J. Nonlinear Var. Anal. 10 (2026), No. 1, pp. 1-40
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.10.2026.1.01

PEAK SOLUTIONS FOR LOGARITHMIC SCALAR FIELD SYSTEMS

XINAN DUAN, ZIJUAN GAO, QING GUO*

College of Science, Minzu University of China, Beijing 100081, China

Abstract. We are concerned with a class of important Schrödinger equations in mathematical physics with logarithmic nonlinearities: $-\varepsilon^2\Delta u+V(y)u=u\log|u|$, $u>0$, in $H^1(\mathbb{R}^N)$, where $N\geq 3$, ε is a small positive parameter, and $V(y)$ denotes the potential function. The main difficulties to apply Lyapunov-Schmidt reduction to logarithmic scalar equations are caused by the non-smooth property and sublinear growth of the logarithmic non-linearity. Our method is fundamentally based on a new type of inner-outer decomposition, setting it apart from conventional gluing techniques that usually require distinct constructions for the inner and outer problems. Rather than this traditional separation, we incorporate the minimization operator for the outer problem directly with the operator related to the fixed-point theorem, enhancing the reduction framework to be applicable. We prove the existence of positive multipeak solutions under certain assumptions on $V(y)$. Finally, we also use the local Pohozaev identities to obtain the non-degenerate of positive multipeak solutions.

Keywords. Inner-outer gluing; Local Pohozaev Identities; Lyapunov-Schmidt reduction method; Logarithmic Schrödinger systems; Multi-peak Solutions; Non-degeneracy.

1. Introduction

We consider the following logarithmic scalar field system:

$$-\varepsilon^2\Delta u+V(y)u=u\log|u|,\quad u>0,\quad \text{in}\quad H^1(\mathbb{R}^N),\tag{1.1}$$

where ε is a small positive parameter and $N\geq 3$. This system arises from the following time-dependent logarithmic Schrödinger system:

$$i\varepsilon\partial_t u+\frac{\varepsilon^2}{2}\Delta u-V(y)u+u\log u=0,\tag{1.2}$$

which was proposed by Bialynicki-Birula and Mycielski [3] as a model of nonlinear wave mechanics. System (1.2) finds extensive applications across various fields, including quantum optics [4], nuclear physics [13], geophysical phenomena like magma transport [9], effective quantum gravity theories, superfluidity, Bose-Einstein condensation, and open quantum systems (see [18] and the references therein). Mathematically, there has been considerable recent interest in the existence and qualitative properties of solutions to nonlinear Schrödinger equations. Studies such as [1, 2, 5, 6, 7] examine the existence and stability of standing waves and address the Cauchy problem within an appropriate functional framework for system (1.2).

*Corresponding author.
E-mail address: guoqing0117@163.com (Q. Guo).
Received 7 January 2025; Accepted 12 May 2025; Published online 1 October 2025.

However, for the systems with logarithmic non-linearity, new challenges arise when applying the Lyapunov-Schmidt reduction method to (1.1). Specifically, the sublinear growth of the logarithmic nonlinearity near zero complicates the identification of a suitable contraction mapping. In this paper, we develop a novel approach to applying Lyapunov-Schmidt reduction to investigate the existence of multiple solutions to (1.1). To introduce our main result, we suppose that $V(y) \in C^2 : \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:
(V_1): $V(y) \in L^\infty(\mathbb{R}^N)$ and $0 < \inf_{\mathbb{R}^N} V(y) \le \sup_{\mathbb{R}^N} V(y) < \infty$.
(V_2): There exist k points $\xi_1, \cdots, \xi_k$ such that $\nabla V(\xi_j) = 0$ and $deg(\nabla V, B_\delta(\xi_j), 0) \neq 0$, for any $j = 1, \cdots, k$.

A function $u \in H^1(\mathbb{R}^N)$ is said to be a (weak) solution to (1.1) if it satisfies

$$\Big| \int_{\mathbb{R}^N} u^2 \log|u| dy \Big| < \infty$$

and

$$\int_{\mathbb{R}^N} (\varepsilon^2 \nabla u \nabla v + V(y) uv) dy = \int_{\mathbb{R}^N} (uv \log|u|) dy, \ \text{ for all } v \in C_0^\infty(\mathbb{R}^N).$$

The primary results are as follows.

Theorem 1.1. *If $N \ge 3$, $V(y)$ satisfies* (V_1) *and* (V_2)*, then problem* (1.1) *has a k-peak solution concentrated at $\xi_1, \cdots, \xi_k$ for $\varepsilon > 0$ sufficiently small.*

Theorem 1.2. *If $N \ge 3$, $V(y)$ satisfies* (V_1) *and x_0 is an isolated local maximum point of $V(y)$, then problem* (1.1) *has a k-peak solution concentrated at x_0, for $\varepsilon > 0$ sufficiently small.*

We consider the following system

$$-\Delta u + V(x^j) u = u \log|u|, \ u > 0, \ \text{in } H^1(\mathbb{R}^N), \tag{1.3}$$

and as $\varepsilon \to 0$, $x^j \to \xi_j$, $j = 1, \cdots, k$. It is known from [8, 14] that system (1.3) has a unique positive solution $U_j(y) := e^{V(x^j) + \frac{N}{2}} e^{-\frac{|y|^2}{4}}$, which is non-degenerate in the sense that if $u \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u^2 |y|^2 dy < \infty$ satisfies

$$-\Delta u + \Big(\frac{|y|^2}{4} - \frac{N}{2} - 1 \Big) u = 0, \ u > 0, \ \text{in } H^1(\mathbb{R}^N), \tag{1.4}$$

then $u \in span\{ \frac{\partial U_j}{\partial y_i} | i = 1, \cdots, N, j = 1, \cdots, k\}$. Actually, the uniqueness of positive solutions to (1.3) was established in [14] for $N \ge 1$ and in [8] for $N \ge 3$. The non-degeneracy of these solutions was proved in [8] for $N \ge 3$, with the proof also valid for $N \le 2$.

For any $x^j \in \mathbb{R}^N$ with $j = 1, \cdots, k$, we denote

$$U_{\varepsilon,j}(y) = e^{V(x^j) + \frac{N}{2}} e^{-\frac{|y - x^j|^2}{4\varepsilon^2}},$$

which is the solution to

$$-\varepsilon^2 \Delta u + V(x^j) u = u \log|u|, \ \text{in } \mathbb{R}^N.$$

To recover smoothness, as in [17], we introduce a family of perturbed problems, for small $\tau > 0$,

$$-\varepsilon^2 \Delta u + V(y) u = g_\tau(u), \quad y \in \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \tag{1.5}$$

where $g_\tau(u) = \tau^{-1}(|u|^\tau u - u)$. Since $g_\tau(s) \to s \log|s|$ in $C_{loc}(\mathbb{R})$ as $\tau \to 0^+$, the solutions to (1.1) can be obtained by taking the limits of those to (1.5) with some uniform estimates. We first solve equation (1.5) with uniform estimates independent of τ. Next, we introduce some notations.

Denote $d := \min_{m\neq j}|x^m - x^j|, m, j = 1, \cdots, k,$

$$\langle u_1,u_2\rangle_\varepsilon = \int_{\mathbb{R}^N} \varepsilon^2\nabla u_1\nabla u_2 + V(y)u_1u_2,\ u_1,u_2\ \in H^1(\mathbb{R}^N),$$
$$\langle u_1,u_2\rangle_{\varepsilon,\Omega} = \int_{\Omega} \varepsilon^2\nabla u_1\nabla u_2 + V(y)u_1u_2,\ u_1,u_2\ \in H^1(\mathbb{R}^N),$$
$$\langle \eta_1,\eta_2\rangle_{\varepsilon,\Omega} = \int_{\Omega} \varepsilon^2\nabla \eta_1\nabla \eta_2 + V(y)\eta_1\eta_2,\ \eta_1,\eta_2\ \in H^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$$

and the norms

$$\|u\|_\varepsilon = \sqrt{\langle u,u\rangle_\varepsilon},\ u\ \in H^1(\mathbb{R}^N),$$
$$\|u\|_{\varepsilon,\Omega} = \sqrt{\langle u,u\rangle_{\varepsilon,\Omega}},\ u\ \in H^1(\mathbb{R}^N),$$
$$\|\eta\|_{\varepsilon,\Omega} = \sqrt{\langle \eta,\eta\rangle_{\varepsilon,\Omega}},\ \eta\ \in H^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N).$$

Let

$$W(y) := W_\varepsilon(y) := W_{\varepsilon,\tau}(y) := \sum_{j=1}^k U_{\varepsilon,j}(y),$$

where $U_{\varepsilon,j}(y) = e^{V(x^j)+\frac{N}{2}}e^{-\frac{|y-x^j|^2}{4\varepsilon^2}}$.

We verify Theorem 1.1 and Theorem 1.2 by proving the following Theorem 1.3 and Theorem 1.4.

Theorem 1.3. *If $N\geq 3$, $V(y)$ satisfies (V_1) and (V_2). There exist $\delta>0, \varepsilon_0>0$ and $\tau_\varepsilon\in(0,e^{-e^{\frac{1}{\varepsilon}}})$ such that, for any $\varepsilon\in(0,\varepsilon_0],\ \tau\in(0,\tau_\varepsilon],\ B_\delta(\xi_m)\cap B_\delta(\xi_j)=\emptyset,\ m,j=1,\cdots,k$, problem (1.5) has a solution $u_{\varepsilon,\tau}$ of the form*

$$u_{\varepsilon,\tau} = W_{\varepsilon,\tau} + \psi_{\varepsilon,\tau},$$

where $x^j\in B_\delta(\xi_j),\ \|\psi_{\varepsilon,\tau}\|_\varepsilon = O(\varepsilon^{\frac{N}{2}+1}),\ \|\psi_{\varepsilon,\tau}\|_{L^\infty}=O(\varepsilon)$.

Theorem 1.4. *If $N\geq 3$, $V(y)$ satisfies (V_1) and suppose x_0 is an isolated local maximum point of $V(y)$. There exist $\varepsilon_0>0$ and $\tau_\varepsilon\in(0,e^{-e^{\frac{1}{\varepsilon}}})$ such that, for any $\varepsilon\in(0,\varepsilon_0],\ \tau\in(0,\tau_\varepsilon]$, problem (1.5) has a solution $u_{\varepsilon,\tau}$ of the form*

$$u_{\varepsilon,\tau} = W_{\varepsilon,\tau} + \psi_{\varepsilon,\tau},$$

where $x^j\to x_0,\ \frac{|x^j-x^m|}{\varepsilon}\to+\infty$ if $m\neq j$ and $\|\psi_{\varepsilon,\tau}\|^2_\varepsilon = o(\varepsilon^N),\ \|\psi_{\varepsilon,\tau}\|^2_{L^\infty}=o(\varepsilon^2)$.

In fact, the non-degeneracy of positive multipeak solutions to the nonlinear Schrödinger equation has garnered significant interest in recent years (see [11] and references therein). The non-degeneracy refers to the stability of these solutions against small perturbations. The study of the non-degeneracy result is inspired by Guo-Musso-Peng-Yan [10]. Before introducing the non-degeneracy result, we need some notations.

Denote

$$H_\varepsilon(\mathbb{R}^N) = \left\{\eta\in H^1(\mathbb{R}^N) : \|\eta\|_* = \sup_{y\in\mathbb{R}^N}\Big(\sum_{j=1}^k e^{-\frac{|y-x^j|^2}{4\varepsilon^2}}\Big)^{-1}|\eta|\right\}.$$

For any $\eta \in H_\varepsilon(\mathbb{R}^N)$, we define

$$\mathscr{L}_\varepsilon(\eta) = -\varepsilon^2 \Delta \eta + V(y)\eta - (1+\log u_\varepsilon)\eta, \tag{1.6}$$

where $u_\varepsilon = W_\varepsilon + \psi_\varepsilon$. From the proof Theorem 1.1, we can know that W_ε and ψ_ε are the limits of $W_{\varepsilon,\tau}$ and $\psi_{\varepsilon,\tau}$ as $\tau \to 0$ in Theorem 1.3.

We obtain the non-degeneracy of the positive k-peak solution of (1.1).

Theorem 1.5. *Let the assumptions of Theorem 1.1 hold. Suppose that potential function $V(y)$ satisfies*

$$\det\left(\left(\frac{\partial^2 V(\xi_j)}{\partial \xi_{j,i} \partial \xi_{j,l}} \right)_{1 \le i,l \le N} \right) \neq 0, \quad \textit{for all } j = 1, \dots, k.$$

Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of positive solutions to problem (1.1), concentrating at the set $\{\xi_1, \dots, \xi_k\} \subset \mathbb{R}^N$. Let $\eta_\varepsilon \in H_\varepsilon(\mathbb{R}^N)$ be a solution to the linearized equation $\mathscr{L}_\varepsilon \eta_\varepsilon = 0$, where $\mathscr{L}_\varepsilon$ is defined as (1.6). Then $\eta_\varepsilon \equiv 0$.

Remark 1.1. *In Theorem 1.5, we address the non-degeneracy of the concentrated solutions with separated k points obtained in Theorem 1.1. As for the case of clustering concentrated solutions in Theorem 1.2, a completely new technical approach is required for the precise handling of interaction terms and so we will treat it as an independent work in a subsequent paper.*

Then, we give a brief introduction to the ideas and methods of the article.

Step (I): The existence of positive multipeak solutions.

We apply the Lyapunov-Schmidt method by solving the perturbed problem:

$$L_\varepsilon \psi_\varepsilon - l_\varepsilon - R_\varepsilon(\psi_\varepsilon) = 0, \tag{1.7}$$

where L_ε is a bounded linear operator in $H^1(\mathbb{R}^N)$ defined by

$$\langle L_\varepsilon \psi, v \rangle_\varepsilon = \int_{\mathbb{R}^N} (\varepsilon^2 \nabla \psi \nabla v + V(y)\psi v - g_\tau'(W)\psi v), \ \forall v \in H^1(\mathbb{R}^N), \tag{1.8}$$

$l_\varepsilon \in H^1(\mathbb{R}^N)$ by

$$\langle l_\varepsilon, v \rangle_\varepsilon = \int_{\mathbb{R}^N} \Big(\sum_{j=1}^k (V(x^j) - V(y)) U_{\varepsilon,j} + \Big(g_\tau(W) - \sum_{j=1}^k g(U_{\varepsilon,j}) \Big) \Big) v, \ \forall v \in H^1(\mathbb{R}^N), \tag{1.9}$$

and $R_\varepsilon(\psi) \in H^1(\mathbb{R}^N)$ by

$$\langle R_\varepsilon(\psi), v \rangle_\varepsilon = \int_{\mathbb{R}^N} (g_\tau(W+\psi) - g_\tau(W) - g_\tau'(W)\psi) v, \ \forall v \in H^1(\mathbb{R}^N). \tag{1.10}$$

We expect K_ε the approximate kernel of L_ε given by

$$K_\varepsilon = span\left\{ \frac{\partial U_{\varepsilon,j}}{\partial y_i}, j = 1, \cdots, k, i = 1, \cdots, N \right\}.$$

What challenges are we facing in this work? To explain this, we denote

$$\begin{aligned} &E := E_\varepsilon := K_\varepsilon^\perp \\ =&\left\{ v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} f'(U_{\varepsilon,j}) \frac{\partial U_{\varepsilon,j}}{\partial y_i} v = \int_{B_{2\varepsilon\sqrt{V(x^j)+\frac{N}{2}+2}}(x^j)} f'(U_{\varepsilon,j}) \frac{\partial U_{\varepsilon,j}}{\partial y_i} v = 0, j = 1, \cdots, k \right\} \end{aligned}$$

where we refer to the definition (2.1) and (2.2) in Section 2.

First, we know that to apply the reduction method, the process to construct a k-peak solution for (1.1) consist of two steps:

Step (i): Finite dimensional reduction: We solve (1.7) up to an approximate kernel K_ε of L_ε. That is, for any given x^j, we prove the existence of $\psi_\varepsilon \in E_\varepsilon$, such that, for some constants $a_{i,j}$,

$$L_\varepsilon \psi_\varepsilon - l_\varepsilon - R_\varepsilon(\psi_\varepsilon) = \sum_{j=1}^{k}\sum_{i=1}^{N} a_{i,j} f'(U_{\varepsilon,j}) \frac{\partial U_{\varepsilon,j}}{\partial y_i}.$$

Step (ii): Solve the finite dimensional problem and it suffices to choose $x_j, j = 1, \cdots, k$ appropriately such that all the constants $a_{i,j} = 0$ in Step (i).

A significant challenge arises during the execution of reduction Step (i). The logarithmic non-linearity maps an infinitesimal to a lower order one, making it infeasible to directly apply the implicit function theorem or the contraction mapping principle to solve $L_\varepsilon \psi = l_\varepsilon + R_\varepsilon(\psi)$, $\psi \in E$, with any uniform bound independent of τ. To address this issue, we will follow a three-step approach. Before that, we first partition the space into inner regions (near the concentration points) and an outer region (far from the concentration points) based on the distribution of the concentration points (refer to Section 3 for details).

Step 1: Although the reduction cannot be performed in the outer region, we can modify the nonlinear terms to make the corresponding functional convex. This involves considering a boundary value problem in the outer region where the solution u matches any given function u_0 in the inner region. By exploiting the convexity of the functional, we obtain an energy-minimizing solution corresponding to the boundary value problem in the outer region, dependent on the boundary function. This establishes a mapping $S(u_0)$ relative to the class of boundary functions, and it can be shown that this mapping exhibits favorable regularity.

Step 2: We combine S with the operator T derived from the fixed point theorem. Using the a priori estimates of the minimizer, it is straightforward to verify that the composite operator TS satisfies the conditions of the fixed point theorem, even though T alone does not. Essentially, we perform the reduction only within the inner region, achieving dimensional reduction, which facilitates the next phase of the reduction method, allowing us to address a finite-dimensional problem.

Step 3: We rigorously establish that the fixed point of the operator TS inherently corresponds to the fixed point of the operator T, thereby confirming the existence of the fixed point for T.

Finally, using the uniform estimates for $\psi_{\varepsilon,\tau}$, Theorem 1.1 can be obtained by taking the limits of $u_{\varepsilon,\tau}$ in Theorem 1.3. Specifically, we find a positive solution u_ε to (1.1) of the form $u_\varepsilon = W_\varepsilon + \psi_\varepsilon$, where W_ε and ψ_ε are the limits of $W_{\varepsilon,\tau}$ and $\psi_{\varepsilon,\tau}$ as $\tau \to 0$. See Section 4 for a rigorous proof.

Step (II): The non-degenerate of positive multipeak solutions.

For using the local Pohozaev identity, estimating the $\|\eta\|_\varepsilon$ is vital. We estimate the $\|\eta\|_\varepsilon$ and the perturbation's term $\|\psi\|_\varepsilon$ smaller by dividing regions, which are inner regions (near the concentration points) and an outer region (far from the concentration points) based on the distribution of the concentration points (refer to Section 5 for details).

In this paper, we denote various generic constants by C. We use $O(A)$, $o(A)$ to mean

$$|O(A)| \le C|A|, \quad \text{and} \quad o(A)/|A| \to 0 \quad \text{as } |A| \to 0,$$

respectively.

his paper is organized as follows. In Section 2, we introduce some notations and provide basic estimates. In Section 3, we modify the reduction framework and reduce the problem to a finite-dimensional one. In Section 4, we present the proofs of Theorem 1.1, Theorem 1.2, Theorem 1.3, and Theorem 1.4. The non-degeneracy of the multi-peak solutions in Theorem 1.5 is established in Section 5. Finally, the Appendix contains fundamental estimates related to the energy expansions.

2. Preliminaries and Modification

Inspired by [17], system (1.5) can be rewritten as $-\varepsilon^2\Delta u + V(y)u + h_\tau(u) = f_\tau(u)$, $u \in H^1(\mathbb{R}^N)$, where $\tau \in (0,1)$ and

$$h_\tau(u) = f_\tau(u) - g_\tau(u) = \begin{cases} \tau^{-1}\big(u - |u|^\tau u\big), & \text{if } |u| \le \left(\frac{1-\tau}{1+\tau}\right)^{\tau^{-1}}, \\ u + \left(\frac{1-\tau}{1+\tau}\right)^{1+\tau^{-1}}, & \text{if } u > \left(\frac{1-\tau}{1+\tau}\right)^{\tau^{-1}}, \\ u - \left(\frac{1-\tau}{1+\tau}\right)^{1+\tau^{-1}}, & \text{if } u < -\left(\frac{1-\tau}{1+\tau}\right)^{\tau^{-1}}, \end{cases}$$

$$f_\tau(u) = \begin{cases} 0, & \text{if } u \le \left(\frac{1-\tau}{1+\tau}\right)^{\tau^{-1}}, \\ \tau^{-1}\big(|u|^\tau u - u\big) + u + \left(\frac{1-\tau}{1+\tau}\right)^{1+\tau^{-1}}, & \text{if } u > \left(\frac{1-\tau}{1+\tau}\right)^{\tau^{-1}}, \\ \tau^{-1}\big(|u|^\tau u - u\big) + u - \left(\frac{1-\tau}{1+\tau}\right)^{1+\tau^{-1}}, & \text{if } u < -\left(\frac{1-\tau}{1+\tau}\right)^{\tau^{-1}}. \end{cases}$$

We also denote

$$h(u) = \begin{cases} -u\log|u|, & \text{if } |u| \le e^{-2}, \\ u + e^{-2}, & \text{if } u > e^{-2}, \\ u - e^{-2}, & \text{if } u < -e^{-2}, \end{cases}$$

$$f(u) = \begin{cases} 0, & \text{if } u \le e^{-2}, \\ u\log|u| + u + e^{-2}, & \text{if } u > e^{-2}, \\ u\log|u| + u - e^{-2}, & \text{if } u < -e^{-2}, \end{cases} \tag{2.1}$$

and

$$g(u) = u\log|u| = -h(u) + f(u),$$

$$H(u) = \int_0^u h(t)dt,\ F(u) = \int_0^u f(t)dt,\ G(u) = \int_0^u g(t)dt,$$

$$H_\tau(u) = \int_0^u h_\tau(t)dt,\ F_\tau(u) = \int_0^u f_\tau(t)dt,\ G_\tau(u) = \int_0^u g_\tau(t)dt.$$

Here, we give some properties of g_τ, h_τ, and f_τ.

Lemma 2.1. [16] *There exists $\tau_0 > 0$ such that, for $\tau \in (0,\tau_0]$, the following conclusions hold*
(i) g_τ, h_τ, and f_τ are odd and C^1 in $\mathbb{R}$; $g_\tau \in C^2$ in $\mathbb{R}\setminus\{0\}$; h_τ is strictly increasing in $\mathbb{R}$ and concave in $(0,\infty)$; h'_τ is locally Lipschitz continuous; H_τ is even, convex and nonnegative in $\mathbb{R}$.
(ii) For any $t_1, t_2 \in \mathbb{R}$ with $t_1 \ge t_2$,

$$t_1 - t_2 \le h_\tau(t_1) - h_\tau(t_2) \le 2h_\tau\Big(\frac{t_1 - t_2}{2}\Big).$$

(iii) Let $\delta \in (0, \frac{1}{2}e^{-2}]$. For any $s \geq \delta, t \in \mathbb{R}$,

$$|h_\tau(s+t) - h_\tau(s)| \leq 2|t \log \delta|,$$
$$|h_\tau(s+t) - h_\tau(s) - h'_\tau(s)t| \leq \frac{8t^2}{\delta}|\log \delta|.$$

Lemma 2.2. *There exists $\tau_\varepsilon \in (0, e^{-e^{\frac{1}{\varepsilon}}})$ such that, for each $\tau \in (0, \tau_\varepsilon)$,*

$$\|g_\tau(u_\varepsilon) - g(u_\varepsilon)\|_{L^\infty(\mathbb{R}^N)} + \|g'_\tau(u_\varepsilon) - g'(u_\varepsilon)\|_{L^\infty(\mathbb{R}^N)} = O\Big(e^{-e^{\frac{1}{\varepsilon}}}\Big),$$

$$\sum_{j=1}^{k} \|g'_\tau(U_{\varepsilon,j}) - g'(U_{\varepsilon,j})\|_{L^\infty(\mathbb{R}^N)} + \|g'_\tau(W) - g'(W)\|_{L^\infty(\mathbb{R}^N)} = O\Big(e^{-e^{\frac{1}{\varepsilon}}}\Big),$$

$$\sum_{j=1}^{k} \|g_\tau(U_{\varepsilon,j}) - g(U_{\varepsilon,j})\|_{L^\infty(\mathbb{R}^N)} + \|g_\tau(W) - g(W)\|_{L^\infty(\mathbb{R}^N)} = O\Big(e^{-e^{\frac{1}{\varepsilon}}}\Big),$$

and

$$\sum_{j=1}^{k} \|g_\tau(U_{\varepsilon,j}) - g(U_{\varepsilon,j})\|_{L^2(\mathbb{R}^N)} + \|g_\tau(W) - g(W)\|_{L^2(\mathbb{R}^N)} = \varepsilon^{\frac{N}{2}} O\Big(e^{-e^{\frac{1}{\varepsilon}}}\Big).$$

Proof. According to the uniform convergence $g_\tau(s) \to g(s)$ on any closed interval as $\tau \to 0$, we can obtain the first three conclusions of the lemma.

Next, we claim that there exists $\tau_\varepsilon \in (0, e^{-e^{\frac{1}{\varepsilon}}})$ such that $\|g_\tau(W) - g(W)\|_{L^2(\mathbb{R}^N)} \leq C\varepsilon^{\frac{N}{2}} e^{-e^{\frac{1}{\varepsilon}}}$ for some $C > 0$ independent of ε and τ.

In fact, for some $C, c > 0$, there holds $|W(y)| \leq Ce^{-c|y|^2}$ for $y \in \mathbb{R}^N \backslash B_{e^{\frac{1}{\varepsilon}}}(0)$. From $|g_\tau(s)| + |g(s)| \leq C\big(|s|^{\frac{2}{3}} + s^2\big)$ for C independent of $\tau \in (0,1)$, we obtain

$$\|g_\tau(W) - g(W)\|^2_{L^2\left(\mathbb{R}^N \backslash B_{e^{\frac{1}{\varepsilon}}}(0)\right)} \leq C\|W\|^2_{L^2\left(\mathbb{R}^N \backslash B_{e^{\frac{1}{\varepsilon}}}(0)\right)} + C\|W\|^{\frac{3}{2}}_{L^{\frac{3}{2}}\left(\mathbb{R}^N \backslash B_{e^{\frac{1}{\varepsilon}}}(0)\right)} \leq C\varepsilon^N e^{-e^{\frac{1}{\varepsilon}}}.$$

On the other hand, as $\tau \to 0$, $g_\tau(W(y)) \to g(W(y))$ uniformly for $y \in B_{e^{\frac{1}{\varepsilon}}}(0)$. So, we find τ_ε, such that, for $\tau \in (0, \tau_\varepsilon]$, $\|g_\tau(W) - g(W)\|^2_{L^2\left(B_{e^{\frac{1}{\varepsilon}}}(0)\right)} \leq C\varepsilon^N e^{-e^{\frac{1}{\varepsilon}}}$. Then

$$\|g_\tau(W) - g(W)\|_{L^2(\mathbb{R}^N)} \leq C\varepsilon^{\frac{N}{2}} e^{-e^{\frac{1}{\varepsilon}}}.$$

Similarly,

$$\sum_{j=1}^{k} \|g_\tau(U_{\varepsilon,j}) - g(U_{\varepsilon,j})\|_{L^2(\mathbb{R}^N)} = \varepsilon^{\frac{N}{2}} O\Big(e^{-e^{\frac{1}{\varepsilon}}}\Big).$$

□

Throughout the paper, we assume that $d = \min_{m \neq j} |x^m - x^j| > 0$ is a constant with $j = 1, \cdots, k$. We denote $s^\pm = \max\{0, \pm s\}$ for $s \in \mathbb{R}$. In what follows, we always assume $\tau \in (0, \tau_\varepsilon]$, where

τ_ε is determined in Lemma 2.2. Define

$$\begin{aligned}&E := E_\varepsilon := K_\varepsilon^\perp\\ =&\Big\{v\in H^1(\mathbb{R}^N): \int_{\mathbb{R}^N} f'(U_{\varepsilon,j})\frac{\partial U_{\varepsilon,j}}{\partial y_i}v = \int_{B_{2\varepsilon\sqrt{V(x^j)+\frac{N}{2}+2}}(x^j)} f'(U_{\varepsilon,j})\frac{\partial U_{\varepsilon,j}}{\partial y_i}v = 0, j=1,\cdots,k\Big\},\end{aligned} \tag{2.2}$$

which is a closed subspace of the Hilbert space $H^1(\mathbb{R}^N)$.

We denote, for each $M\in(0,\frac{1}{2}d)$,

$$Q_M = \bigcup_{j=1}^k B_M(x^j) \ and\ E_M = E\cap H_0^1(Q_M).$$

We remark that if $M\in\Big(2\varepsilon\sqrt{V(x^j)+\frac{N}{2}+2},\frac{1}{2}d\Big)$, then E_M and $H_0^1\left(\mathbb{R}^N\backslash Q_M\right)$ can be considered as closed subspaces of E.

Remark 2.1.

$$\sum_{j\neq m}^k \|U_{\varepsilon,j}\|_{L^\infty(B_{\frac{1}{2}d}(x^m))} + \sum_{j\neq m}^k \|U_{\varepsilon,j}\log U_{\varepsilon,j}\|_{L^\infty(B_{\frac{1}{2}d}(x^m))} = O\Big(e^{-\frac{d^2}{32\varepsilon^2}}\Big). \tag{2.3}$$

In fact, to show (2.3), we only need to notice that, for $y\in B_{\frac{1}{2}d}(x^m))$ and for any $j\neq m$, we have $|y-x^j|>\frac{1}{2}d$, so

$$\sum_{j\neq m}^k U_{\varepsilon,j} \le \sum_{j\neq m}^k |U_{\varepsilon,j}\log U_{\varepsilon,j}| = \sum_{j\neq m}^k e^{V(x^j)+\frac{N}{2}} e^{-\frac{|y-x^j|^2}{4\varepsilon^2}}\Big|V(x^j)+\frac{N}{2}-\frac{|y-x^j|^2}{4\varepsilon^2}\Big| = O\Big(e^{-\frac{d^2}{32\varepsilon^2}}\Big).$$

Lemma 2.3. *For any $\alpha>0,\beta>0$ and $l\neq j$, there exists a constant $\sigma>0$ such that $\int_{\mathbb{R}^N} U_{\varepsilon,l}^\alpha U_{\varepsilon,j}^\beta \le C\varepsilon^N e^{-\sigma\frac{|x^l-x^j|^2}{\varepsilon^2}}$.*

Proof. We may suppose $\alpha\ge\beta$. Recalling the expression of $U_{\varepsilon,j}$, by direct calculation, we have

$$\begin{aligned}\int_{\mathbb{R}^N} U_{\varepsilon,l}^\alpha U_{\varepsilon,j}^\beta =& \int_{\mathbb{R}^N} e^{(\alpha V(x^l)+\beta V(x^j)+\frac{N}{2}(\alpha+\beta))} e^{-\frac{\alpha|y-x^l|^2+\beta|y-x^j|^2}{4\varepsilon^2}}\\ \le& C\varepsilon^N\int_{\mathbb{R}^N} e^{(\alpha V(x^l)+\beta V(x^j)+\frac{N}{2}(\alpha+\beta))} e^{-\alpha|x|^2} e^{-\frac{\beta|2\varepsilon x+x^l-x^j|^2}{4\varepsilon^2}}\\ =&\begin{cases} C\varepsilon^N e^{(\alpha V(x^l)+\beta V(x^j)+\frac{N}{2}(\alpha+\beta))} e^{-\beta\frac{|x^l-x^j|^2}{8\varepsilon^2}}, \text{ if } \alpha>\beta,\\ C\varepsilon^N e^{(\alpha V(x^l)+\beta V(x^j)+\frac{N}{2}(\alpha+\beta))} e^{-(\beta-\theta)\frac{|x^l-x^j|^2}{8\varepsilon^2}}, \text{ if } \alpha=\beta,\end{cases}\end{aligned}$$

where $C>0$ is independent of ε and τ and $\theta>0$ is arbitrarily small. □

In what follows, we always assume $\tau\in(0,\tau_\varepsilon]$, where τ_ε is determined in Lemma 2.2.

Lemma 2.4. *There exist some $C>0$ independent of ε and τ, $\tau\in(0,\tau_\varepsilon)$, such that*

$$\|g_\tau(W)-\sum_{j=1}^k g(U_{\varepsilon,j})\|_{L^\infty(Q_{\frac{1}{8}d})} = O\Big(e^{-\frac{49d^2}{512\varepsilon^2}}\Big), \tag{2.4}$$

$$\|g_\tau(W)-\sum_{j=1}^{k}g(U_{\varepsilon,j})\|_{L^\infty(\mathbb{R}^N)}=O\Big(e^{-\frac{d^2}{256\varepsilon^2}}\Big), \tag{2.5}$$

$$\sup_{\psi\in H^1(Q_{\frac{1}{8}d}),\|\psi\|_\varepsilon=1}\int_{\mathbb{R}^N}(g_\tau(W)-\sum_{j=1}^{k}g(U_{\varepsilon,j}))\psi=\varepsilon^{\frac{N}{2}}O\Big(e^{-\frac{d^2}{16\varepsilon^2}}\Big), \tag{2.6}$$

and

$$\sup_{\psi\in H^1(\mathbb{R}^N),\|\psi\|_\varepsilon=1}\int_{\mathbb{R}^N}(g_\tau(W)-\sum_{j=1}^{k}g(U_{\varepsilon,j}))\psi=\varepsilon^{\frac{N}{2}}O\Big(e^{-\frac{d^2}{32\varepsilon^2}}\Big). \tag{2.7}$$

Proof. (i) In $B_{\frac{1}{8}d}(x^m)$, we have

$$\begin{aligned}
&g(W)-\sum_{j=1}^{k}g(U_{\varepsilon,j})\\
=&W\log|W|-\sum_{j=1}^{k}U_{\varepsilon,j}\log|U_{\varepsilon,j}|\\
=&(\sum_{j=1}^{k}U_{\varepsilon,j})\log\Big|\sum_{j=1}^{k}U_{\varepsilon,j}\Big|-\sum_{j=1}^{k}U_{\varepsilon,j}\log|U_{\varepsilon,j}|\\
=&(U_{\varepsilon,m}+\sum_{j\neq m}U_{\varepsilon,j})\log\Big|U_{\varepsilon,m}+\sum_{j\neq m}U_{\varepsilon,j}\Big|-U_{\varepsilon,m}\log|U_{\varepsilon,m}|-\sum_{j\neq m}U_{\varepsilon,j}\log|U_{\varepsilon,j}|\\
=&(U_{\varepsilon,m}+\sum_{j\neq m}U_{\varepsilon,j})\log\Big|1+U_{\varepsilon,m}^{-1}\sum_{j\neq m}U_{\varepsilon,j}\Big|+\sum_{j\neq m}U_{\varepsilon,j}\log(U_{\varepsilon,m}U_{\varepsilon,j}^{-1})\\
=&\sum_{j\neq m}U_{\varepsilon,j}+U_{\varepsilon,m}^{-1}(\sum_{j\neq m}U_{\varepsilon,j})^2+\sum_{j\neq m}U_{\varepsilon,j}\log(U_{\varepsilon,m}U_{\varepsilon,j}^{-1})+U_{\varepsilon,m}\mathscr{R}(U_{\varepsilon,m}^{-1}(\sum_{j\neq m}U_{\varepsilon,j})),
\end{aligned}$$

where $\mathscr{R}(s)=(1+s)\log|1+s|-s(1+s)$. Since, for each $s\in\mathbb{R}$, $|\mathscr{R}(s)|\leq 2s^2$, then it holds that

$$\Big|U_{\varepsilon,m}\mathscr{R}(U_{\varepsilon,m}^{-1}(\sum_{j\neq m}U_{\varepsilon,j}))\Big|\leq 2U_{\varepsilon,m}^{-1}(\sum_{j\neq m}U_{\varepsilon,j})^2.$$

Note that, in $B_{\frac{1}{8}d}(x^m)$, $U_{\varepsilon,j}<U_{\varepsilon,m}$ $(j\neq m)$, so $U_{\varepsilon,j}\log(U_{\varepsilon,m}U_{\varepsilon,j}^{-1})$ is positive. Hence

$$\Big|g(W)-\sum_{j=1}^{k}g(U_{\varepsilon,j})\Big|\leq\sum_{j\neq m}U_{\varepsilon,j}+U_{\varepsilon,m}^{-1}(\sum_{j\neq m}U_{\varepsilon,j})^2+\sum_{j\neq m}U_{\varepsilon,j}\log(U_{\varepsilon,m}U_{\varepsilon,j}^{-1}).$$

In $B_{\frac{1}{8}d}(x^m)$, there holds

$$\begin{aligned}
&\sum_{j\neq m}U_{\varepsilon,j}=\sum_{j\neq m}e^{V(x^j)+\frac{N}{2}-\frac{|y-x^j|^2}{4\varepsilon^2}}\text{ and }U_{\varepsilon,m}^{-1}\Big(\sum_{j\neq m}U_{\varepsilon,j}\Big)^2\leq C\sum_{j\neq m}e^{2V(x^j)-V(x^m)+\frac{N}{2}}e^{-\frac{|y-x^j|^2}{4\varepsilon^2}},\\
&\sum_{j\neq m}U_{\varepsilon,j}\log(U_{\varepsilon,m}U_{\varepsilon,j}^{-1})\leq C\sum_{j\neq m}e^{V(x^j)+\frac{N}{2}}e^{-\frac{|y-x^j|^2}{4\varepsilon^2}}\Big(V(x^m)-V(x^j)+\frac{|x^m-x^j|^2}{4\varepsilon^2}\Big).
\end{aligned}$$

When $j\neq m$, for $y\in B_{\frac{1}{8}d}(x^m)$, we have $|y-x^j|>\frac{7}{8}d$, so

$$\begin{aligned}
&\Big\|g(W)-\sum_{j=1}^{k}g(U_{\varepsilon,j})\Big\|_{L^\infty(B_{\frac{1}{8}d}(x^m))}=O\Big(e^{-\frac{49d^2}{512\varepsilon^2}}\Big),\\
&\Big\|g(W)-\sum_{j=1}^{k}g(U_{\varepsilon,j})\Big\|_{L^\infty(\cup B_{\frac{1}{8}d}(x^m))}=O\Big(e^{-\frac{49d^2}{512\varepsilon^2}}\Big).
\end{aligned}\tag{2.8}$$

By Lemma 2.2 and (2.8), we obtain (2.4).

(ii) Since, in $\mathbb{R}^N\setminus\bigcup_{m=1}^k B_{\frac{1}{8}d}(x^m)$, we have

$$\begin{aligned}
g(W)-\sum_{m=1}^{k}g(U_{\varepsilon,m})=\sum_{m=1}^{k}U_{\varepsilon,m}\log\Big|\frac{\sum_{s=1}^k U_{\varepsilon,s}}{U_{\varepsilon,m}}\Big|&\le C\sum_{m=1}^{k}U_{\varepsilon,m}^{\frac12}\Big(\sum_{s=1}^{k}U_{\varepsilon,s}\Big)^{\frac12}\\
&\le C\sum_{m=1}^{k}U_{\varepsilon,m}+C\sum_{s\neq m}\sum_{m=1}^{k}U_{\varepsilon,m}^{\frac12}U_{\varepsilon,s}^{\frac12}=O\Big(e^{-\frac{d^2}{256\varepsilon^2}}\Big),
\end{aligned}\tag{2.9}$$

which together with (2.8) and Lemma 2.2 yields

$$\|g_\tau(W)-\sum_{j=1}^{k}g(U_{\varepsilon,j})\|_{L^\infty(\mathbb{R}^N)}=O\Big(e^{-\frac{d^2}{256\varepsilon^2}}\Big).$$

(iii) For any $\psi\in H^1(\mathbb{R}^N)$ with $\|\psi\|_{L^2(B_{\frac{1}{8}d}(x^m))}\le\|\psi\|_\varepsilon=1$, by the Hölder inequality, in $B_{\frac{1}{8}d}(x^m)$, there holds

$$\begin{aligned}
\Big|\int_{B_{\frac{1}{8}d}(x^m)}(g(W)-\sum_{j=1}^{k}g(U_{\varepsilon,j}))\psi\Big|&\le C\Big(\int_{B_{\frac{1}{8}d}(x^m)}|\psi|^2\Big)^{\frac12}\|g(W)-\sum_{j=1}^{k}g(U_{\varepsilon,j})\|_{L^2(B_{\frac{1}{8}d}(x^m))}\\
&\le C\|g(W)-\sum_{j=1}^{k}g(U_{\varepsilon,j})\|_{L^2(B_{\frac{1}{8}d}(x^m))}.
\end{aligned}$$

When $j\neq m$, for $y\in B_{\frac{1}{8}d}(x^m)$, we have

$$\begin{aligned}
\sum_{j\neq m}\int_{B_{\frac{1}{8}d}(x^m)}U_{\varepsilon,j}^2&=\sum_{j\neq m}\int_{B_{\frac{1}{8}d}(x^m)}e^{2V(x^j)+N}e^{-\frac{|y-x^j|^2}{2\varepsilon^2}}\\
&\le C\varepsilon^N\int_{B_{\frac{d}{16\varepsilon}}(0)}e^{2V(x^j)+N}e^{-|x|^2}e^{-\frac{|x^m-x^j|^2}{4\varepsilon^2}}=\varepsilon^N O\Big(e^{-\frac{d^2}{4\varepsilon^2}}\Big),
\end{aligned}$$

$$
\begin{aligned}
&\sum_{j\neq m}\int_{B_{\frac{1}{8}d}(x^m)}(U_{\varepsilon,j}\log U_{\varepsilon,j})^2\\
&=\sum_{j\neq m}\int_{B_{\frac{1}{8}d}(x^m)}\Big(V(x^j)+\frac{N}{2}-\frac{|y-x^j|^2}{4\varepsilon^2}\Big)^2 e^{2V(x^j)+N}e^{-\frac{|y-x^j|^2}{2\varepsilon^2}}\\
&\leq C\varepsilon^N\int_{B_{\frac{d}{16\varepsilon}}(0)}\Big(V(x^j)+\frac{N}{2}-\frac{1}{2}|x|^2-\frac{|x^m-x^j|^2}{8\varepsilon^2}\Big)^2 e^{2V(x^j)+N}e^{-|x|^2}e^{-\frac{|x^m-x^j|^2}{4\varepsilon^2}}\\
&=\varepsilon^N O\Big(e^{-\frac{d^2}{8\varepsilon^2}}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{j\neq m}\int_{B_{\frac{1}{8}d}(x^m)}(U_{\varepsilon,j}\log(U_{\varepsilon,m}U_{\varepsilon,j}^{-1}))^2\\
&=\sum_{j\neq m}\int_{B_{\frac{1}{8}d}(x^m)}\Big(V(x^m)-V(x^j)+\frac{|y-x^j|^2-|y-x^m|^2}{4\varepsilon^2}\Big)^2 e^{2V(x^j)+N}e^{-\frac{|y-x^j|^2}{2\varepsilon^2}}\\
&\leq C\varepsilon^N\int_{B_{\frac{d}{16\varepsilon}}(0)}\Big(V(x^m)-V(x^j)+\frac{1}{2}|x|^2-\frac{|x^m-x^j|^2}{8\varepsilon^2}\Big)^2 e^{2V(x^j)+N}e^{-|x|^2}e^{-\frac{|x^m-x^j|^2}{4\varepsilon^2}}\\
&=\varepsilon^N O\Big(e^{-\frac{d^2}{8\varepsilon^2}}\Big).
\end{aligned}
$$

When $j\neq m$, for $y\in B_{\frac{1}{8}d}(x^m)$, there holds $|y-x^j|>\frac{7}{8}d$ and

$$
\begin{aligned}
&\Big\|g(W)-\sum_{j=1}^k g(U_{\varepsilon,j})\Big\|_{L^2(B_{\frac{1}{8}d}(x^m))}=\varepsilon^{\frac{N}{2}}O(e^{-\frac{d^2}{16\varepsilon^2}}),\\
&\Big\|g(W)-\sum_{j=1}^k g(U_{\varepsilon,j})\Big\|_{L^2(\cup B_{\frac{1}{8}d}(x^m))}=\varepsilon^{\frac{N}{2}}O\Big(e^{-\frac{d^2}{16\varepsilon^2}}\Big).
\end{aligned}
\tag{2.10}
$$

By Lemma 2.2, we obtain (2.6) from (2.10).

(iv) In $\mathbb{R}^N\setminus\bigcup_{m=1}^k B_{\frac{1}{8}d}(x^m)$, from (2.9), we also have

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N\setminus B_{\frac{1}{8}d}(x^m)}(g(W)-\sum_{j=1}^k g(U_{\varepsilon,j}))\psi\Big|\\
&\leq\Big(\int_{\mathbb{R}^N\setminus B_{\frac{1}{8}d}(x^m)}|\psi|^2\Big)^{\frac{1}{2}}\|g(W)-\sum_{j=1}^k g(U_{\varepsilon,j})\|_{L^2(\mathbb{R}^N\setminus B_{\frac{1}{8}d}(x^m))}\\
&\leq C\|g(W)-\sum_{j=1}^k g(U_{\varepsilon,j})\|_{L^2(\mathbb{R}^N\setminus B_{\frac{1}{8}d}(x^m))}=\varepsilon^{\frac{N}{2}}O\Big(e^{-\frac{d^2}{32\varepsilon^2}}\Big),
\end{aligned}
$$

which together with (2.10) and Lemma 2.2 yields

$$
\sup_{\psi\in H^1(\mathbb{R}^N),\|\psi\|_\varepsilon=1}\int_{\mathbb{R}^N}(g_\tau(W)-\sum_{j=1}^k g(U_{\varepsilon,j}))\psi=\varepsilon^{\frac{N}{2}}O\Big(e^{-\frac{d^2}{32\varepsilon^2}}\Big).
$$

□

We end this section with an estimate of L^∞. This is a variant of the De Giorgi-Nash-Moser's iteration.

Lemma 2.5. [16] *Assume $R>0$, $l\in\mathbb{N}\setminus\{0\}$, $\{u_i\}_{i=1}^l\subset C_0^1(B_R(0))$ with $\int_{B_R(0)}u_iu_j=\int_{B_R(0)}u_i=0$ $i,j=1,\cdots,l, i\neq j$. If $\phi\in H^1\big(B_{R+2}(0)\big)$ satisfies $\int_{B_{R+2}(0)}\nabla\phi\nabla v+b_1\phi v=\int_{B_{R+2}(0)}b_2v$ for all $v\in H_n$, where $H_n=\Big\{v\in H_0^1(B_{R+2}(0)):\int_{B_R(0)}vu_i=0, i=1,\cdots,l\Big\}$ and $b_1,\ b_2\in L^q(B_{R+2}(0))$ for some $q>\frac{N}{2}$, then $\|\phi\|_{L^\infty(B_{R+1}(0))}\le C\big(\|\phi\|_{H^1(B_{R+2}(0))}+\|b_2\|_{L^q(B_{R+2}(0))}\big)$, where $C>0$ is a constant depending only on $N,q,\|b_1\|_{L^q(B_R(0))}$, and $\sup_{y\in B_{R+1}(0)}\|b_1^-\|_{L^q(B_1(y))}$. In addition, if $u_i=0$ for all $i=1,\cdots,l$, then the constant C depends only on N,q, and $\sup_{y\in B_{R+1}(0)}\|b_1^-\|_{L^q(B_1(y))}$.*

3. REDUCTION OF THE PERTURBED PROBLEM

System (1.5) can also be rewritten as the following equation about ψ:

$$\begin{cases}L_\varepsilon\psi=l_\varepsilon+R_\varepsilon(\psi), & \text{in } \mathbb{R}^N,\\ \psi\in H^1(\mathbb{R}^N),\end{cases}$$

where L_ε is a bounded linear operator in $H^1(\mathbb{R}^N)$, defined by

$$\langle L_\varepsilon\psi,v\rangle_\varepsilon=\int_{\mathbb{R}^N}(\varepsilon^2\nabla\psi\nabla v+V(y)\psi v-g_\tau'(W)\psi v),\ \forall v\in H^1(\mathbb{R}^N),$$

$l_\varepsilon\in H^1(\mathbb{R}^N)$ satisfies

$$\langle l_\varepsilon,v\rangle_\varepsilon=\int_{\mathbb{R}^N}\Big(\sum_{j=1}^k(V(x^j)-V(y))U_{\varepsilon,j}+\Big(g_\tau(W)-\sum_{j=1}^k g(U_{\varepsilon,j})\Big)\Big)v,\ \forall v\in H^1(\mathbb{R}^N),$$

and $R_\varepsilon(\psi)\in H^1(\mathbb{R}^N)$ satisfies

$$\langle R_\varepsilon(\psi),v\rangle_\varepsilon=\int_{\mathbb{R}^N}(g_\tau(W+\psi)-g_\tau(W)-g_\tau'(W)\psi)v,\ \forall v\in H^1(\mathbb{R}^N).$$

We define

$$d_\varepsilon:=\varepsilon^{\frac{N}{2}+1-\theta}\text{ and }\dot{d}_\varepsilon:=\varepsilon^{1-\theta},\tag{3.1}$$

where $\theta>0$ is a small enough constant.

We also define the projection P_ε from $H^1(\mathbb{R}^N)$ to E as follows:

$$P_\varepsilon u=u-\sum_{j=1}^k\sum_{i=1}^N\int_{B_{2\varepsilon\sqrt{V(x^j)+\frac{N}{2}+2}}(x^j)}\Big(f'(U_{\varepsilon,j})\frac{\partial U_{\varepsilon,j}}{\partial y_i}u\Big)\frac{\partial U_{\varepsilon,j}}{\partial y_i}.$$

Our aim in this section is to solve $P_\varepsilon L_\varepsilon\psi=P_\varepsilon l_\varepsilon+P_\varepsilon R_\varepsilon(\psi)$ in

$$\Lambda=\big\{\psi\in E\cap L^\infty(\mathbb{R}^N)\mid\|\psi\|_\varepsilon\le d_\varepsilon\text{ and }\|\psi\|_{L^\infty(\mathbb{R}^N)}\le\dot{d}_\varepsilon\big\}.$$

Before the statement of main idea of the proof, we give some notations. Denote $\sigma_\varepsilon:=\frac{1}{2}\varepsilon\sqrt{|\ln\varepsilon|}$. For $i=1,2,3$, we set $D_i:=\bigcup_{j=1}^k B_{i\sigma_\varepsilon}(x^j)$. Take $\chi\in E\cap C_0^1(\mathbb{R}^N)$ such that

$$0\le\chi\le1,\ |\nabla\chi|\le2\sigma_\varepsilon^{-1},\ \chi=\begin{cases}1, & \text{in } D_2,\\ 0, & \text{in } \mathbb{R}^N\setminus D_3.\end{cases}\tag{3.2}$$

Then, for any $v\in E$, we have $\chi v\in E\cap H_0^1(D_3)$ and $(1-\chi)v\in E\cap H_0^1(\mathbb{R}^N\setminus D_3)$.

3.1. **The Minimization Problem.** In this section, we are to solve the following boundary problem

$$\begin{cases} -\varepsilon^2\Delta(W+\phi)+V(W+\phi)+h_\tau(W+\phi)=0 \text{ in } \mathbb{R}^N\backslash D_1, \\ \phi=\psi, \text{ in } D_1. \end{cases} \tag{3.3}$$

For each $\psi \in E$, we denote a set $E_\psi=\{u\in E|\ u=W+\psi \text{ in } D_1\}$ and define a functional $\Gamma: E_\psi \to \mathbb{R}$ as

$$\Gamma(u)=\frac{1}{2}\int_{\mathbb{R}^N}(\varepsilon^2|\nabla u|^2+Vu^2)+\int_{\mathbb{R}^N}H_\tau(u),\ u\in E_\psi.$$

Noting that H_τ is a strictly convex function and $H_\tau(t)\geq\dfrac{t^2}{2}$ for any $t\in\mathbb{R}$, there holds that Γ is a strictly convex functional on E_ψ and $\Gamma(u)\to\infty$, as $\|u\|_\varepsilon\to\infty$. Then there exists a unique minimizer to the following minimization problem

$$\inf_{u\in E_\psi}\Gamma(u). \tag{3.4}$$

We define the operator $S:E\to E$ as follows:

Definition 3.1. *For each $\psi\in E$, let $u_\psi\in E_\psi$ be the unique minimizer to (3.4). Define $S(\psi)=u_\psi-W$.*

It is clear that $S(\psi)=\psi$ in D_1. Moreover, $\phi=S(\psi)$ if and only if ϕ is the unique weak solution to (3.3). We next give some estimates on S restricted to Λ.

Lemma 3.1. *There exist $\theta>0$ small enough and $\varepsilon_0>0$ such that, for any $\varepsilon\in(0,\varepsilon_0)$, $\tau\in(0,\tau_\varepsilon)$ and some constants $C_1>0$ independent of ε and τ, the following statements hold:*

(i) If $\psi\in\Lambda$, then $\|S(\psi)\|_{\varepsilon,\mathbb{R}^N\backslash D_1}\leq C_1\frac{\sigma_\varepsilon^2}{\varepsilon^2}d_\varepsilon$ and

$$\|S(\psi)\|_{\varepsilon,\mathbb{R}^N\backslash D_2}\leq C_1\varepsilon^\theta d_\varepsilon,\ \|S(\psi)\|_{L^\infty(\mathbb{R}^N\backslash D_2)}\leq C_1\varepsilon^\theta\dot{d}_\varepsilon.$$

Furthermore, if $\|\psi\|_{L^\infty(D_2\backslash D_1)}\leq\frac{1}{2}\dot{d}_\varepsilon$, then $\|S(\psi)\|_{L^\infty(\mathbb{R}^N\backslash D_1)}\leq C_1\dot{d}_\varepsilon$.
(ii) If $\psi_i\in\Lambda$ and $\|\psi_i\|_{L^\infty(D_2\backslash D_1)}\leq\dot{d}_\varepsilon$, $i=1,2$, then

$$\|S(\psi_1)-S(\psi_2)\|_{\varepsilon,\mathbb{R}^N\backslash D_1}\leq C_1\frac{\sigma_\varepsilon^2}{\varepsilon^2}\|\psi_1-\psi_2\|_{\varepsilon,D_3\backslash D_1},$$
$$\|S(\psi_1)-S(\psi_2)\|_{\varepsilon,\mathbb{R}^N\backslash D_2}\leq C_1\varepsilon^\theta\|\psi_1-\psi_2\|_{\varepsilon,D_3\backslash D_1},$$
$$\|S(\psi_1)-S(\psi_2)\|_{L^\infty(\mathbb{R}^N\backslash D_1)}\leq C_1\frac{\sigma_\varepsilon^2}{\varepsilon^2}\|\psi_1-\psi_2\|_{L^\infty(D_3\backslash D_1)},$$
$$\|S(\psi_1)-S(\psi_2)\|_{L^\infty(\mathbb{R}^N\backslash D_2)}\leq C_1\varepsilon^\theta\|\psi_1-\psi_2\|_{L^\infty(D_3\backslash D_1)}.$$

Proof. (i) In $\mathbb{R}^N\backslash D_1$, we have $\|W\|_{L^\infty(\mathbb{R}^N\backslash D_1)}\leq\sum_{j=1}^k e^{V(x^j)+\frac{N}{2}}e^{-\frac{\sigma_\varepsilon^2}{4\varepsilon^2}}$. So, for all small ε, $f_\tau(W)=0$ in $\mathbb{R}^N\backslash D_1$ and for $v\in E\cap H_0^1(\mathbb{R}^N\backslash D_1)$, $\tau\in(0,\tau_\varepsilon]$, by (2.7), there holds

$$\int_{\mathbb{R}^N}\Big(\sum_{j=1}^k g(U_{\varepsilon,j})-g_\tau(W)\Big)v=O\Big(\varepsilon^{-(1+\theta)}e^{-\frac{d^2}{32\varepsilon^2}}d_\varepsilon\Big)\|v\|_\varepsilon$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N}\big(VW-\sum_{j=1}^{k}V(x^j)U_{\varepsilon,j}\big)v\\
=&\int_{\mathbb{R}^N}\big((V(x^j)+\langle\nabla V(x^j),y-x^j\rangle+O(|y-x^j|^2))W-\sum_{j=1}^{k}V(x^j)U_{\varepsilon,j}\big)v\\
=&\int_{\mathbb{R}^N}\big(\langle\nabla V(x^j),y-x^j\rangle+O(|y-x^j|^2)\big)Wv\\
=&O\big(\varepsilon^{\theta}d_\varepsilon\big)\|v\|_\varepsilon.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{\mathbb{R}^N}\big(\varepsilon^2\nabla W\nabla v+VWv+h_\tau(W)v\big)=&\int_{\mathbb{R}^N}\big(\varepsilon^2\nabla W\nabla v+VWv+h_\tau(W)v-f_\tau(W)v\big)\\
=&\int_{\mathbb{R}^N}\big(\sum_{j=1}^{k}g(U_{\varepsilon,j})-g_\tau(W)\big)v+\big(VW-\sum_{j=1}^{k}V(x^j)U_{\varepsilon,j}\big)v\\
=&O\big(\varepsilon^{\theta}d_\varepsilon\big)\|v\|_\varepsilon.
\end{aligned}\tag{3.5}$$

Setting $\phi=S(\psi)$, by (3.3) and (3.5), we obtain, for each $v\in E\cap H_0^1(\mathbb{R}^N\backslash D_1)$,

$$\int_{\mathbb{R}^N}\Big(\varepsilon^2\nabla\phi\nabla v+V\phi v+(h_\tau(W+\phi)-h_\tau(W))v\Big)=O\big(\varepsilon^{\theta}d_\varepsilon\big)\|v\|_\varepsilon.\tag{3.6}$$

For $n=1,2,\cdots,\Big[\frac{1}{2\varepsilon}\sqrt{|\ln\varepsilon|}\Big]-1$, we take $\tilde{\eta}_n\in E\cap C_0^1(\mathbb{R}^N)$ such that

$$0\le\tilde{\eta}_n\le1,\ |\nabla\tilde{\eta}_n|\le2,\ \tilde{\eta}_n=\begin{cases}0,\ \text{in } Q_{\sigma_\varepsilon+(n-1)\varepsilon^2},\\ 1,\ \text{in } \mathbb{R}^N\backslash Q_{\sigma_\varepsilon+n\varepsilon^2}.\end{cases}\tag{3.7}$$

Setting $v=\tilde{\eta}_n\phi$ in (3.6), we obtain from Lemma 2.1 (ii) that

$$\begin{aligned}
\|\phi\|^2_{\varepsilon,\mathbb{R}^N\backslash Q_{\sigma_\varepsilon+n\varepsilon^2}}\le&\int_{\mathbb{R}^N}\big(\varepsilon^2|\nabla\phi|^2+V\phi^2\big)\tilde{\eta}_n\\
\le&\int_{\mathbb{R}^N}\big(\varepsilon^2|\nabla\phi|^2+V\phi^2+(h_\tau(W+\phi)-h_\tau(W))\phi\big)\tilde{\eta}_n\\
=&-\int_{\mathbb{R}^N}\varepsilon^2\big(\nabla\phi\nabla\tilde{\eta}_n\big)\phi+O\big(\varepsilon^{\theta}d_\varepsilon\big)\|\tilde{\eta}_n\phi\|_\varepsilon\\
\le&\int_{Q_{\sigma_\varepsilon+n\varepsilon^2}\backslash Q_{\sigma_\varepsilon+(n-1)\varepsilon^2}}(\varepsilon^2|\nabla\phi|^2+\varepsilon^2\phi^2)+O\big(\varepsilon^{\theta}d_\varepsilon\big)\|\phi\|_{\varepsilon,\mathbb{R}^N\backslash Q_{\sigma_\varepsilon+(n-1)\varepsilon^2}}\\
\le&2e^{-\frac{1}{2}}\|\phi\|^2_{\varepsilon,\mathbb{R}^N\backslash Q_{\sigma_\varepsilon+(n-1)\varepsilon^2}}-\|\phi\|^2_{\varepsilon,\mathbb{R}^N\backslash Q_{\sigma_\varepsilon+n\varepsilon^2}}+O\big(\varepsilon^{2\theta}d_\varepsilon^2\big).
\end{aligned}$$

Therefore, it holds that

$$\begin{aligned}
\|\phi\|^2_{\varepsilon,\mathbb{R}^N\backslash Q_{2\sigma_\varepsilon-\varepsilon^2}}\le&e^{-\frac{\sqrt{|\ln\varepsilon|}}{4\varepsilon}-1}\|\phi\|^2_{\varepsilon,\mathbb{R}^N\backslash D_1}+(1+\sum_{n=1}^{\infty}e^{-\frac{n}{2}})O\big(\varepsilon^{2\theta}d_\varepsilon^2\big)\\
=&e^{-\frac{\sqrt{|\ln\varepsilon|}}{4\varepsilon}-1}\|\phi\|^2_{\varepsilon,\mathbb{R}^N\backslash D_1}+O\big(\varepsilon^{2\theta}d_\varepsilon^2\big).
\end{aligned}\tag{3.8}$$

Let $\chi \in E\cap C_0^1(\mathbb{R}^N)$ be the truncation function satisfying (3.2). Substituting $v=\chi(\phi-\psi)\in E\cap H_0^1(D_3\backslash D_1)$ in (3.6), we have

$$
\begin{aligned}
\|\phi\|^2_{\varepsilon,D_2\backslash D_1} &\le \int_{D_3\backslash D_1}\big(\varepsilon^2|\nabla\phi|^2+V\phi^2\big)\chi\\
&\le \int_{D_3\backslash D_1}\big(\varepsilon^2|\nabla\phi|^2+V\phi^2+(h_\tau(W+\phi)-h_\tau(W)\phi)\big)\chi\\
&=-\int_{D_3\backslash D_1}\varepsilon^2(\nabla\phi\nabla\chi)\phi+\int_{D_3\backslash D_1}\varepsilon^2\nabla\phi\nabla(\chi\psi)+V\phi\psi\chi\\
&\quad+\int_{D_3\backslash D_1}\big(h_\tau(W+\phi)-h_\tau(W)\big)\psi\chi+O\big(\varepsilon^\theta d_\varepsilon\big)\|\phi-\psi\|_{\varepsilon,D_3\backslash D_1}.
\end{aligned}\tag{3.9}
$$

Since $\sum_{j=1}^k e^{V(x^j)+\frac{N}{2}}e^{-\frac{9\sigma_\varepsilon^2}{4\varepsilon^2}}\le W<e^{-2}$ in $D_3\backslash D_1$, by Lemma 2.1 (iii), we have

$$
|h_\tau(W+\phi)-h_\tau(W)|\le 2\Big|\phi\log\Big(\sum_{j=1}^k e^{V(x^j)+\frac{N}{2}}e^{-\frac{9\sigma_\varepsilon^2}{4\varepsilon^2}}\Big)\Big|\le\frac{9\sigma_\varepsilon^2}{2\varepsilon^2}|\phi| \text{ in } D_3\backslash D_1. \tag{3.10}
$$

Then by (3.9), (3.10) and the Young's inequality, we have

$$
\begin{aligned}
\|\phi\|^2_{\varepsilon,D_2\backslash D_1} \le&2\varepsilon\sigma_\varepsilon^{-1}\|\phi\|^2_{\varepsilon,\mathbb{R}^N\backslash D_1}+\frac{\sigma_\varepsilon}{\varepsilon}\|\psi\|^2_{\varepsilon,D_3\backslash D_1}+\int_{(D_3\backslash D_1)}5\frac{\sigma_\varepsilon^2}{\varepsilon^2}|\phi\psi|\\
&+O\big(\varepsilon^\theta d_\varepsilon\big)\big(\|\phi\|_{\varepsilon,D_3\backslash D_1}+\|\psi\|_{\varepsilon,D_3\backslash D_1}\big)\\
\le&\big(2\varepsilon\sigma_\varepsilon^{-1}+\frac{1}{3}\big)\|\phi\|^2_{\varepsilon,\mathbb{R}^N\backslash D_1}+C\frac{\sigma_\varepsilon^4}{\varepsilon^4}\|\psi\|^2_{\varepsilon,D_3\backslash D_1}+O\big(\varepsilon^{2\theta}d_\varepsilon^2\big).
\end{aligned}\tag{3.11}
$$

From (3.8) and (3.11), we obtain, for ε sufficiently small,

$$
\|\phi\|_{\varepsilon,\mathbb{R}^N\backslash D_1}\le C\frac{\sigma_\varepsilon^2}{\varepsilon^2}\|\psi\|_{\varepsilon,D_3\backslash D_1}+O\big(\varepsilon^\theta d_\varepsilon\big)\le C\frac{\sigma_\varepsilon^2}{\varepsilon^2}d_\varepsilon. \tag{3.12}
$$

Moreover, recalling that $\sigma_\varepsilon^2=\frac{1}{4}\varepsilon^2|\ln\varepsilon|$, by (3.8) and (3.12), we obtain, for some $C,c>0$,

$$
\begin{aligned}
\|\phi\|_{\varepsilon,\mathbb{R}^N\backslash D_2}\le\|\phi\|_{\varepsilon,\mathbb{R}^N\backslash Q_{2\sigma_\varepsilon-\varepsilon^2}}\le&C\frac{\sigma_\varepsilon^2}{\varepsilon^2}e^{-\frac{\sqrt{|\ln\varepsilon|}}{4\varepsilon}-1}\|\psi\|_{\varepsilon,D_3\backslash D_1}+O\big(\varepsilon^\theta d_\varepsilon\big)\\
\le&Ce^{-c\frac{\sigma_\varepsilon}{\varepsilon^2}}d_\varepsilon+O\big(\varepsilon^\theta d_\varepsilon\big)\le C\varepsilon^\theta d_\varepsilon.
\end{aligned}
$$

Note by (3.3) that ϕ weakly solves

$$
-\varepsilon^2\Delta\phi+V\phi+h_\tau(W+\phi)-h_\tau(W)=\sum_{j=1}^k V(x^j)U_{\varepsilon,j}-VW+g_\tau(W)-\sum_{j=1}^k g(U_{\varepsilon,j}),
$$

in $H^1_{loc}(\mathbb{R}^N\backslash D_1)$. By monotonicity of h and (2.5), we have

$$
\frac{h_\tau(W+\phi)-h_\tau(W)}{\phi}\ge 0
$$

and

$$
\Big\|\Big(\sum_{j=1}^k V(x^j)U_{\varepsilon,j}-VW\Big)+g_\tau(W)-\sum_{j=1}^k g(U_{\varepsilon,j})\Big\|_{L^\infty(\mathbb{R}^N\backslash D_1)}=O\big(\varepsilon^\theta d_\varepsilon\big).
$$

Thus, as ε is small, $|\phi|$ weakly solves

$$-\varepsilon^2\Delta|\phi|+V|\phi|\le \varepsilon^\theta \dot{d}_\varepsilon,\ \text{in } H^1_{loc}(\mathbb{R}^N\backslash D_1).$$

We get by the local L^∞ estimate ([12], Theorem 8.17) that, for some $C(N)>0$ depending only on N,

$$\|\phi\|_{L^\infty(\mathbb{R}^N\backslash D_2)}\le C(N)\Big(\|\phi\|^2_{\varepsilon,\mathbb{R}^N\backslash Q_{2\sigma_\varepsilon-\varepsilon^2}}+\varepsilon^\theta\dot{d}_\varepsilon\Big)\le C\varepsilon^\theta\dot{d}_\varepsilon.$$

Now, we assume further that $\|\psi\|_{L^\infty(D_2\backslash D_1)}\le\frac{1}{2}\dot{d}_\varepsilon$. Setting $\tilde{\phi}(x)=\phi(\sigma_\varepsilon x+x^j)$, there holds

$$-\varepsilon^2\Delta|\tilde{\phi}|+V(\sigma_\varepsilon x+x^j)|\tilde{\phi}|\le\sigma_\varepsilon^2\varepsilon^\theta\dot{d}_\varepsilon,\ \text{in } B_2(0)\backslash B_1(0).$$

By the global L^∞ estimate ([12], Theorem 8.16), one has

$$\begin{aligned}\|\phi\|_{L^\infty(D_2\backslash D_1)}&=\|\tilde{\phi}\|_{L^\infty(B_2(0)\backslash B_1(0))}\\&\le\sup_{\partial B_2(0)\cup\partial B_1(0)}|\tilde{\phi}|+C(N)\sigma_\varepsilon^2\varepsilon^\theta\dot{d}_\varepsilon\\&=\frac{1}{2}\dot{d}_\varepsilon+C(N)\sigma_\varepsilon^2\varepsilon^\theta\dot{d}_\varepsilon\\&\le\frac{1}{2}\dot{d}_\varepsilon.\end{aligned}$$

(ii) Note that $\phi_1=S(\psi_1)$ and $\phi_2=S(\psi_2)$ satisfy $\phi_1-\phi_2=\psi_1-\psi_2$ in D_1 and for all $v\in H^1_0(\mathbb{R}^N\backslash D_1)$,

$$\int_{\mathbb{R}^N}\varepsilon^2\nabla(\phi_1-\phi_2)\nabla v+\int_{\mathbb{R}^N}\Big(V(\phi_1-\phi_2)+h_\tau(W+\phi_1)-h_\tau(W+\phi_2)\Big)v=0. \tag{3.13}$$

Set $v=\tilde{\eta}_n(\phi_1-\phi_2)$ in (3.13) where, $\tilde{\eta}_n,\ n=1,2,\cdots,\left[\frac{1}{2\varepsilon}\sqrt{|\ln\varepsilon|}\right]-1$, are truncation functions taken as (3.7). Then similar to (3.8), we obtain for some $C,c>0$

$$\|\phi_1-\phi_2\|^2_{\varepsilon,\mathbb{R}^N\backslash D_2}\le\|\phi_1-\phi_2\|^2_{\varepsilon,\mathbb{R}^N\backslash Q_{2\sigma_\varepsilon-\varepsilon^2}}\le Ce^{-c\frac{\sigma_\varepsilon}{\varepsilon^2}}\|\phi_1-\phi_2\|^2_{\varepsilon,\mathbb{R}^N\backslash D_1}. \tag{3.14}$$

On the other hand, let χ be the truncation function as (3.2) and set

$$v=\chi(\phi_1-\phi_2)-\chi(\psi_1-\psi_2)\in H^1_0(D_3\backslash D_1).$$

By (i), $|\phi_1|+|\phi_2|\le\dot{d}_\varepsilon\le\frac{W}{2}$ in $D_3\backslash D_1$, we have in $D_3\backslash D_1$,

$$|h_\tau(W+\phi_1)-h_\tau(W+\phi_2)|\le\Big|h'_\tau(\frac{W}{2})\Big||\phi_1-\phi_2|\le C\frac{\sigma_\varepsilon^2}{\varepsilon^2}|\phi_1-\phi_2|.$$

Similar to (3.11), it follows that

$$\|\phi_1-\phi_2\|^2_{\varepsilon,D_2\backslash D_1}\le\Big(2\varepsilon\sigma_\varepsilon^{-1}+\frac{1}{3}\Big)\|\phi_1-\phi_2\|^2_{\varepsilon,\mathbb{R}^N\backslash D_1}+C\frac{\sigma_\varepsilon^4}{\varepsilon^4}\|\psi_1-\psi_2\|^2_{\varepsilon,D_3\backslash D_1}. \tag{3.15}$$

Thus, (3.14) and (3.15) imply

$$\begin{aligned}&\|\phi_1-\phi_2\|^2_{\varepsilon,\mathbb{R}^N\backslash D_1}\le C\frac{\sigma_\varepsilon^4}{\varepsilon^4}\|\psi_1-\psi_2\|^2_{\varepsilon,D_3\backslash D_1},\\&\|\phi_1-\phi_2\|^2_{\varepsilon,\mathbb{R}^N\backslash Q_{2\sigma_\varepsilon-\varepsilon^2}}\le C\frac{\sigma_\varepsilon^4}{\varepsilon^4}e^{-c\frac{\sigma_\varepsilon}{\varepsilon^2}}\|\psi_1-\psi_2\|^2_{\varepsilon,D_3\backslash D_1}\le C\varepsilon^\theta\|\psi_1-\psi_2\|^2_{\varepsilon,D_3\backslash D_1}.\end{aligned} \tag{3.16}$$

To obtain the L^∞ estimate, we note that by Lemma 2.1 (ii) $|h_\tau(W+\phi_1)-h_\tau(W+\phi_2)| \geq |\phi_1-\phi_2|$. Therefore, (3.3) implies that $|\phi_1-\phi_2|$ weakly solves

$$-\varepsilon^2\Delta|\phi_1-\phi_2|+(V+1)|\phi_1-\phi_2| \leq 0, \text{ in } \mathbb{R}^N\backslash D_1.$$

Then the conclusion of (ii) follows from (3.16) and the local L^∞ estimate ([12], Theorem 8.17). □

The following lemma proves that linear operator $P_\varepsilon L_\varepsilon$ is bounded and invertible from E to E.

Lemma 3.2. *There exist $\varepsilon_0>0,\ \delta>0,\ \gamma>0$ such that, for any $\varepsilon\in(0,\varepsilon_0)$, $\tau\in(0,\tau_\varepsilon)$ and $x^j\in B_\delta(\xi_j)$, $\|P_\varepsilon L_\varepsilon\psi\|_\varepsilon \geq \gamma\|\psi\|_\varepsilon$ for all $\psi\in E$.*

Proof. Arguing by contradiction, we assume that there exist $\varepsilon_n\to 0$, $x^{j,\varepsilon_n}\to\xi_j$, $\tau_{\varepsilon_n}\to 0$ and $\psi_{\varepsilon_n}\in E_{\varepsilon_n}$ such that $\|P_{\varepsilon_n}L_{\varepsilon_n}\psi_n\|_\varepsilon = o(1)\|\psi_n\|_\varepsilon$. For simplicity, we denote $P_\varepsilon L_\varepsilon$ for $P_{\varepsilon_n}L_{\varepsilon_n}$. We may assume $\|\psi_n\|_\varepsilon^2 = 2^N\varepsilon_n^N$, so

$$\begin{aligned}&\int_{\mathbb{R}^N}\varepsilon_n^2\nabla\psi_n\nabla v+(V(y)+\bar\tau_n^{-1})\psi_n v-\left(1+\bar\tau_n^{-1}\right)W^{\bar\tau_n}\psi_n v\\ =&\langle L_\varepsilon\psi_n,v\rangle_{\varepsilon_n}=\langle P_\varepsilon L_\varepsilon\psi_n,v\rangle_{\varepsilon_n}=o(1)\|\psi_n\|_{\varepsilon_n}\|v\|_{\varepsilon_n}=o\left(2^{\frac{N}{2}}\varepsilon_n^{\frac{N}{2}}\right)\|v\|_{\varepsilon_n},\ \forall v\in E.\end{aligned}\tag{3.17}$$

In particular,

$$\int_{\mathbb{R}^N}\varepsilon_n^2|\nabla\psi_n|^2+(V(y)+\bar\tau_n^{-1})\psi_n^2-\left(1+\bar\tau_n^{-1}\right)W^{\bar\tau_n}\psi_n^2=o\left(2^N\varepsilon_n^N\right)\tag{3.18}$$

and

$$\int_{\mathbb{R}^N}\varepsilon_n^2|\nabla\psi_n|^2+V(y)\psi_n^2=2^N\varepsilon_n^N.\tag{3.19}$$

Let $\bar\psi_n(x)=\psi_n(2\varepsilon_n x+x^{j,\varepsilon_n})$. Then $\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla\bar\psi_n|^2+V(2\varepsilon_n x+x^{j,\varepsilon_n})\psi_n^2\leq 1$. Without loss of generality, we may assume there exist $\psi\in H^1(\mathbb{R}^N)$ such that

$$\bar\psi_n\rightharpoonup\psi \text{ weakly in } H^1_{loc}(\mathbb{R}^N) \textit{ and } \bar\psi_n\to\psi \text{ strongly in } L^2_{loc}(\mathbb{R}^N).$$

In view of

$$\int_{\mathbb{R}^N}f'(U_{\varepsilon_n,j})Z_j v=\int_{B_{2\varepsilon_n\sqrt{V(x^j)+\frac{N}{2}+2}}(x^j)}f'(U_{\varepsilon_n,j})Z_j v=0,$$

we have

$$\int_{\mathbb{R}^N}f'(U_j)\frac{\partial U_j}{\partial x_i}\bar v_n=\int_{B_{\sqrt{V(x^j)+\frac{N}{2}+2}}(0)}f'(U_j)\frac{\partial U_j}{\partial x_i}\bar v_n=0,\ i=1,\cdots,N.$$

Thus $\int_{\mathbb{R}^N}f'(U_j)\frac{\partial U_j}{\partial x_i}v=0$.

Next, we claim that

$$\int_{\mathbb{R}^N}|y|^2\psi^2dy<\infty.\tag{3.20}$$

To show (3.20), we have $\|W\|_{L^\infty(\mathbb{R}^N)}\leq\sum_{j=1}^k e^{V(x^{j,\varepsilon_n})+\frac{N}{2}}$. By (3.18), we have, for some $C>0$ independent of ε_n,

$$\int_{\mathbb{R}^N}\varepsilon_n^2|\nabla\psi_n|^2+V(y)\psi_n^2+h'_{\bar\tau_n}(W)\psi_n^2=\int_{\mathbb{R}^N}f'_{\bar\tau_n}(W)\psi_n^2+o(2^N\varepsilon_n^N)\leq C\int_{\mathbb{R}^N}\psi_n^2+2^N\varepsilon_n^N.$$

From (3.19), we have $0 \le \int_{\mathbb{R}^N} h'_{\bar{\tau}_n}(W)\psi_n^2 \le C\varepsilon_n^N$. On the other hand, there exists $R_1 > 0$ independent of ε_n such that $0 < W(y) \le \frac{1}{2}e^{-2}$ for $y \in B_{\frac{1}{4}d(x^{j,\varepsilon_n})} \backslash B_{2\varepsilon_n R_1(x^{j,\varepsilon_n})}$. So, for any $R \ge R_1$, we have $B_{2\sqrt{\varepsilon_n}R(x^{j,\varepsilon_n})} \subset B_{\frac{1}{4}d(x^{j,\varepsilon_n})}$ for all small ε_n and

$$\begin{aligned}&\limsup_{n\to\infty}\int_{B_{2\sqrt{\varepsilon_n}R(x^{j,\varepsilon_n})}\backslash B_{2\varepsilon_n R_1(x^{j,\varepsilon_n})}} \frac{1-W^{\bar{\tau}_n}}{\bar{\tau}_n}\psi_n^2 \\ \le&\limsup_{n\to\infty}\int_{B_{\frac{1}{4}d(x^{j,\varepsilon_n})}\backslash B_{2\varepsilon_n R_1(x^{j,\varepsilon_n})}} (h'_{\bar{\tau}_n}(W)+1)\psi_n^2 \le C\varepsilon_n^N.\end{aligned}$$

That is,

$$\limsup_{n\to\infty}\int_{B_{\frac{R}{\sqrt{\varepsilon_n}}(0)}\backslash B_{R_1(0)}} \frac{1-\left(U_j(x)+\sum_{m\neq j}U_m(2\varepsilon_n x+x^{j,\varepsilon_n})\right)^{\bar{\tau}_n}}{\bar{\tau}_n}\psi_n^2 \le C, \tag{3.21}$$

where $C > 0$ is independent of R and ε_n. Hence, as $n \to \infty$, we have

$$\frac{1-\left(U_j(x)+\sum_{m\neq j}U_m(2\varepsilon_n x+x^{j,\varepsilon_n})\right)^{\bar{\tau}_n}}{\bar{\tau}_n} \to -\log U_j. \tag{3.22}$$

Note that in $B_{\frac{R}{\sqrt{\varepsilon_n}}(0)} \backslash B_{R_1(0)}$, $U_j(x) = e^{V(x^j)+\frac{N}{2}}e^{-|x|^2} \le W(2\varepsilon_n x + x^{j,\varepsilon_n}) \le \frac{1}{2}e^{-2}$. Then by (3.21), (3.22) and Fatou s Lemma, we have $\int_{B_{\frac{R}{\sqrt{\varepsilon_n}}(0)}\backslash B_{R_1(0)}} -(\log U_j)\psi^2 \le C$, for some C independent of R. Since $U_j(x) = e^{V(x^j)+\frac{N}{2}}e^{-|x|^2}$, we obtain (3.20) by letting $\varepsilon_n \to 0$.

Next, we clam $\psi = 0$. Let

$$\bar{E} = \left\{\psi \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} f'(U_j)\frac{\partial U_j}{\partial x_i}\psi = 0,\ \int_{\mathbb{R}^N}|y|^2\psi^2 dy < \infty\right\}$$

be the Banach space with the norm $\|v\|_1^2 = \|v\|_{\varepsilon_n}^2 + \int_{\mathbb{R}^N}|y|^2 v^2$ for all $v \in \bar{E}$. For any $R > 0$, let $v \in C_0^\infty(B_R(0)) \cap \bar{E}$, $v_n(y) := v\left(\frac{y-x^{j,\varepsilon_n}}{2\varepsilon_n}\right) \in C_0^\infty\left(B_{2R\varepsilon_n}(x^{j,\varepsilon_n})\right)$, then we find from (3.17) and (3.22) that

$$\int_{\mathbb{R}^N} \nabla v \nabla \psi + V(\xi_j)v\psi - v\psi - (\log U_j)v\psi = 0. \tag{3.23}$$

By the density of $C_0^\infty(\mathbb{R}^N)$ in $v \in \bar{E}$, (3.23) holds for any $v \in \bar{E}$. However, (3.23) also holds for $v = \frac{\partial U_j}{\partial x_i}$. Thus, for any $v \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}|y|^2v^2 < \infty$ (3.23) holds. Or equivalently, ψ solves (1.4). Since U_j is non-degenerate, then $\psi = \sum_{i=1}^N C_i \frac{\partial U_j}{\partial x_i}$. We obtain $C_i = 0$, so $\psi = 0$. Thus

$$\int_{B_{2R\varepsilon_n}(x^{j,\varepsilon_n})} \psi_n^2 = o(2^N\varepsilon_n^N),\ \forall R > 0.$$

Since

$$W(y) = \sum_{j=1}^k U_{\varepsilon_n,j}(y) = \sum_{j=1}^k e^{V(x^{j,\varepsilon_n})+\frac{N}{2}}e^{-\frac{|y-x^{j,\varepsilon_n}|^2}{4\varepsilon_n^2}},$$

we fix an $R > 0$ such that $W(y) \le e^{-3}$ for $y \in \mathbb{R}^N \backslash \cup_{j=1}^k B_{2\varepsilon_n R(x^{j,\varepsilon_n})}$ and $W(y) \le 2e^{V(x^{j,\varepsilon_n})+\frac{N}{2}}$ for $y \in \cup_{j=1}^k B_{2\varepsilon_n R(x^{j,\varepsilon_n})}$. Then, for n sufficiently large,

$$\bar{\tau}_n^{-1} - (1+\bar{\tau}_n^{-1})W^{\bar{\tau}_n} \ge \bar{\tau}_n^{-1} - (1+\bar{\tau}_n^{-1})e^{-3\bar{\tau}_n} \ge 1 \text{ in } y \in \mathbb{R}^N \backslash \cup_{j=1}^k B_{2\varepsilon_n R(x^{j,\varepsilon_n})}$$

and

$$\bar{\tau}_n^{-1} - (1+\bar{\tau}_n^{-1})W^{\bar{\tau}_n} \geq -V(x^{j,\varepsilon_n}) - \frac{N}{2} - 2\log 2 - 2 \text{ in } y \in B_{2\varepsilon_n R(x^{j,\varepsilon_n})}.$$

Thus

$$\begin{aligned}
& o(2^N\varepsilon_n^N) \\
=& \int_{\mathbb{R}^N} \varepsilon_n^2|\nabla\psi_n|^2 + (V(y)+\bar{\tau}_n^{-1})\psi_n^2 - \big(1+\bar{\tau}_n^{-1}\big)W^{\bar{\tau}_n}\psi_n^2 \\
=& \int_{\mathbb{R}^N\setminus\cup_{j=1}^k B_{2\varepsilon_n R(x^{j,\varepsilon_n})}} \varepsilon_n^2|\nabla\psi_n|^2 + (V(y)+\bar{\tau}_n^{-1})\psi_n^2 - \big(1+\bar{\tau}_n^{-1}\big)W^{\bar{\tau}_n}\psi_n^2 \\
& + \int_{\cup_{j=1}^k B_{2\varepsilon_n R(x^{j,\varepsilon_n})}} \varepsilon_n^2|\nabla\psi_n|^2 + (V(y)+\bar{\tau}_n^{-1})\psi_n^2 - \big(1+\bar{\tau}_n^{-1}\big)W^{\bar{\tau}_n}\psi_n^2 \\
\geq& \int_{\mathbb{R}^N} \varepsilon_n^2|\nabla\psi_n|^2 + \int_{\mathbb{R}^N\setminus\cup_{j=1}^k B_{2\varepsilon_n R(x^{j,\varepsilon_n})}} \psi_n^2 + \sum_{j=1}^k \int_{B_{2\varepsilon_n R(x^{j,\varepsilon_n})}} \big(-V(x^{j,\varepsilon_n}) - \frac{1}{2}N - 2\log 2 - 2\big)\psi_n^2 \\
& + o(2^N\varepsilon_n^N) \\
=& 2^N\varepsilon_n^N + o(2^N\varepsilon_n^N).
\end{aligned}$$

This is a contradiction to (3.18). □

We define the operators B and T as follows:

Definition 3.2. *Define $B(\psi) := P_\varepsilon l_\varepsilon + P_\varepsilon R_\varepsilon(\psi)$. From Lemmas 3.2, we can define $T(\psi) := (P_\varepsilon L_\varepsilon)^{-1}B(\psi) = (P_\varepsilon L_\varepsilon)^{-1}P_\varepsilon l_\varepsilon + (P_\varepsilon L_\varepsilon)^{-1}P_\varepsilon R_\varepsilon(\psi)$.*

3.2. **Finding the Fixed Point of TS.** Due to the sublinear nonlinear term, it is very difficult or even impossible to directly prove the existence of a fixed point for the operator T. Therefore, we utilize the a priori estimates obtained for the minimizer to first find a fixed point of the composite operator TS, and then demonstrate that it is indeed a fixed point of T. To apply the fixed point theorem to the composite operator TS, we require the following estimates.

Lemma 3.3. *There exist $\theta > 0$ small enough and $C > 0$ independent of ε and τ, such that $\|l_\varepsilon\|_\varepsilon \leq C\varepsilon^\theta d_\varepsilon$ and $\|l_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq C\varepsilon^\theta \dot{d}_\varepsilon$.*

Proof. Recall that $\langle l_\varepsilon, v\rangle_\varepsilon = \int_{\mathbb{R}^N}\Big(\sum_{j=1}^k (V(x^j)-V(y))U_{\varepsilon,j} + \Big(g_\tau(W) - \sum_{j=1}^k g(U_{\varepsilon,j})\Big)\Big)v$. One has

$$\begin{aligned}
& \Big|\int_{\mathbb{R}^N}(V(y)-V(x^j))U_{\varepsilon,j}v\Big| \\
\leq& C\Big(\int_{\mathbb{R}^N}(V(y)-V(x^j))^2U_{\varepsilon,j}^2\Big)^{\frac{1}{2}}\|v\|_\varepsilon \\
=& C\Big(2^N\varepsilon^N\int_{\mathbb{R}^N}(V(2\varepsilon x+x^j)-V(x^j))^2U_{\varepsilon,j}^2(2\varepsilon x+x^j)\Big)^{\frac{1}{2}}\|v\|_\varepsilon \\
\leq& C\varepsilon^{\frac{N}{2}}\Big(\int_{\mathbb{R}^N}(\varepsilon|\nabla V(x^j)||x| + \varepsilon^2|x|^2)^2U_{\varepsilon,j}^2(2\varepsilon x+x^j)\Big)^{\frac{1}{2}}\|v\|_\varepsilon \\
\leq& C\varepsilon^{\frac{N}{2}}(\varepsilon|\nabla V(x^j)| + \varepsilon^2)\|v\|_\varepsilon = C\varepsilon^\theta d_\varepsilon\|v\|_\varepsilon.
\end{aligned} \tag{3.24}$$

On the other hand, from lemma 2.1 and lemma 2.5, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^N}\Big(g_\tau(W)-\sum_{j=1}^k g(U_{\varepsilon,j})\Big)v\Big| \le &\int_{\mathbb{R}^N}\Big|\sum_{j=1}^k g(U_{\varepsilon,j})-g_\tau(W)\Big||v| \\ &\le C\varepsilon^{-(1+\theta)}e^{-\frac{d^2}{32\varepsilon^2}}d_\varepsilon\|v\|_\varepsilon \le C\varepsilon^\theta d_\varepsilon\|v\|_\varepsilon.\end{aligned} \tag{3.25}$$

From (3.24) and (3.25), we see that $\|l_\varepsilon\|_\varepsilon \le C\varepsilon^\theta d_\varepsilon$. Similarly, $\|l_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \le C\varepsilon^\theta \dot{d}_\varepsilon$. □

Next, it is more difficult to estimate the error term $R_\varepsilon(S(\psi))$.

Lemma 3.4. *There exist $\theta>0$ small enough and $C>0$ independent of ε and τ, such that $\|R_\varepsilon(S(\psi))\|_\varepsilon \le C\varepsilon^\theta d_\varepsilon$ and $\|R_\varepsilon(S(\psi))\|_{L^\infty(\mathbb{R}^N)} \le C\varepsilon^\theta \dot{d}_\varepsilon$.*

Proof. Recall that $\langle R_\varepsilon(\psi),v\rangle_\varepsilon = \int_{\mathbb{R}^N}(g_\tau(W+\psi)-g_\tau(W)-g'_\tau(W)\psi)v$. In D_2, for any $\bar\theta\in(0,1)$, from Lemma 3.1, we direct calculate

$$\begin{aligned}|\langle R_\varepsilon(S(\psi)),v\rangle_\varepsilon| =&\Big|\int_{D_2}\big(g_\tau(W+S(\psi))-g_\tau(W)-g'_\tau(W)S(\psi)\big)v\Big| \\ =&\Big|\int_{D_2} g''_\tau(W+\bar\theta S(\psi))\big(S(\psi)\big)^2 v\Big| \\ \le&\int_{D_1}\Big|g''_\tau\Big(e^{-\frac{\sigma_\varepsilon^2}{4\varepsilon^2}}\Big)\big(S(\psi)\big)^2 v\Big| + \int_{D_2\setminus D_1}\Big|g''_\tau\Big(e^{-\frac{\sigma_\varepsilon^2}{\varepsilon^2}}\Big)\big(S(\psi)\big)^2 v\Big| \\ \le& Ce^{\frac{\sigma_\varepsilon^2}{4\varepsilon^2}}\dot{d}_\varepsilon d_\varepsilon\|v\|_\varepsilon + Ce^{\frac{\sigma_\varepsilon^2}{\varepsilon^2}}\dot{d}_\varepsilon d_\varepsilon\|v\|_\varepsilon \le C\varepsilon^\theta d_\varepsilon\|v\|_\varepsilon.\end{aligned} \tag{3.26}$$

On the other hand, in $\mathbb{R}^N\setminus D_2$, we use the fact that $S(\psi)$ is a solution of system (3.3). Since system (3.3) is equivalent to $L_\varepsilon S(\psi) = l_\varepsilon + R_\varepsilon(S(\psi))$, then, by Lemma 3.1, there holds

$$\begin{aligned}\|R_\varepsilon(S(\psi))\|^2_{\varepsilon,\mathbb{R}^N\setminus D_2} =&\|L_\varepsilon S(\psi)-l_\varepsilon\|^2_{\varepsilon,\mathbb{R}^N\setminus D_2} \le C\|S(\psi)\|^2_{\varepsilon,\mathbb{R}^N\setminus D_2}+C\|l_\varepsilon\|^2_{\varepsilon,\mathbb{R}^N\setminus D_2} \\ \le& CC_1^2\varepsilon^{2\theta}d_\varepsilon^2 + C\varepsilon^{2\theta}d_\varepsilon^2 \le C\varepsilon^{2\theta}d_\varepsilon^2.\end{aligned}$$

From (3.26) and (3.26), we have $\|R_\varepsilon(S(\psi))\|_\varepsilon \le C\varepsilon^\theta d_\varepsilon$. Similarly, we can obtain

$$\|R_\varepsilon(S(\psi))\|_{L^\infty(\mathbb{R}^N)} = O\big(\varepsilon^\theta \dot{d}_\varepsilon\big).$$

□

Based on the above estimates, we have the following result.

Proposition 3.1. *There exist $\delta>0$, $\theta>0$ small enough and $\varepsilon_0>0$ such that, for any $\varepsilon\in(0,\varepsilon_0]$, $B_\delta(\xi_m)\cap B_\delta(\xi_j)=\emptyset$, $m,j=1,\cdots,k$, there exist a unique C^1-map $\psi\in E$ satisfying*

$$\psi = TS(\psi) \text{ and } \|\psi\|_\varepsilon \le d_\varepsilon,\ \|\psi\|_{L^\infty} \le \dot{d}_\varepsilon.$$

Proof. Recalling that the operator of T is defined in 3.2. In view of Lemma 3.2, there holds

$$\psi = TS(\psi) = (P_\varepsilon L_\varepsilon)^{-1}P_\varepsilon l_\varepsilon + (P_\varepsilon L_\varepsilon)^{-1}P_\varepsilon R_\varepsilon(S(\psi)).$$

Now, we apply the fixed point theorem in $\Lambda = \big\{\psi\in E\cap L^\infty(\mathbb{R}^N)\mid \|\psi\|_\varepsilon \le d_\varepsilon \text{ and } \|\psi\|_{L^\infty(\mathbb{R}^N)} \le \dot{d}_\varepsilon\big\}$.

(i) TS maps Λ onto Λ. In fact, for any $\psi\in\Lambda$, it follows from Lemmas 3.3 and 3.4 that

$$\|TS(\psi)\|_\varepsilon \le C\|l_\varepsilon\|_\varepsilon + C\|R_\varepsilon(S(\psi))\|_\varepsilon \le \frac{1}{2}d_\varepsilon. \tag{3.27}$$

Similarly, $\|TS(\psi)\|_{L^\infty(\mathbb{R}^N)} \le \frac{1}{2}\dot{d}_\varepsilon$.

(ii) TS is a contraction map. To prove this claim, from Lemma 3.1, for any $\psi_1,\psi_2 \in \Lambda$ and $\bar\theta \in (0,1)$, in D_1, we have

$$
\begin{aligned}
&|R_\varepsilon(S(\psi_1)) - R_\varepsilon(S(\psi_2))|\\
=&\Big|\Big(\big(g_\tau(W+S(\psi_1)) - \big(g_\tau(W+S(\psi_2))\Big) - g_\tau'(W)\Big(S(\psi_1)-S(\psi_2)\Big)\Big|\\
=&\Big|\Big(g_\tau'(W+S(\psi_2)+\bar\theta(S(\psi_1)-S(\psi_2)\Big)\Big(S(\psi_1)-S(\psi_2)\Big) - g_\tau'(W)\Big(S(\psi_1)-S(\psi_2)\Big)\Big|\\
=&\Big|\Big(g_\tau'(W+S(\psi_2)+\bar\theta(S(\psi_1)-S(\psi_2))) - g_\tau'(W)\Big)\Big(S(\psi_1)-S(\psi_2)\Big)\Big|\\
=&\Big|\Big(g_\tau''(W+\bar\theta(S(\psi_2)+\bar\theta(S(\psi_1)-S(\psi_2))))(S(\psi_2)+\bar\theta(S(\psi_1)-S(\psi_2)))\Big)\Big(S(\psi_1)-S(\psi_2)\Big)\Big|\\
=&\Big|\Big(g_\tau''(W+S(\psi_2)+\bar\theta S(\psi_1)+(\bar\theta-\bar\theta^2)S(\psi_2))(\bar\theta S(\psi_1)+(1-\bar\theta)S(\psi_2))\Big)\Big(S(\psi_1)-S(\psi_2)\Big)\Big|\\
\le&\Big|g_\tau''\Big(e^{-\frac{\sigma_\varepsilon^2}{4\varepsilon^2}}\Big)\big(S(\psi_1)-S(\psi_2)\big)\big(S(\psi_1)+S(\psi_2)\big)\Big|\\
\le& Ce^{\frac{\sigma_\varepsilon^2}{4\varepsilon^2}}\dot{d}_\varepsilon(\psi_1-\psi_2).
\end{aligned}
$$

Similarly, in $D_2\backslash D_1$, there holds

$$
\begin{aligned}
|R_\varepsilon(S(\psi_1)) - R_\varepsilon(S(\psi_2))| \le&\Big|g_\tau''\Big(e^{-\frac{\sigma_\varepsilon^2}{\varepsilon^2}}\Big)\big(S(\psi_1)-S(\psi_2)\big)\big(S(\psi_1)+S(\psi_2)\big)\Big|\\
\le& Ce^{\frac{\sigma_\varepsilon^2}{\varepsilon^2}}\dot{d}_\varepsilon(S(\psi_1)-S(\psi_2)).
\end{aligned}
\tag{3.28}
$$

Recalling the definitions of σ_ε and $\dot{d}_\varepsilon$, from Lemmas 3.1, we have

$$
\begin{aligned}
&\|TS(\psi_1)-TS(\psi_2)\|_\varepsilon^2\\
\le& C\|R_\varepsilon(S(\psi_1)) - R_\varepsilon(S(\psi_2))\|_\varepsilon^2\\
=& C\|R_\varepsilon(S(\psi_1)) - R_\varepsilon(S(\psi_2))\|_{\varepsilon,D_1}^2 + C\|R_\varepsilon(S(\psi_1)) - R_\varepsilon(S(\psi_2))\|_{\varepsilon,D_2\backslash D_1}^2\\
&+C\|R_\varepsilon(S(\psi_1)) - R_\varepsilon(S(\psi_2))\|_{\varepsilon,\mathbb{R}^N\backslash D_2}^2\\
\le& Ce^{\frac{\sigma_\varepsilon^2}{4\varepsilon^2}}\dot{d}_\varepsilon\|S(\psi_1)-S(\psi_2)\|_{\varepsilon,D_1}^2 + Ce^{\frac{\sigma_\varepsilon^2}{\varepsilon^2}}\dot{d}_\varepsilon\|S(\psi_1)-S(\psi_2)\|_{\varepsilon,D_2\backslash D_1}^2\\
&+C\|S(\psi_1)-S(\psi_2)\|_{\varepsilon,\mathbb{R}^N\backslash D_2}^2\\
\le&\frac{1}{4}\|\psi_1-\psi_2\|_{\varepsilon,D_1}^2 + C\frac{\sigma_\varepsilon^4}{\varepsilon^4}e^{\frac{\sigma_\varepsilon^2}{\varepsilon^2}}\dot{d}_\varepsilon\|\psi_1-\psi_2\|_{\varepsilon,D_3\backslash D_1}^2 + CC_1^2\varepsilon^{2\theta}\|\psi_1-\psi_2\|_{\varepsilon,D_3\backslash D_2}^2\\
\le&\frac{1}{4}\|\psi_1-\psi_2\|_\varepsilon^2.
\end{aligned}
$$

Therefore, $\|TS(\psi_1)-TS(\psi_1)\|_\varepsilon \le \frac{1}{2}\|\psi_1-\psi_2\|_\varepsilon$. Similarly, $\|TS(\psi_1)-TS(\psi_2)\|_{L^\infty(\mathbb{R}^N)} \le \frac{1}{2}\|\psi_1-\psi_2\|_{L^\infty(\mathbb{R}^N)}$. By the fixed point theorem, we conclude that, for any $\varepsilon\in(0,\varepsilon_0]$, $B_\delta(\xi_m)\cap B_\delta(\xi_j)=\emptyset$, $m,j=1,\cdots,k$, there exists $\psi\in\Lambda$, depending on x^j and ε, satisfying $\psi = TS(\psi)$. Similar to (3.27), we obtain

$$
\|\psi\|_\varepsilon = \|TS(\psi)\|_\varepsilon \le C\|l_\varepsilon\|_\varepsilon + C\|R_\varepsilon(S(\psi))\|_\varepsilon \le \frac{1}{2}d_\varepsilon
$$

and

$$\|\psi\|_{L^\infty(\mathbb{R}^N)} = \|TS(\psi)\|_{L^\infty(\mathbb{R}^N)} \le C\|l_\varepsilon\|_{L^\infty(\mathbb{R}^N)} + C\|R_\varepsilon(S(\psi))\|_{L^\infty(\mathbb{R}^N)} \le \frac{1}{2}\dot{d}_\varepsilon.$$

□

3.3. **The Fixed Point of T.** Based on the fixed point of the composite operator TS and the solution of the minimizer problem, we have the following result.

Proposition 3.2. *There exist $\delta > 0$, $\theta > 0$ small enough and $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0,\varepsilon_0]$, $B_\delta(\xi_m) \cap B_\delta(\xi_j) = \emptyset$, $m, j = 1,\cdots,k$, there exists a unique C^1-map $\psi \in E$, which is given in Proposition 3.1, satisfying $\psi = T\psi$.*

Proof. Recalling the definition of operator B, there holds $B(S(\psi)) = P_\varepsilon l_\varepsilon + P_\varepsilon R_\varepsilon(S(\psi))$. For any $\psi \in \Lambda$, we claim $\psi = TS(\psi) = T\psi$. Since $S(\psi)$ is a solution of (3.3), then $P_\varepsilon L_\varepsilon S(\psi) = P_\varepsilon l_\varepsilon + P_\varepsilon R_\varepsilon(S(\psi))$. For any $v \in H_0^1(\mathbb{R}^N \backslash D_1)$, we have

$$\langle B(S(\psi)) - P_\varepsilon L_\varepsilon S(\psi), v\rangle_\varepsilon = 0. \tag{3.29}$$

On the other hand, ψ is the fixed point of TS, so $P_\varepsilon L_\varepsilon \psi = P_\varepsilon L_\varepsilon TS(\psi) = P_\varepsilon l_\varepsilon + P_\varepsilon R_\varepsilon(S(\psi))$. For any $v \in H_0^1(\mathbb{R}^N \backslash D_1)$, we have

$$\langle B(S(\psi)) - P_\varepsilon L_\varepsilon \psi, v\rangle_\varepsilon = 0. \tag{3.30}$$

From (3.29) and (3.30), we obtain $\langle P_\varepsilon L_\varepsilon(\psi - S(\psi)), v\rangle_\varepsilon = 0$. In particular, $\langle P_\varepsilon L_\varepsilon(\psi - S(\psi)), \psi - S(\psi)\rangle_\varepsilon = 0$. Therefore, in $\mathbb{R}^N \backslash D_1$, by Lemma 3.1, we obtain $\psi = S(\psi)$. In fact, $\psi = S(\psi)$ in D_1. Hence, $\psi = TS(\psi) = T\psi$ and we complete the proof. □

Completion of Proof of Proposition 3.2. Let $\psi \in \Lambda$ be the fixed point of T in Proposition 3.2. We remark that ψ is well-defined by the uniqueness. In light of the implicit function theorem, it suffices to prove that the operator $L_\varepsilon : E \to E$ defined as follows is invertible:

$$\langle L_\varepsilon u, v\rangle = \int_{\mathbb{R}^N} \varepsilon^2 \nabla u \nabla v + (V(y) - g_\tau'(W+\psi))uv, \ u, v \in E.$$

For any $u \in E$, let $u = u_1 + u_2$ with $u_1 = \chi u \in E_{2\sigma_\varepsilon}$ and $u_2 = (1-\chi)u \in E \cap H_0^1(\mathbb{R}^N \backslash D_1)$, where $\chi \in E \cap C_0^1(\mathbb{R}^N)$ is the truncation function satisfying (3.2). By the choice of χ, we check that

$$\|u\|_\varepsilon \le \|u_1\|_\varepsilon + \|u_2\|_\varepsilon \le 2\|u\|_\varepsilon \tag{3.31}$$

On one hand,

$$\begin{aligned}
\|L_\varepsilon u\|_\varepsilon &= \sup_{v\in E, \|v\|_\varepsilon = 1} \langle L_\varepsilon u, v\rangle \ge \sup_{v\in E_{2\sigma_\varepsilon}, \|v\|_\varepsilon = 1} \langle Lu, v\rangle = \sup_{v\in E_{2\sigma_\varepsilon}, \|v\|_\varepsilon = 1} \langle Lu, \chi v\rangle \\
&= \sup_{v\in E_{2\sigma_\varepsilon}, \|v\|_\varepsilon = 1} \int_{\mathbb{R}^N} \varepsilon^2 \nabla u \nabla(\chi v) + (V(y) - g_\tau'(W+\psi))\chi uv \\
&\ge \sup_{v\in E_{2\sigma_\varepsilon}, \|v\|_\varepsilon = 1} \int_{\mathbb{R}^N} \varepsilon^2 \nabla(\chi u) \nabla v + (V(y) - g_\tau'(W+\psi))\chi uv - 4\varepsilon\sigma_\varepsilon^{-1}\|u\|_\varepsilon \\
&\ge \sup_{v\in E_{2\sigma_\varepsilon}, \|v\|_\varepsilon = 1} \langle L_{2\sigma_\varepsilon} u_1, v\rangle - \|g_\tau'(W+\psi) - g_\tau'(W)\|_{L^2(D_2)}\|u\|_\varepsilon - 4\varepsilon\sigma_\varepsilon^{-1}\|u\|_\varepsilon,
\end{aligned}$$

where $\langle L_{2\sigma_\varepsilon}u_1,v\rangle$ is defined as

$$\begin{aligned}\langle L_{2\sigma_\varepsilon}u_1,v\rangle_\varepsilon &= \int_{Q_{2\sigma_\varepsilon}} \varepsilon^2\nabla u_1\nabla v+(V(y)+\tau^{-1})u_1v-\big(1+\tau^{-1}\big)W^\tau u_1v \\ &= \int_{\mathbb{R}^N} \varepsilon^2\nabla u_1\nabla v+\big(V(y)-g'_\tau(W)\big)u_1v,\ u_1,v\in E_{2\sigma_\varepsilon}.\end{aligned} \tag{3.32}$$

In view of $\|\psi\|_{L^\infty(\mathbb{R}^N)}\le \dot{d}_\varepsilon$, as ε is small, we obtain

$$\|Lu\|_\varepsilon \ge \gamma\|u_1\|_\varepsilon - 5\varepsilon\sigma_\varepsilon^{-1}\|u\|_\varepsilon. \tag{3.33}$$

On the other hand, since $-g'_\tau(W+\psi)=h'_\tau(W+\psi)\ge 1$ in $\mathbb{R}^N\backslash D_1$, we have

$$\begin{aligned}\langle L_\varepsilon u,u_2\rangle_\varepsilon &= \int_{\mathbb{R}^N}\varepsilon^2\nabla u\nabla u_2+(V(y)-g'_\tau(W+\psi))(1-\chi)u^2 \\ &\ge \|u_2\|_\varepsilon^2+\int_{\mathbb{R}^N}\varepsilon^2\nabla u_1\nabla u_2 \\ &= \|u_2\|_\varepsilon^2+\int_{\mathbb{R}^N}\varepsilon^2(u(1-2\chi)\nabla u\nabla\chi-u^2|\nabla\chi|^2)+\int_{\mathbb{R}^N}\varepsilon^2\chi(1-\chi)|\nabla u|^2 \\ &\ge \|u_2\|_\varepsilon^2-\int_{\mathbb{R}^N}\varepsilon^2(|u||\nabla\chi||\nabla u|+|\nabla\chi|^2u^2) \\ &\ge \|u_2\|_\varepsilon^2-2\varepsilon\sigma_\varepsilon^{-1}\|u\|_\varepsilon.\end{aligned}$$

Therefore,

$$\|L_\varepsilon u\|_\varepsilon\|u_2\|_\varepsilon \ge \|u_2\|_\varepsilon^2-2\varepsilon\sigma_\varepsilon^{-1}\|u\|_\varepsilon. \tag{3.34}$$

By (3.31) C(3.34), we have

$$2\|L_\varepsilon u\|_\varepsilon\|u_2\|_\varepsilon \ge \|L_\varepsilon u\|(\|u_1\|_\varepsilon+\|u_2\|_\varepsilon)\ge \frac{1}{2}\min\{1,\gamma\}\|u\|_\varepsilon^2-7\varepsilon\sigma_\varepsilon^{-1}\|u\|_\varepsilon^2.$$

Thus L_ε is invertible, and we have completed the proof. □

4. Proof the Existence of the Solutions

4.1. **proof of Theorem 1.1 and Theorem 1.3.** For $\varepsilon\in(0,\varepsilon_0)$ and $\tau\in(0,\tau_\varepsilon]$, let ψ be given in Proposition 3.1. To show our main theorems, we need some results as follows. Section 3 implies that

$$L_\varepsilon\psi-l_\varepsilon-R_\varepsilon(\psi)=\sum_{j=1}^k\sum_{i=1}^N a_{i,j}f'(U_{\varepsilon,j})\frac{\partial U_{\varepsilon,j}}{\partial y_i} \tag{4.1}$$

for some constants $a_{i,j}$. Every $a_{i,j}$ is determined by the following equations The second step is to choose x^j suitably, such that $a_{i,j}=0,\ i=1,\cdots,N,\ j=1,\cdots,k$. The function in the right hand side of (4.1) belongs to

$$E^\perp = span\Big\{\frac{\partial U_{\varepsilon,j}}{\partial y_i},\ i=1,\cdots,N,\ j=1,\cdots,k\Big\}.$$

Therefore, if the left hand side of (4.1) belongs to E, then the function in the right hand side of (4.1) must be zero. Recall that $u_{\varepsilon,\tau}=W_\varepsilon+\psi_{\varepsilon,\tau}=\sum_{j=1}^k U_{\varepsilon,j}+\psi_{\varepsilon,\tau}$. Then one has

$$\langle L_\varepsilon\psi-l_\varepsilon-R_\varepsilon(\psi),v\rangle_\varepsilon=\int_{\mathbb{R}^N}\varepsilon^2\nabla u_{\varepsilon,\tau}\nabla v+V(y)u_{\varepsilon,\tau}v-g_\tau(u_{\varepsilon,\tau})v,\ \forall v\in H^1(\mathbb{R}^N).$$

Let $\chi_0 \in C_0^1(\mathbb{R}^N)$ be a fixed truncation function such that

$$0 \le \chi_0 \le 1,\ |\nabla \chi_0| \le 2\sigma_\varepsilon^{-1},\ |\Delta \chi_0| \le 2\sigma_\varepsilon^{-2},\ \chi_0 = \begin{cases} 1, \text{ in } D_1, \\ 0, \text{ in } \mathbb{R}^N \backslash D_2. \end{cases} \tag{4.2}$$

Lemma 4.1. *If x^j satisfies*

$$\int_{\mathbb{R}^N} \Big(-\varepsilon^2 \Delta u_{\varepsilon,\tau} + V(y) u_{\varepsilon,\tau} - g_\tau(u_{\varepsilon,\tau}) \Big) \chi_0 \frac{\partial U_{\varepsilon,j}}{\partial y_i} = 0,\ i = 1, \cdots, N,\ j = 1, \cdots, k, \tag{4.3}$$

then $a_{i,j} = 0,\ i = 1, \cdots, N,\ j = 1, \cdots, k.$

Proof. If (4.3) holds, then

$$\sum_{j=1}^{k} \sum_{h=1}^{N} a_{i,j} \int_{\mathbb{R}^N} f'(U_{\varepsilon,j}) \frac{\partial U_{\varepsilon,j}}{\partial y_i} \chi_0 \frac{\partial U_{\varepsilon,j}}{\partial y_h} = 0,\ i = 1, \cdots, N,\ j = 1, \cdots, k.$$

By direct calculation, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} f'(U_{\varepsilon,j}) \frac{\partial U_{\varepsilon,j}}{\partial y_i} \chi_0 \frac{\partial U_{\varepsilon,j}}{\partial y_h} \\
=&\int_{B_{2\varepsilon\sqrt{V(x^j)+\frac{N}{2}+2}}(x^j)} (2 + \log U_{\varepsilon,j}) \frac{\partial U_{\varepsilon,j}}{\partial y_i} \frac{\partial U_{\varepsilon,j}}{\partial y_h} \\
=&2 \int_{B_{2\varepsilon\sqrt{V(x^j)+\frac{N}{2}+2}}(x^j)} \frac{\partial U_{\varepsilon,j}}{\partial y_i} \frac{\partial U_{\varepsilon,j}}{\partial y_h} + \int_{B_{2\varepsilon\sqrt{V(x^j)+\frac{N}{2}+2}}(x^j)} \log U_{\varepsilon,j} \frac{\partial U_{\varepsilon,j}}{\partial y_i} \frac{\partial U_{\varepsilon,j}}{\partial y_h} \\
=&2 \int_{B_{2\varepsilon\sqrt{V(x^j)+\frac{N}{2}+2}}(x^j)} \frac{\partial U_{\varepsilon,j}}{\partial y_i} \frac{\partial U_{\varepsilon,j}}{\partial y_h} + \int_{B_{2\varepsilon\sqrt{V(x^j)+\frac{N}{2}+2}}(x^j)} \Big| V(x^j) + \frac{N}{2} - \frac{|y - x^j|^2}{4\varepsilon^2} \Big| \frac{\partial U_{\varepsilon,j}}{\partial y_i} \frac{\partial U_{\varepsilon,j}}{\partial y_h} \\
=&2^{N+1} \varepsilon^N \int_{B_{\sqrt{V(x^j)+\frac{N}{2}+2}}(0)} \frac{\partial U_{\varepsilon,j}(2\varepsilon z + x^j)}{\partial z_i} \frac{\partial U_{\varepsilon,j}(2\varepsilon z + x^j)}{\partial z_h} \\
&+ 2^N \varepsilon^N \int_{B_{\sqrt{V(x^j)+\frac{N}{2}+2}}(0)} \Big| V(x^j) + \frac{N}{2} - |z| \Big| \frac{\partial U_{\varepsilon,j}(2\varepsilon z + x^j)}{\partial z_i} \frac{\partial U_{\varepsilon,j}(2\varepsilon z + x^j)}{\partial z_h} \\
=&\varepsilon^{N-4} (\delta_{hi} c_j + o(1)),
\end{aligned}$$

where $\delta_{hi} = 0$ if $h = i$, and $\delta_{ii} = 1, c_j > 0$ is a constant. Moreover, it follows from Lemma 2.3 that

$$\int_{\mathbb{R}^N} f'(U_{\varepsilon,j}) \frac{\partial U_{\varepsilon,j}}{\partial y_i} \chi_0 \frac{\partial U_{\varepsilon,m}}{\partial y_h} = \varepsilon^{N-4} O\Big(e^{-\sigma \frac{|x^m - x^j|^2}{\varepsilon^2}} \Big),\ j \ne m.$$

So, we conclude that $a_{i,j} = 0,\ i = 1, \cdots, N,\ j = 1, \cdots, k.$ □

Now, we are ready to complete our main results.

Proof of Theorem 1.3. We only need to solve (4.3). The main task is to find the main term for the function in the left hand side of (4.3). Recall that $W_\varepsilon = \sum_{j=1}^{k} U_{\varepsilon,j}(y)$, where $U_{\varepsilon,j}(y) =$

$e^{V(x^j)+\frac{N}{2}}e^{-\frac{|y-x^j|^2}{4\varepsilon^2}}$. Note that χ_0 is defined in (4.2). Then the function in the left hand side of (4.3) become

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\Big(-\varepsilon^2\Delta u_{\varepsilon,\tau}+V(y)u_{\varepsilon,\tau}-g_\tau(u_{\varepsilon,\tau})\Big)\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\\
=&\int_{\mathbb{R}^N}\Big(-\varepsilon^2\Delta W_\varepsilon+V(y)W_\varepsilon-g_\tau(W_\varepsilon)\Big)\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\\
&+\int_{\mathbb{R}^N}\Big(-g_\tau(u_{\varepsilon,\tau})+g_\tau(W_\varepsilon)+g'(U_{\varepsilon,j})\psi_{\varepsilon,\tau}\Big)\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\\
&+\int_{\mathbb{R}^N}(V(y)-V(x^j))\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\psi_{\varepsilon,\tau}+\int_{\mathbb{R}^N}\Big(-2\varepsilon^2\nabla\chi_0\nabla\frac{\partial U_{\varepsilon,j}}{\partial y_i}\psi_{\varepsilon,\tau}-\varepsilon^2\psi_{\varepsilon,\tau}\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Delta\chi_0\Big)\\
=:&A_1+A_2+A_3+A_4.
\end{aligned}
$$

Firstly, applying the symmetry of $U_{\varepsilon,j}$ and Lemma 2.3, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}(V(y)-V(x^j))W_\varepsilon\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\\
=&\Big|\int_{\mathbb{R}^N}(V(y)-V(x^j))\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}U_{\varepsilon,j}\Big|+\sum_{j\neq m}\Big|\int_{\mathbb{R}^N}(V(y)-V(x^j))\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}U_{\varepsilon,m}\Big|\\
=&\Big(\int_{D_2}(V(y)-V(x^j))^2\Big(\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big)^2\Big)^{\frac{1}{2}}\|U_{\varepsilon,j}\|_{L^2(\mathbb{R}^N)}+\varepsilon^{N-1}O\Big(e^{-\frac{c}{\varepsilon^2}}\Big)\\
=&C\Big(2^N\varepsilon^N\int_{\mathbb{R}^N}(V(2\varepsilon x+x^j)-V(x^j))^2\Big(\frac{\partial U_{\varepsilon,j}(2\varepsilon x+x^j)}{\partial x_i}\Big)^2\Big)^{\frac{1}{2}}\|U_{\varepsilon,j}\|_{L^2(\mathbb{R}^N)}+\varepsilon^{N-1}O\Big(e^{-\frac{c}{\varepsilon^2}}\Big)\\
=&C\varepsilon^{\frac{N}{2}}\Big(\int_{\mathbb{R}^N}(\varepsilon|\nabla V(x^j)||x|+\varepsilon^2|x|^2)^2\Big(\frac{\partial U_{\varepsilon,j}(2\varepsilon x+x^j)}{\partial x_i}\Big)^2\Big)^{\frac{1}{2}}\|U_{\varepsilon,j}\|_{L^2(\mathbb{R}^N)}+\varepsilon^{N-1}O\Big(e^{-\frac{c}{\varepsilon^2}}\Big)\\
\leq&C\varepsilon^N(|\nabla V(x^j)|+\varepsilon),
\end{aligned}
$$

for some $c>0$. Therefore, by Lemmas 2.2, we have

$$
\begin{aligned}
A_1=&\int_{\mathbb{R}^N}(V(y)-V(x^j))W_\varepsilon\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}+\int_{\mathbb{R}^N}\Big(\sum_{i=1}^k g(U_{\varepsilon,i})-g_\tau(W_\varepsilon)\Big)\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\\
=&C\varepsilon^N(|\nabla V(x^j)|+\varepsilon)+\varepsilon^{N-\theta}O\Big(e^{-e^{\frac{1}{\varepsilon}}}\Big)\\
\leq&C\varepsilon^N(|\nabla V(x^j)|+\varepsilon).
\end{aligned}
$$

For A_2. Next, we claim that the contribution of the error term $\psi_{\varepsilon,\tau}$ to the function in the left hand side of (4.3) is negligible. Then

$$
\begin{aligned}
|A_2|\leq&\Big|\int_{\mathbb{R}^N}\Big(g_\tau(u_{\varepsilon,\tau})-g_\tau(W_\varepsilon)-g'(U_{\varepsilon,j})\psi_{\varepsilon,\tau}\Big)\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big|+\Big|\int_{\mathbb{R}^N}\Big(g(W_\varepsilon)-g_\tau(W_\varepsilon)\Big)\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big|\\
\leq&\Big|\int_{\mathbb{R}^N}\Big(g_\tau(W_\varepsilon+\psi_{\varepsilon,\tau})-g_\tau(W_\varepsilon)-g'(W_\varepsilon)\psi_{\varepsilon,\tau}\Big)\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big|\\
&+\Big|\int_{\mathbb{R}^N}\Big(g'(W_\varepsilon)\psi_{\varepsilon,\tau}-g'(U_{\varepsilon,j})\psi_{\varepsilon,\tau}\Big)\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big|=A_{21}+A_{22}.
\end{aligned}
$$

Since $\psi_{\varepsilon,\tau}\in\Lambda$. By Lemmas 2.2, we obtain

$$\begin{aligned}
A_{21}\le&\Big|\int_{\mathbb{R}^N}\Big(g_\tau(W_\varepsilon+\psi_{\varepsilon,\tau})-g_\tau(W_\varepsilon)-g'_\tau(W_\varepsilon)\psi_{\varepsilon,\tau}\Big)\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big|\\
&+\Big|\int_{\mathbb{R}^N}\Big(g'_\tau(W_\varepsilon)-g'(W_\varepsilon)\Big)\psi_{\varepsilon,\tau}\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big|\\
=&\Big|\int_{\mathbb{R}^N}\Big(g_\tau(W_\varepsilon+\psi_{\varepsilon,\tau})-g_\tau(W_\varepsilon)-g'_\tau(W_\varepsilon)\psi_{\varepsilon,\tau}\Big)\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big|+\varepsilon^{N-\theta}O\Big(e^{-e^{\frac{1}{\varepsilon}}}\Big)\\
=&\Big|\int_{D_2}\Big(g_\tau(W_\varepsilon+\psi_{\varepsilon,\tau})-g_\tau(W_\varepsilon)-g'_\tau(W_\varepsilon)\psi_{\varepsilon,\tau}\Big)\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big|+\varepsilon^{N-\theta}O\Big(e^{-e^{\frac{1}{\varepsilon}}}\Big)\\
\le&\Big|\int_{D_2}g''_\tau(e^{-\frac{\sigma_\varepsilon^2}{\varepsilon^2}})(\psi_{\varepsilon,\tau})^2\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big|+\varepsilon^{N-\theta}O\Big(e^{-e^{\frac{1}{\varepsilon}}}\Big)\\
\le&C\varepsilon^{\frac{N}{2}-1}e^{\frac{\sigma_\varepsilon^2}{\varepsilon^2}}d_\varepsilon\|\psi_{\varepsilon,\tau}\|_\varepsilon+C\varepsilon^{\frac{N}{2}-1}d_\varepsilon\|\psi_{\varepsilon,\tau}\|_\varepsilon+\varepsilon^{N-\theta}O\Big(e^{-e^{\frac{1}{\varepsilon}}}\Big)\\
=&O(\varepsilon^{N+\frac{3}{4}-2\theta}).
\end{aligned}$$

There exists a constant $\delta>0$ such that

$$\begin{aligned}
A_{22}=&\Big|\int_{\mathbb{R}^N}(\log W_\varepsilon-\log U_{\varepsilon,j})\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\psi_{\varepsilon,\tau}\Big|\le\int_{\mathbb{R}^N}\Big|(\log W_\varepsilon-\log U_{\varepsilon,j})\frac{\partial U_{\varepsilon,j}}{\partial y_i}\psi_{\varepsilon,\tau}\Big|\\
=&\int_{B_\delta(x^j)}\Big|\Big(\log(1+\frac{\sum_{s\neq j}U_{\varepsilon,s}}{U_{\varepsilon,j}})\Big)\frac{\partial U_{\varepsilon,j}}{\partial y_i}\psi_{\varepsilon,\tau}\Big|+\int_{\mathbb{R}^N\setminus B_\delta(x^j)}\Big|\Big(\log\frac{\sum_{s=1}^kU_{\varepsilon,s}}{U_{\varepsilon,j}}\Big)\frac{\partial U_{\varepsilon,j}}{\partial y_i}\psi_{\varepsilon,\tau}\Big|\\
=&O\Big(\varepsilon^{-1}\int_{B_\delta(x^j)}(\sum_{s\neq j}U_{\varepsilon,s})\psi_{\varepsilon,\tau}+\int_{\mathbb{R}^N\setminus B_\delta(x^j)}U_{\varepsilon,j}^{\frac{1}{2}-\theta}(\sum_{s=1}^kU_{\varepsilon,s})^{\frac{1}{2}}\psi_{\varepsilon,\tau}\Big)\\
=&O\Big(\varepsilon^{-1}e^{-\frac{c}{\varepsilon^2}}\|\psi_{\varepsilon,\tau}\|_\varepsilon\Big).
\end{aligned}$$

By Lemmas 2.2, we see that

$$\begin{aligned}
|A_3|=&\Big|\int_{\mathbb{R}^N}(V(y)-V(x^j))\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i}\psi_{\varepsilon,\tau}\Big|\\
\le&C\Big(\int_{D_2}(V(y)-V(x^j))^2\Big(\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big)^2\Big)^{\frac{1}{2}}\|\psi_{\varepsilon,\tau}\|_\varepsilon\\
=&C\Big(2^N\varepsilon^N\int_{\mathbb{R}^N}(V(2\varepsilon x+x^j)-V(x^j))^2\Big(\frac{\partial U_{\varepsilon,j}(2\varepsilon x+x^j)}{\partial x_i}\Big)^2\Big)^{\frac{1}{2}}\|\psi_{\varepsilon,\tau}\|_\varepsilon\\
\le&C\varepsilon^{\frac{N}{2}}\Big(\int_{\mathbb{R}^N}(\varepsilon|\nabla V(x^j)||x|+\varepsilon^2|x|^2)^2\Big(\frac{\partial U_{\varepsilon,j}(2\varepsilon x+x^j)}{\partial x_i}\Big)^2\Big)^{\frac{1}{2}}\|\psi_{\varepsilon,\tau}\|_\varepsilon\\
\le&C\varepsilon^{N+1-\theta}(|\nabla V(x^j)|+\varepsilon).
\end{aligned}$$

In $D_2\backslash D_1$, we know that $\left|\nabla \frac{\partial U_{\varepsilon,j}}{\partial y_i}\right| = O\Big(\varepsilon^{-\frac{31}{16}}|\ln\varepsilon|\Big)$ and $\left|\frac{\partial U_{\varepsilon,j}}{\partial y_i}\right| = O\Big(\varepsilon^{-\frac{15}{16}}\sqrt{|\ln\varepsilon|}\Big)$. Recall the definition of σ_ε and $\psi_{\varepsilon,\tau}\in\Lambda$. Finally, we obtain

$$\begin{aligned}|A_4| \le & \Big|\int_{\mathbb{R}^N} 2\varepsilon^2\nabla\chi_0\nabla\frac{\partial U_{\varepsilon,j}}{\partial y_i}\psi_{\varepsilon,\tau}\Big| + \Big|\int_{\mathbb{R}^N}\varepsilon^2\psi_{\varepsilon,\tau}\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Delta\chi_0\Big|\\ \le & 4\int_{D_2\backslash D_1}\varepsilon^2\sigma_\varepsilon^{-1}\Big|\nabla\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big||\psi_{\varepsilon,\tau}| + 2\int_{D_2\backslash D_1}\varepsilon^2\sigma_\varepsilon^{-2}|\psi_{\varepsilon,\tau}|\Big|\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big|\\ = & O\Big(\sigma_\varepsilon^N\varepsilon^2\sigma_\varepsilon^{-1}\Big|\nabla\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big||\psi_{\varepsilon,\tau}| + \sigma_\varepsilon^N\varepsilon^2\sigma_\varepsilon^{-2}|\psi_{\varepsilon,\tau}|\Big|\frac{\partial U_{\varepsilon,j}}{\partial y_i}\Big|\Big) = O(\varepsilon^{N+\theta}).\end{aligned}$$

In conclusion, we have

$$\int_{\mathbb{R}^N}\Big(-\varepsilon^2\Delta u_{\varepsilon,\tau} + V(y)u_{\varepsilon,\tau} - g_\tau(u_{\varepsilon,\tau})\Big)\chi_0\frac{\partial U_{\varepsilon,j}}{\partial y_i} = C\varepsilon^N(|\nabla V(x^j)|+\varepsilon) + O(\varepsilon^{N+\theta}).$$

Therefore, system (4.3) is equivalent to

$$\nabla V(x^j) = O(\varepsilon^\theta),\ j=1,\cdots,k. \tag{4.4}$$

By the assumption that $deg(\nabla V, B_\delta(\xi_j),0)\neq 0$, we deduce that (4.4) has a solution $x^j\in B_\delta(\xi_j)$, and $|x^j-\xi_j| = O(\varepsilon^\theta)$. For $\varepsilon\in(0,\varepsilon_0)$ and $\tau\in(0,\tau_\varepsilon]$, we prove that $W_\varepsilon+\psi_{\varepsilon,\tau}$ is a solution to (1.5). □

Proof of Theorem 1.1. By Theorem 1.3, for $\varepsilon\in(0,\varepsilon_0)$ and $\tau\in(0,\tau_\varepsilon]$, there exists $W_{\varepsilon,\tau}+\psi_{\varepsilon,\tau}$ which is a solution to (1.5). In addition, $\|\psi_{\varepsilon,\tau}\|_\varepsilon \le \frac{1}{2}d_\varepsilon$. Then, up to a subsequence, we may assume as $\tau\to 0$,

$$\psi_{\varepsilon,\tau} \rightharpoonup \psi_\varepsilon \text{ weakly in } H^1(\mathbb{R}^N),$$
$$W_{\varepsilon,\tau}\to W_\varepsilon \text{ strongly in } \ H^1(\mathbb{R}^N).$$

We have that $W_{\varepsilon,\tau}+\psi_{\varepsilon,\tau}\rightharpoonup W_\varepsilon+\psi_\varepsilon := u_\varepsilon$ and u_ε is a solution to (1.1).

Next, we prove that u_ε is positive. As $\varepsilon\to 0$, there holds $\|u_\varepsilon^-\|_{L^p}\le\|\psi_\varepsilon\|_{L^p}\to 0$ for all $p\in(2,2^*)$. However, by (1.1) and the Sobolev inequality, we have

$$\|u_\varepsilon^-\|_{L^p}^2 \le C_p\int_{\mathbb{R}^N}\Big(\varepsilon^2|\nabla u_\varepsilon^-|^2 + V(u_\varepsilon^-)^2\Big) \le C_p\int_{\mathbb{R}^N}(u_\varepsilon^-)^2\log(u_\varepsilon^-) \le \|u_\varepsilon^-\|_{L^p}^p,$$

for some $C_p>0$ independent of ε. Therefore, $u_\varepsilon^-=0$ for small ε and the maximum principle of [15] implies $u_\varepsilon>0$. As a result, $W_\varepsilon+\psi_\varepsilon$ is a positive solution to (1.1). The proof for Theorem 1.1 is completed. □

4.2. **proof of Theorem 1.2 and Theorem 1.4.** We outline the steps of the proof of Theorem 1.2 and Theorem 1.4. Similar procedures to Theorem 1.1 are not repeated here. Note that x_0 is an isolated local maximum point of $V(y)$. We take k points $x^1,\cdots,x^k$ such that $x^j\to x_0$, $j=1,\cdots,k$, as $\varepsilon\to 0$. In this case, taking $d:=\min_{m\neq j}|x^m-x^j| = \sqrt{\varepsilon\ln\frac{1}{\varepsilon}}, m,j=1,\cdots,k$, Lemma 2.2 and Lemma 2.4 are also available.

For any fixed $\delta>0$ small enough, we always assume

$$\mathbf{x}\in S_\varepsilon := \Big\{\mathbf{x}: x^j\in\overline{B_\delta(x_0)},\ j=1,\cdots,k,\ |x^m-x^j|\ge\sqrt{\varepsilon\ln\frac{1}{\varepsilon}},\ m\neq j\Big\}. \tag{4.5}$$

Step (i): For each $\mathbf{x} \in S_\varepsilon$ and $\psi \in E$, we set $J(\psi) = I(W+\psi)$. Thus

$$\Big\langle \frac{\partial J(\psi)}{\partial \psi}, v \Big\rangle_\varepsilon = \int_{\mathbb{R}^N} \varepsilon^2 \nabla (W+\psi)\nabla v + \int_{\mathbb{R}^N} (V(W+\psi) - g_\tau(W+\psi))v, \ v \in E,$$

which can be written as

$$\Big\langle \frac{\partial J(\psi)}{\partial \psi}, v \Big\rangle_\varepsilon = \langle L_\varepsilon \psi - l_\varepsilon - R_\varepsilon(\psi), v\rangle_\varepsilon, \ v \in E,$$

where L_ε, l_ε and $R_\varepsilon(\psi)$ are defined in (1.8), (1.9), and (1.10). Similar to the proof of Theorem 1.3 in Section 3, by replacing ξ_j with x_0, $j = 1, \cdots, k$, we can solve $\frac{\partial J(\psi)}{\partial \psi} = 0$ in

$$\Lambda = \big\{\psi \in E \cap L^\infty(\mathbb{R}^N) \mid \|\psi\|_\varepsilon \le d_\varepsilon \text{ and } \|\psi\|_{L^\infty(\mathbb{R}^N)} \le \dot{d}_\varepsilon\big\},$$

where d_ε and $\dot{d}_\varepsilon$ are defined in (3.1).

Step (ii): For $\varepsilon \in (0, \varepsilon_0)$, $\mathbf{x} \in S_\varepsilon$, and $\tau \in (0, \tau_\varepsilon]$, we have reduced the perturbed problem to

$$\frac{\partial J(\psi)}{\partial \psi} = L_\varepsilon \psi - l_\varepsilon - R_\varepsilon(\psi) = \sum_{j=1}^{k}\sum_{i=1}^{N} a_{i,j} f'(U_{\varepsilon,j}) \frac{\partial U_{\varepsilon,j}}{\partial y_i},$$

for some constants $a_{i,j}$. Therefore, we need to choose x^j suitably such that $a_{i,j} = 0$, $i = 1, \cdots, N$, $j = 1, \cdots, k$. For this purpose, we use the following result, which can be proved by standard analysis.

Lemma 4.2. *Denoting $F(\mathbf{x})$ for $I(W+\psi)$, suppose that $\mathbf{x}^*$ is a critical point of $F(\mathbf{x})$. Then $a_{i,j} = 0$ for $i = 1, \cdots, N$ and $j = 1, \cdots, k$.*

From Lemma 4.2, we only need to prove that $F(\mathbf{x})$ has a critical point in S_ε. To show our main theorems, we need some energy expansions as follows.

Lemma 4.3. *There exits $\theta > 0$ small enough such that $J(\psi) = I(W) + O\big(\varepsilon^{N+1-2\theta}\big)$.*

Proof. **Step (i)**: To obtain the lower bound for $J(\psi)$, we expand $J(\psi)$ as follows

$$J(\psi) = I(W+\psi) = I(W) + I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \frac{1}{2}\int_{\mathbb{R}^N} (\varepsilon^2 |\nabla \psi|^2 + V\psi^2) + \int_{\mathbb{R}^N} \Big(\sum_{j=1}^{k} g(U_{\varepsilon,j}) - g_\tau(W)\Big)\psi, \\ I_2 &= \int_{\mathbb{R}^N} (H_\tau(W+\psi) - H_\tau(W) - h_\tau(W)\psi), \\ I_3 &= -\int_{\mathbb{R}^N} (F_\tau(W+\psi) - F_\tau(W) - f_\tau(W)\psi). \end{aligned}$$

From Lemma 2.4, one sees that $I_1 = O\big(\varepsilon^{N+1-2\theta}\big)$. By the mean value theorem and Lemma 2.1 (ii), there holds $I_2 = \int_{\mathbb{R}^N} (h_\tau(W+\bar{\theta}\psi) - h_\tau(W))\psi \ge 0$, where $\bar{\theta} \in (0,1)$. On the other hand, we have

$$|I_3| \le \|f_\tau'(2|W|)\|_{L^\infty} \|\psi\|_\varepsilon^2 = O\big(\varepsilon^{N+1-2\theta}\big).$$

Therefore,

$$J(\psi) \ge I(W) + O\big(\varepsilon^{N+1-2\theta}\big).$$

Step (ii): We estimate the upper bound. Let $\chi \in E \cap C_0^1(\mathbb{R}^N)$ be as in (3.2). We can know $\chi\psi \in E \cap H_0^1(D_3)$. Since ψ is the fixed point of S, by Lemma 3.1 and the definition of d_ε, one fnds

$$\begin{aligned}\|\chi\psi\|_\varepsilon^2 &= \|\psi\|_{\varepsilon,D_2}^2 + \|\chi\psi\|_{\varepsilon,\mathbb{R}^N\setminus D_2}^2 \\ &\le \|\psi\|_{\varepsilon,D_2}^2 + \|\psi\|_{\varepsilon,\mathbb{R}^N\setminus D_2}^2 \\ &\le \frac{1}{4}d_\varepsilon^2 + \|S(\psi)\|_{\varepsilon,\mathbb{R}^N\setminus D_2} \le \frac{1}{4}\varepsilon^{N+2-2\theta} + C_1^2\varepsilon^{2\theta}d_\varepsilon^2 \le \frac{1}{4}\varepsilon^{N+2-2\theta}.\end{aligned} \tag{4.6}$$

By Proposition 3.2 and the definition of S, it holds that $\psi = S(\psi) = S(\chi\psi)$, where S be the operator given in Definition 3.1. Then by Lemma 3.1 (i), there holds

$$\begin{aligned}J(\psi) = J(S(\psi)) &= \inf_{u\in E_\psi}\Gamma(u) \\ &= J(S(\chi\psi)) = I(W + S(\chi\psi)) \\ &= \Gamma(W + S(\chi\psi)) \\ &\le \Gamma(W+\chi\psi) = I(W+\chi\psi) = I(W) + \bar{I}_1 + \bar{I}_2,\end{aligned}$$

where

$$\begin{aligned}\bar{I}_1 &= \frac{1}{2}\int_{\mathbb{R}^N}(\varepsilon^2|\nabla\chi\psi|^2 + V(\chi\psi)^2) + \int_{\mathbb{R}^N}\Big(\sum_{j=1}^k g(U_{\varepsilon,j}) - g_\tau(W)\Big)\chi\psi, \\ \bar{I}_2 &= -\int_{\mathbb{R}^N}(G_\tau(W+\chi\psi) - G_\tau(W) - g_\tau(W)\chi\psi),\end{aligned}$$

and the last inequality holds because

$$J(S(\psi)) = \Gamma(S(\psi)) = \inf_{u\in E_\psi}\Gamma(u) \le \Gamma(W+\chi\psi).$$

By Lemma 2.4 again, $\bar{I}_1 = O\Big(\varepsilon^{N+1-2\theta}\Big)$.

On the other hand,

$$\|\chi\psi\|_{L^\infty(D_2)} \le \frac{1}{2}d_\varepsilon < \frac{1}{2}\inf_{D_2}|W|. \tag{4.7}$$

Therefore, it holds from (4.6) and (4.7) that

$$\begin{aligned}|\bar{I}_2| &= \Big|\int_{D_2}(G_\tau(W+\chi\psi) - G_\tau(W) - g_\tau(W)\chi\psi)\Big| \\ &\le \int_{D_2}|g'_\tau\Big(\frac{1}{2}\inf_{D_2}|W|\Big)|(\chi\psi)^2 \le C\frac{\sigma_\varepsilon^2}{\varepsilon^2}\varepsilon^{N+2-2\theta} = O\Big(\varepsilon^{N+1-2\theta}\Big).\end{aligned}$$

□

Now, we are ready to complete our main results.

Proof of Theorem 1.4. With S_ε defined in (4.5), we consider the following maximization problem

$$\max_{\mathbf{x}\in S_\varepsilon} F(\mathbf{x}).$$

Then it is achieved by $\mathbf{x}^* \in S_\varepsilon$. In order to prove that $\mathbf{x}^*$ is a critical point of $F(\mathbf{x})$, it suffices to show that $\mathbf{x}^*$ is an interior point of S_ε. We take $x^j, j = 1,\cdots,k$, satisfying $|x^j - x_0| \le \varepsilon^\beta$

and $|x^m - x^j| \geq \left(\sqrt{\varepsilon \ln \frac{1}{\varepsilon}}\right)^{1-\beta}$, $m \neq j$, where $1 \gg \beta > 0$ is a small fixed constant. Then, for $\mathbf{x}^* = (x^1, \cdots, x^k) \in S_\varepsilon$, one has

$$F(\mathbf{x}^*) = \frac{1}{4} Ak\varepsilon^N e^{2V(x_0)+N} + \varepsilon^N O\big(\varepsilon^{2\beta}\big).$$

Suppose that there exists x^{j_0} such that, for $x^{j_0} \in \partial B_\delta(x_0)$,

$$F(\mathbf{x}) \leq \frac{1}{4}\varepsilon^N (2\pi)^{\frac{N}{2}} e^{2V(x^{\varepsilon,j_0})+N} + \frac{1}{4}\sum_{j\neq j_0} \varepsilon^N (2\pi)^{\frac{N}{2}} e^{2V(x_0)+N} + o\big(\varepsilon^N\big) < F(\mathbf{x}^*).$$

But this contradicts the fact that $\mathbf{x}^*$ is a maximum point of $F(\mathbf{x})$ in S_ε.

Suppose that there exist x^{m_0} and x^{j_0}, $m_0 \neq j_0$, such that $|x^{m_0} - x^{j_0}| = \sqrt{\varepsilon \ln \frac{1}{\varepsilon}}$. Then

$$F(\mathbf{x}) \leq \frac{1}{4} k\varepsilon^N (2\pi)^{\frac{N}{2}} e^{2V(x_0)+N} - B\varepsilon^{N+\frac{1}{8\varepsilon}} e^{2V(x_0)+N} + \varepsilon^N O(\varepsilon) < F(\mathbf{x}^*),$$

for some $B > 0$, which is also a contradiction. For $\varepsilon \in (0, \varepsilon_0)$, $\mathbf{x} \in S_\varepsilon$ and $\tau \in (0, \tau_\varepsilon]$, we have proved that $\mathbf{x}^*$ is a maximum point of $F(\mathbf{x})$ in S_ε. Then $W_{\varepsilon,\tau} + \psi_{\varepsilon,\tau}$ is a critical point of I and a solution to (1.5).

□

Proof of Theorem 1.2. By simply repeating the proof process of Theorem 1.1, we can derive Theorem 1.2 immediately.

□

5. The Non-degeneracy of the Solutions

In this section, we prove the non-degeneracy of positive multi-peak solutions Theorem 1.5. From Section 4, we find a positive solution u_ε to (1.1) of the form $u_\varepsilon = W_\varepsilon + \psi_\varepsilon$, where W_ε and ψ_ε are the limits of $W_{\varepsilon,\tau}$ and $\psi_{\varepsilon,\tau}$ as $\tau \to 0$. In order to obtain some important estimates, system (1.1) can also be rewritten as the following equation about ψ_ε :

$$\begin{cases} \tilde{L}_\varepsilon \psi_\varepsilon = \tilde{l}_\varepsilon + \tilde{R}_\varepsilon(\psi_\varepsilon), & \text{in } \mathbb{R}^N, \\ \psi_\varepsilon \in H^1(\mathbb{R}^N), \end{cases}$$

where $\tilde{L}_\varepsilon$ is a bounded linear operator in $H^1(\mathbb{R}^N)$, defined by

$$\langle \tilde{L}_\varepsilon \psi_\varepsilon, v\rangle_\varepsilon = \int_{\mathbb{R}^N} (\varepsilon^2 \nabla \psi_\varepsilon \nabla v + V(y)\psi_\varepsilon v - g'(W)\psi_\varepsilon v), \ \forall v \in H^1(\mathbb{R}^N),$$

$\tilde{l}_\varepsilon \in H^1(\mathbb{R}^N)$ satisfies

$$\langle \tilde{l}_\varepsilon, v\rangle_\varepsilon = \int_{\mathbb{R}^N} \Big(\sum_{j=1}^k (V(x^j) - V(y))U_{\varepsilon,j} + \Big(g(W) - \sum_{j=1}^k g(U_{\varepsilon,j})\Big)\Big) v, \ \forall v \in H^1(\mathbb{R}^N),$$

and $\tilde{R}_\varepsilon(\psi_\varepsilon) \in H^1(\mathbb{R}^N)$ satisfies

$$\langle \tilde{R}_\varepsilon(\psi_\varepsilon), v\rangle_\varepsilon = \int_{\mathbb{R}^N} (g(W + \psi_\varepsilon) - g(W) - g'(W)\psi_\varepsilon)v, \ \forall v \in H^1(\mathbb{R}^N).$$

5.1. **Pohozaev identities.** The crucial Pohozaev type identities we will use are as follows.

Proposition 5.1. *Let u be the solution of (1.1), $\mathscr{L}_\varepsilon(\eta)=0$. Then the following local Pohozaev identities hold:*

$$\int_\Omega \frac{\partial V(y)}{\partial y_i}u^2 = -2\varepsilon^2\int_{\partial\Omega}\frac{\partial u}{\partial y_i}\frac{\partial u}{\partial \nu}+\varepsilon^2\int_{\partial\Omega}|\nabla u|^2\nu_i-\int_{\partial\Omega}u^2\log|u|\nu_i+\int_{\partial\Omega}(V(y)+\frac{1}{2})u^2\nu_i, \tag{5.1}$$

and

$$\int_\Omega \frac{\partial V(y)}{\partial y_i}u\eta = -\varepsilon^2\int_{\partial\Omega}(\frac{\partial u}{\partial \nu}+\frac{\partial \eta}{\partial y_i})+\int_{\partial\Omega}(\varepsilon^2\langle\nabla u,\nabla\eta\rangle+V(y)u\eta)\nu_i-\int_{\partial\Omega}\log|u|u\eta\nu_i, \tag{5.2}$$

where $\nu=(\nu_1,\cdots,\nu_2)$ is the unit outward normal of $\partial\Omega$.

Proof. Identity (5.1) is obtained by multiplying $\frac{\partial u(y)}{\partial y_i}$ on both sides of (1.1) and integrating on $\partial\Omega$. While the identity in (5.2) is obtained by multiplying $\frac{\partial \eta(y)}{\partial y_i}$ and $\frac{\partial u(y)}{\partial y_i}$ on both sides of (1.1) and $\mathscr{L}_\varepsilon(\eta)=0$, respectively, and integrating on $\partial\Omega$. We omit the details. □

5.2. **Some estimates on the multipeak solutions.** First, we need to estimate u_ε on $\partial B_\delta(x^{j,\varepsilon})$ as the following Lemma.

Lemma 5.1. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a positive solution to (1.1) concentrating at $\xi_1,\cdots,\xi_k\subset\mathbb{R}^N$. For any fixed $R>>1$, there exist constants $c>0$ enough small and $C>0$ such that*

$$|u_\varepsilon(y)|\le C\sum_{j=1}^k e^{-c\frac{|y-x^{j,\varepsilon}|}{\varepsilon}},\ \forall\, y\in\mathbb{R}^N,\ j=1,\cdots,k. \tag{5.3}$$

and

$$|\nabla u_\varepsilon(y)|\le C\sum_{j=1}^k e^{-c\frac{R}{\varepsilon}},\ \forall y\in\mathbb{R}^N``B_{R\varepsilon}(x^{j,\varepsilon}),\ j=1,\cdots,k. \tag{5.4}$$

Proof. Set $V_m=\frac{1}{2}\inf_{y\in\mathbb{R}^N}V(y)$ and write $-\varepsilon^2\Delta u_\varepsilon+(V(y)-\log|u_\varepsilon|)u_\varepsilon=0$. For fixed R enough big and ε enough small, for any $\alpha\in(0,V_m)$, there exists $R>0$ such that

$$V(y)-\log|u_\varepsilon|\ge\alpha,\ y\in\mathbb{R}^N\setminus\sum_{j=1}^k B_{R\varepsilon}(x^{j,\varepsilon}).$$

Therefore, we have $-\varepsilon^2\Delta u_\varepsilon+\alpha u_\varepsilon\le 0$ in $\mathbb{R}^N\setminus\sum_{j=1}^k B_{R\varepsilon}(x^{j,\varepsilon})$. Let $\bar{L}_\varepsilon v=-\varepsilon^2\Delta v+\alpha v$, $v\in H^1(\mathbb{R}^N)$. For $v_j(y)=e^{-\frac{\sqrt{\alpha}|y-x^{j,\varepsilon}|}{\varepsilon}}$, $j=1,\cdots,k$, it follows that

$$\bar{L}_\varepsilon v_j(y)=-\varepsilon^2\Big(\frac{\alpha}{\varepsilon^2}-\frac{(N-1)\sqrt{\alpha}}{\varepsilon|y-x^{j,\varepsilon}|}\Big)v_j(y)+\alpha v_j(y)\ge 0.$$

Taking $\tilde{v}_j(y)=cv_j(y)-u_\varepsilon(y)$, one has $\bar{L}_\varepsilon\tilde{v}_j(y)=c\bar{L}_\varepsilon v_j(y)-\bar{L}_\varepsilon u_\varepsilon(y)\ge 0$, in $\mathbb{R}^N\setminus\sum_{j=1}^k B_{R\varepsilon}(x^{j,\varepsilon})$. Since $u_\varepsilon\in C_{loc}(\mathbb{R}^N)$, in $\partial B_{R\varepsilon}(x^{j,\varepsilon})$, then there exists $M>0$ such that $|u_\varepsilon|\le M$. When $c=Me^{\sqrt{\alpha}R}$, in $\partial B_{2R\varepsilon}(x^{j,\varepsilon})$, one has $\tilde{v}_j(y)=cv_j(y)-u_\varepsilon(y)\ge ce^{-\frac{\sqrt{\alpha}|y-x^{j,\varepsilon}|}{\varepsilon}}-M\ge 0$. Therefore,

$$\begin{cases}\bar{L}_\varepsilon\tilde{v}_j(y)\ge 0, & \text{in } \mathbb{R}^N``\sum_{j=1}^k B_{R\varepsilon}(x^{j,\varepsilon}),\\ \tilde{v}_j(y)\ge 0, & \text{in } \partial B_{R\varepsilon}(x^{j,\varepsilon}),\\ \tilde{v}_j(y)\to 0, & \text{as } y\to\infty.\end{cases}$$

Hence, by the comparison theorem, we obtain $\tilde{v}_j(y) \geq 0$ in $\mathbb{R}^N \setminus \sum_{j=1}^k B_{R\varepsilon}(x^{j,\varepsilon})$. While in $B_{R\varepsilon}(x^{j,\varepsilon})$, we have the estimate $Me^{\sqrt{\alpha}R}\sum_{j=1}^k e^{-\sqrt{\alpha}\frac{|y-x^{j,\varepsilon}|}{\varepsilon}} \geq M \geq u_\varepsilon$. This completes the proof of (5.3). (5.4) is similar to (5.3). □

Corollary 5.1. *If ψ_ε in Theorem 1.1 satisfies $\|\psi_\varepsilon\|_\varepsilon = O\big(\varepsilon^{\frac{N}{2}+1}\big)$, for any fixed $\delta > 0$ enough small, then there exist constants $c > 0$ enough small and $C > 0$ such that $|\psi_\varepsilon| \leq C\sum_{j=1}^k e^{-c\frac{|y-x^{j,\varepsilon}|}{\varepsilon}}$ for all $y \in \mathbb{R}^N$ and $|\nabla\psi_\varepsilon| \leq Ce^{-\frac{c}{\varepsilon}}$ for all $y \in \partial B_\delta(x^j)$, $j = 1, \cdots, k$.*

For using the crucial Pohozaev type identity (5.1), we also need the following result.

Proposition 5.2. *Let $u_\varepsilon = \sum_{j=1}^k U_{\varepsilon,j} + \psi_\varepsilon$ be the solution of (1.1). Then $\|\psi_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{N}{2}+2})$.*

Proof. From Lemma 2.2 and the definition of L_ε, we see that

$$\begin{aligned}\langle L_\varepsilon\psi_\varepsilon, \psi_\varepsilon\rangle_\varepsilon &= \int_{\mathbb{R}^N} \varepsilon^2|\nabla\psi_\varepsilon|^2 + V(y)(\psi_\varepsilon)^2 - g'_\tau(W)(\psi_\varepsilon)^2\\ &= \int_{\mathbb{R}^N} \varepsilon^2|\nabla\psi_\varepsilon|^2 + V(y)(\psi_\varepsilon)^2 - g'(W)(\psi_\varepsilon)^2 + O(e^{-e^{\frac{1}{\varepsilon}}}\int_{\mathbb{R}^N}(\psi_\varepsilon)^2)\\ &= \langle\tilde{L}_\varepsilon\psi_\varepsilon, \psi_\varepsilon\rangle_\varepsilon + O(e^{-e^{\frac{1}{\varepsilon}}}\int_{\mathbb{R}^N}(\psi_\varepsilon)^2).\end{aligned}$$

Note that $\tilde{\gamma}\|\psi_\varepsilon\|_\varepsilon^2 \leq \langle L_\varepsilon\psi_\varepsilon, \psi_\varepsilon\rangle_\varepsilon \leq \langle\tilde{L}_\varepsilon\psi_\varepsilon, \psi_\varepsilon\rangle_\varepsilon$, $\psi_\varepsilon \in E_\varepsilon$. We mainly estimate $\langle\tilde{L}_\varepsilon\psi_\varepsilon, \psi_\varepsilon\rangle_\varepsilon$.

$$\langle\tilde{L}_\varepsilon\psi_\varepsilon, \psi_\varepsilon\rangle_\varepsilon = \int_{\mathbb{R}^N} \tilde{l}_\varepsilon\psi_\varepsilon + \int_{\mathbb{R}^N} \tilde{R}_\varepsilon(\psi_\varepsilon)\psi_\varepsilon,$$

Under the condition (V_2), we obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}^N}(V(y)-V(x^j))U_{\varepsilon,j}\psi_\varepsilon\Big| &\leq C\Big(2^N\varepsilon^N\int_{\mathbb{R}^N}(V(2\varepsilon x+x^j)-V(x^j))^2U_{\varepsilon,j}^2(2\varepsilon x+x^j)\Big)^{\frac{1}{2}}\|\psi_\varepsilon\|_\varepsilon\\ &\leq C\varepsilon^{\frac{N}{2}}\Big(\int_{\mathbb{R}^N}(\varepsilon|\nabla V(x^j)||x|+\varepsilon^2|x|^2)^2U_{\varepsilon,j}^2(2\varepsilon x+x^j)\Big)^{\frac{1}{2}}\|\psi_\varepsilon\|_\varepsilon\\ &\leq C\varepsilon^{\frac{N}{2}+2}\|\psi_\varepsilon\|_\varepsilon.\end{aligned} \tag{5.5}$$

On the other hand, from Lemma 2.2, we have

$$\Big|\int_{\mathbb{R}^N}\Big(g_\tau(W)-\sum_{j=1}^k g(U_{\varepsilon,j})\Big)\psi_\varepsilon\Big| \leq \int_{\mathbb{R}^N}\Big|\sum_{j=1}^k g(U_{\varepsilon,j})-g_\tau(W)\Big||\psi_\varepsilon| \leq Ce^{-e^{\frac{1}{\varepsilon}}}\|\psi_\varepsilon\|_\varepsilon. \tag{5.6}$$

Recalling the definition of $\tilde{l}_\varepsilon$, we see from Lemma 2.2, (5.5), and (5.6) that

$$\langle\tilde{l}_\varepsilon, \psi_\varepsilon\rangle_\varepsilon = \langle l_\varepsilon, \psi_\varepsilon\rangle_\varepsilon + O(e^{-e^{\frac{1}{\varepsilon}}}\int_{\mathbb{R}^N}\psi_\varepsilon) \leq C\varepsilon^{\frac{N}{2}+2}\|\psi_\varepsilon\|_\varepsilon.$$

In D_2, for any $\bar{\theta} \in (0,1)$, we direct calculate

$$\begin{aligned}&\Big|\int_{D_2}(g_\tau(W+\psi_\varepsilon)-g_\tau(W)-g'_\tau(W)\psi_\varepsilon)\psi_\varepsilon\Big|\\ =&\Big|\int_{D_2}g''_\tau(W+\bar{\theta}\psi_\varepsilon)(\psi_\varepsilon)^2\psi_\varepsilon\Big| \leq \int_{D_2}\Big|g''_\tau\Big(e^{-\frac{\sigma_\varepsilon^2}{\varepsilon^2}}\Big)(\psi_\varepsilon)^2\psi_\varepsilon\Big| \leq Ce^{\frac{\sigma_\varepsilon^2}{\varepsilon^2}}\varepsilon\|\psi_\varepsilon\|_\varepsilon^2 \leq C\varepsilon^{\frac{3}{4}}\|\psi_\varepsilon\|_\varepsilon^2.\end{aligned} \tag{5.7}$$

On the other hand, in $\mathbb{R}^N\backslash D_2$, we use the fact that ψ_ε is a solution of system (1.1). Since system (1.1) is equivalent to $\tilde{L}_\varepsilon\psi_\varepsilon = \tilde{l}_\varepsilon + \tilde{R}_\varepsilon(\psi_\varepsilon)$, there holds

$$\int_{\mathbb{R}^N\backslash D_2} \tilde{R}_\varepsilon(\psi_\varepsilon)\psi_\varepsilon = \int_{\mathbb{R}^N\backslash D_2} (\tilde{L}_\varepsilon\psi_\varepsilon - \tilde{l}_\varepsilon)\psi_\varepsilon \leq C\varepsilon^{\frac{N}{2}+2}\|\psi_\varepsilon\|_{\varepsilon,\mathbb{R}^N\backslash D_2}. \tag{5.8}$$

On the other hand, from Lemma 2.2, (5.7) and (5.8), we obtain

$$\langle \tilde{R}_\varepsilon(\psi_\varepsilon), \psi_\varepsilon\rangle_\varepsilon = \langle R_\varepsilon(\psi_\varepsilon), \psi_\varepsilon\rangle_\varepsilon + O(e^{-e^{\frac{1}{\varepsilon}}}\|\psi_\varepsilon\|_\varepsilon) = o(1)\|\psi_\varepsilon\|_\varepsilon^2 + O(e^{-e^{\frac{1}{\varepsilon}}}\|\psi_\varepsilon\|_\varepsilon).$$

Finally, we obtain $\|\psi_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{N}{2}+2})$.

□

From the crucial Pohozaev type identity (5.1) and Proposition 5.2, we have the following known result.

Lemma 5.2. *Let u_ε be the solution of (1.1) concentrating at k, $k \geq 2$, different non-degenerate critical points $\{\xi_1, \cdots, \xi_k\} \subset \mathbb{R}^N$ of $V(y)$. Then it holds*

$$|x^{j,\varepsilon} - \xi_j| = O(\varepsilon), \text{ for } j = 1, \cdots, k. \tag{5.9}$$

Proof. Applying (5.1) to u_ε with $\Omega = B_\delta(x^{j,\varepsilon})$, where $\delta > 0$ enough small, we have

$$\begin{aligned}\int_{B_\delta(x^{j,\varepsilon})} \frac{\partial V(y)}{\partial y_i} u_\varepsilon^2 = &-2\varepsilon^2 \int_{\partial B_\delta(x^{j,\varepsilon})} \frac{\partial u_\varepsilon}{\partial y_i}\frac{\partial u_\varepsilon}{\partial \nu} + \varepsilon^2 \int_{\partial B_\delta(x^{j,\varepsilon})} |\nabla u_\varepsilon|^2\nu_i \\ &- \int_{\partial B_\delta(x^{j,\varepsilon})} u_\varepsilon^2 \log|u_\varepsilon|\nu_i + \int_{\partial B_\delta(x^{j,\varepsilon})} (V(y) + \frac{1}{2})u_\varepsilon^2\nu_i,\end{aligned} \tag{5.10}$$

From Lemma 5.1, we know that $|u_\varepsilon| + |\nabla u_\varepsilon| \leq Ce^{-\frac{c}{\varepsilon}}$ for all $y \in B_\delta(x^{j,\varepsilon})$, $j = 1, \cdots, k$, where $c > 0$ is a constant. Then,

$$u_\varepsilon^2 \log u_\varepsilon \leq Ce^{-\frac{2c}{\varepsilon}}(\frac{c}{\varepsilon} - \log C) = O(e^{-\frac{c}{\varepsilon}}).$$

Therefore, (5.10) equivalent to

$$\int_{B_\delta(x^{j,\varepsilon})} \frac{\partial V(y)}{\partial y_i} u_\varepsilon^2 = O(e^{-\frac{c}{\varepsilon}}),\ i = 1, \cdots, N.$$

On the other hand,

$$\begin{aligned}&\int_{B_\delta(x^{j,\varepsilon})} \Big(\frac{\partial V(y)}{\partial y_i} - \frac{\partial V(x^{j,\varepsilon})}{\partial y_i}\Big)u_\varepsilon^2 \\ &= \int_{B_\delta(x^{j,\varepsilon})} \langle \nabla^2 V(x^{j,\varepsilon}), y - x^{j,\varepsilon}\rangle u_\varepsilon^2 + O(\int_{B_\delta(x^{j,\varepsilon})} \Big(|y - x^{j,\varepsilon}|^2\Big)u_\varepsilon^2) \\ &= \int_{B_\delta(x^{j,\varepsilon})} \langle \nabla^2 V(x^{j,\varepsilon}), y - x^{j,\varepsilon}\rangle (U_{\varepsilon,j}^2 + 2U_{\varepsilon,j}\psi_\varepsilon + (\psi_\varepsilon)^2) + O(e^{-\frac{c}{\varepsilon}} + \varepsilon^{N+1})\end{aligned}$$

Now, by the symmetry of $U_{\varepsilon,j}$, we have $\int_{B_\delta(x^{j,\varepsilon})} \langle \nabla^2 V(x^{j,\varepsilon}), y - x^{j,\varepsilon}\rangle U_{\varepsilon,j}^2 = 0$, $i = 1, \cdots, N$. By Hölder's inequality, we have

$$\int_{B_\delta(x^{j,\varepsilon})} \langle \nabla^2 V(x^{j,\varepsilon}), y - x^{j,\varepsilon}\rangle (2U_{\varepsilon,j}\psi_\varepsilon + (\psi_\varepsilon)^2) = o(\varepsilon^{N+1}).$$

Therefore, $\int_{B_\delta(x^{j,\varepsilon})}\frac{\partial V(y)}{\partial y_i}u_\varepsilon^2=o(\varepsilon^{N+1})$. Then, for $i=1,\cdots,N$,

$$\int_{B_\delta(x^{j,\varepsilon})}\langle\frac{\partial^2 V(\xi_j)}{\partial x_i\partial x_l},y_l-x^{j,\varepsilon,l}\rangle u_\varepsilon^2=o(\varepsilon^{N+1}).$$

So, combining the condition (V_2) and $\int_{B_\delta(x^{j,\varepsilon})}u_\varepsilon^2=O(\varepsilon^N)$, we obtain (5.9). □

5.3. **Non-degeneracy result.** Let u_ε be a solution of (1.1) constructed in Theorem 1.1. Suppose that there exist $\varepsilon_m\to 0$, satisfying $\eta_{\varepsilon_m}\in H^1(\mathbb{R}^N)$, $\|\eta_{\varepsilon_m}\|_{L^\infty(\mathbb{R}^N)}=1$, and $\mathscr{L}_{\varepsilon_m}\eta_{\varepsilon_m}=0$. Let $\eta_{\varepsilon_m,j}(x)=\eta_{\varepsilon_m}(2\varepsilon_m x+x^{j,\varepsilon_m})$. Now we study the asymptotic behavior of $\eta_{\varepsilon_m,j}(x)$.

Lemma 5.3. *It holds $\eta_{\varepsilon_m,j}(x)\to\sum_{i=1}^N a_{j,i}\frac{\partial U_j}{\partial x_i}$, uniformly in $C^1(B_R(0))$ for any $R>0$, where $a_{j,i}$ are some constants.*

Proof. In view of $|\eta_{\varepsilon_m,j}|\le C$, we may assume that $\eta_{\varepsilon_m,j}\to\eta_j$ in $C_{loc}(\mathbb{R}^N)$. Then η_j satisfies

$$-\Delta\eta_j+V(\xi_j)\eta_j=(1+\log U_j)\eta_j,\ \text{in } \mathbb{R}^N,$$

which implies $\eta_j=\sum_{i=1}^N a_{j,i}\frac{\partial U_j}{\partial x_i}$. □

We decompose

$$\eta_{\varepsilon_m,j}(x)=\sum_{i=1}^N a_{m,j,i}\frac{\partial U_j}{\partial x_i}+\eta^*_{\varepsilon_m,j}.$$

As in the proof of Lemma 5.1, it is standard to obtain the following two lemmas.

Lemma 5.4. *There are constants $c>0$ and $\delta>0$ enough small, such that $|\eta_\varepsilon|+|\nabla\eta_\varepsilon|=O(e^{-\frac{c}{\varepsilon}})$ for all $y\in\partial B_\delta(x^j)$, $j=1,\cdots,k$.*

Lemma 5.5. *There exist $\varepsilon_0>0$ and $\delta>0$ enough small, $R>0$ enough big and $c>0$ enough small, for any $\varepsilon\in(0,\varepsilon_0)$, such that $|\eta^*_{\varepsilon_m,j}|+|\nabla\eta^*_{\varepsilon_m,j}|=O(e^{-\frac{c}{\varepsilon}})$ for all $y\in\partial B_\delta(x^j)$, $j=1,\cdots,k$.*

For using the crucial Pohozaev type identity (5.2), we also need the following result.

Proposition 5.3. *For η_ε satisfying $\mathscr{L}_\varepsilon(\eta)=0$, it holds $\|\eta_\varepsilon\|_\varepsilon=O(\varepsilon^{\frac{N}{2}})$.*

Proof. From $-\varepsilon^2\Delta\eta(y)+V(y)\eta-(1+\log u_\varepsilon)\eta(y)=0$, we have

$$\|\eta_\varepsilon\|_\varepsilon^2=\int_{\mathbb{R}^N}(1+\log u_\varepsilon)(\eta_\varepsilon)^2=\int_{\mathbb{R}^N}g'_\tau(u_\varepsilon)(\eta_\varepsilon)^2+O(e^{-e^{\frac{1}{\varepsilon}}}\int_{\mathbb{R}^N}(\eta_\varepsilon)^2).$$

Since $|\eta_\varepsilon|\le 1$ and $|\psi_\varepsilon|<\varepsilon$, then

$$\begin{aligned}\Big|\int_{D_2}g''_\tau(W+\bar\theta\psi_\varepsilon)(\psi_\varepsilon)^2(\eta_\varepsilon)^2\Big|\le&\int_{D_2}\Big|g''_\tau\Big(e^{-\frac{\sigma_\varepsilon^2}{\varepsilon^2}}\Big)(\psi_\varepsilon)^2(\eta_\varepsilon)^2\Big|\\ \le&\frac{1}{2}\int_{D_2}\Big[\frac{1}{\varepsilon^2}\Big(g''_\tau\Big(e^{-\frac{\sigma_\varepsilon^2}{\varepsilon^2}}\Big)\Big)^2(\psi_\varepsilon)^4+\varepsilon^2(\eta_\varepsilon)^4\Big]\\ \le&Ce^{\frac{2\sigma_\varepsilon^2}{\varepsilon^2}}\varepsilon^{N+4}+\varepsilon^2\int_{D_2}(\eta_\varepsilon)^2\le C\varepsilon^N+\varepsilon^2\|\eta_\varepsilon\|^2_{\varepsilon,D_2}.\end{aligned}$$

For ε enough small, for $y\in B_{2\sigma_\varepsilon}(x^j)$, we have $W(y)\le 2e^{\frac{N}{2}}$. For $y\in B_{2\sigma_\varepsilon}(x^j)$, there holds

$$g'_\tau(W)=\tau^{-1}((1+\tau)|W|^\tau-1)\le V(x^j)+\frac{N}{2}+\log 2+2.$$

By Cauchy's inequality and the fact $|\eta_\varepsilon|\le 1$, we see that

$$\begin{aligned}\Big|\int_{D_2} g'_\tau(W)\psi_\varepsilon(\eta_\varepsilon)^2\Big| \le&\frac{1}{2}\int_{D_2}\Big[\frac{1}{\varepsilon^2}\big(g'_\tau(W)\big)^2\big(\psi_\varepsilon\big)^2+\varepsilon^2(\eta_\varepsilon)^4\Big]\\ \le& C\varepsilon^{N+2}+\varepsilon^2\int_{D_2}(\eta_\varepsilon)^2\le C\varepsilon^N+\varepsilon^2\|\eta_\varepsilon\|^2_{\varepsilon,D_2}.\end{aligned}$$

Therefore, for any $\bar\theta\in(0,1)$, it holds

$$\begin{aligned}\int_{D_2} g'_\tau(u_\varepsilon)(\eta_\varepsilon)^2 =& \int_{D_2}\Big(g'_\tau(W+\psi_\varepsilon)(\eta_\varepsilon)^2-g'_\tau(W)\psi_\varepsilon(\eta_\varepsilon)^2\Big)+\int_{D_2} g'_\tau(W)\psi_\varepsilon(\eta_\varepsilon)^2\\ =&\int_{D_2} g''_\tau(W+\bar\theta\psi_\varepsilon)(\psi_\varepsilon)^2(\eta_\varepsilon)^2+\int_{D_2} g'_\tau(W)\psi_\varepsilon(\eta_\varepsilon)^2\le C\varepsilon^N+\varepsilon^2\|\eta_\varepsilon\|^2_{\varepsilon,D_2}.\end{aligned}$$

Therefore,

$$\|\eta_\varepsilon\|^2_{\varepsilon,D_2}=\int_{D_2} g'_\tau(u_\varepsilon)(\eta_\varepsilon)^2+O(e^{-e^{\frac{1}{\varepsilon}}}\int_{D_2}(\eta_\varepsilon)^2)\le C\varepsilon^N+\varepsilon^2\|\eta_\varepsilon\|^2_{\varepsilon,D_2}. \tag{5.11}$$

By Cauchy's inequality and the fact $|\eta_\varepsilon|\le 1$, we see that

$$\begin{aligned}\int_{\mathbb{R}^N\setminus D_2} g'(u_\varepsilon)(\eta_\varepsilon)^2 <& C\int_{\mathbb{R}^N\setminus D_2} g'(u_\varepsilon)e^{-\frac{\sigma^2_\varepsilon}{\varepsilon^2}}\|\eta\|_*\eta_\varepsilon\\ <& C\int_{\mathbb{R}^N\setminus D_2} e^{-\frac{c}{\varepsilon}}\|\eta\|_*\eta_\varepsilon\\ \le& C\varepsilon^N+\varepsilon^2\|\eta_\varepsilon\|^2_{\varepsilon,\mathbb{R}^N\setminus D_2}.\end{aligned}$$

So, $\|\eta_\varepsilon\|^2_{\varepsilon,\mathbb{R}^N\setminus D_2}\le C\varepsilon^N+\varepsilon^2\|\eta_\varepsilon\|^2_{\varepsilon,\mathbb{R}^N\setminus D_2}$, which together with (5.11) finishes the proof. □

To see a contradiction, we also need the following crucial result.

Proposition 5.4. *Let $a_{j,i}$ be defined as in Lemma 5.3. Then $a_{j,i}=0$ for $j=1,\cdots,k$, $i=1,\cdots,N$.*

Proof. Applying (5.2) to u_ε with $\Omega=B_\delta(x^{j,\varepsilon})$, where $\delta>0$ enough small, we have

$$\begin{aligned}&\int_{B_\delta(x^{j,\varepsilon})}\frac{\partial V(y)}{\partial y_i}u_\varepsilon\eta_\varepsilon\\ =&-\varepsilon^2\int_{\partial B_\delta(x^{j,\varepsilon})}(\frac{\partial u_\varepsilon}{\partial\nu}+\frac{\partial\eta_\varepsilon}{\partial y_i})+\int_{\partial B_\delta(x^{j,\varepsilon})}(\varepsilon^2\langle\nabla u_\varepsilon,\nabla\eta_\varepsilon\rangle+V(y)u_\varepsilon\eta_\varepsilon)\nu_i\\ &-\int_{\partial B_\delta(x^{j,\varepsilon})}\log|u_\varepsilon|u_\varepsilon\eta_\varepsilon\nu_i,\end{aligned} \tag{5.12}$$

where $\nu=(\nu_1,\cdots,\nu_2)$ is the unit outward normal of $\partial B_\delta(x^{j,\varepsilon})$, $i=1,\cdots,N$. From Lemma 5.1, we see that

$$u_\varepsilon\log u_\varepsilon\le Ce^{-\frac{c}{\varepsilon}}\Big(\frac{c}{\varepsilon}-\log C\Big)=O(e^{-\frac{c}{\varepsilon}}).$$

In view of $|\eta_\varepsilon|\le 1$, one has $\int_{\partial B_\delta(x^{j,\varepsilon})}\log|u_\varepsilon|u_\varepsilon\eta_\varepsilon\nu_i=O(e^{-\frac{c}{\varepsilon}})$. From Lemmas 5.1 and 5.4, we have

$$\int_{\partial B_\delta(x^{j,\varepsilon})}(\varepsilon^2\langle\nabla u_\varepsilon,\nabla\eta_\varepsilon\rangle+V(y)u_\varepsilon\eta_\varepsilon)\nu_i=O(e^{-\frac{c}{\varepsilon}}),$$

and

$$\varepsilon^2\int_{\partial B_\delta(x^{j,\varepsilon})}(\frac{\partial u_\varepsilon}{\partial\nu}+\frac{\partial\eta_\varepsilon}{\partial y_i})=O(e^{-\frac{c}{\varepsilon}}).$$

So, (5.12) is equivalent to $\int_{B_\delta(x^{j,\varepsilon})}\frac{\partial V(y)}{\partial y_i}u_\varepsilon\eta_\varepsilon=O(e^{-\frac{c}{\varepsilon}})$, $i=1,\cdots,N$. As a result,

$$\int_{B_{\frac{\delta}{2\varepsilon}}(0)}\frac{\partial V(2\varepsilon x+x^{j,\varepsilon})}{\partial x_i}u_\varepsilon(2\varepsilon x+x^{j,\varepsilon})\eta_\varepsilon(2\varepsilon x+x^{j,\varepsilon})=O(\varepsilon^{-N}e^{-\frac{c}{\varepsilon}}),\ i=1,\cdots,N.$$

Then

$$\int_{B_{\frac{\delta}{2\varepsilon}}(0)}\Big(\frac{\partial V(2\varepsilon x+x^{j,\varepsilon})}{\partial x_i}-\frac{\partial V(x^{j,\varepsilon})}{\partial x_i}u_\varepsilon\Big)u_\varepsilon(2\varepsilon x+x^{j,\varepsilon})\eta_\varepsilon(2\varepsilon x+x^{j,\varepsilon})=O(\varepsilon^{-N}e^{-\frac{c}{\varepsilon}}),$$

which also implies that

$$\varepsilon\sum_{l=1}^{N}\int_{B_{\frac{\delta}{2\varepsilon}}(0)}\Big(\frac{\partial^2 V(x^{j,\varepsilon})}{\partial x_i\partial x_l}\Big)x_lu_\varepsilon(2\varepsilon x+x^{j,\varepsilon})\eta_\varepsilon(2\varepsilon x+x^{j,\varepsilon})=o(\varepsilon).\tag{5.13}$$

Letting $\varepsilon\to 0$ in (5.13), one sees that

$$\sum_{l=1}^{N}\int_{B_R(0)}\Big(\frac{\partial^2 V(y)}{\partial x_i\partial x_l}|_{y=\xi_j}\Big)x_lU_j\sum_{i=1}^{K}a_{j,i}\frac{\partial U_j}{\partial x_i}=\sum_{i=1}^{K}\sum_{l=1}^{N}\Big(\frac{\partial^2 V(y)}{\partial x_i\partial x_l}|_{y=\xi_j}\Big)a_{j,i}\int_{\mathbb{R}^N}x_lU_j\frac{\partial U_j}{\partial x_i}=0,\tag{5.14}$$

but

$$\int_{\mathbb{R}^N}x_lU_j\frac{\partial U_j}{\partial x_i}=\frac{1}{2}\int_{\mathbb{R}^N}x_l\frac{\partial^2 U_j}{\partial x_i}=-\frac{1}{2}\int_{\mathbb{R}^N}U_j^2<0.$$

We obtain from (5.14) that

$$\Big(\frac{\partial^2 V(y)}{\partial x_i\partial x_l}|_{y=\xi_j}\Big)_{N\times N}(a_{j,i})^{\mathsf{T}}=(0,\cdots,0)^{\mathsf{T}},\ j=1,\cdots,k,$$

where $a_{j,i}=(a_{j,i,1},\cdots,a_{j,i,N})$. By the non-degeneracy of ξ_j, we conclude that $a_{j,i}=0, j=1,\cdots,k,\ i=1,\cdots,N$. $\square$

Proof of Theorem 1.5. In conclusion, we have proved $\eta_\varepsilon=o(1)$ in $B_{R\varepsilon}(x^{j,\varepsilon}), j=1,\cdots,k$, which, together with Lemma 5.4, gives $\|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^N)}=o(1)$. This is a contradiction to $\|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^N)}=1$. $\square$

Appendix A. Energy Expansions

In this section, we expand $I(W)$, where

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}(\varepsilon^2|\nabla u|^2+V(y)u^2)dy-\int_{\mathbb{R}^N}G_\tau(u)dy,\ u\in H^1(\mathbb{R}^N).$$

Proposition A.1. *It holds following estimate*

$$\begin{aligned}I(W)=&\frac{1}{4}\sum_{m=1}^{k}\varepsilon^N(2\pi)^{\frac{N}{2}}e^{2V(x^m)+N}-\sum_{j\neq m}\varepsilon^N(2\pi)^{\frac{N}{2}}e^{V(x^m)+V(x^j)+N}e^{-\frac{|x^j-x^m|^2}{16\varepsilon^2}}\\&+\varepsilon^NO\Big(\sum_{j\neq m}e^{-\frac{(1-2\sqrt{\varepsilon})^2|x^j-x^m|^2}{8\varepsilon^2}}+\varepsilon\sum_{m=1}^{k}\nabla V(x^m)\Big).\end{aligned}$$

Proof. We note that

$$\begin{aligned} I(W) =&\frac{1}{2}\int_{\mathbb{R}^N}(\varepsilon^2|\nabla W|^2+V(y)W^2)-\int_{\mathbb{R}^N}G_\tau(W)\\ =&\frac{1}{2}\int_{\mathbb{R}^N}\Big(\sum_{j=1}^k g(U_{\varepsilon,j})-g_\tau(W)\Big)W+\frac{1}{2}\int_{\mathbb{R}^N}\Big(\sum_{j=1}^k (V(y)-V(x^j))U_{\varepsilon,j}\Big)W \\ &+\frac{1}{2(2+\tau)}\int_{\mathbb{R}^N}W^{\tau+2}. \end{aligned} \tag{A.1}$$

Recall that $d:=\min_{m\neq j}|x^m-x^j|, m,j=1,\cdots,k$. When $j\neq m$, for any $y\in B_{2\sqrt{\varepsilon}d}(x^m)$, there holds

$$\begin{aligned} &\int_{B_{2\sqrt{\varepsilon}d}(x^m)}\Big(g(W)-\sum_{j=1}^k g(U_{\varepsilon,j})\Big)W\\ \leq&\int_{B_{2\sqrt{\varepsilon}d}(x^m)}U_{\varepsilon,m}\sum_{j\neq m}U_{\varepsilon,j}+4\Big(\sum_{j\neq m}U_{\varepsilon,j}\Big)^2+3U_{\varepsilon,m}^{-1}\Big(\sum_{j\neq m}U_{\varepsilon,j}\Big)^3+\sum_{j\neq m}U_{\varepsilon,m}U_{\varepsilon,j}\log(U_{\varepsilon,m}U_{\varepsilon,j}^{-1})\\ &+\Big(\sum_{j\neq m}U_{\varepsilon,j}\Big)\Big(\sum_{j\neq m}U_{\varepsilon,j}\log(U_{\varepsilon,m}U_{\varepsilon,j}^{-1})\Big)\\ =&\sum_{j\neq m}\varepsilon^N(2\pi)^{\frac{N}{2}}e^{V(x^m)+V(x^j)+N}e^{-\frac{d^2}{16\varepsilon^2}}+\varepsilon^N O\Big(e^{-\frac{d^2}{8\varepsilon^2}}\Big). \end{aligned}$$

Hence,

$$\int_{\cup_{m=1}^k B_{2\sqrt{\varepsilon}d}(x^m)}\Big(g(W)-\sum_{j=1}^k g(U_{\varepsilon,j})\Big)W=\sum_{j\neq m}\varepsilon^N(2\pi)^{\frac{N}{2}}e^{V(x^m)+V(x^j)+N}e^{-\frac{d^2}{16\varepsilon^2}}+\varepsilon^N O\Big(e^{-\frac{d^2}{8\varepsilon^2}}\Big). \tag{A.2}$$

From (A.2) and Lemma 2.2, we have

$$\begin{aligned} &\int_{\mathbb{R}^N}\Big(\sum_{j=1}^k g(U_{\varepsilon,j})-g_\tau(W)\Big)W\\ =&-\int_{\mathbb{R}^N}\Big(g(W)-\sum_{j=1}^k g(U_{\varepsilon,j})\Big)W+\varepsilon^N O(e^{-e^{\frac{1}{\varepsilon}}})\\ =&-\int_{\cup_{m=1}^k B_{2\sqrt{\varepsilon}d}(x^m)}\Big(g(W)-\sum_{j=1}^k g(U_{\varepsilon,j})\Big)W\\ &-\int_{\mathbb{R}^N\setminus\cup_{m=1}^k B_{2\sqrt{\varepsilon}d}(x^m)}\Big(g(W)-\sum_{j=1}^k g(U_{\varepsilon,j})\Big)W+\varepsilon^N O(e^{-e^{\frac{1}{\varepsilon}}})\\ =&-\sum_{j\neq m}\varepsilon^N(2\pi)^{\frac{N}{2}}e^{2V(x_0)+N}e^{-\frac{d^2}{16\varepsilon^2}}+\varepsilon^N O\Big(e^{-\frac{(1-2\sqrt{\varepsilon})^2d^2}{8\varepsilon^2}}\Big). \end{aligned} \tag{A.3}$$

When $j \neq m$, for any $y \in B_{2\sqrt{\varepsilon}d}(x^m)$, we obtain

$$\begin{aligned}
&\int_{B_{2\sqrt{\varepsilon}d}(x^m)} \big(V(y)-V(x^m)\big)U_{\varepsilon,m}W \\
=&\int_{B_{2\sqrt{\varepsilon}d}(x^m)} \big(V(y)-V(x^m)\big)U_{\varepsilon,m}^2 + \int_{B_{2\sqrt{\varepsilon}d}(x^m)} \big(V(y)-V(x^m)\big)U_{\varepsilon,m}\sum_{j\neq m} U_{\varepsilon,j} \\
&+\int_{B_{2\sqrt{\varepsilon}d}(x^m)} \big(V(y)-V(x^m)\big)\sum_{j\neq m\neq l} U_{\varepsilon,j}U_{\varepsilon,l} \\
\leq & C\varepsilon^N \int_{B_{\frac{d}{\sqrt{\varepsilon}}}(0)} \big(V(2\varepsilon x+x^m)-V(x^m)\big)e^{2V(x^m)+N}e^{-2|x|^2} \\
&+\sum_{j\neq m}\varepsilon^N \int_{B_{\frac{d}{\sqrt{\varepsilon}}}(0)} \big(V(2\varepsilon x+x^m)-V(x^m)\big)e^{2V(x_0)+N}e^{-\frac{3}{2}|x|^2}e^{-\frac{|x^m-x^j|^2}{8\varepsilon^2}} \\
&+\sum_{j\neq m\neq l}\varepsilon^N \int_{B_{\frac{d}{\sqrt{\varepsilon}}}(0)} \big(V(2\varepsilon x+x^m)-V(x^m)\big)e^{2V(x_0)+N}e^{-\frac{3}{2}|x|^2}e^{-\frac{|x^j-x^l|^2}{8\varepsilon^2}} \\
=&\varepsilon^{N+1}O\Big(\nabla V(x^m)+\varepsilon\Big).
\end{aligned}$$

Hence,

$$\int_{\cup_{m=1}^k B_{2\sqrt{\varepsilon}d}(x^m)} \sum_{m=1}^k \big(V(y)-V(x^m)\big)U_{\varepsilon,m}W = \varepsilon^{N+1}O\Big(\sum_{m=1}^k \nabla V(x^m)+\varepsilon\Big). \tag{A.4}$$

From (A.4), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \sum_{j=1}^k \big(V(y)-V(x^j)\big)U_{\varepsilon,j}W \\
=&\int_{\cup_{m=1}^k B_{2\sqrt{\varepsilon}d}(x^m)} \sum_{m=1}^k (V(y)-V(x^m))U_{\varepsilon,m}W + \int_{\mathbb{R}^N \setminus \cup_{m=1}^k B_{2\sqrt{\varepsilon}d}(x^m)} \sum_{m=1}^k \big(V(y)-V(x^m)\big)U_{\varepsilon,m}W \\
=&\int_{\cup_{m=1}^k B_{2\sqrt{\varepsilon}d}(x^m)} \sum_{j=m}^k (V(y)-V(x^m))U_{\varepsilon,m}W + \varepsilon^{N+1}e^{-\frac{d^2}{8\varepsilon^2}}O(\nabla V(x^m)+\varepsilon) \\
=&\varepsilon^{N+1}O(\sum_{m=1}^k \nabla V(x^m)+\varepsilon).
\end{aligned} \tag{A.5}$$

Finally, due to the choice of τ_{ε_n}, there holds

$$\begin{aligned}
\frac{1}{2(2+\tau)}\int_{B_{2\sqrt{\varepsilon}d}(x^m)} W^{\tau+2} =&\frac{1}{4}\Big(1+O\Big(e^{-e^{\frac{1}{\varepsilon}}}\Big)\Big)\int_{B_{2\sqrt{\varepsilon}d}(x^m)} \Big(U_{\varepsilon,m}+\sum_{j\neq m} U_{\varepsilon,m}\Big)^2 \\
=&\frac{1}{4}\int_{B_{2\sqrt{\varepsilon}d}(x^m)} U_{\varepsilon,m}^2 + \varepsilon^N O\Big(e^{-\frac{d^2}{8\varepsilon^2}}\Big) \\
=&\frac{1}{4}\varepsilon^N(2\pi)^{\frac{N}{2}}e^{2V(x^m)+N} + \varepsilon^N O\Big(e^{-\frac{d^2}{8\varepsilon^2}}\Big).
\end{aligned}$$

Thus,

$$\frac{1}{2(2+\tau)}\int_{\cup_{m=1}^{k}B_{2\sqrt{\varepsilon}d}(x^m)}W^{\tau+2}=\frac{1}{4}\sum_{i=1}^{m}\varepsilon^N(2\pi)^{\frac{N}{2}}e^{2V(x^m)+N}+\varepsilon^N O\Big(e^{-\frac{d^2}{8\varepsilon^2}}\Big). \tag{A.6}$$

As a result,

$$\begin{aligned}\frac{1}{2(2+\tau)}\int_{\mathbb{R}^N}W^{\tau+2}=&\frac{1}{2(2+\tau)}\int_{\cup_{m=1}^{k}B_{2\sqrt{\varepsilon}d}(x^m)}W^{\tau+2}+\frac{1}{2(2+\tau)}\int_{\mathbb{R}^N``\cup_{m=1}^{k}B_{2\sqrt{\varepsilon}d}(x^m)}W^{\tau+2}\\=&\frac{1}{2(2+\tau)}\int_{\cup_{m=1}^{k}B_{2\sqrt{\varepsilon}d}(x^m)}W^{\tau+2}+\varepsilon^N O\Big(e^{-\frac{(1-2\sqrt{\varepsilon})^2d^2}{8\varepsilon^2}}\Big)\\=&\frac{1}{4}\sum_{i=1}^{m}\varepsilon^N(2\pi)^{\frac{N}{2}}e^{2V(x^m)+N}+\varepsilon^N O\Big(e^{-\frac{(1-2\sqrt{\varepsilon})^2d^2}{8\varepsilon^2}}\Big).\end{aligned} \tag{A.7}$$

Thus, the result of Proposition A.1 follows from (A.3), (A.5), and (A.7). □

Acknowledgments
The author is grateful to the reviewers for useful suggestions which improved the contents of this paper. Qing Guo has been supported by the National Natural Science Foundation of China (Grant No. 12271539), and this research was also supported by the China Scholarship Council (CSC) during the visit of the third author to Sapienza University of Rome.

REFERENCES

[1] A.H. Ardila, Existence and stability of standing waves for nonlinear fractional Schrödinger equation with logarithmic nonlinearity, Nonlinear Anal. 155 (2107) 52-64.
[2] A.H. Ardila, Orbital stability of Gausson solutions to logarithmic Schrödinger equations, Electron. J. Differential Equations, 335 (2016) 1-9.
[3] I. Bialynicki-Birula, J. Mycielski, Nonlinear wave mechanics, Ann. Phys. 100 (1976) 62-93.
[4] H. Buljan, A. Siber, M. Soljacic, T. Schwartz, M. Segev, D.N. Christodoulides, Incoherent white light solitons in logarithmically saturable noninstantaneous nonlinear media, Phys. Rev. E 68 (2003) 036607.
[5] T. Cazenave, Stable solutions of the logarithmic Schrödinger equation, Nonlinear. Anal. 7 (1983) 1127-1140.
[6] T. Cazenave, A. Haraux, Equations dévolution avec non-linéarité logarithmique, Ann. Fac. Sci. Toulouse Math. 2 (1980) 21-51.
[7] T. Cazenave, P.L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, Comm. Math. Phys. 85 (1982) 549-561.
[8] P. D'Avenia, E. Montefusco, M. Squassina, On the logarithmic Schrödinger equation, Commun. Contemp. Math. 16 (2014) 1350032.
[9] S. De Martino, M. Falanga, C. Godano, G. Lauro, Logarithmic Schrödinger-like equation as a model for magma transport, Europhys Lett. 63 (2003) 472-475.
[10] Y. Guo, M. Musso, S. Peng, S. Yan, Non-degeneracy of multi-bubbling solutions for the prescribed scalar curvature equations and applications, J. Funct. Anal. (2020) 108553.
[11] Q. Guo, L. Zhao, Non-degeneracy of synchronized vector solutions for weakly coupled nonlinear Schrödiner systems, Proc. Edinb. Math. Soc. 65 (2022) 441-459.
[12] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, springer, 2001.
[13] E.F. Hefter, Application of the nonlinear Schrödinger equation with a logarithmic inhomogeneous term to nuclear physics, Phys. Rev. 32 (1985), 1201-1204.
[14] W.C. Troy, Uniqueness of positive ground state solutions of the logarithmic Schrödinger equation, Arch. Ration. Mech. Anal. 222 (2016) 1581-1600.

[15] J.L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12 (1984) 191-202.
[16] Z.Q. Wang, C. Zhang, Z. Zhang, Multiple bump solutions to logarithmic scalar field equations, Adv. Differential Equations 28 (2023) 981-1036.
[17] Z.Q. Wang, C. Zhang, Convergence from power-law to logarithm-law in nonlinear scalar field equations, Arch. Ration. Mech. Anal. 231 (2019) 45-61.
[18] K.G. Zloshchastiev, pontaneous symmetry breaking and mass generation as built-in phenomena in logarithmic nonlinear quantum theory, Acta Physica Polonica. B 42 (2011) 261-292.