $x \in \mathbb{R}$,

$$D_r^hG_q\,(\bar{x},\bar{y},\bar{u},\bar{v})\,(x) = \begin{cases} \{(0,0)\}, & \text{if } r = [0,+\infty), \\ \mathbb{R}^2_+, & \text{if } r = +\infty. \end{cases}$$

3. Calculus Rules

In this section, we study some calculus rules for the higher-order tangent derivative. The main idea in this section is inspired by [4, 17, 24, 30].

3.1. **Derivatives of compositions.** For the set-valued mappings $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $G\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^p$, we study the composition set-valued mapping

$$(G \circ F)(x) := \cup_{y \in F(x)} G(y).$$

Denote by $G^{-1}(z) := \{y \in \mathbb{R}^m \mid z \in G(y)\}$, the resultant set-valued map is $H : \mathbb{R}^n \times \mathbb{R}^p \rightrightarrows \mathbb{R}^m$ by $H(x,z) := F(x) \cap G^{-1}(z)$.

Definition 3.1. Given $((\bar{x},\bar{z}),\bar{y}) \in \operatorname{gph}(\operatorname{Cl} H)$, $(u,w) \in \mathbb{R}^n \times \mathbb{R}^p$, and $r \in \overline{\mathbb{R}}_+$, the $\bar{y}$-higher-order tangent derivative of $G \circ F$ at $(\bar{x},\bar{z})$ in the direction (u,w) with index r is the set-valued map $D^h_r(G \circ_{\bar{y}} F)(\bar{x},\bar{z},u,w) : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ defined by

$$D^h_r(G \circ_{\bar{y}} F)(\bar{x},\bar{z},u,w)(x) := \Big\{ z \in \mathbb{R}^p \mid \exists t_k \searrow 0, \exists s_k \searrow 0, t_k s_k^{-1} \to r,\ \exists (x_k,\bar{y}_k,z_k) \to (x,\bar{y},z) : \bar{y}_k \in H\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k, \bar{z} + t_k w + \frac{1}{2} t_k s_k z_k\right), \forall k \in \mathbb{N} \Big\}.$$

Motivated by the work in [17], we use the following definition of directionally semi-compact.

Definition 3.2. Let $H : \mathbb{R}^n \times \mathbb{R}^p \rightrightarrows \mathbb{R}^m$, $(\bar{x},\bar{z}) \in \operatorname{dom}(\operatorname{Cl} H)$, $(u,w) \in \mathbb{R}^n \times \mathbb{R}^p$, and $r \in \overline{\mathbb{R}}_+$. H is directionally semi-compact at $(\bar{x},\bar{z})$ with respect to (u,w) in the direction (x,y) iff, for all $t_k \searrow 0, s_k \searrow 0, t_k s_k^{-1} \to r, (x_k,z_k) \to (x,z)$, any sequence $\bar{y}_k \in H\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k, \bar{z} + t_k w + \frac{1}{2} t_k s_k z_k\right)$ has a convergent subsequence.

Proposition 3.1. *Given $(\bar{x},\bar{z}) \in \operatorname{gph}(G \circ F)$, $(u,w) \in \mathbb{R}^n \times \mathbb{R}^p$, and $r \in \overline{\mathbb{R}}_+$. Assume that H is directionally semi-compact at $(\bar{x},\bar{z})$. Then,*

$$D^h_r(G \circ F)(\bar{x},\bar{z},u,w)(x) = \bigcup_{\bar{y} \in (\operatorname{Cl} H)(\bar{x},\bar{z})} D^h_r(G \circ_{\bar{y}} F)(\bar{x},\bar{z},u,w)(x). \tag{3.1}$$

Proof. Let $\bar{y} \in (\operatorname{Cl} H)(\bar{x},\bar{z})$ and $z \in D^h_r(G \circ_{\bar{y}} F)(\bar{x},\bar{z},u,w)(x)$. Then there exist sequences $t_k \searrow 0, s_k \searrow 0$ with $t_k s_k^{-1} \to r$, and $(x_k,\bar{y}_k,z_k) \to (x,\bar{y},z)$ such that

$$\bar{y}_k \in H\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k, \bar{z} + t_k w + \frac{1}{2} t_k s_k z_k\right).$$

By the definition of H, one has

$$\bar{z} + t_k w + \frac{1}{2} t_k s_k z_k \in G(\bar{y}_k) \subseteq (G \circ F)\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k\right).$$

Hence, we arrive at $z \in D^h_r(G \circ F)(\bar{x},\bar{z},u,w)(x)$.

For the converse, pick any $z \in D^h_r(G \circ F)(\bar{x},\bar{z},u,w)(x)$. Then we find sequences $t_k \searrow 0$ and $s_k \searrow 0$ with $\frac{t_k}{s_k} \to r$, and $(x_k,z_k) \to (x,z)$ such that

$$\bar{z} + t_k w + \frac{1}{2} t_k s_k z_k \in (G \circ F)\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k\right).$$

Hence, there exists $\bar{y}_k \in F\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k\right)$ with $\bar{z} + t_k w + \frac{1}{2} t_k s_k z_k \in G(\bar{y}_k)$ for all $n \in \mathbb{N}$. This gives that

$$\begin{aligned} \bar{y}_k &\in G^{-1}\left(\bar{z} + t_k w + \frac{1}{2} t_k s_k z_k\right) \bigcap F\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k\right) \\ &= H\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k, \bar{z} + t_k w + \frac{1}{2} t_k s_k z_k\right). \end{aligned} \tag{3.2}$$

From the assumption that H is directionally semi-compact, we can assume that $\{\bar{y}_k\}$ converges to some $\bar{y}$ (using a subsequence if necessary). Since $\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k, \bar{z} + t_k w + \frac{1}{2} t_k s_k z_k, \bar{y}_k\right) \to$

$(\bar{x},\bar{z},\bar{y})$, we have $\bar{y} \in (\mathrm{Cl}\,H)(\bar{x},\bar{z})$. This together with (3.2) derives that

$$z \in D_r^h(G \circ_{\bar{y}} F)(\bar{x},\bar{z},u,w)(x) \subseteq \bigcup_{\bar{y} \in (\mathrm{Cl}\,H)(\bar{x},\bar{z})} D_r^h(G \circ_{\bar{y}} F)(\bar{x},\bar{z},u,w)(x).$$

The proof is done. □

The following example shows that the directional semi-compactness of H in Proposition 3.1 is essential.

Example 3.1. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$, and $G : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by

$$F(x) = \begin{cases} \{y \in \mathbb{R} \mid y = x^{3/2}\}, & \text{if } x \geq 0, \\ \emptyset, & \text{if } x < 0, \end{cases} \quad G(y) = \{z \in \mathbb{R} \mid z = -y\}.$$

Then

$$(G \circ F)(x) = \begin{cases} \{z \in \mathbb{R} \mid z = -x^{3/2}\}, & \text{if } x \geq 0, \\ \emptyset, & \text{if } x < 0, \end{cases}$$

$$H(x,z) = F(x) \cap G^{-1}(z) = \begin{cases} \{0\}, & \text{if } x = z = 0, \\ \emptyset, & \text{otherwise }. \end{cases}$$

Let $(\bar{x},\bar{z}) = (0,0)$, $\bar{y} = 0 \in (\mathrm{Cl}\,H)(\bar{x},\bar{z})$, and $(u,w) = (1,0)$. Hence

$$D_r^h(G \circ F)(\bar{x},\bar{z},u,w)(x) = \begin{cases} \mathbb{R}_+, & \text{if } r = 0, \\ \emptyset, & \text{if } r = (0,+\infty], \end{cases} \quad D_r^h(G \circ_{\bar{y}} F)(\bar{x},\bar{z},u,w)(x) = \{0\}.$$

Then

$$\bigcup_{\bar{y} \in (\mathrm{Cl}\,H)(\bar{x},\bar{z})} D_r^h(G \circ_{\bar{y}} F)(\bar{x},\bar{z},u,w)(x) \subseteq D_r^h(G \circ F)(\bar{x},\bar{z},u,w)(x).$$

However, the converse is not true. The reason is that H is not directionally semi-compact at $(0,0)$.

Proposition 3.2. *Let* $(\bar{x},\bar{z}) \in \mathrm{gph}(G \circ F)$, $(u,v,w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$, *and* $r \in \overline{\mathbb{R}}_+$. *If for all* $(x,z) \in \mathbb{R}^n \times \mathbb{R}^p$, *the following condition holds*

$$D_r^h F(\bar{x},\bar{y},u,v)(x) \cap D_r^h G^{-1}(\bar{z},\bar{y},w,v)(z) \subseteq D_r^h H((\bar{x},\bar{z}),\bar{y},(u,w),v)(x,z), \tag{3.3}$$

then

$$\left(D_r^h G(\bar{y},\bar{z},v,w) \circ D_r^h F(\bar{x},\bar{y},u,v)\right)(x) \subseteq D_r^h(G \circ_{\bar{y}} F)(\bar{x},\bar{z},u,w)(x). \tag{3.4}$$

Proof. Let z be in the left-hand side of (3.4). Then there exists $y \in D_r^h F(\bar{x},\bar{y},u,v)(x)$ with $y \in D_r^h G^{-1}(\bar{z},\bar{y},w,v)(z)$. In view of (3.3), we have $y \in D_r^h H((\bar{x},\bar{y}),\bar{z},(u,v),w)(x,z)$. Thus there exist sequences $t_k \searrow 0, s_k \searrow 0$ with $t_k s_k^{-1} \to r$ and $(x_k,z_k,y_k) \to (x,z,y)$ such that

$$\bar{y} + t_k v + \frac{1}{2} t_k s_k y_k \in H\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k, \bar{z} + t_k w + \frac{1}{2} t_k s_k z_k\right).$$

It follows from the formula of H that

$$\bar{y} + t_k v + \frac{1}{2} t_k s_k y_k \in F\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k\right), \tag{3.5}$$

$$\bar{z} + t_k w + \frac{1}{2} t_k s_k z_k \in G\left(\bar{y} + t_k v + \frac{1}{2} t_k s_k y_k\right). \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\bar{z}+t_kw+\frac{1}{2}t_ks_kz_k \in (G\circ F)\left(\bar{x}+t_ku+\frac{1}{2}t_ks_kx_k\right),$$

which means $z \in (G\circ F)(\bar{x},\bar{z},u,w)$. This together with (3.1) gives that $z \in (G\circ_{\bar{y}} F)(\bar{x},\bar{z},u,w)$. □

Proposition 3.3. *Let* $(\bar{x},\bar{z}) \in \operatorname{gph}(G\circ F)$, $(u,v,w) \in \mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p$, *and* $r \in \overline{\mathbb{R}}_+$. *If the following condition holds*

$$D_r^h H((\bar{x},\bar{z}),\bar{y},(u,w),v)(0,0) = \{0\}, \tag{3.7}$$

then

$$D_r^h(G\circ_{\bar{y}} F)(\bar{x},\bar{z},u,w)(x) \subseteq \left(D_r^h G(\bar{y},\bar{z},v,w)\circ D_r^h F(\bar{x},\bar{y},u,v)\right)(x).$$

Proof. Let $z \in D_r^h(G\circ_{\bar{y}} F)(\bar{x},\bar{z},u,w)(x)$. Then there exist sequences $t_k \searrow 0, s_k \searrow 0$ with $t_ks_k^{-1} \to r$ and $(x_k,\bar{y}_k,z_k) \to (x,\bar{y},z)$ such that

$$\bar{y}_k \in H\left(\bar{x}+t_ku+\frac{1}{2}t_ks_kx_k,\bar{z}+t_kw+\frac{1}{2}t_ks_kz_k\right).$$

There are two cases to be considered.

Case 1. If $\bar{y}_k = \bar{y}$ for infinitely many times k, then

$$0 \in D_r^h F(\bar{x},\bar{y},u,v)(x) \text{ and } z \in D_r^h G(\bar{y},\bar{z},v,w)(0).$$

Case 2. If $\bar{y}_k \neq \bar{y}$ for all k, we denote $q_k := \dfrac{\|\bar{y}_k-\bar{y}-t_kv\|}{t_k}$ and $\widehat{q}_k := \dfrac{\bar{y}_k-\bar{y}-t_kv}{t_kq_k}$. We see that $\widehat{q}_k \to \widehat{q} \in \mathbb{R}^m$ with $\|\widehat{q}\| = 1$. Moreover, we have

$$\bar{y}+t_kv+\frac{1}{2}t_kq_k\widehat{q}_k = \bar{y}_k \in H\left(\bar{x}+t_ku+\frac{1}{2}t_ks_kx_k,\bar{z}+t_kw+\frac{1}{2}t_ks_kz_k\right). \tag{3.8}$$

Suppose that $\dfrac{s_k}{q_k} \to 0$. It follows from (3.8) that

$$\bar{y}+t_kv+\frac{1}{2}t_kq_k\widehat{q}_k = \bar{y}_k \in H\left(\bar{x}+t_ku+\frac{1}{2}t_kq_k\left(\frac{s_k}{q_k}x_k\right),\bar{z}+t_kw+\frac{1}{2}t_kq_k\left(\frac{s_k}{q_k}z_k\right)\right),$$

which means that $\widehat{q} \in D_r^h H((\bar{x},\bar{z}),\bar{y},(u,w),v)(0,0)$, a contradiction to (3.7). Hence, $\frac{q_k}{s_k}$ has a bounded subsequence. Taking a subsequence, one may assume that $\frac{q_k}{s_k}$ has a limit $l \in \mathbb{R}_+$. It follows from (3.8) that

$$\bar{y}+t_kv+\frac{1}{2}t_ks_k\left(\frac{q_k}{s_k}\widehat{q}_k\right) = \bar{y}_k \in H\left(\bar{x}+t_ku+\frac{1}{2}t_ks_kx_k,\bar{z}+t_kw+\frac{1}{2}t_ks_kz_k\right).$$

By the definition of H, we arrive at $l\widehat{q} \in D_r^h F(\bar{x},\bar{y},u,v)(x)$ and $z \in D_r^h G(\bar{y},\bar{z},v,w)(l\widehat{q})$. □

To illustrate Propositions 3.2 and 3.3, we consider the following example.

Example 3.2. Let $F:\mathbb{R}\rightrightarrows\mathbb{R}$ and $G:\mathbb{R}\rightrightarrows\mathbb{R}$ be given by

$$F(x) = \{y\in\mathbb{R} \mid y \geq x^6\} \quad \text{and} \quad G(y) = \{z\in\mathbb{R} \mid z \geq \sqrt[3]{y}\}.$$

Then

$$H(x,z) = F(x) \cap G^{-1}(z) = \begin{cases} \{(x,z) \in \mathbb{R}^2 \mid x^6 \le y \le z^3\}, & \text{if } x \in [0,1] \text{ and } z \in [0,1], \\ \emptyset, & \text{otherwise .} \end{cases}$$

Consider $(\bar{x},\bar{z}) = (0,0)$, $\bar{y} = 0 \in (\mathrm{Cl}\,H)(\bar{x},\bar{z})$, and $(u,v,w) = (1,0,0)$.

- For $r = [0,+\infty)$, we have

$$D_r^h F\,(\bar{x},\bar{y},u,v)\,(0) = \mathbb{R}_+, D_r^h G\,(\bar{y},\bar{z},v,w)\,(0) = \mathbb{R}_+, D_r^h G^{-1}\,(\bar{z},\bar{y},w,v)\,(0) = \mathbb{R}_-,$$
$$D_r^h H\,((\bar{x},\bar{z}),\bar{y},(u,w),v)\,(0,0) = \{0\}, D_r^h(G \circ_{\bar{y}} F)\,(\bar{x},\bar{z},u,w)\,(0) = \mathbb{R}_+.$$

Then we can check that (3.3) and (3.7) hold and

$$\left(D_r^h G\,(\bar{y},\bar{z},v,w) \circ D_r^h F\,(\bar{x},\bar{y},u,v)\right)(0) = D_r^h(G \circ_{\bar{y}} F)\,(\bar{x},\bar{z},u,w)\,(0).$$

- For $r = +\infty$, the above derivatives are empty.

Consequently, the conclusions of Propositions 3.2 and 3.3 are fulfilled.

3.2. **Derivatives of sums.** Now we study the sum rule for two set-valued mappings $P,Q : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. For I being the identity map on $\mathbb{R}^n$ and $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, we denote $F = I \times P$ and $G(x,y) = y + Q(x)$. For $(x,z) \in \mathbb{R}^n \times \mathbb{R}^m$, we set $K(x,z) := P(x) \cap (z - Q(x))$. The resultant set-valued mapping $H : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ associated to F, G is $H(x,z) = \{x\} \times K(x,z)$.

Definition 3.3. Given $((\bar{x},\bar{z}),\bar{y}) \in \mathrm{gph}(\mathrm{Cl}\,K)$, $(u,w) \in \mathbb{R}^n \times \mathbb{R}^p$, and $r \in \overline{\mathbb{R}}_+$, the $\bar{y}$-higher-order tangent derivative of $P+Q$ at $(\bar{x},\bar{z})$ in the direction (u,w) with index r is the set-valued mapping $D_r^h(P +_{\bar{y}} Q)(\bar{x},\bar{z},u,w) \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is

$$D_r^h(P +_{\bar{y}} Q)(\bar{x},\bar{z},u,w)(x) := \Big\{z \in \mathbb{R}^p \mid \exists t_k \searrow 0, \exists s_k \searrow 0, t_k s_k^{-1} \to r,\ \exists (x_k,\bar{y}_k,z_k) \to (x,\bar{y},z) :$$
$$\bar{y}_k \in K\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k, \bar{z} + t_k w + \frac{1}{2} t_k s_k z_k\right), \forall k \in \mathbb{N}\Big\}.$$

We observe that

$$D_r^h\,(P +_{\bar{y}} Q)\,(\bar{x},\bar{z},u,w) = D_r^h\,(G \circ_{\bar{y}} F)\,(\bar{x},\bar{z},u,w).$$

Consequently, this subsection can be presented in an analogous way as the previous one.

Proposition 3.4. *Give $(\bar{x},\bar{z}) \in \mathrm{gph}(P+Q)$, $(u,w) \in \mathbb{R}^n \times \mathbb{R}^m$, and $r \in \overline{\mathbb{R}}_+$. Suppose that K is directionally semi-compact at $(\bar{x},\bar{z})$. Then,*

$$D_r^h(P+Q)(\bar{x},\bar{z},u,w)(x) = \bigcup_{\bar{y} \in (\mathrm{Cl}\,K)(\bar{x},\bar{z})} D_r^h(P +_{\bar{y}} Q)(\bar{x},\bar{z},u,w)(x). \tag{3.9}$$

Proof. Let $\bar{y} \in (\mathrm{Cl}\,K)(\bar{x},\bar{z})$ and $z \in D_r^h(P +_{\bar{y}} Q)(\bar{x},\bar{z},u,w)(x)$. There exist sequences $t_k \searrow 0, s_k \searrow 0$ with $t_k s_k^{-1} \to r$ and $(x_k,\bar{y}_k,z_k) \to (x,\bar{y},z)$ such that

$$\bar{y}_k \in K\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k, \bar{z} + t_k w + \frac{1}{2} t_k s_k z_k\right).$$

By the definition of K, we have

$$\bar{z}_k := \bar{z} + t_k w + \frac{1}{2} t_k s_k z_k = \bar{y}_k + (\bar{z}_k - \bar{y}_k) \in (P+Q)\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k\right).$$

Consequently, $z \in D_r^h(P+Q)(\bar{x},\bar{z},u,w)(x)$.

For the converse, taking $z \in D_r^h(P+Q)(\bar{x},\bar{z},u,w)(x)$, we see that there exist sequences $t_k \searrow 0, s_k \searrow 0$ with $t_k s_k^{-1} \to r$ and $(x_k, z_k) \to (x,z)$ such that $\bar{z}_k \in (P+Q)\left(\bar{x}+t_k u+\frac{1}{2}t_k s_k x_k\right)$. Hence, there exists $\bar{y}_k \in P\left(\bar{x}+t_k u+\frac{1}{2}t_k s_k x_k\right)$ satisfying $\bar{z}_k - \bar{y}_k \in Q\left(\bar{x}+t_k u+\frac{1}{2}t_k s_k x_k\right)$ for all $k \in \mathbb{N}$. This implies that

$$\bar{y}_k \in P\left(\bar{x}+t_k u+\frac{1}{2}t_k s_k x_k\right) \bigcap \left(\bar{z}_k - Q\left(\bar{x}+t_k u+\frac{1}{2}t_k s_k x_k\right)\right) = K\left(\bar{x}+t_k u+\frac{1}{2}t_k s_k x_k, \bar{z}_k\right). \tag{3.10}$$

By the compactness of K, we can assume that $\{\bar{y}_k\}$ converges to some $\bar{y}$. On the one hand, we see that $\left(\bar{x}+t_k u+\frac{1}{2}t_k s_k x_k, \bar{z}_k, \bar{y}_k\right) \to (\bar{x},\bar{z},\bar{y})$, and hence $\bar{y} \in (\mathrm{Cl}\,K)(\bar{x},\bar{z})$. This together with (3.10) asserts that

$$z \in D_r^h(P+_{\bar{y}}Q)(\bar{x},\bar{z},u,w)(x) \subseteq \bigcup_{\bar{y}\in(\mathrm{Cl}\,K)(\bar{x},\bar{z})} D_r^h(P+_{\bar{y}}Q)(\bar{x},\bar{z},u,w)(x).$$

□

Example 3.3. Let $P, Q : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by

$$P(x) = \begin{cases} [0,1], & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0, \end{cases} \quad Q(x) = \begin{cases} \{0\}, & \text{if } x \neq 0, \\ [0,1], & \text{if } x = 0. \end{cases}$$

Then, $(P+Q)(x) = [0,1]$ for all $x \in \mathbb{R}$ and

$$K(x,z) = P(x) \cap (z - Q(x)) = \begin{cases} \{z\}, & \text{if } x \neq 0 \text{ and } z \in [0,1], \\ \{0\}, & \text{if } x = 0 \text{ and } z \in [0,1], \\ \emptyset, & \text{otherwise.} \end{cases}$$

Consider $(\bar{x},\bar{z}) = (0,1)$ and $(u,w) = (1,0)$. Then, $(\mathrm{Cl}\,K)(\bar{x},\bar{z}) = \{0\}$ and hence $\bar{y} = 0$. By calculating, we have

$$D_r^h(P+Q)(\bar{x},\bar{z},u,w)(x) = D_r^h(P+_{\bar{y}}Q)(\bar{x},\bar{z},u,w)(x) = \mathbb{R}_-.$$

Thus we have that the relation (3.9) in Proposition 3.4 holds.

Proposition 3.5. *Give* $((\bar{x},\bar{z}),\bar{y}) \in \operatorname{gph} K, (u,v,w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$, *and* $r \in \overline{\mathbb{R}}_+$. *If for all* $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, *the following condition is satisfied*

$$D_r^h P(\bar{x},\bar{y},u,v)(x) \cap \left(y - D_r^h Q(\bar{x},\bar{z}-\bar{y},u,w-v)(x)\right) \subseteq D_r^h K((\bar{x},\bar{z}),\bar{y},(u,w),v)(x,y), \tag{3.11}$$

then

$$D_r^h P(\bar{x},\bar{y},u,v)(x) + D_r^h Q(\bar{x},\bar{z}-\bar{y},u,w-v)(x) \subseteq D_r^h(P+_{\bar{y}}Q)(\bar{x},\bar{z},u,w)(x). \tag{3.12}$$

Proof. Let z be in the left-hand side of (3.12). Then there exists $z' \in D_r^h P(\bar{x},\bar{y},u,v)(x)$ such that

$$z' \in z - D_r^h Q(\bar{x},\bar{z}-\bar{y},u,w-v)(x).$$

From condition (3.11), $z' \in D_r^h K((\bar{x},\bar{z}),\bar{y},(u,w),v)(x,z)$. Then there exist sequences $t_k \searrow 0, s_k \searrow 0$ with $t_k s_k^{-1} \to r$ and $(x_k, z_k, z'_k) \to (x,z,z')$ satisfying

$$\bar{y} + t_k v + \frac{1}{2} t_k s_k z'_k \in K\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k, \bar{z} + t_k w + \frac{1}{2} t_k s_k z_k\right).$$

Therefore, we arrive at $z \in D_r^h(P +_{\bar{y}} Q)(\bar{x},\bar{z},u,w)(x)$. □

Proposition 3.6. *Give $((\bar{x},\bar{z}),\bar{y}) \in \operatorname{gph} K$, $(u,v,w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$, and $r \in \overline{\mathbb{R}}_+$. If the following condition is fulfilled*

$$D_r^h K((\bar{x},\bar{z}),\bar{y},(u,w),v)(0,0) = \{0\}, \tag{3.13}$$

then

$$D_r^h(P +_{\bar{y}} Q)(\bar{x},\bar{z},u,w)(x) \subseteq D_r^h P(\bar{x},\bar{y},u,v)(x) + D_r^h Q(\bar{x},\bar{z}-\bar{y},u,w-v)(x). \tag{3.14}$$

Proof. Let $z \in D_r^h(P +_{\bar{y}} Q)(\bar{x},\bar{z},u,w)(x)$. Then there exist sequences $t_k \searrow 0, s_k \searrow 0$ with $t_k s_k^{-1} \to r$ and $(x_k, \bar{y}_k, z_k) \to (x, \bar{y}, z)$ such that

$$\bar{y}_k \in K\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k, \bar{z} + t_k w + \frac{1}{2} t_k s_k z_k\right).$$

It suffices to investigate two following cases.

Case 1. If $\bar{y}_k = \bar{y}$ for infinitely many times k, then

$$0 \in D_r^h P(\bar{x},\bar{y},u,v)(x) \quad \text{and} \quad z \in D_r^h Q(\bar{x},\bar{z}-\bar{y},u,w-v)(0).$$

Case 2. If $\bar{y}_k \neq \bar{y}$ for all k, we denote $q_k := \dfrac{\|\bar{y}_k - \bar{y} - t_k v\|}{t_k}$ and $\widehat{q}_k := \dfrac{\bar{y}_k - \bar{y} - t_k v}{t_k q_k}$. Thus $\widehat{q}_k$ has a subsequence converging to some $\widehat{q} \in \mathbb{R}^m$ with $\|\widehat{q}\| = 1$. Moreover, we see that

$$\bar{y} + t_k v + \frac{1}{2} t_k q_k \widehat{q}_k = \bar{y}_k \in K\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k, \bar{z} + t_k w + \frac{1}{2} t_k s_k z_k\right). \tag{3.15}$$

If $\dfrac{s_k}{q_k} \to 0$, it follows from (3.15) that

$$\bar{y} + t_k v + \frac{1}{2} t_k q_k \widehat{q}_k = \bar{y}_k \in K\left(\bar{x} + t_k u + \frac{1}{2} t_k q_k \left(\frac{s_k}{q_k} x_k\right), \bar{z} + t_k w + \frac{1}{2} t_k q_k \left(\frac{s_k}{q_k} z_k\right)\right).$$

Hence, $\widehat{q} \in D_r^h K((\bar{x},\bar{z}),\bar{y},(u,w),v)(0,0)$, a contradiction to (3.13). Thus $\frac{q_k}{s_k}$ has a bounded subsequence. Taking a subsequence, $\frac{q_k}{s_k}$ has a limit $l \in \mathbb{R}_+$. From (3.15), we have

$$\bar{y} + t_k v + \frac{1}{2} t_k s_k \left(\frac{q_k}{s_k} \widehat{q}_k\right) = \bar{y}_k \in K\left(\bar{x} + t_k u + \frac{1}{2} t_k s_k x_k, \bar{z} + t_k w + \frac{1}{2} t_k s_k z_k\right).$$

By definition of K, we obtain

$$l\widehat{q} \in D_r^h P(\bar{x},\bar{y},u,v)(x) \quad \text{and} \quad z - l\widehat{q} \in D_r^h Q(\bar{x},\bar{z}-\bar{y},u,w-v)(x).$$

Therefore, relation (3.14) are proved. □

Example 3.4. Let $P, Q : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined by

$$P(x) = \left\{(y_1,y_2) \in \mathbb{R}^2 \mid y_1 \geq x^2, y_2 = 0\right\} \quad \text{and} \quad Q(x) = \{(y_1,y_2) \in \mathbb{R}^2 \mid y_1 \geq x^4, y_2 = 0\}.$$

After some directed computations, we have

$$K(x,z)=\begin{cases}\{(y_1,y_2)\in\mathbb{R}^2 \mid x^2\le y_1\le z_1-x^4, y_2=0\}\\ \text{ if } (x,z)\in\{(x,z)\in\mathbb{R}\times\mathbb{R}^2 \mid z_1-x^4\ge x^2, z_2=0\},\\ \emptyset, \text{ if } \text{ otherwise.}\end{cases}$$

Consider $(\bar{x},\bar{z})=(0,(0,0))$ and $(u,v,w)=(1,(0,0),(0,0))$. Then, $(\mathrm{Cl}\,K)(\bar{x},\bar{z})=\{(0,0)\}$. Directed calculations yield that the following higher-order tangent derivatives.

• If $r=0$. Then, for all $x\in\mathbb{R}$ and $z_1\in\mathbb{R}_+$,

$$D_0^h K((\bar{x},\bar{z}),\bar{y},(u,w),v)(x,z)=\left\{(y_1,y_2)\in\mathbb{R}^2 \mid 0\le y_1\le z_1, y_2=0\right\},$$

$$D_0^h(P+_{\bar{y}}Q)(\bar{x},\bar{z},u,w)(x)=\left\{(h_1,h_2)\in\mathbb{R}^2 \mid h_1\ge 0, h_2=0\right\},$$

and

$$D_0^h P(\bar{x},\bar{y},u,v)(x)=D_0^h Q(\bar{x},\bar{z}-\bar{y},u,w-v)(x)=\left\{(y_1,y_2)\in\mathbb{R}^2 \mid y_1\ge 0, y_2=0\right\}.$$

Letting $(x,z)=(0,(0,0))$, we have $D_0^h K((\bar{x},\bar{z}),\bar{y},(u,w),v)(0,(0,0))=\{(0,0)\}$ and

$$D_0^h P(\bar{x},\bar{y},u,v)(0)\cap\Big((0,0)-D_0^h Q(\bar{x},\bar{z}-\bar{y},u,w-v)(0)\Big)\subseteq D_0^h K((\bar{x},\bar{z}),\bar{y},(u,w),v)(0,(0,0)).$$

Hence, we have

$$D_0^h P(\bar{x},\bar{y},u,v)(0)+D_0^h Q(\bar{x},\bar{z}-\bar{y},u,w-v)(0)=D_0^h(P+_{\bar{y}}Q)(\bar{x},\bar{z},u,w)(0).$$

Thus, the conclusion of Proposition 3.6 holds.

• If $r=(0,+\infty)$. Then, for all $x\in\mathbb{R}$ and $z_1\in\mathbb{R}_+$,

$$\begin{aligned}D_r^h K((\bar{x},\bar{z}),\bar{y},(u,w),v)(x,z)&=\left\{(y_1,y_2)\in\mathbb{R}^2 \mid 2r\le y_1\le z_1, y_2=0\right\},\\ D_r^h(P+_{\bar{y}}Q)(\bar{x},\bar{z},u,w)(x)&=\left\{(h_1,h_2)\in\mathbb{R}^2 \mid h_1\ge 0, h_2=0\right\},\\ D_r^h P(\bar{x},\bar{y},u,v)(x)&=\left\{(y_1,y_2)\in\mathbb{R}^2 \mid y_1\ge 2r, y_2=0\right\},\\ D_r^h Q(\bar{x},\bar{z}-\bar{y},u,w-v)(x)&=\left\{(y_1,y_2)\in\mathbb{R}^2 \mid y_1\ge 0, y_2=0\right\}.\end{aligned}$$

It is easy to check that (3.11) and (3.12) are satisfied. However, (3.14) is not fulfilled. The reason is that condition (3.13) does not hold since $D_r^h K((\bar{x},\bar{z}),\bar{y},(u,w),v)(0,(0,0))=\emptyset$.

Later, we directly apply Propositions 3.5 and 3.6 to see an estimation of the higher-order tangent derivative for the profile map, G_-.

Corollary 3.1. *Let $G:\mathbb{R}^n\rightrightarrows\mathbb{R}^m, K(x,z):=(-C)\cap(z-G(x)), ((\bar{x},\bar{z}),\bar{y})\in\mathrm{gph}\,K$, $(u,v,w)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m$, and $r\in\overline{\mathbb{R}}_+$. Then,*

$$D_r^h G(\bar{x},\bar{z},u,w)(x)-C\subseteq D_r^h(G-_{\bar{y}}C)(\bar{x},\bar{z},u,w)(x).$$

This inclusion becomes an equality if $D_r^h K((\bar{x},\bar{z}),\bar{y},(u,w),v)(0,0)=\{0\}$.

3.3. **Higher order tangent derivatives of the profile mapping.** In this subsection, we always assume that C, the ordering cone, has a compact base Q.

Proposition 3.7. *Let* $(\bar{x},\bar{y}) \in \operatorname{gph} G$, $(\bar{u},\bar{v}) \in \mathbb{R}^n \times \mathbb{R}^m$ *and* $r \in \overline{\mathbb{R}}_+$. *For any* $x \in \operatorname{dom}\left(D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})\right)$,

$$\operatorname{Max}\left(D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x), C\right) \subseteq D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x). \tag{3.16}$$

Proof. Let $x \in \operatorname{dom}\left(D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})\right)$. If $\operatorname{Max}\left(D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x), C\right) = \emptyset$, then (3.16) is trivial. Take any $y \in \operatorname{Max}\left(D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x), C\right)$. Using the definition, we find $y \in D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x)$ such that

$$\left(D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x) - y\right) \cap (C \backslash \{0\}) = \emptyset. \tag{3.17}$$

As $y \in D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x)$, there exists sequences $t_k \searrow 0, s_k \searrow 0$ with $t_k s_k^{-1} \to r$, $(x_k, y_k) \to (x,y)$ and $c_k \in C$ satisfying

$$\bar{y} + t_k \bar{v} + \frac{1}{2} t_k s_k \left(y_k + \frac{2c_k}{t_k s_k}\right) \in G\left(\bar{x} + t_k \bar{u} + \frac{1}{2} t_k s_k x_k\right). \tag{3.18}$$

We first prove that $\dfrac{c_k}{t_k s_k} \to 0$ (using a subsequence if necessary). As the base Q of C is compact, there are $\alpha_k \geq 0$ and $q_k \in Q : q_k \to q$ (for some $q \in Q \backslash \{0\}$) such that $c_k = \alpha_k q_k$.

- If $\alpha_k = 0$ for infinitely many times $k \in \mathbb{N}$, it is obvious that $\dfrac{c_k}{t_k s_k} \to 0$.
- If $\alpha_k > 0$ for all k, then $\dfrac{c_k}{t_k s_k} = \dfrac{\alpha_k q_k}{t_k s_k} \to 0$ if and only if $\dfrac{\alpha_k}{t_k s_k} \to 0$. Suppose that $\frac{c_k}{t_k s_k}$ does not

converge to 0. Then, for some $\varepsilon > 0$, we may assume that $\dfrac{\alpha_k}{t_k s_k} \geq \varepsilon$, for all n. Set $\hat{c}_k = \varepsilon \dfrac{t_k s_k c_k}{\alpha_k}$. Then $c_k \in \hat{c}_k + C$. Combining this and (3.18), we have

$$\bar{y} + t_k \bar{v} + \frac{1}{2} t_k s_k \left(y_k + \frac{2\hat{c}_k}{t_k s_k}\right) \in G\left(\bar{x} + t_k \bar{u} + \frac{1}{2} t_k s_k x_k\right) - C.$$

One has $\dfrac{2\hat{c}_k}{t_k s_k} \to 2\varepsilon q \neq 0$. Then, $y_k + \dfrac{2\hat{c}_k}{t_k s_k} \to y + 2\varepsilon q$, i.e., $y + 2\varepsilon q \in D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x)$. Therefore, we obtain

$$2\varepsilon q \in \left(D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x) - y\right) \cap C,$$

which contradicts (3.17). Hence, $\dfrac{c_k}{t_k s_k} \to 0$. By (3.18), we arrive at $y \in D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$. □

Example 3.5. Let $C = \mathbb{R}_+$ and $G : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by $G(x) = \left\{y \in \mathbb{R} \mid y = x^2\right\}$. Consider $(\bar{x},\bar{y}) = (0,0)$ and $(\bar{u},\bar{v}) = (1,0)$. Directed computations yield that, for $x \in \mathbb{R}$,

$$D_r^h G\left(\bar{x},\bar{y},\bar{u},\bar{v}\right)(x) = \begin{cases} \{2r\}, & \text{if } r = [0,+\infty), \\ \emptyset, & \text{if } r = +\infty, \end{cases}$$

$$D_r^h G_-\left(\bar{x},\bar{y},\bar{u},\bar{v}\right)(x) = \begin{cases} (-\infty, 2r], & \text{if } r = [0,+\infty), \\ \emptyset, & \text{if } r = +\infty. \end{cases}$$

Hence, for all $x \in \mathbb{R}$, we have $\operatorname{Max}\left(D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x), C\right) \subseteq D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$.

Definition 3.4. (see [12]) Let $(\bar{x},\bar{y}) \in \operatorname{gph} G$, $(\bar{u},\bar{v}) \in \mathbb{R}^n \times \mathbb{R}^m$, and $r \in \overline{\mathbb{R}}_+$. G is x-directionally calm at $(\bar{x},\bar{y})$ in the direction $(\bar{u},\bar{v})$ with index r if there exist $c>0$ and $\rho>0$ such that, for all $t,s \in (0,\rho) : ts^{-1} \in \mathbb{B}^1(r,\rho)$, $x' \in \mathbb{B}^n(x,\rho)$,

$$G\left(\bar{x}+t\bar{u}+\frac{1}{2}tsx'\right) \subseteq \{\bar{y}+t\bar{v}\}+\frac{1}{2}cts\left\|x'\right\|\mathbb{B}^m.$$

The following lemma is necessary in our study.

Lemma 3.1. *([26]) Let $C \subset \mathbb{R}^m$ be a convex regular cone. Let $\Omega \in \mathbb{R}^m$ be a closed and C-upper bounded. Then,* $\operatorname{Max}(\Omega,C) \neq \emptyset$ *and* $\Omega \subseteq \operatorname{Max}(\Omega,C)-C$.

Proposition 3.8. *Let $(\bar{x},\bar{y}) \in \operatorname{gph} G$, $(\bar{u},\bar{v}) \in \mathbb{R}^n \times \mathbb{R}^m$, and $r \in \overline{\mathbb{R}}_+$. Suppose that G is x-directionally calm at $(\bar{x},\bar{y})$ in the direction $(\bar{u},\bar{v})$ with index r and the set $D^h_r G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x)$ has a C-upper bound for every $x \in \operatorname{dom}\left(D^h_r G_-(\bar{x},\bar{y},\bar{u},\bar{v})\right)$. Then,*

$$D^h_r G(\bar{x},\bar{y},\bar{u},\bar{v})(x)-C = D^h_r G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x), \tag{3.19}$$

$$\operatorname{Max}\left(D^h_r G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x),C\right)-C = D^h_r G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x). \tag{3.20}$$

Proof. Let $y \in D^h_r G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$ and $c \in C$ be arbitrarily chosen. Then, the there exist sequences $t_k \searrow 0, s_k \searrow 0, t_k s_k^{-1} \to r$, $(x_k,y_k) \to (x,y)$ such that

$$\bar{y}+t_k\bar{v}+\frac{1}{2}t_ks_ky_k \in G\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right).$$

By setting $\bar{y}_k := y_k - c$, we have that

$$\bar{y}+t_k\bar{v}+\frac{1}{2}t_ks_k\bar{y}_k = \bar{y}+t_k\bar{v}+\frac{1}{2}t_ks_ky_k-\frac{1}{2}t_ks_kc \in G\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right)-C.$$

As $\bar{y}_k$ converges to $y-c$, we deduce that $y-c \in D^h_r G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x)$.

Conversely, let $y \in D^h_r G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x)$. Then there exist sequences $t_k \searrow 0, s_k \searrow 0, t_k s_k^{-1} \to r$, $(x_k,y_k) \to (x,y)$, and $c_k \in C$, satisfying

$$\bar{y}+t_k\bar{v}+\frac{1}{2}t_ks_k\left(y_k+\frac{2c_k}{t_ks_k}\right) \in G\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right). \tag{3.21}$$

Then, by the calmness of G, (3.21) gives us, for k large enough,

$$\bar{y}+t_k\bar{v}+\frac{1}{2}t_ks_k\left(y_k+\frac{2c_k}{t_ks_k}\right) \in \{\bar{y}+t_k\bar{v}\}+\frac{1}{2}ct_ks_k\left\|x_k\right\|\mathbb{B}^m,$$

which means that $\left\|y_k+\dfrac{2c_k}{t_ks_k}\right\| \leq c\left\|x_k\right\|$. In view of the finite dimension of the space, we have $x_k \to x$ and $y_k+\dfrac{2c_k}{t_ks_k} \to \hat{y}$. It follows that $\hat{y} \in D^h_r G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$ and $\dfrac{2c_k}{t_ks_k} \to \hat{y}-y \in C$. Thus $y \in D^h_r G(\bar{x},\bar{y},\bar{u},\bar{v})(x)-C$. By using (3.16), (3.19), and Lemma 3.1, the desired result is obtained. □

Proposition 3.9. *Let $(\bar{x},\bar{y}) \in \operatorname{gph} G$, $(\bar{u},\bar{v}) \in \mathbb{R}^n \times \mathbb{R}^m$ and $r \in \overline{\mathbb{R}}_+$. Suppose that $D^h_r G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x)$ has a C-upper bound for all $x \in \operatorname{dom}\left(D^h_r G_-(\bar{x},\bar{y},\bar{u},\bar{v})\right)$. Then,*

$$\operatorname{Max}\left(D^h_r G(\bar{x},\bar{y},\bar{u},\bar{v})(x),C\right) = \operatorname{Max}\left(D^h_r G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x),C\right)$$

and

$$\mathrm{WMax}\left(D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x),C\right)=\mathrm{WMax}\left(D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x),C\right).$$

Proof. In view of Proposition 3.8, we have $D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x)-C=D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x)$. Consequently, for any $x\in\mathrm{dom}\left(D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})\right)$, we obtain

$$\mathrm{Max}\left(D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x),C\right)=\mathrm{Max}\left(D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x),C\right)$$

and

$$\mathrm{WMax}\left(D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x),C\right)=\mathrm{WMax}\left(D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x),C\right).$$

□

The C-upper bound property in Proposition 3.9 plays a key role as seen in the next example.

Example 3.6. Let $C=\mathbb{R}^2_+$ and $G:\mathbb{R}\rightrightarrows\mathbb{R}^2$ be defined by

$$G(x)=\begin{cases}\{(0,0)\}, & \text{if } x\le 0,\\ \left\{\left(x,x^{1/2}\right)\right\}, & \text{if } x>0.\end{cases}$$

Then, we have

$$G_-(x)=\begin{cases}\mathbb{R}^2_-, & \text{if } x\le 0,\\ \{(y_1,y_2)\mid y_1\le x, y_2\le x^{1/2}\}, & \text{if } x>0.\end{cases}$$

Take $(\bar{x},\bar{y})=(0,(0,0))$ and $(\bar{u},\bar{v})=(1,(0,0))$. Then, for any $x\in\mathbb{R}$,

$$D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x)=\{(0,0)\},\quad\text{and}\quad D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x)=\begin{cases}\mathbb{R}^2, & \text{if } r=[0,+\infty),\\ \emptyset, & \text{if } r=+\infty.\end{cases}$$

We can check that Proposition 3.9 do not satisfy in this example, because the $D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x)$ is not C-upper bound for any $x\in\mathbb{R}$.

4. Higher-order Differential Properties of a Class of Set-Valued Maps

In this section, let K be a compact subset of $\mathbb{R}^n$ and $F:\mathbb{R}^n\to L(\mathbb{R}^n,\mathbb{R}^m)$ be twice Fréchet differentiable. We consider a set-valued mapping $G:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ given by

$$G(x):=\left\langle F(x),x-K\right\rangle=\bigcup_{z\in K}\langle F(x),x-z\rangle,\quad\forall x\in K.$$

For $\bar{x}\in\mathbb{R}^n$, we denote $\overline{F}(x):=\langle F(\bar{x}),\bar{x}-x\rangle$. The following theorem is inspired of the work of [19].

Theorem 4.1. *Let* $\bar{y}=\langle F(\bar{x}),\bar{x}-\hat{x}\rangle\in G(\bar{x})$, $(\bar{u},\bar{v})\in T(\mathrm{gph}\,G,(\bar{x},\bar{y}))$, *and* $r\in\mathbb{R}_+$. *Assume that*

$$\lim_{\|z\|\to+\infty}\|\langle F(\bar{x}),z\rangle\|=+\infty. \tag{4.1}$$

Then, for any $x\in\mathrm{dom}\left(D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})\right)$,

$$\begin{aligned}D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x)=\bigcup_{x^*\in T_r^h(K,\hat{x},w)}&[\langle F(\bar{x}),x-x^*\rangle+\langle\nabla F(\bar{x})\bar{u},2r(\bar{u}-w)\rangle]\\&+\left\langle\nabla^2F(\bar{x})(\bar{u},\bar{u}),r(\bar{x}-\hat{x})\right\rangle+\langle\nabla F(\bar{x})x,\bar{x}-\hat{x}\rangle,\end{aligned}$$

where w *satisfies* $\langle F(\bar{x}),w\rangle=-\bar{v}+\langle\nabla F(\bar{x})\bar{u},\bar{x}-\hat{x}\rangle+\langle F(\bar{x}),\bar{u}\rangle$.

Proof. Let $y \in D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$. Then, there exist sequences $t_k \searrow 0, s_k \searrow 0, t_k s_k^{-1} \to r$, and $(x_k,y_k) \to (x,y)$ such that

$$\bar{y} + t_k\bar{v} + \frac{1}{2}t_k s_k y_k \in G\left(\bar{x} + t_k\bar{u} + \frac{1}{2}t_k s_k x_k\right).$$

It follows from the definition that there exist $\hat{x}_k \in K$ such that

$$\bar{y} + t_k\bar{v} + \frac{1}{2}t_k s_k y_k = \left\langle F\left(\bar{x} + t_k\bar{u} + \frac{1}{2}t_k s_k x_k\right), \bar{x} + t_k\bar{u} + \frac{1}{2}t_k s_k x_k - \hat{x}_k\right\rangle. \tag{4.2}$$

Since F is twice differentiable, it follows from the Taylor's polynomial that

$$\begin{aligned} F\left(\bar{x} + t_k\bar{u} + \frac{1}{2}t_k s_k x_k\right) =& F(\bar{x}) + t_k\nabla F(\bar{x})\bar{u} + \frac{1}{2}t_k s_k \nabla F(\bar{x})x_k \\ &+ \frac{1}{2}t_k^2\nabla^2 F(\bar{x})\left(\bar{u} + \frac{1}{2}s_k x_k, \bar{u} + \frac{1}{2}s_k x_k\right) + \circ\left(\left\|t_k\bar{u} + \frac{1}{2}t_k s_k x_k\right\|^2\right). \end{aligned} \tag{4.3}$$

As K is compact, we can assume that $\hat{x}_k \to x' \in K$. By the continuity of F and (4.2), we have $\bar{y} = \langle F(\bar{x}), \bar{x} - x'\rangle = \overline{F}(x')$. By (4.2) and (4.3), we have

$$\begin{aligned} \bar{y} + t_k\bar{v} + \frac{1}{2}t_k s_k y_k =& \langle F(\bar{x}), \bar{x} - \hat{x}_k\rangle + t_k\langle F(\bar{x}), \bar{u}\rangle + t_k\langle \nabla F(\bar{x})\bar{u}, \bar{x} - \hat{x}_k\rangle \\ &+ \frac{1}{2}t_k s_k\langle F(\bar{x}), x_k\rangle + \frac{1}{2}t_k^2\langle \nabla F(\bar{x})\bar{u}, 2\bar{u} + s_k x_k\rangle \\ &+ \frac{1}{2}t_k s_k\left\langle \nabla F(\bar{x})x_k, \bar{x} + t_k\bar{u} + \frac{1}{2}t_k s_k x_k - \hat{x}_k\right\rangle \\ &+ \frac{1}{2}t_k^2\left\langle \nabla^2 F(\bar{x})\left(\bar{u} + \frac{1}{2}s_k x_k, \bar{u} + \frac{1}{2}s_k x_k\right), \bar{x} + t_k\bar{u} + \frac{1}{2}t_k s_k x_k - \hat{x}_k\right\rangle \\ &+ \frac{1}{2}t_k^2\left\langle \frac{\circ\left(\left\|t_k\bar{u} + \frac{1}{2}t_k s_k x_k\right\|^2\right)}{\frac{1}{2}t_k^2}, \bar{x} + t_k\bar{u} + \frac{1}{2}t_k s_k x_k - \hat{x}_k\right\rangle. \end{aligned} \tag{4.4}$$

We obtain from (4.4) that

$$\left\langle F(\bar{x}), \frac{\hat{x}_k - \hat{x}}{t_k}\right\rangle \to -\bar{v} + \langle F(\bar{x}), \bar{u}\rangle + \langle \nabla F(\bar{x})\bar{u}, \bar{x} - \hat{x}\rangle. \tag{4.5}$$

We next show that $\left\{\frac{\hat{x}_k - \hat{x}}{t_k}\right\}$ is bounded. Suppose that $\left\{\frac{\hat{x}_k - \hat{x}}{t_k}\right\} \to +\infty$. By assumption (4.1), we have

$$\left\|\left\langle F(\bar{x}), \frac{\hat{x}_k - \hat{x}}{t_k}\right\rangle\right\| \to +\infty,$$

which is a contradiction to (4.5). Hence, $\left\{\frac{\hat{x}_k - \hat{x}}{t_k}\right\}$ is bounded, and

$$\frac{\hat{x}_k - \hat{x}}{t_k} \to w \in T(K, \hat{x}).$$

As $t_k \searrow 0$, we have $\hat{x}_k \to \hat{x}$ and $x' = \hat{x}$. It follows from (4.5) that

$$\langle F(\bar{x}), w\rangle = -\bar{v} + \langle F(\bar{x}), \bar{u}\rangle + \langle \nabla F(\bar{x})\bar{u}, \bar{x} - \hat{x}\rangle.$$

By (4.4) and (4.5), we see that

$$\left\langle F(\bar{x}), \frac{\frac{\hat{x}_k-\hat{x}}{t_k}-w}{\frac{1}{2}s_k}\right\rangle = -y_k - \frac{t_k}{s_k}\left\langle \nabla F(\bar{x})\bar{u}, \frac{\hat{x}_k-\hat{x}}{\frac{1}{2}t_k}\right\rangle + \langle F(\bar{x}), x_k\rangle$$
$$+\frac{t_k}{s_k}\langle \nabla F(\bar{x})\bar{u}, 2\bar{u}+s_kx_k\rangle + \left\langle \nabla F(\bar{x})x_k, \bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k-\hat{x}_k\right\rangle$$
$$+\frac{t_k}{s_k}\left\langle \nabla^2 F(\bar{x})\left(\bar{u}+\frac{1}{2}s_kx_k, \bar{u}+\frac{1}{2}s_kx_k\right), \bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k-\hat{x}_k\right\rangle$$
$$+\frac{t_k}{s_k}\left\langle \frac{\circ\left(\left\|t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right\|^2\right)}{\frac{1}{2}t_k^2}, \bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k-\hat{x}_k\right\rangle.$$

As $\dfrac{\circ\left(\left\|t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right\|^2\right)}{\frac{1}{2}t_k^2} \to 0$, we have

$$\left\langle F(\bar{x}), \frac{\frac{\hat{x}_k-\hat{x}}{t_k}-w}{\frac{1}{2}s_k}\right\rangle$$
$$\to -y + \langle F(\bar{x}), x\rangle + \langle \nabla F(\bar{x})\bar{u}, 2r(\bar{u}-w)\rangle + \left\langle \nabla F(\bar{x})x + \nabla^2 F(\bar{x})(\bar{u},\bar{u}), r(\bar{x}-\hat{x})\right\rangle.$$

With the same arguments, we see that the sequence $\left\{\frac{\frac{\hat{x}_k-\hat{x}}{t_k}-w}{\frac{1}{2}s_k}\right\}$ is bounded. Hence, we can assume that $\dfrac{\frac{\hat{x}_k-\hat{x}}{t_k}-w}{\frac{1}{2}s_k} \to x^* \in T_r^h(K,\hat{x},w)$. Consequently, we obtain

$$y \in \bigcup_{x^*\in T_r^h(K,\hat{x},w)} [\langle F(\bar{x}), x-x^*\rangle + \langle \nabla F(\bar{x})\bar{u}, 2r(\bar{u}-w)\rangle]$$
$$+\langle \nabla F(\bar{x})x, \bar{x}-\hat{x}\rangle + \left\langle \nabla^2 F(\bar{x})(\bar{u},\bar{u}), r(\bar{x}-\hat{x})\right\rangle.$$

For the converse, we now take $x^* \in T_r^h(K,\hat{x},w)$ and

$$y = \langle F(\bar{x}), x-x^*\rangle + \langle \nabla F(\bar{x})\bar{u}, 2r(\bar{u}-w)\rangle + \langle \nabla F(\bar{x})x, \bar{x}-\hat{x}\rangle + \left\langle \nabla^2 F(\bar{x})(\bar{u},\bar{u}), r(\bar{x}-\hat{x})\right\rangle.$$

Then, there exist sequences $\{\hat{x}_k\} \subset K$ with $\hat{x}_k \to \hat{x}$ and $t_k \searrow 0, s_k \searrow 0$ satisfying

$$\frac{\hat{x}_k-\hat{x}}{t_k} \to w \quad \text{and} \quad \frac{\frac{\hat{x}_k-\hat{x}}{t_k}-w}{\frac{1}{2}s_k} \to x^*.$$

Since F is twice continuously Fréchet differentiable, we can take the sequences $\{x_k\}$ and $\{y_k\}$ such that $x_k \to x$ and

$$y_k = \left\langle F\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right), x_k - \frac{\frac{\hat{x}_k-\hat{x}}{t_k} - w}{\frac{1}{2}s_k}\right\rangle + \langle \nabla F(\bar{x})x_k, \bar{x}-\hat{x}\rangle$$
$$+ \left\langle \frac{F\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right) - F(\bar{x})}{s_k}, 2(\bar{u}-w)\right\rangle$$
$$+ \left\langle \frac{F\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right) - F(\bar{x}) - \nabla F(\bar{x})\left(t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right)}{\frac{1}{2}t_ks_k}, \bar{x}-\hat{x}\right\rangle.$$

Hence, we see that $y_k \to y$ and

$$\bar{y}+t_k(\langle \nabla F(\bar{x})\bar{u}, \bar{x}-\hat{x}\rangle + \langle F(\bar{x}), \bar{u}-w\rangle) + \frac{1}{2}t_ks_ky_k$$
$$= \left\langle F\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right), \bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k-\hat{x}_k\right\rangle \in G\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right).$$

As $\bar{v} = \langle \nabla F(\bar{x})\bar{u}, \bar{x}-\hat{x}\rangle + \langle F(\bar{x}), \bar{u}-w\rangle$, we obtain $y \in D^h_r G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$. This completes the proof. ☐

Remark 4.1. (i) For $r=0$, Theorem 4.1 gives that, for any $x \in \operatorname{dom}\left(D^h_0 G(\bar{x},\bar{y},\bar{u},\bar{v})\right)$,

$$D^h_0 G(\bar{x},\bar{y},\bar{u},\bar{v})(x) = \bigcup_{x^*\in T^h_0(K,\hat{x},w)} \langle F(\bar{x}), x-x^*\rangle + \langle \nabla F(\bar{x})x, \bar{x}-\hat{x}\rangle,$$

where w satisfies $\langle F(\bar{x}), w\rangle = -\bar{v} + \langle \nabla F(\bar{x})\bar{u}, \bar{x}-\hat{x}\rangle + \langle F(\bar{x}), \bar{u}\rangle$. This formula is new and gives the behavior of the asymptotic tangent derivative of the map G.

(ii) For $r=1$, the second-order contingent derivative $D^h_1 G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$ was studied in [19]. Theorem 4.1 extends and generalizes the corresponding results in [19].

We next give an upper estimation for $D^h_{+\infty}G$ as follows.

Theorem 4.2. *Suppose that* $\nabla F(\bar{x}) = 0$ *and* $\nabla^2 F(\bar{x}) = 0$. *Then, for any* $x \in \operatorname{dom}\left(D^h_{+\infty}G(\bar{x},\bar{y},\bar{u},\bar{v})\right)$,

$$D^h_{+\infty}G(\bar{x},\bar{y},\bar{u},\bar{v})(x) \subseteq \bigcup_{x^*\in T^h_{+\infty}(K,\hat{x},w)} \langle F(\bar{x}), x-x^*\rangle + \mathbb{R}^m_+,$$

where w *satisfies* $\langle F(\bar{x}), w\rangle = -\bar{v} + \langle F(\bar{x}), \bar{u}\rangle$.

Proof. Pick any $y \in D^h_{+\infty}G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$. With the same arguments in proof of Theorem 4.1 and assumptions $\nabla F(\bar{x}) = 0$ and $\nabla^2 F(\bar{x}) = 0$, (4.4) is rewrite as

$$\bar{y}+t_k\bar{v}+\frac{1}{2}t_ks_ky_k = \langle F(\bar{x}), \bar{x}-\hat{x}_k\rangle + t_k\langle F(\bar{x}), \bar{u}\rangle + \frac{1}{2}t_ks_k\langle F(\bar{x}), x_k\rangle$$
$$+ \frac{1}{2}t_k^2\left\langle \frac{\circ\left(\left\|t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right\|^2\right)}{\frac{1}{2}t_k^2}, \bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k-\hat{x}_k\right\rangle. \tag{4.6}$$

It follows from (4.6) that

$$\left\langle F(\bar{x}), \frac{\hat{x}_k-\hat{x}}{t_k}\right\rangle \to -\bar{v} + \langle F(\bar{x}), \bar{u}\rangle$$

and

$$\left\langle F(\bar{x}), \frac{\frac{\hat{x}_k-\hat{x}}{t_k}-w}{\frac{1}{2}s_k} \right\rangle \to -y + \langle F(\bar{x}), x\rangle + c,$$

for some $c \in \mathbb{R}^m_+$. Consequently, we obtain the desired result immediately. □

Example 4.1. Let $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x-1$, $K = \{x \in \mathbb{R} \mid -1 \le x \le 1\}$. Consider $(\bar{x}, \bar{y}) = (0,1)$, $(\bar{u}, \bar{v}) = (1,-2)$, and $r \in \mathbb{R}_+$. We have

$$G(x) = \bigcup_{z\in K} \langle F(x), x-z\rangle = \bigcup_{z\in K} \langle x-1, x-z\rangle = \left[x^2-1, (x-1)^2\right],$$

and condition (4.1) holds. After some calculations, for $x \in \operatorname{dom}\left(D^h_r G(\bar{x},\bar{y},\bar{u},\bar{v})\right) = \mathbb{R}$,

$$D^h_r G(\bar{x},\bar{y},\bar{u},\bar{v})(x) = \{y \in \mathbb{R} \mid y \le -2x+2r\}. \tag{4.7}$$

On the other hand, $\overline{F}(x) = \langle F(\bar{x}), \bar{x}-x\rangle = x$ and $\hat{x} = 1$ satisfies $\langle F(\bar{x}), \bar{x}-\hat{x}\rangle = \bar{y}$. Then, we see that $w = 0$, $T^h_r(K,\hat{x},w) = \mathbb{R}_-$, $\langle F(\bar{x}), x-x^*\rangle = -x+x^*$, and

$$\langle \nabla F(\bar{x})\bar{u}, 2r(\bar{u}-w)\rangle = 2r,\ \langle \nabla F(\bar{x})x, \bar{x}-\hat{x}\rangle = -x,$$
$$\left\langle \nabla^2 F(\bar{x})(\bar{u},\bar{u}), r(\bar{x}-\hat{x})\right\rangle = 0.$$

Consequently, for any $x \in \mathbb{R}$,

$$\begin{aligned}\bigcup_{x^*\in T^h_r(K,\hat{x},w)} &[\langle F(\bar{x}), x-x^*\rangle + \langle \nabla F(\bar{x})\bar{u}, 2r(\bar{u}-w)\rangle] \\ &+ \langle \nabla F(\bar{x})x, \bar{x}-\hat{x}\rangle + \left\langle \nabla^2 F(\bar{x})(\bar{u},\bar{u}), r(\bar{x}-\hat{x})\right\rangle = \{y \in \mathbb{R} \mid y \le -2x+2r\}.\end{aligned} \tag{4.8}$$

By (4.7) and (4.8), we see that the conclusion of Theorem 4.1 is fulfilled.

The next example shows that the inclusion in Theorem 4.2 could be strict.

Example 4.2. Let $K = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$, $(\bar{x},\bar{y}) = (0,0)$, $(\bar{u},\bar{v}) = (1,0)$, and $r = +\infty$. Consider a map $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x^3$. We can easily check that, for any $x \in K$,

$$G(x) = \bigcup_{z\in K} \langle F(x), x-z\rangle = \bigcup_{z\in K} \langle x^3, x-z\rangle = \left[x^3(x-1), x^4\right].$$

By direct calculations, we see that, for all $x \in \mathbb{R}$,

$$D^h_{+\infty} G(\bar{x},\bar{y},\bar{u},\bar{v})(x) = \{0\} \subsetneq \bigcup_{x^*\in T^h_{+\infty}(K,\hat{x},w)} \langle F(\bar{x}), x-x^*\rangle + \mathbb{R}_+ = \mathbb{R}_+.$$

5. Differential Sensitivity of Weak Vector Variational Inequalities

In this section, we consider a weak vector variational inequality (WVVI), which is of

$$\text{finding } \hat{x} \in K \text{ such that } \langle F(\hat{x}), x-\hat{x}\rangle \notin -\operatorname{int} C, \quad \forall x \in K,$$

where F and K are mentioned in Section 4 and C is a closed convex cone with nonempty interior. We first recall the gap function for (WVVI), as follows.

Definition 5.1. (see [5]) A set-valued map $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be a gap function of (WVVI) if the following conditions hold

(i) $0_{\mathbb{R}^m} \in W(\bar{x})$ iff $\bar{x}$ solves the WVVI;
(ii) $W(x) \cap (-\operatorname{int} C) = \emptyset, \forall x \in K$.

The following theorem is useful in our analysis.

Theorem 5.1. (see [5]) *The set-valued map $W:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$, defined by $W(x) := \mathrm{WMax}\,(G(x), \mathrm{int}\,C)$, is a gap function for (WVVI).*

We first present the relationship between higher-order tangent derivatives $D_r^h W$ and $D_r^h G$.

Theorem 5.2. *Let $\bar{x}\in K, \bar{y}\in G(\bar{x}), (\bar{u},\bar{v})\in T(\mathrm{gph}\,G,(\bar{x},\bar{y}))$, and $r\in\overline{\mathbb{R}}_+$. Suppose that*

$$D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x) = D_r^{hi} G(\bar{x},\bar{y},\bar{u},\bar{v})(x), \quad \forall x\in \mathrm{dom}\left(D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})\right). \tag{5.1}$$

Then, for any $x\in\mathrm{dom}\left(D_r^h W(\bar{x},\bar{y},\bar{u},\bar{v})\right)$,

$$D_r^h W(\bar{x},\bar{y},\bar{u},\bar{v})(x) \subseteq W\,\mathrm{Max}\left(D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x), \mathrm{int}\,C\right).$$

Proof. Take any $y\in D_r^h W(\bar{x},\bar{y},\bar{u},\bar{v})(x)$. It follows from the definition that $y\in D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$. If $y\notin \mathrm{WMax}\left(D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x), \mathrm{int}C\right)$, then there exists $\hat{y}\in D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$ such that

$$\hat{y}-y\in\mathrm{int}\,C. \tag{5.2}$$

As $y\in D_r^h W(\bar{x},\bar{y},\bar{u},\bar{v})(x)$, there exist sequences $t_k\searrow 0, s_k\searrow 0$, $t_k s_k^{-1}\to r$, and $(x_k,y_k)\to(x,y)$ satisfying

$$\bar{y}+t_k\bar{v}+\frac{1}{2}t_k s_k y_k \in W\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_k s_k x_k\right). \tag{5.3}$$

It follows from $\hat{y}\in D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$ and (5.1) that, for the preceding sequences t_k and s_k, there exist sequences $(\hat{x}_k,\hat{y}_k)\to(x,\hat{y})$ such that

$$\bar{y}+t_k\bar{v}+\frac{1}{2}t_k s_k \hat{y}_k \in G\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_k s_k \hat{x}_k\right).$$

Hence, there exists $x_k'\in K$ such that

$$\bar{y}+t_k\bar{v}+\frac{1}{2}t_k s_k \hat{y}_k = \left\langle F\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_k s_k \hat{x}_k\right), \bar{x}+t_k\bar{u}+\frac{1}{2}t_k s_k \hat{x}_k - x_k'\right\rangle. \tag{5.4}$$

Since $F:\mathbb{R}^n\to L(\mathbb{R}^n,\mathbb{R}^m)$ is twice continuously Fréchet differentiable, we have

$$\begin{aligned} F\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_k s_k \hat{x}_k\right) &= F(\bar{x})+t_k\nabla F(\bar{x})\left(\bar{u}+\frac{1}{2}s_k\hat{x}_k\right) \\ &\quad +\frac{1}{2}t_k^2\nabla^2 F(\bar{x})\left(\bar{u}+\frac{1}{2}s_k\hat{x}_k, \bar{u}+\frac{1}{2}s_k\hat{x}_k\right)+\circ\left(\left\|t_k\bar{u}+\frac{1}{2}t_k s_k\hat{x}_k\right\|^2\right). \end{aligned} \tag{5.5}$$

As $\{x_k\}$ and $\{\hat{x}_k\}$ are two convergent sequences, one has

$$\frac{\circ\left(\left\|t_k\bar{u}+\frac{1}{2}t_k s_k x_k\right\|^2\right)}{\frac{1}{2}t_k s_k}\to 0 \quad \text{and} \quad \frac{\circ\left(\left\|t_k\bar{u}+\frac{1}{2}t_k s_k \hat{x}_k\right\|^2\right)}{\frac{1}{2}t_k s_k}\to 0.$$

Then, it gives that

$$o(t_k s_k) = \circ\left(\left\|t_k\bar{u}+\frac{1}{2}t_k s_k\hat{x}_k\right\|^2\right) - \circ\left(\left\|t_k\bar{u}+\frac{1}{2}t_k s_k x_k\right\|^2\right).$$

By (4.3) and (5.5), we can deduce that

$$\begin{aligned}
&\left\langle F\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_k\hat{x}_k\right), \bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_k\hat{x}_k-x_k'\right\rangle \\
&=\left\langle F\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right), \bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k-x_k'\right\rangle \\
&+\frac{1}{2}t_ks_k\left\langle F\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right), \hat{x}_k-x_k\right\rangle \\
&+\frac{1}{2}t_ks_k\left\langle \nabla F(\bar{x})(\hat{x}_k-x_k), \bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_k\hat{x}_k-x_k'\right\rangle \\
&+\frac{1}{4}t_k^2s_k\left\langle \nabla^2 F(\bar{x})(\hat{x}_k,\hat{x}_k)-\nabla^2 F(\bar{x})(x_k,x_k), \bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_k\hat{x}_k-x_k'\right\rangle \\
&+\frac{1}{2}t_ks_k\left\langle \frac{\circ(t_ks_k)}{\frac{1}{2}t_ks_k}, \bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_k\hat{x}_k-x_k'\right\rangle.
\end{aligned}$$

Set

$$\begin{aligned}
\alpha(k) :=&\left\langle F\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right), \hat{x}_k-x_k\right\rangle \\
&+\left\langle \nabla F(\bar{x})(\hat{x}_k-x_k), \bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_k\hat{x}_k-x_k'\right\rangle \\
&+\left\langle \frac{1}{2}t_k\nabla^2 F(\bar{x})(\hat{x}_k,\hat{x}_k)-\frac{1}{2}t_k\nabla^2 F(\bar{x})(x_k,x_k), \bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_k\hat{x}_k-x_k'\right\rangle \\
&+\left\langle \frac{\circ(t_ks_k)}{\frac{1}{2}t_ks_k}, \bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_k\hat{x}_k-x_k'\right\rangle.
\end{aligned} \tag{5.6}$$

Since $\hat{x}_k-x_k\to 0$ and $\dfrac{\circ(t_ks_k)}{\frac{1}{2}t_ks_k}\to 0$, we see that $\alpha(k)\to 0$. With the aid of (5.4) and (5.6), we have

$$\begin{aligned}
\bar{y}+t_k\bar{v}+\frac{1}{2}t_ks_k[\hat{y}_k-\alpha(k)] &= \left\langle F\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right), \bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k-x_k'\right\rangle \\
&\in G\left(\bar{x}+t_k\bar{u}+\frac{1}{2}t_ks_kx_k\right).
\end{aligned}$$

By definition of W and (5.3), we obtain

$$\left[\bar{y}+t_k\bar{v}+\frac{1}{2}t_ks_k(\hat{y}_k-\alpha(k))\right]-\left[\bar{y}+t_k\bar{v}+\frac{1}{2}t_ks_ky_k\right]\notin \operatorname{int} C,$$

which yields that $\hat{y}_k-\alpha(k)-y_k\notin \operatorname{int} C$. Consequently, we arrive at $\hat{y}-y\notin \operatorname{int} C$, which contradicts (5.2). The proof is done. □

Example 5.1. Let $K=\{x\in\mathbb{R} \mid -1\le x\le 1\}$, $(\bar{x},\bar{y})=(0,0)$, $(\bar{u},\bar{v})=(1,0)$, and $r\in\overline{\mathbb{R}}_+$. Consider a map $F:\mathbb{R}\to\mathbb{R}$ with $F(x)=x^2$. We can easily check that, for any $x\in K$,

$$G(x)=\bigcup_{z\in K}\langle F(x),x-z\rangle=\bigcup_{z\in K}\langle x^2,x-z\rangle=\left[x^2(x-1),x^2(x+1)\right].$$

Then, $W(x) = \text{WMax}\,(G(x), \text{int}\, C) = x^2(x+1)$ for all $x \in K$. By calculating, we see that, for any $x \in \mathbb{R}$,

$$D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x) = D_r^{hi} G(\bar{x},\bar{y},\bar{u},\bar{v})(x) = \begin{cases} [-2r, 2r], & \text{if } r = [0,+\infty), \\ \emptyset, & \text{if } r = +\infty, \end{cases}$$

which fulfill condition (5.1). We have that, for any $x \in \mathbb{R}$,

$$D_r^h W(\bar{x},\bar{y},\bar{u},\bar{v})(x) = \begin{cases} \{2r\}, & \text{if } r = [0,+\infty), \\ \emptyset, & \text{if } r = +\infty, \end{cases} \quad W\,\text{Max}\left(D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x), \text{int}\, C\right) = \{2r\}.$$

Consequently, the conclusion of Theorem 5.2 holds.

Definition 5.2. We say that G is C-maxicomplete by W near $\bar{x} \in K$ if there exists a neighborhood U of $\bar{x}$ such that $G(x) \subseteq W(x) - C$, for all $x \in U$.

Let C be a convex cone. As $W(x) \subseteq G(x)$, the C-maxicompleteness of G by W near $\bar{x} \in K$ implies that $W(x) - C = G(x) - C$, for any $x \in U$. Hence, if G is C-maxicomplete by W near $\bar{x} \in K$, then, for any $\bar{y} \in W(\bar{x})$, $D_r^h W_-(\bar{x},\bar{y},\bar{u},\bar{v})(x) = D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x)$.

Theorem 5.3. *Let all the assumptions of Proposition 3.8 hold and G be C-maxicomplete by W near $\bar{x} \in K$. Then,*

$$W\,\text{Max}\left(D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x), \text{int}\, C\right) \subseteq D_r^h W(\bar{x},\bar{y},\bar{u},\bar{v})(x), \quad \forall x \in \text{dom}\left(D_r^h W(\bar{x},\bar{y},\bar{u},\bar{v})\right).$$

Proof. Since K is a compact set, $G(x)$ is also compact for any $x \in K$. Then, it follows from Propositions 3.7 and 3.9, and the above remark, we have that

$$\begin{aligned} \text{WMax}\left(D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x), \text{int}\, C\right) &= \text{WMax}\left(D_r^h G_-(\bar{x},\bar{y},\bar{u},\bar{v})(x), \text{int}\, C\right) \\ &= \text{WMax}\left(D_r^h W_-(\bar{x},\bar{y},\bar{u},\bar{v})(x), \text{int}\, C\right) \\ &= \text{WMax}\left(D_r^h W(\bar{x},\bar{y},\bar{u},\bar{v})(x), \text{int}\, C\right) \\ &\subseteq D_r^h W(\bar{x},\bar{y},\bar{u},\bar{v})(x). \end{aligned}$$

□

By directly applying Theorems 4.1, 5.2, and 5.3, we obtain the following results.

Theorem 5.4. *Suppose that the conditions of Proposition 3.8, Theorems 4.1 and 5.2 hold and G is C-maxicomplete by W near $\bar{x} \in K$. Then, for any $x \in \text{dom}\left(D_r^h W(\bar{x},\bar{y},\bar{u},\bar{v})\right)$,*

$$\begin{aligned} D_r^h W(\bar{x},\bar{y},\bar{u},\bar{v})(x) =& W\,\text{Max}\left(D_r^h G(\bar{x},\bar{y},\bar{u},\bar{v})(x), \text{int}\, C\right) \\ =& W\,\text{Max}\left(\bigcup_{x^* \in T_r^h(K,\hat{x},w)} [\langle F(\bar{x}), x - x^* \rangle + \langle \nabla F(\bar{x})\bar{u}, 2r(\bar{u} - w)\rangle], \text{int}\, C\right) \\ &+ \left\langle \nabla^2 F(\bar{x})(\bar{u},\bar{u}), r(\bar{x} - \hat{x})\right\rangle + \langle \nabla F(\bar{x})x, \bar{x} - \hat{x}\rangle, \end{aligned}$$

where w satisfies $\langle F(\bar{x}), w\rangle = -\bar{v} + \langle \nabla F(\bar{x})\bar{u}, \bar{x} - \hat{x}\rangle + \langle F(\bar{x}), \bar{u}\rangle$.

Remark 5.1. By using the second-order contingent derivative, [19, Corollary 4.1] gave us results on the weak vector variational inequalities with only index $r = 1$. In our study, we not only consider index $r = 0$ but also with index $r \in (0, +\infty)$. Therefore, our results extend and generalize in [19, Corollary 4.1].

Finally, we provide the following example to explain Theorem 5.4.

Example 5.2. Let $K = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$, $(\bar{x}, \bar{y}) = (1, 2)$, $(\bar{u}, \bar{v}) = (1, 5)$, and $r \in \mathbb{R}_+$. Consider a map $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x^2$. We can easily check that, for any $x \in K$,

$$G(x) = \bigcup_{z \in K} \langle F(x), x - z \rangle = \bigcup_{z \in K} \langle x^2, x - z \rangle = \left[x^2(x-1), x^2(x+1) \right],$$

and condition (4.1) holds. Then $W(x) = \text{WMax}\,(G(x), \text{int}\,C) = x^2(x+1)$ for all $x \in K$. By direct calculations, one see that, for all $x \in \mathbb{R}$,

$$D_r^h G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = D_r^{hi} G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = D_r^h G_-(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = \{y \in \mathbb{R} \mid y \leq 5x + 8r\},$$

$$D_r^h W(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = W\,\text{Max}\left(D_r^h G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x), \text{int}\,C \right) = \{y \in \mathbb{R} \mid y = 5x + 8r\}.$$

Hence, conditions (3.19), (3.20), and (5.1) are satisfied.

On the other hand, $\overline{F}(x) = \langle F(\bar{x}), \bar{x} - x \rangle = 1 - x$, and $\hat{x} = -1$ satisfy $\langle F(\bar{x}), \bar{x} - \hat{x} \rangle = \bar{y}$. Then, we see that $w = 0$, $T_r^h(K, \hat{x}, w) = \mathbb{R}_+$, $\langle F(\bar{x}), x - x^* \rangle = x - x^*$, and

$$\langle \nabla F(\bar{x})\bar{u}, 2r(\bar{u} - w) \rangle = 4r, \; \langle \nabla F(\bar{x})x, \bar{x} - \hat{x} \rangle = 4x,$$
$$\left\langle \nabla^2 F(\bar{x})(\bar{u}, \bar{u}), r(\bar{x} - \hat{x}) \right\rangle = 4r.$$

Hence, for any $x \in \mathbb{R}$, we see that

$$W\,\text{Max}\left(\bigcup_{x^* \in T_r^h(K, \hat{x}, w)} [\langle F(\bar{x}), x - x^* \rangle + \langle \nabla F(\bar{x})\bar{u}, 2r(\bar{u} - w) \rangle], \text{int}\,C \right)$$
$$+ \left\langle \nabla^2 F(\bar{x})(\bar{u}, \bar{u}), r(\bar{x} - \hat{x}) \right\rangle + \langle \nabla F(\bar{x})x, \bar{x} - \hat{x} \rangle = \{y \in \mathbb{R} \mid y = 5x + 8r\} = D_r^h W(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x).$$

Therefore, the conclusion of Theorem 5.4 is satisfied.

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