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RIEMANN-STIELTJES OPERATORS ON BERGMAN-MORREY SPACES IN THE UNIT BALL OF $\mathbb{C}^n$

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Abstract. In this paper, the Bergman-Morrey space $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ is introduced in the open unit ball of $\mathbb{C}^n$ and the identity operator from $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ into a tent space $\mathscr{T}^{\infty}_{p,\frac{\lambda(n+1)}{n}}(\mu)$ is characterized. Furthermore, the boundedness, compactness, and essential norm of the Riemann-Stieltjes operators V_g and U_g on $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ spaces are investigated.

Keywords. Bergman-Morrey space; Carleson measure; Essential norm; Riemann-Stieltjes operator.

1. Introduction

Let $\mathbb{B}_n$ and $\mathbb{S}_n$ denote the open unit ball and its boundary, respectively, in $\mathbb{C}^n$. In the case of $n=1$, the open unit ball $\mathbb{B}_n$ reduces to the open unit disc $\mathbb{D}$ in $\mathbb{C}$. We use $H(\mathbb{B}_n)$ to denote the space of all holomorphic functions on $\mathbb{B}_n$. For a function $f \in H(\mathbb{B}_n)$, we define the complex gradient and radial derivative of f as follows:

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots \frac{\partial f}{\partial z_n}(z)\right), \ \mathscr{R}f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z), \ z \in \mathbb{B}_n.$$

For any $a \in \mathbb{B}_n \backslash \{0\}$, let σ_a denote the biholomorphic mapping of $\mathbb{B}_n$, that is,

$$\sigma_a(z) = \frac{a - P_a(z) - \sqrt{1-|a|^2}(z - P_a(z))}{1 - \langle z, a \rangle}, \ z \in \mathbb{B}_n,$$

where $P_a(z) = \frac{\langle z,a \rangle a}{|a|^2}$. It is known that $\sigma_a \circ \sigma_a(z) = z$ and

$$1 - |\sigma_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a \rangle|^2}.$$

For $0 < p < \infty$, the Bergman space $\mathscr{A}^p(\mathbb{B}_n)$ is the space of all functions $f \in H(\mathbb{B}_n)$ such that

$$\|f\|^p_{\mathscr{A}^p(\mathbb{B}_n)} = \int_{\mathbb{B}_n} |f(z)|^p dV(z) < \infty,$$

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where dV denotes the normalized Lebesgue measure on $\mathbb{B}_n$. It is known that $f \in \mathscr{A}^p(\mathbb{B}_n)$ if and only if $(1-|z|^2)\mathscr{R}f(z) \in L^p(\mathbb{B}_n, dV)$. Moreover,

$$\|f\|^p_{\mathscr{A}^p(\mathbb{B}_n)} \approx |f(0)|^p + \int_{\mathbb{B}_n} |\mathscr{R}f(z)|^p (1-|z|^2)^p dV(z).$$

We refer to [41, 42] for more results about Bergman spaces.

For $0 < \alpha < \infty$, we define the α-Bloch space $\mathscr{B}^\alpha$ as the set of all $f \in H(\mathbb{B}_n)$ satisfying (see [42])

$$\|f\|_{\mathscr{B}^\alpha} = \sup_{z \in \mathbb{B}_n} |\mathscr{R}f(z)|(1-|z|^2)^\alpha < \infty.$$

It is evident that $\mathscr{B}^\alpha$ is a Banach space under the norm $\|f\| = |f(0)| + \|f\|_{\mathscr{B}^\alpha}$. In particular, the classical Bloch space $\mathscr{B}$ is just $\mathscr{B}^1$. We denote the little Bloch space by $\mathscr{B}_0$, which consists of all $f \in \mathscr{B}$ such that $\lim_{|z|\to 1}(1-|z|^2)|\mathscr{R}f(z)| = 0$. We use $H^\infty = H^\infty(\mathbb{B}_n)$ to represent the space of all bounded holomorphic functions on $\mathbb{B}_n$.

Let $0 < p < \infty$, $-(n+1) < q < \infty$, and $0 \le s < \infty$ be such that $q + s > -1$. The general space $F(p,q,s)$ consists of all $f \in H(\mathbb{B}_n)$ satisfying the following norm condition:

$$\|f\|^p_{F(p,q,s)} = |f(0)|^p + \sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} |\nabla f(z)|^p (1-|z|^2)^q g^s(z,a) dV(z) < \infty,$$

where $g(z,a) = \log \frac{1}{|\sigma_a(z)|}$. According to [39, Theorem 3.1], we have

$$\|f\|^p_{F(p,q,s)} \approx |f(0)|^p + \sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} |\mathscr{R}f(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s dV(z).$$

The space $F(p,q,s)$ was originally introduced by Zhao on $\mathbb{D}$ in [40]. Special values of p,q,s correspond to some classical function spaces, such as weighted Dirichlet spaces, weighted Bergman spaces, BMOA, Q_s spaces, and others (see, e.g., [14, 25]; for the unit disk setting, see [39]). It is worth noting that $F(p,q,s)$ is a subspace of $\mathscr{B}^{(n+1+q)/p}$, and when $s > n$, $F(p,q,s)$ is equivalent to the space $\mathscr{B}^{(n+1+q)/p}$. In [18], Morrey introduced the Morrey space, which has been utilized in the investigation of partial differential equation solutions' regularity and harmonic analysis on Euclidean spaces. In [31], Wu and Xie proposed and examined the holomorphic Morrey space denoted by $\mathscr{L}^{2,\lambda}$ on the unit disk. In [15], Liu and Lou provided a characterization of the Carleson measure for $\mathscr{L}^{2,\lambda}$.

Let $p > 0$, $0 \le \lambda \le 2$. The Bergman-Morrey space $\mathscr{A}^{p,\lambda}(\mathbb{D})$ is defined as the set of all functions $f \in H(\mathbb{D})$ satisfying the following norm condition:

$$\|f\|_{\mathscr{A}^{p,\lambda}(\mathbb{D})} = |f(0)| + \sup_{a \in \mathbb{D}} (1-|a|^2)^{\frac{2-\lambda}{p}} \|f \circ \sigma_a - f(a)\|_{\mathscr{A}^p(\mathbb{D})} < \infty.$$

This space was originally introduced by Yang and Liu in [36]. They provided a characterization of the identity operator from $\mathscr{A}^{p,\lambda}(\mathbb{D})$ into a tent space, as well as the boundedness and essential norm of the corresponding Volterra operators on the space $\mathscr{A}^{p,\lambda}(\mathbb{D})$. In recent decades, scholars introduced and studied various Morrey type spaces. For more information on these spaces and their properties, we refer to [5, 15, 30, 31, 32, 34, 36, 38] and the references therein.

Based on the work of [36], we introduce the Bergman-Morrey space $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ on the unit ball as follows: for $0 < \lambda < 1 < p < \infty$, $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ is the space of all $f \in H(\mathbb{B}_n)$ satisfying

$$\|f\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} = |f(0)| + \sup_{a \in \mathbb{B}_n} (1-|a|^2)^{\frac{(1-\lambda)(n+1)}{p}} \|f \circ \sigma_a - f(a)\|_{\mathscr{A}^p(\mathbb{B}_n)} < \infty.$$

We note that $\mathscr{A}^{p,0}(\mathbb{B}_n)=\mathscr{A}^p(\mathbb{B}_n)$ and $\mathscr{A}^{p,1}(\mathbb{B}_n)=\mathscr{B}$. Furthermore, it is straightforward to verify that $\mathscr{B}\subset\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\subset\mathscr{A}^p(\mathbb{B}_n)$, $0<\lambda<1$.

The organization of this paper is as follows. Section 2 presents a characterization of the boundedness and compactness of the identity operator from $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ into $\mathscr{T}^{\infty}_{p,\frac{\lambda(n+1)}{n}}(\mu)$, a tent space, which is defined in the same section. Section 3 investigates the boundedness of the Riemann-Stieltjes operators V_g, U_g, and the multiplication operator M_g on $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$. Additionally, Section 4, the last section, covers the determination of the essential norm of the operators V_g and U_g. In this paper, we use the notation $A\approx B$ to denote that $A\lesssim B\lesssim A$ for two functions. Specifically, $B\lesssim A$ means that there exists a positive constant C such that $B\leq CA$.

2. Embedding Theorem

This section begins with the definition of the Carleson measure, which was first proposed by Carleson [2] in the unit disk and has numerous applications. For $0<t<\infty$, a positive Borel measure μ on $\mathbb{B}_n$ is called a t-Carleson measure if

$$\|\mu\|_{\mathscr{CM}_t}=\sup\left\{\frac{\mu(\mathscr{Q}_\delta(\zeta))}{\delta^{nt}};\zeta\in\mathbb{S}_n,\delta>0\right\}<\infty,$$

where $\mathscr{Q}_\delta(\zeta)=\{z\in\mathbb{B}_n:|1-\langle z,\zeta\rangle|<\delta\}$ for any $\zeta\in\mathbb{S}_n$ and $\delta>0$. When $\zeta=\frac{a}{|a|}$ and $\delta=\sqrt{1-|a|^2}$, we write $\mathscr{Q}_\delta(\zeta)=\mathscr{Q}(a)$. The measure μ is called a vanishing t-Carleson measure if

$$\lim_{\delta\to 0}\frac{\mu(\mathscr{Q}_\delta(\zeta))}{\delta^{nt}}=0 \text{ for } \zeta\in\mathbb{S}_n.$$

An equivalent description of t-Carleson measure is stated as follows (see [41, Theorem 45]).

Lemma 2.1. *Let $t,r>0$ and μ be a positive Borel measure on $\mathbb{B}_n$. Then μ is a t-Carleson measure if and only if*

$$\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{(1-|a|^2)^r}{|1-\langle z,a\rangle|^{nt+r}}d\mu(z)<\infty.$$

Further,

$$\|\mu\|_{\mathscr{CM}_t}\approx\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{(1-|a|^2)^r}{|1-\langle z,a\rangle|^{nt+r}}d\mu(z).$$

For $0<p,s<\infty$, the non-isotropic tent type space $\mathscr{T}^\infty_{p,s}(\mu)$ is defined as the set of all μ-measurable functions f on $\mathbb{B}_n$ such that

$$\|f\|^p_{\mathscr{T}^\infty_{p,s}(\mu)}=\sup\left\{\delta^{-ns}\int_{\mathscr{Q}_\delta(\zeta)}|f(z)|^p d\mu(z);\zeta\in\mathbb{S}_n,\delta>0\right\}<\infty.$$

From [20, Proposition 2.1], we obtain the following equivalent characterization for space $\mathscr{T}^\infty_{p,s}(\mu)$.

Lemma 2.2. *Let $p,s,m>0$ and μ be a positive Borel measure on $\mathbb{B}_n$. Then a function $f\in H(\mathbb{B}_n)$ belongs to $\mathscr{T}^\infty_{p,s}(\mu)$ if and only if*

$$\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|f(z)|^p\frac{(1-|a|^2)^m}{|1-\langle z,a\rangle|^{ns+m}}d\mu(z)<\infty.$$

Moreover,

$$\|f\|^p_{\mathscr{T}^\infty_{p,s}(\mu)} \approx \sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|f(z)|^p\frac{(1-|a|^2)^m}{|1-\langle z,a\rangle|^{ns+m}}d\mu(z).$$

From [38, Theorem 1], we immediately obtain the following conclusion.

Lemma 2.3. *Let* $0<\lambda<1<p<\infty$ *and* $f\in H(\mathbb{B}_n)$. *Then* $f\in\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ *if and only if*

$$\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|\mathscr{R}f(z)|^p(1-|z|^2)^{p-\lambda(n+1)}(1-|\sigma_a(z)|^2)^{\lambda(n+1)}dV(z)<\infty.$$

Lemma 2.4. *Let* $0<\lambda<1<p<\infty$ *and* $f\in\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$. *Then*

$$|f(a)|\lesssim\frac{\|f\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}}{(1-|a|^2)^{\frac{(1-\lambda)(n+1)}{p}}},\ a\in\mathbb{B}_n.$$

Proof. Assume that $f\in\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$. By [39, Lemma 2.1], we have

$$|\mathscr{R}f(a)|\lesssim\frac{1}{(1-|a|^2)^{\frac{(1-\lambda)(n+1)}{p}+1}}\|f\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}.$$

Integrating both sides of the above inequality, the desired result can be obtained immediately. □

The following lemma can be found in [20, Corollary 2.1].

Lemma 2.5. *For* $m>-1$, $r,t>0$ *with* $0<r+t-m-n-1<r$, *there exists a constant* $C>0$ *such that*

$$\int_{\mathbb{B}_n}\frac{(1-|z|^2)^m}{|1-\langle z,a\rangle|^r|1-\langle z,d\rangle|^t}dV(z)\leq\frac{C}{(1-|a|^2)^{r+t-m-n-1}}.$$

Next, we describe the boundedness and compactness of the identity operator from $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ into space $\mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu)$. The following two theorems are the main conclusions of this section.

Theorem 2.1. *Let* $0<\lambda<1<p<\infty$ *and* μ *be a positive Borel measure on* $\mathbb{B}_n$. *Then the identity operator* $I_d:\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu)$ *is bounded if and only if* μ *is an* $\frac{n+1}{n}$*-Carleson measure.*

Proof. First we suppose that the operator $I_d:\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu)$ is bounded. For any $\zeta\in\mathbb{S}_n$ and $0<\delta<1$, we set

$$f_{\zeta,\delta}(z)=\frac{1}{(1-\langle z,(1-\delta)\zeta\rangle)^{\frac{(1-\lambda)(n+1)}{p}}}.$$

By Lemma 2.5, we have

$$\begin{aligned}
&\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|\mathscr{R}f_{\zeta,\delta}(z)|^p(1-|z|^2)^{p-\lambda(n+1)}(1-|\sigma_a(z)|^2)^{\lambda(n+1)}dV(z)\\
\lesssim&\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{|(1-\delta)\zeta|^p}{|1-\langle z,(1-\delta)\zeta\rangle|^{(1-\lambda)(n+1)+p}}(1-|z|^2)^{p-\lambda(n+1)}(1-|\sigma_a(z)|^2)^{\lambda(n+1)}dV(z)\\
\leq&\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^p(1-|a|^2)^{\lambda(n+1)}}{|1-\langle z,a\rangle|^{2\lambda(n+1)}|1-\langle z,(1-\delta)\zeta\rangle|^{(1-\lambda)(n+1)+p}}dV(z)\\
<&\infty.
\end{aligned}$$

Then Lemma 2.3 yields that $f_{\zeta,\delta}\in\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$. Since $|1-\langle z,(1-\delta)\zeta\rangle|\approx\delta$ for all $z\in\mathscr{Q}_\delta(\zeta)$, one has

$$\infty>\|f_{\zeta,\delta}\|^p_{\mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu)}\geq\delta^{-\lambda(n+1)}\int_{\mathscr{Q}_\delta(\zeta)}|f_{\zeta,\delta}(z)|^pd\mu(z)\gtrsim\delta^{-(n+1)}\mu(\mathscr{Q}_\delta(\zeta)),$$

which means that μ is an $\frac{n+1}{n}$-Carleson measure by the arbitrariness of δ.

Conversely, we ssume that μ is an $\frac{n+1}{n}$-Carleson measure. From Lemma 2.2, we just need to show that, for any $f\in\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$, there exists a $t>0$ such that

$$\|f\|^p_{\mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{p}}(\mu)}\approx\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|f(z)|^p\frac{(1-|a|^2)^t}{|1-\langle z,a\rangle|^{\lambda(n+1)+t}}d\mu(z)<\|f\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}.$$

Let $p-1<\varepsilon<2p$. Choose a constant t such that $t>\max\{(2-\lambda)(n+1),(2-\lambda)(n+1)-p-n+\varepsilon\}$ and $t<(2-\lambda)(n+1)+p$. Then

$$\begin{aligned}
\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|f(z)|^p\frac{(1-|a|^2)^t}{|1-\langle z,a\rangle|^{\lambda(n+1)+t}}d\mu(z)\lesssim&\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|f(z)-f(a)|^p\frac{(1-|a|^2)^t}{|1-\langle z,a\rangle|^{\lambda(n+1)+t}}d\mu(z)\\
&+\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|f(a)|^p\frac{(1-|a|^2)^t}{|1-\langle z,a\rangle|^{\lambda(n+1)+t}}d\mu(z)\\
:=&E+F.
\end{aligned}$$

Employing Lemmas 2.1 and 2.4 and noting that $t>(1-\lambda)(n+1)$, we see that

$$F\lesssim\|f\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{(1-|a|^2)^{t-(1-\lambda)(n+1)}}{|1-\langle z,a\rangle|^{\lambda(n+1)+t}}d\mu(z)<\infty.$$

Since μ is an $\frac{n+1}{n}$-Carleson measure, [41, Theorem 50] yields that $\mathscr{A}^p\subset L^p_\mu$. Therefore,

$$\begin{aligned}
E\lesssim&\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}\left|\mathscr{R}\left(\frac{(f(z)-f(a))(1-|a|^2)^{\frac{t}{p}}}{(1-\langle z,a\rangle)^{\frac{\lambda(n+1)+t}{p}}}\right)\right|^p(1-|z|^2)^pdV(z)+\|f\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\\
\lesssim&\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{|\mathscr{R}f(z)|^p(1-|a|^2)^t}{|1-\langle z,a\rangle|^{\lambda(n+1)+t}}(1-|z|^2)^pdV(z)\\
&+\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{|f(z)-f(a)|^p(1-|a|^2)^t}{|1-\langle z,a\rangle|^{\lambda(n+1)+t+p}}(1-|z|^2)^pdV(z)+\|f\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\\
=:&E_1+E_2+\|f\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}.
\end{aligned}$$

In view of $t > \lambda(n+1)$, one has

$$\begin{aligned} E_1 &\lesssim \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|\mathscr{R}f(z)|^p(1-|a|^2)^{\lambda(n+1)}}{|1-\langle z,a\rangle|^{2\lambda(n+1)}}(1-|z|^2)^p dV(z) \\ &= \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\mathscr{R}f(z)|^p(1-|z|^2)^{p-\lambda(n+1)}(1-|\sigma_a(z)|^2)^{\lambda(n+1)} dV(z) \\ &\approx \|f\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} < \infty. \end{aligned}$$

From [42, page 51], Hölder's inequality and [42, Theorem 1.12], for any $a \in \mathbb{B}_n$ and $f \in H(\mathbb{B}_n)$, one has

$$\begin{aligned} &|f\circ\sigma_a(z) - f\circ\sigma_a(0)|^p \\ \lesssim &\left(\int_{\mathbb{B}_n} |\mathscr{R}(f\circ\sigma_a)(w)| \frac{(1-|w|^2)^\beta}{|1-\langle z,w\rangle|^{n+\beta}} dV(w)\right)^p \\ = &\left(\int_{\mathbb{B}_n} |\mathscr{R}(f\circ\sigma_a)(w)| \frac{(1-|w|^2)^{\frac{\beta+\varepsilon}{p}}}{|1-\langle z,w\rangle|^{\frac{n+\beta}{p}}} \cdot \frac{(1-|w|^2)^{\frac{\beta(p-1)}{p}-\frac{\varepsilon}{p}}}{|1-\langle z,w\rangle|^{\frac{(n+\beta)(p-1)}{p}}} dV(w)\right)^p \\ \lesssim &\int_{\mathbb{B}_n} |\mathscr{R}(f\circ\sigma_a)(w)|^p \frac{(1-|w|^2)^{\beta+\varepsilon}}{|1-\langle z,w\rangle|^{n+\beta}} dV(w) \left(\int_{\mathbb{B}_n} \frac{(1-|w|^2)^{\beta-\frac{\varepsilon}{p-1}}}{|1-\langle z,w\rangle|^{n+\beta}} dV(w)\right)^{p-1} \\ \lesssim &(1-|z|^2)^{p-1-\varepsilon} \int_{\mathbb{B}_n} |\mathscr{R}(f\circ\sigma_a)(w)|^p \frac{(1-|w|^2)^{\beta+\varepsilon}}{|1-\langle z,w\rangle|^{n+\beta}} dV(w), \end{aligned}$$

where β is large enough. Therefore, using Lemma 2.5 and the change of variable $z = \sigma_a(u)$, we have

$$\begin{aligned} E_2 &= \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z)-f(a)|^p(1-|a|^2)^t}{|1-\langle z,a\rangle|^{\lambda(n+1)+t+p}}(1-|z|^2)^p dV(z) \\ &= \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |f\circ\sigma_a(u) - f\circ\sigma_a(0)|^p \frac{(1-|a|^2)^t(1-|\sigma_a(u)|^2)^p}{|1-\langle\sigma_a(u),a\rangle|^{\lambda(n+1)+p+t}} \left(\frac{1-|a|^2}{|1-\langle u,a\rangle|^2}\right)^{n+1} dV(u) \\ &= \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |f\circ\sigma_a(u) - f\circ\sigma_a(0)|^p \frac{(1-|u|^2)^p(1-|a|^2)^{(1-\lambda)(n+1)}}{|1-\langle u,a\rangle|^{(2-\lambda)(n+1)+p-t}} dV(u) \\ &\lesssim \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\mathscr{R}(f\circ\sigma_a)(w)|^p(1-|w|^2)^{\beta+\varepsilon} \int_{\mathbb{B}_n} \frac{(1-|u|^2)^{2p-1-\varepsilon}(1-|a|^2)^{(1-\lambda)(n+1)}}{|1-\langle u,w\rangle|^{n+\beta}|1-\langle u,a\rangle|^{(2-\lambda)(n+1)+p-t}} dV(u)dV(w) \\ &\lesssim \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\mathscr{R}(f\circ\sigma_a)(w)|^p(1-|w|^2)^{-(2-\lambda)(n+1)+p+t}(1-|a|^2)^{(1-\lambda)(n+1)} dV(w) \\ &\lesssim \sup_{a\in\mathbb{B}_n} (1-|a|^2)^{(1-\lambda)(n+1)} \int_{\mathbb{B}_n} |\mathscr{R}(f\circ\sigma_a)(w)|^p(1-|w|^2)^p dV(w) \\ &\lesssim \sup_{a\in\mathbb{B}_n} (1-|a|^2)^{(1-\lambda)(n+1)} \|f\circ\sigma_a - f(a)\|^p_{\mathscr{A}^p(\mathbb{B}_n)} \\ &\approx \|f\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} < \infty. \end{aligned}$$

Then

$$\|f\|^p_{\mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{p}}(\mu)} \approx \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |f(z)|^p \frac{(1-|a|^2)^t}{|1-\langle z,a\rangle|^{\lambda(n+1)+t}} d\mu(z) \lesssim \|f\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} < \infty,$$

which means that $I_d : \mathscr{A}^{p,\lambda}(\mathbb{B}_n) \to \mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu)$ is bounded. □

The identity operator $I_d : \mathscr{A}^{p,\lambda}(\mathbb{B}_n) \to \mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu)$ is said to be compact if $\|f_k\|_{\mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu)} \to 0$ as $k\to\infty$, when $\{f_k\}$ is a bounded sequence in $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ that converges to 0 uniformly on every compact subset of $\mathbb{B}_n$.

Theorem 2.2. *Let $0<\lambda<1<p<\infty$ and μ be a positive Borel measure on $\mathbb{B}_n$ such that the point evaluations are bounded functionals on $\mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu)$. Then the identity operator $I_d : \mathscr{A}^{p,\lambda}(\mathbb{B}_n) \to \mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu)$ is compact if and only if μ is a vanishing $\frac{n+1}{n}$-Carleson measure.*

Proof. First we suppose that $I_d : \mathscr{A}^{p,\lambda}(\mathbb{B}_n) \to \mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu)$ is compact. For any $\zeta\in\mathbb{S}_n$ and $\delta_k\to 0$ as $k\to\infty$, we set

$$f_{\zeta,k}(z) = \frac{1-|1-\delta_k|^2}{(1-\langle z,(1-\delta_k)\zeta\rangle)^{\frac{(1-\lambda)(n+1)}{p}+1}}.$$

It is easy to see that $\{f_{\zeta,k}\}$ is a bounded sequence in $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ and converges to 0 uniformly on every compact set of $\mathbb{B}_n$ as $k\to\infty$. Using the fact that $|1-\langle z,(1-\delta_k)\zeta\rangle| \geq \delta_k$ when $z\in\mathscr{Q}_{\delta_k}(\zeta)$, we have

$$\frac{\mu(\mathscr{Q}_{\delta_k}(\zeta))}{\delta_k^{\,n+1}} \lesssim \delta_k^{\,-\lambda(n+1)} \int_{\mathscr{Q}_{\delta_k}(\zeta)} |f_{\zeta,k}(z)|^p d\mu(z) \lesssim \|f_{\zeta,k}\|^p_{\mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu)} \to 0$$

as $k\to\infty$, which means that μ is a vanishing $\frac{n+1}{n}$-Carleson measure.

Conversely, we assume that μ is a vanishing $\frac{n+1}{n}$-Carleson measure. Let $\{f_k\}$ be a bounded sequence in $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ and converge to 0 uniformly on every compact set of $\mathbb{B}_n$. Then the definition of $\mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu)$ and Theorem 2.1 imply

$$\delta^{-\lambda(n+1)} \int_{\mathscr{Q}_\delta(\zeta)} |f_k(z)|^p d\mu(z) \lesssim \|\mu\|_{\mathscr{CM}_{\frac{n+1}{n}}} \|f_k\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}.$$

Therefore, for any $0<r<1$, we have

$$\begin{aligned}
&\delta^{-\lambda(n+1)} \int_{\mathscr{Q}_\delta(\zeta)} |f_k(z)|^p d\mu(z)\\
=&\delta^{-\lambda(n+1)} \int_{\mathscr{Q}_\delta(\zeta)} |f_k(z)|^p \chi_{\{z\in\mathbb{B}_n:|z|\leq r\}} d\mu(z) + \delta^{-\lambda(n+1)} \int_{\mathscr{Q}_\delta(\zeta)} |f_k(z)|^p \chi_{\{z\in\mathbb{B}_n:|z|>r\}} d\mu(z)\\
\lesssim&\delta^{-\lambda(n+1)} \int_{\mathscr{Q}_\delta(\zeta)} |f_k(z)|^p \chi_{\{z\in\mathbb{B}_n:|z|\leq r\}} d\mu(z) + \|f_k\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} \|\mu_r\|_{\mathscr{CM}_{\frac{n+1}{n}}},
\end{aligned}$$

where $d\mu_r(z) = \chi_{\{z\in\mathbb{B}_n:|z|>r\}} d\mu(z)$. Since $\{f_k\}$ converges to 0 uniformly on $\{z\in\mathbb{B}_n : |z|\leq r\}$, one has

$$\delta^{-\lambda(n+1)} \int_{\mathscr{Q}_\delta(\zeta)} |f_k(z)|^p \chi_{\{z\in\mathbb{B}_n:|z|\leq r\}} d\mu(z) \to 0$$

as $k \to \infty$. Now, we just need to proof that $\|\mu_r\|_{\mathscr{CM}_{\frac{n+1}{n}}} \to 0$ when $r \to 1$ (It is worth noting that the proof of this fact was given in [4, Lemma 3.1], but here we present an alternative proof). For $\zeta \in \mathbb{S}_n$ and $\delta > 0$, let

$$\tilde{\mathscr{Q}}_\delta(\zeta) = \{\eta \in \mathbb{S}_n : |1 - \langle \eta, \zeta \rangle| < \delta\}.$$

From the proof of [10, Theorem 4.1], it is clear that $\mathscr{Q}_\delta(\zeta) \subset \hat{\mathscr{Q}}_{4\delta}(\zeta) \subset \mathscr{Q}_{16\delta}(\zeta)$, where

$$\hat{\mathscr{Q}}_\delta(\zeta) = \left\{ z \in \mathbb{B}_n : \frac{z}{|z|} \in \tilde{\mathscr{Q}}_\delta(\zeta), 1 - \delta < |z| < 1 \right\}.$$

Thus $\mathscr{Q}_\delta(\zeta)$ in the definition of (vanishing) t-Carleson measure can be replaced by $\hat{\mathscr{Q}}_\delta(\zeta)$. Then, for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that $\mu(\hat{\mathscr{Q}}_\delta(\zeta)) < \varepsilon \delta^{n+1}$ for all $\delta \le \delta_0$ and $\zeta \in \mathbb{S}_n$. If $\delta > \delta_0$, we choose a natural number m satisfying $\frac{\delta}{\delta_0} < m < \frac{2\delta}{\delta_0}$ for any $\delta \le 2$. From [19, Lemma 3.3], we have that $\tilde{\mathscr{Q}}_\delta$ can be covered by N balls $\tilde{\mathscr{Q}}_{\frac{\delta}{m}}$ on $\mathbb{S}_n$, where $N \le Cm^n$, $C > 1$. Therefore, $\hat{\mathscr{Q}}_\delta \cap \{z \in \mathbb{B}_n : |z| > r_0\} \subset \bigcup_N \hat{\mathscr{Q}}_{\frac{\delta}{m}}$ with $r_0 = 1 - \frac{\delta_0}{m}$. Then

$$\begin{aligned}\mu_{r_0}(\hat{\mathscr{Q}}_\delta) &\le \mu_{r_0}(\bigcup_N \hat{\mathscr{Q}}_{\frac{\delta}{m}}) \le \mu_{r_0}(\bigcup_N \hat{\mathscr{Q}}_{\delta_0}) \le \sum_N \mu(\hat{\mathscr{Q}}_{\delta_0}) \\ &\le N\varepsilon \delta_0^{n+1} < C\left(\frac{2\delta}{\delta_0}\right)^n \varepsilon \delta_0^{n+1} < C\varepsilon \delta^n \delta_0^{n+1-n} < C\varepsilon \delta^{n+1},\end{aligned}$$

which implies that, for $r > r_0$, $\mu_r(\hat{\mathscr{Q}}_\delta) < C\varepsilon \delta^{n+1}$. Therefore,

$$\delta^{-\lambda(n+1)} \int_{\mathscr{Q}_\delta(\zeta)} |f_k(z)|^p d\mu(z) \lesssim \varepsilon.$$

Thus $I_d : \mathscr{A}^{p,\lambda}(\mathbb{B}_n) \to \mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu)$ is compact. □

3. Riemann-Stieltjes Operators

This section aims to study the boundedness of the Riemann-Stiltjes operators V_g, U_g and the multiplication operator M_g on $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$. Now we recall the definitions of these operators. Let $g \in H(\mathbb{B}_n)$. The Riemann-Stieltjes operator V_g is defined by

$$V_g f(z) = \int_0^1 f(sz)\mathscr{R}g(sz)\frac{ds}{s},\ f \in H(\mathbb{B}_n),\ z \in \mathbb{B}_n.$$

The operator V_g was first introduced and studied by Hu in [7]. A related operator U_g is defined by

$$U_g f(z) = \int_0^1 \mathscr{R}f(sz)g(sz)\frac{ds}{s},\ f \in H(\mathbb{B}_n),\ z \in \mathbb{B}_n.$$

Clearly, the multiplication operator M_g is determined by

$$M_g f(z) = f(z)g(z) = f(0)g(0) + V_g f(z) + U_g f(z),\ f \in H(\mathbb{B}_n),\ z \in \mathbb{B}_n.$$

Some information on these integral-type operators and related operators on the unit disk and the polydisk and many references up to the end of 2006 can be found in [3, 22]. For some later results on the operators and their extensions in [23] and [24] we refer to, for example, [1, 3, 8, 9, 11, 12, 13, 16, 17, 19, 20, 21, 25, 27, 28, 33] and the references therein.

Now we state and prove the main results in this section.

Theorem 3.1. *Let* $0<\lambda<1<p<\infty$ *and* $g\in H(\mathbb{B}_n)$. *Then* $V_g:\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ *is bounded if and only if* $g\in\mathscr{B}$.

Proof. Assume first that $V_g:\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ is bounded. Since $\mathscr{R}(V_gf)(z)=f(z)\mathscr{R}g(z)$, we find by Lemma 2.3 that

$$\begin{aligned}\|V_gf\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}&\approx\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|f(z)|^p|\mathscr{R}g(z)|^p(1-|z|^2)^{p-\lambda(n+1)}(1-|\sigma_a(z)|^2)^{\lambda(n+1)}dV(z)\\&=\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|f(z)|^p\frac{(1-|a|^2)^{\lambda(n+1)}}{|1-\langle z,a\rangle|^{2\lambda(n+1)}}d\mu_g(z)\\&\lesssim\|f\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)},\end{aligned}$$

where $d\mu_g=|\mathscr{R}g(z)|^p(1-|z|^2)^pdV(z)$. This with Lemma 2.2 implies that $I_d:\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu_g)$ is bounded. Using Theorem 2.1, we obtain that μ_g is an $\frac{n+1}{n}$-Carleson measure. Lemma 2.1 yields that μ_g is an $\frac{n+1}{n}$-Carleson measure if and only if

$$\begin{aligned}\infty&>\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{(1-|a|^2)^{n+1}}{|1-\langle z,a\rangle|^{2(n+1)}}d\mu_g(z)\\&=\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|\mathscr{R}g(z)|^p(1-|z|^2)^{p-(n+1)}(1-|\sigma_a(z)|^2)^{n+1}dV(z),\end{aligned}$$

which implies that $g\in F(p,p-(n+1),(n+1))=\mathscr{B}$.

Conversely, we suppose that $g\in\mathscr{B}=F(p,p-(n+1),(n+1))$. From the definition of $F(p,p-(n+1),(n+1))$, we see that μ_g is an $\frac{n+1}{n}$-Carleson measure. By Theorem 2.1, we obtain that $I_d:\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{T}^\infty_{p,\frac{\lambda(n+1)}{n}}(\mu_g)$ is bounded. Then Lemma 2.2 gives that

$$\begin{aligned}\|V_gf\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}&\approx\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|f(z)|^p|\mathscr{R}g(z)|^p(1-|z|^2)^{p-\lambda(n+1)}(1-|\sigma_a(z)|^2)^{\lambda(n+1)}dV(z)\\&=\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|f(z)|^p\frac{(1-|a|^2)^{\lambda(n+1)}}{|1-\langle z,a\rangle|^{2\lambda(n+1)}}d\mu_g(z)<\infty.\end{aligned}$$

The proof is complete. □

Theorem 3.2. *Let* $0<\lambda<1<p<\infty$ *and* $g\in H(\mathbb{B}_n)$. *Then* $U_g:\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ *is bounded if and only if* $g\in H^\infty$.

Proof. Suppose that $U_g:\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ is bounded. For $a\in\mathbb{B}_n$ with $|a|>\frac{3}{4}$, we set

$$f_a(z)=\frac{1-|a|^2}{\left(\frac{(1-\lambda)(n+1)}{p}+1\right)(1-\langle z,a\rangle)^{\frac{(1-\lambda)(n+1)}{p}+1}},\ z\in\mathbb{B}_n.$$

Note that [20, Lemma 2.4] yields that $\|f_a\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\lesssim 1$. It is known that

$$V(D(a,\frac{1}{2}))\approx(1-|a|^2)^{n+1},\ 1-|a|^2\approx 1-|z|^2\approx|1-\langle z,a\rangle|,$$

for all $z \in D(a, \frac{1}{2})$, where $D(a,r)$ denotes the Bergman metric ball. Noting that, for $z \in D(a, \frac{1}{2})$, $|\langle z,a\rangle| \gtrsim 1$. Employing the subharmonicity of $|g|^p$ (see [42, Lemma 2.24]), we obtain that

$$\begin{aligned}
|g(a)|^p &\lesssim \frac{1}{V(D(a,\frac{1}{2}))}\int_{D(a,\frac{1}{2})}|g(z)|^p dV(z)\\
&\lesssim \frac{1}{(1-|a|^2)^{n+1+p}}\int_{D(a,\frac{1}{2})}|g(z)|^p(1-|z|^2)^p dV(z)\\
&\lesssim \int_{D(a,\frac{1}{2})}\left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^{\frac{(1-\lambda)(n+1)}{p}+2}}\right)^p |g(z)|^p(1-|z|^2)^p\frac{(1-|a|^2)^{\lambda(n+1)}}{(1-\langle z,a\rangle)^{2\lambda(n+1)}}dV(z)\\
&\lesssim \int_{D(a,\frac{1}{2})}\left(\frac{(1-|a|^2)|\langle z,a\rangle|}{|1-\langle z,a\rangle|^{\frac{(1-\lambda)(n+1)}{p}+2}}\right)^p |g(z)|^p(1-|z|^2)^p\frac{(1-|a|^2)^{\lambda(n+1)}}{(1-\langle z,a\rangle)^{2\lambda(n+1)}}dV(z)\\
&\lesssim \int_{D(a,\frac{1}{2})}|\mathscr{R}f_a(z)|^p|g(z)|^p(1-|z|^2)^p\frac{(1-|a|^2)^{\lambda(n+1)}}{(1-\langle z,a\rangle)^{2\lambda(n+1)}}dV(z)\\
&\lesssim \sup_{b\in\mathbb{B}_n}\int_{\mathbb{B}_n}|\mathscr{R}f_a(z)|^p|g(z)|^p(1-|z|^2)^{p-\lambda(n+1)}(1-|\sigma_b(z)|^2)^{\lambda(n+1)}dV(z)\\
&\lesssim \|U_g f_a\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} \lesssim \|U_g\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} < \infty,
\end{aligned}$$

which means that $g \in H^\infty$.

Conversely, we assume that $g \in H^\infty$. Let $f \in \mathscr{A}^{p,\lambda}(\mathbb{B}_n)$. Using Lemma 2.3 twice, we have

$$\begin{aligned}
\|U_g f\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} &\lesssim \sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|\mathscr{R}f(z)|^p|g(z)|^p(1-|z|^2)^{p-\lambda(n+1)}(1-|\sigma_a(z)|^2)^{\lambda(n+1)}dV(z)\\
&\lesssim \|g\|^p_\infty \sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}|\mathscr{R}f(z)|^p(1-|z|^2)^{p-\lambda(n+1)}(1-|\sigma_a(z)|^2)^{\lambda(n+1)}dV(z)\\
&\lesssim \|g\|^p_\infty\|f\|^p_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)},
\end{aligned}$$

which implies that $U_g : \mathscr{A}^{p,\lambda}(\mathbb{B}_n) \to \mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ is bounded. □

Theorem 3.3. *Let $0 < \lambda < 1 < p < \infty$ and $g \in H(\mathbb{B}_n)$. Then $M_g : \mathscr{A}^{p,\lambda}(\mathbb{B}_n) \to \mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ is bounded if and only if $g \in H^\infty$.*

Proof. First, we suppose that $g \in H^\infty$. Using Theorems 3.1 and 3.2 and the fact that $H^\infty \subset \mathscr{B}$, we obtain that V_g and U_g are bounded on $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$. Therefore, $M_g : \mathscr{A}^{p,\lambda}(\mathbb{B}_n) \to \mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ is bounded by the fact that $M_g f(z) = f(0)g(0) + V_g f(z) + U_g f(z)$.

Conversely, we suppose that $M_g : \mathscr{A}^{p,\lambda}(\mathbb{B}_n) \to \mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ is bounded. For any $a \in \mathbb{D}$ with $|a| > 1/2$, we set

$$f_a(z) = \frac{(1-|a|^2)^2}{(1-\langle z,a\rangle)^{\frac{(1-\lambda)(n+1)}{p}+2}} - \frac{1-|a|^2}{(1-\langle z,a\rangle)^{\frac{(1-\lambda)(n+1)}{p}+1}},\ z \in \mathbb{B}_n.$$

Then

$$\mathscr{R}f_a(z) = \frac{\left(\frac{(1-\lambda)(n+1)}{p}+2\right)(1-|a|^2)^2\langle z,a\rangle}{(1-\langle z,a\rangle)^{\frac{(1-\lambda)(n+1)}{p}+3}} - \frac{\left(\frac{(1-\lambda)(n+1)}{p}+1\right)(1-|a|^2)\langle z,a\rangle}{(1-\langle z,a\rangle)^{\frac{(1-\lambda)(n+1)}{p}+2}}.$$

It is clear that

$$f_a(a) = 0,\ \mathscr{R}f_a(a) = |a|^2(1-|a|^2)^{-\frac{(1-\lambda)(n+1)}{p}-1}.$$

Thus we see that $f_a \in \mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ and $\sup_{a\in\mathbb{B}_n} \|f_a\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} < \infty$. Since $\mathscr{A}^{p,\lambda}(\mathbb{B}_n) \subset \mathscr{B}^{1+\frac{(1-\lambda)(n+1)}{p}}$, we see that

$$\begin{aligned}\infty > &\|M_g f_a\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} \gtrsim \|M_g f_a\|_{\mathscr{B}^{1+\frac{(1-\lambda)(n+1)}{p}}}\\ \gtrsim &\sup_{z\in\mathbb{B}_n} |\mathscr{R}g(z)f_a(z) + g(z)\mathscr{R}f_a(z)|(1-|z|^2)^{1+\frac{(1-\lambda)(n+1)}{p}}\\ \gtrsim &|\mathscr{R}g(a)f_a(a) + g(a)\mathscr{R}f_a(a)|(1-|a|^2)^{1+\frac{(1-\lambda)(n+1)}{p}}\\ \gtrsim &|g(a)||a|^2,\end{aligned}$$

from which it easily follows that $g \in H^\infty$. □

4. Essential Norm

This section is devoted to the essential norm of the operators V_g and U_g on $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$. Let A and B be two Banach spaces. The essential norm of a bounded linear operator $T : A \to B$ is defined as

$$\|T\|_{e,A\to B} = \inf_K\{\|T-K\|_{A\to B} : K \text{ is compact from } A \text{ to } B\}.$$

As we know, $T : A \to B$ is compact if and only if $\|T\|_{e,A\to B} = 0$. For recent research on the estimating essential norm of some integral-type operators, we refer to [4, 6, 26, 27, 35].

Let Q and B be two Banach spaces such that Q is a subspace of B. Given $f \in B$, the distance of f to Q is defined as $\mathrm{dist}_B(f,Q) = \inf_{g\in Q}\|f-g\|_B$. From [4, Lemma 4.1], the distance formula for the function in the space $\mathscr{B}$ to the space $\mathscr{B}_0$ is stated as follows.

Lemma 4.1. *Let $g \in \mathscr{B}$. Then*

$$\mathrm{dist}_{\mathscr{B}}(g,\mathscr{B}_0) \approx \limsup_{r\to 1^-}\|g-g_r\|_{\mathscr{B}} \approx \limsup_{|z|\to 1^-}|\mathscr{R}g(z)|(1-|z|^2).$$

Similar to the proof of [37, Lemma 5], we have the following result.

Lemma 4.2. *Let $0 < \lambda < 1 < p < \infty$ and $g \in H(\mathbb{B}_n)$. If $0 < r < 1$ and $g \in \mathscr{B}$, then $V_{g_r} : \mathscr{A}^{p,\lambda}(\mathbb{B}_n) \to \mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ is compact.*

Tjani, in [29], established the following findings within the open unit disc $\mathbb{D}$, which are fundamental in investigating the essential norm of integral operators on different analytic function spaces. It is noteworthy that these findings also hold true in the open unit ball $\mathbb{B}_n$.

Lemma 4.3. *Let X,Y be two Banach spaces of analytic functions on $\mathbb{B}_n$. Suppose that*

(i) The point evaluation functionals on Y are continuous.
(ii) The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.
(iii) $T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.

Then T is compact if and only if for every bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on every compact set of $\mathbb{B}_n$, then the sequence $\{Tf_n\}$ converges to 0 in the norm of Y.

Theorem 4.1. *Let* $0<\lambda<1<p<\infty$ *and* $g\in H(\mathbb{B}_n)$. *If* $V_g:\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ *is bounded, then*

$$\|V_g\|_{e,\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\approx \mathrm{dist}_{\mathscr{B}}(g,\mathscr{B}_0)\approx \limsup_{|z|\to 1^-}|\mathscr{R}g(z)|(1-|z|^2).$$

Proof. From the proof of Theorem 3.1, it is easy to see that $g\in\mathscr{B}$ and

$$\|V_g\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\approx\|g\|_{\mathscr{B}}.$$

From Lemma 4.2, we see that $V_{g_r}:\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ is compact. Hence

$$\begin{aligned}\|V_g\|_{e,\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}&\le\|V_g-V_{g_r}\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\\&=\|V_{g-g_r}\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\approx\|g-g_r\|_{\mathscr{B}}.\end{aligned}$$

Since $\mathscr{R}g_r\in H^\infty$, we see that $g_r\in\mathscr{B}_0$. So,

$$\|V_g\|_{e,\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\lesssim\limsup_{r\to 1^-}\|g-g_r\|_{\mathscr{B}}\approx\limsup_{|z|\to 1^-}|\mathscr{R}g(z)|(1-|z|^2).$$

Next, we estimate the lower bounded. Let $\{a_k\}$ be a bounded sequence in $\mathbb{B}_n$ such that $\lim_{k\to\infty}|a_k|=1$. Set

$$f_k(z)=\frac{(1-|a_k|^2)^{\frac{(1-\lambda)(n+1)}{p}}}{(1-\langle z,a_k\rangle)^{2\frac{(1-\lambda)(n+1)}{p}}},\ z\in\mathbb{B}_n.$$

Then $\{f_k\}$ is bounded in $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ and converges to zero uniformly on every compact subset of $\mathbb{B}_n$ as $k\to\infty$. Let $K:\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ be a compact operator. By Lemma 4.3, we have

$$\lim_{k\to\infty}\|Kf_k\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}=0.$$

Therefore, using [42, Lemma 2.24], we have

$$\begin{aligned}&\|V_g-K\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\\&\gtrsim\limsup_{k\to\infty}\|(V_g-K)(f_k)\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\\&\gtrsim\limsup_{k\to\infty}(\|V_gf_k\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}-\limsup_{k\to\infty}\|Kf_k\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)})\\&=\limsup_{k\to\infty}\|V_gf_k\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\\&\gtrsim\limsup_{k\to\infty}\left(\int_{\mathbb{B}_n}|f_k(z)|^p|\mathscr{R}g(z)|^p(1-|z|^2)^{p-\lambda(n+1)}(1-|\sigma_{a_k}(z)|^2)^{\lambda(n+1)}dV(z)\right)^{\frac{1}{p}}\\&=\limsup_{k\to\infty}\left(\int_{\mathbb{B}_n}\frac{(1-|a_k|^2)^{(1-\lambda)(n+1)+\lambda(n+1)}}{|1-\langle z,a_k\rangle|^{2(1-\lambda)(n+1)+2\lambda(n+1)}}|\mathscr{R}g(z)|^p(1-|z|^2)^pdV(z)\right)^{\frac{1}{p}}\\&\gtrsim\limsup_{k\to\infty}\left(\int_{D(a_k,r)}|\mathscr{R}g(z)|^p(1-|z|^2)^{p-(n+1)}dV(z)\right)^{\frac{1}{p}}\\&\gtrsim\limsup_{k\to\infty}|\mathscr{R}g(a_k)|(1-|a_k|^2),\end{aligned}$$

which implies that

$$\|V_g\|_{e,\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} \gtrsim \limsup_{|z|\to 1^-} |\mathscr{R}g(z)|(1-|z|^2).$$

The proof is complete. □

Theorem 4.2. *Let* $0<\lambda<1<p<\infty$ *and* $g\in H(\mathbb{B}_n)$. *If* $U_g:\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ *is bounded, then*

$$\|U_g\|_{e,\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} \approx \|g\|_\infty.$$

Proof. We define K and $\{a_k\}$ as in the proof of Theorem 4.1. Set

$$f_k(z)=\frac{(1-|a_k|^2)^{\frac{(1-\lambda)(n+1)}{p}}}{(1-\langle z,a_k\rangle)^{2\frac{(1-\lambda)(n+1)}{p}}},\quad z\in\mathbb{B}_n,$$

where $\{a_k\}$ satisfies $|a_k|\geq 3/4$ and $|a_k|\to 1$ as $k\to\infty$. Then $\{f_k\}$ is a bounded sequence in $\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ and converges to zero uniformly on every compact subset of $\mathbb{B}_n$ as $k\to\infty$. Hence

$$\begin{aligned}\|U_g-K\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} &\gtrsim \limsup_{k\to\infty}\|(U_g-K)(f_k)\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\\ &\gtrsim \limsup_{k\to\infty}\|U_gf_k\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}-\limsup_{k\to\infty}\|Kf_k\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\\ &=\limsup_{k\to\infty}\|U_gf_k\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}.\end{aligned}$$

From the proof of Theorem 3.2, we see that $\|U_gf_k\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)}\gtrsim|g(a_k)|$, which implies that

$$\|U_g\|_{e,\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} \gtrsim \|g\|_{H^\infty}.$$

Conversely, we have by Theorem 3.2 that

$$\|U_g\|_{e,\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} \lesssim \|U_g\|_{\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)} \lesssim \|g\|_{H^\infty}.$$

This finishes the proof. □

From Theorems 4.1 and 4.2, we have the following corollary.

Corollary 4.1. *Let* $0<\lambda<1<p<\infty$ *and* $g\in H(\mathbb{B}_n)$. *Then the following statements hold.*

(i) The operator $V_g:\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ *is compact if and only if* $g\in\mathscr{B}_0$;
(ii) The operator $U_g:\mathscr{A}^{p,\lambda}(\mathbb{B}_n)\to\mathscr{A}^{p,\lambda}(\mathbb{B}_n)$ *is compact if and only if* $g=0$.

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